Asymptotic behavior of angular integrals in the massless limit

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We investigate the small-mass asymptotics of a class of massive d-dimensional angular integrals. These integrals arise in a wide range of perturbative quantum field theory calculations. We derive expressions characterizing their behavior in the vicinity of the massless limit for all cases with up to two denominators. The results established in this work are applicable to phase-space calculations where an integration over virtuality including the massless limit is required.

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I. INTRODUCTION

Angular integrals [1–5] are ubiquitous to phase-space calculations in perturbative quantum field theory [6–20]. Examples from QCD include theoretical predictions for the Drell-Yan (DY) process [12,21–24], deep-inelastic scattering (DIS) [25,26], semi-inclusive deep-inelastic scattering (SIDIS) [27,28], prompt-photon production [29], hadron-hadron scattering [30], heavy-quark production [3], and single-spin asymmetries [31,32].

When massless particles are present, the angular integration contains collinear singularities. To regularize these divergencies, the calculations are performed in $d = 4 - 2\varepsilon$ dimensions [33,34].

Following the notation from Refs. [4,5] we define the angular integral with two denominators as

$$I^{(m)}_{j_1,j_2}(v_{12}, v_{11}, v_{22}; \varepsilon) \equiv \int \mathrm{d}\Omega_k \frac{1}{(v_1 \cdot k)^{j_1} (v_2 \cdot k)^{j_2}} \qquad (1)$$

with normalized d-vectors

$$k = (1, \ldots, \sin\theta_1 \cos\theta_2, \cos\theta_1),$$
$$v_1 = (1, \mathbf{0}_{d-2}, \beta_1),$$
$$v_2 = (1, \mathbf{0}_{d-3}, \beta_2 \sin\chi, \beta_2 \cos\chi),$$

kinematic invariants $v_{ij} = v_i \cdot v_j$, and integration measure $\mathrm{d}\Omega_k = \mathrm{d}\theta_1 \sin^{1-2\varepsilon}\theta_1 \mathrm{d}\theta_2 \sin^{-2\varepsilon}\theta_2$. The denominator powers j_1, j_2 are assumed to be integers in the following. The superscript $m = 0$, 1, 2 characterizes the number of nonzero *masses* v_{11}, v_{22}. For convenience, zero indices and masses will be dropped from the notation, i.e., we will write for example $I^{(1)}_{j_1}(v_{11}; \varepsilon)$ instead of $I^{(1)}_{j_1,0}(v_{12}, v_{11}, v_{22}; \varepsilon)$ and $I^{(0)}_{j_1,j_2}(v_{12}; \varepsilon)$ instead of $I^{(0)}_{j_1,j_2}(v_{12}, 0, 0; \varepsilon)$. By partial fraction decomposition a wide range of phase-space integrals can be cast into the form of Eq. (1) [5,25].

In this manuscript we investigate the asymptotic behavior of integrals of the form $I^{(m)}_{j_1,j_2}(v_{12}, v_{11}, v_{22}; \varepsilon)$ in the limit of one or both masses going to zero. In principle, the expansion of all two-denominator angular integrals with integer powers j_1, j_2 is known to all orders in the dimensional regularization parameter ε [3–5]. However, these expansions are not always sufficient, since in general the ε-expansion does not commute with the massless limit.

As an illustration of the potential issue occurring in the massless limit, let us look at the double-massive angular integral with $j_1 = j_2 = 1$. It has the well-known ε-expansion [1,3–5],

$$I^{(2)}_{1,1}(v_{12}, v_{11}, v_{22}; \varepsilon) = \frac{\pi}{\sqrt{X}} \log\left(\frac{v_{12} + \sqrt{X}}{v_{12} - \sqrt{X}}\right) + \mathcal{O}(\varepsilon), \qquad (2)$$

with $X = v_{12}^2 - v_{11} v_{22}$.

One readily sees that the massless limit $v_{11} \to 0$ is ill-defined at the level of the ε-expansion, since $v_{12} - \sqrt{X}$ approaches zero. This is a problem if we were to consider an integral of the form,

$$\int_0^{v_{11}^{\max}} \mathrm{d}v_{11} v_{11}^{-1-\varepsilon} I^{(2)}_{1,1}(v_{12}, v_{11}, v_{22}; \varepsilon). \qquad (3)$$

Here, we would like to replace $I^{(2)}_{1,1}(v_{12}, v_{11}, v_{22}; \varepsilon)$ by its ε-expansion under the integral and employ the distributional identity [35–37],

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$$v_{11}^{-1-n\varepsilon} = -\frac{1}{n\varepsilon}\delta(v_{11}) + \sum_{n=0}^{\infty}\frac{(-n\varepsilon)^n}{n!}\left[\frac{\log^n v_{11}}{v_{11}}\right]_+, \quad (4)$$

on $v_{11}^{-1-\varepsilon}$. However, we cannot use the form of Eq. (2) due to its divergence in the $v_{11} \to 0$ limit. Instead, to properly perform the integration one has to extract the asymptotic behavior of $I_{1,1}^{(2)}(v_{12}, v_{11}, v_{22}; \varepsilon)$ near $v_{11} = 0$ beforehand, resulting in additional powers of $v_{11}^{-\varepsilon}$ entering Eq. (4). It is these $v_{11}^{-\varepsilon}$ terms which spoil the commutativity of the ε-expansion with the massless limit.

The aim of this work is to provide ε-expansions for all massive angular integrals with up to two propagators, where the asymptotic behavior in the massless limit is manifest and which are hence suitable for usage within integrals of the form (3).

Using recursion relations derived from integration-by-parts (IBP) identities, the powers j_1 and j_2 can always be reduced to the cases $j_{1,2} = 0$ or 1. The explicit form of the required recursion relations can be obtained from Sec. 3.3.4 of [5]. Hence, it suffices to consider the master integrals $I_1^{(1)}(v_{11}; \varepsilon)$, $I_{1,1}^{(1)}(v_{12}, v_{11}; \varepsilon)$, and $I_{1,1}^{(2)}(v_{12}, v_{11}, v_{22}; \varepsilon)$.

The remainder of this manuscript is organized as follows. In Sec. II. we recall the two-point splitting lemma which we subsequently use to establish the asymptotic behavior of the master integrals in the massless limits v_{11}, $v_{22} \to 0$. Section III. concludes the paper.

II. ASYMPTOTIC BEHAVIOR IN THE MASSLESS LIMIT

The main tool for the extraction of the asymptotic behavior will be the two-point splitting lemma [5]. Using the notation

$$\Delta_k(v_i, v_j) \equiv \frac{1}{v_i \cdot k v_j \cdot k}, \quad (5)$$

it states that for any two vectors v_1 and v_2, we can choose any scalar λ and construct the linear combination $v_3 = (1-\lambda)v_1 + \lambda v_2$ to obtain the identity,

$$\Delta_k(v_1, v_2) = \lambda\Delta_k(v_1, v_3) + (1-\lambda)\Delta_k(v_2, v_3). \quad (6)$$

This allows us to express a given angular integral in terms of other angular integrals where a new auxiliary vector v_3 has been inserted. By choosing appropriate values for λ, the vector v_3 can be given desirable properties, most importantly being massless. This idea has been fruitfully employed in reference [5] for the calculation of the all-order ε-expansion of the double-massive integral.

A. Asymptotic form of the massive one-denominator integral

We start with the investigation of the massive one-denominator master integral

$$I_1^{(1)}(v_{11}; \varepsilon) = \int \mathrm{d}\Omega_k \frac{1}{v_1 \cdot k}. \quad (7)$$

Its ε-expansion is [3–5]

$$I_1^{(1)}(v_{11}; \varepsilon) = \frac{\pi}{\sqrt{1-v_{11}}}\log\left(\frac{1+\sqrt{1-v_{11}}}{1-\sqrt{1-v_{11}}}\right) + \mathcal{O}(\varepsilon), \quad (8)$$

which is singular in the $v_{11} \to 0$ limit.

To extract the massless limit from Eq. (7) explicitly, we define the auxiliary "zero" vector $v_0 = (1, \mathbf{0}_{d-1})$ with $1/(v_0 \cdot k) = 1$ and set $v_2 = (1-\lambda)v_0 + \lambda v_1$. Demanding v_2 to be massless, i.e., $v_{22} = 0$, we find $\lambda = 1/\sqrt{1-v_{11}}$. We observe that v_1 indeed approaches v_2 in the limit $v_{11} \to 0$. A graphical illustration of this construction is given in Fig. 1.

The two-point splitting lemma (6) provides us with the identity

$$\Delta_k(v_0, v_1) = \lambda\Delta_k(v_0, v_2) + (1-\lambda)\Delta_k(v_1, v_2). \quad (9)$$

Integrating Eq. (9) and substituting the value for λ we get

$$I_1^{(1)}(v_{11}; \varepsilon) = \frac{I_1^{(0)}(\varepsilon)}{\sqrt{1-v_{11}}} - \frac{1-\sqrt{1-v_{11}}}{\sqrt{1-v_{11}}} I_{1,1}^{(1)}(v_{12}, v_{11}; \varepsilon). \quad (10)$$

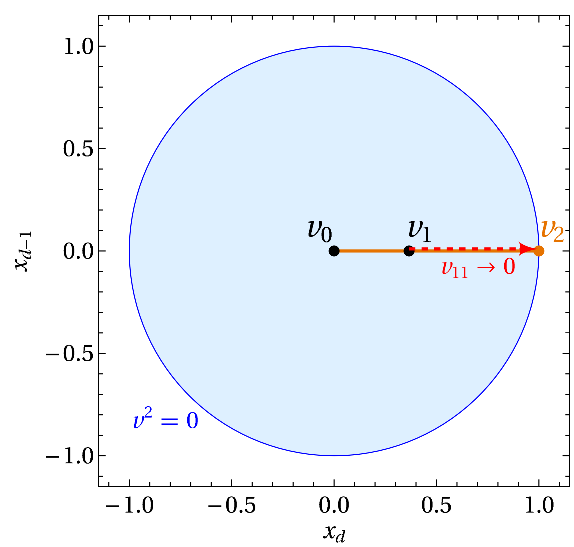

FIG. 1. Sketch illustrating the splitting of the massive one-denominator integral in Eq. (9). The figure shows the slice $x_0 = 1$ of Minkowski space, the blue circle indicates the intersection with the light-cone where $v^2 = 0$.

Hence, we have transformed the massive one-denominator integral into the sum of a massless one-denominator integral and a single-massive two-denominator integral, where the coefficient of the latter vanishes in the massless limit. It is $I_1^{(0)}(\varepsilon) = -\pi/\varepsilon$ and the ε-expansion of the single-massive two-denominator integral is [3–5]

$$I_{1,1}^{(1)}(v_{12}, v_{11}; \varepsilon) = \frac{\pi}{v_{12}} \left(\frac{v_{11}}{v_{12}^2}\right)^{\varepsilon} \left[-\frac{1}{\varepsilon} - 2\varepsilon(\mathrm{Li}_2(\omega_{12}^+) + \mathrm{Li}_2(\omega_{12}^-)) + \mathcal{O}(\varepsilon^2)\right], \tag{11}$$

with $\omega_{12}^{\pm} = 1 - v_{12}/(1 \pm \sqrt{1 - v_{11}})$. For the one-denominator kinematics we have $v_{12} = 1 - \sqrt{1 - v_{11}}$, $\omega_{12}^+ = 2\sqrt{1 - v_{11}}/(1 + \sqrt{1 - v_{11}})$, and $\omega_{12}^- = 0$.

Plugging the ε-expansions into Eq. (10), we receive

$$I_1^{(1)}(v_{11}; \varepsilon) = -\frac{\pi}{\sqrt{1 - v_{11}}} \left\{\frac{1}{\varepsilon} + v_{11}^{-\varepsilon}(1 + \sqrt{1 - v_{11}})^{2\varepsilon} \times \left[-\frac{1}{\varepsilon} - 2\varepsilon \mathrm{Li}_2\left(\frac{2\sqrt{1 - v_{11}}}{1 + \sqrt{1 - v_{11}}}\right) + \mathcal{O}(\varepsilon^2)\right]\right\}. \tag{12}$$

In this form the asymptotic behavior for $v_{11} \to 0$ is explicit. We observe that $I_1^{(1)}(v_{11}; \varepsilon)$ has a part constant in the massless limit and a part proportional to $v_{11}^{-\varepsilon}$. It is the latter that causes the logarithmic divergence in Eq. (8). Note that both parts have a $1/\varepsilon$ pole which cancels between the two for finite v_{11}.

B. Asymptotic form of the single-massive two-denominator integral

The second master integral we look at, is the single-massive two-denominator integral,

$$I_{1,1}^{(1)}(v_{12}, v_{11}; \varepsilon) = \int \mathrm{d}\Omega_k \frac{1}{v_1 \cdot k v_2 \cdot k}. \tag{13}$$

We have already encountered its ε-expansion in Eq. (11). We observe that the expansion is singular in the limit $v_{11} \to 0$ because the variable ω_{12}^- diverges.

To extract the asymptotic behavior of the single-massive integral for small masses, we want to separate off its massless limit. To this end we define the auxiliary vector $v_3 = (1 - \lambda)v_1 + \lambda v_2$. For v_3 to be massless, i.e., $v_{33} = 0$, we set $\lambda = v_{11}/(v_{11} - 2v_{12})$. A graphical illustration of the splitting construction is given in Fig. 2.

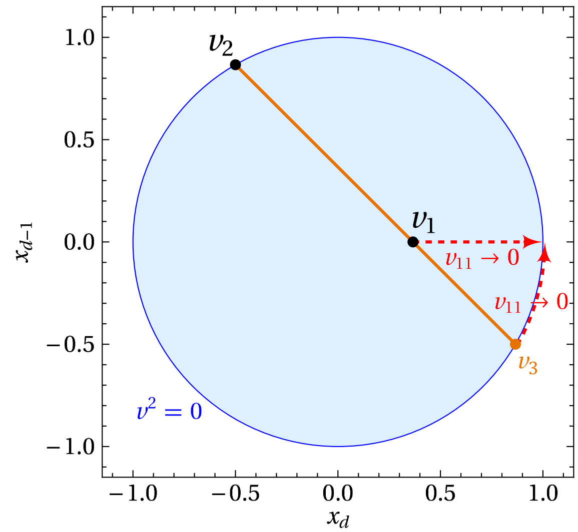

FIG. 2. Sketch illustrating the splitting of the single-massive two-denominator integral in Eq. (14). The figure shows the slice $x_0 = 1$ of Minkowski space, the blue circle indicates the intersection with the light cone where $v^2 = 0$.

Upon integration of the associated two-point splitting identity, which is of the form of Eq. (6), we obtain

$$I_{1,1}^{(1)}(v_{12}, v_{11}; \varepsilon) = \frac{2v_{12} I_{1,1}^{(0)}(v_{23}; \varepsilon)}{2v_{12} - v_{11}} + \frac{v_{11} I_{1,1}^{(1)}(v_{13}, v_{11}; \varepsilon)}{v_{11} - 2v_{12}}, \tag{14}$$

with the scalar products $v_{13} = v_{11}v_{12}/(2v_{12} - v_{11})$ and $v_{23} = 2v_{12}^2/(2v_{12} - v_{11})$.

Hence, we have transformed the single-massive two-denominator integral into the sum of a massless two-denominator integral and a single-massive two-denominator angular integral, where the coefficient of the latter vanishes in the massless limit.

The ε-expansion of the massless two-denominator integral reads [3–5],

$$I_{1,1}^{(0)}(v_{12}; \varepsilon) = \pi \left(\frac{v_{12}}{2}\right)^{-1-\varepsilon} \left[-\frac{1}{\varepsilon} - \varepsilon \mathrm{Li}_2\left(1 - \frac{v_{12}}{2}\right) + \mathcal{O}(\varepsilon^2)\right], \tag{15}$$

for the expansion of $I_{1,1}^{(1)}(v_{13}, v_{11}; \varepsilon)$ we can again use Eq. (11). Plugging these into (14), we receive

$$I_{1,1}^{(1)}(v_{12}, v_{11}; \varepsilon) = -\frac{2\pi}{v_{12}} \left\{ v_{12}^{-\varepsilon}(2\nu)^{\varepsilon} \left[\frac{1}{\varepsilon} + \varepsilon\left(\mathrm{Li}_2\left(1 - \frac{v_{12}}{2\nu}\right)\right) + \mathcal{O}(\varepsilon^2)\right] - v_{11}^{-\varepsilon}(2\nu)^{2\varepsilon}\left[\frac{1}{2\varepsilon} + \varepsilon(\mathrm{Li}_2(\omega_{13}^+) + \mathrm{Li}_2(\omega_{13}^-)) + \mathcal{O}(\varepsilon^2)\right]\right\}, \tag{16}$$

with $\omega_{13}^{\pm} = (v_{12}(1 \pm \sqrt{1 - v_{11}}) - v_{11})/(2v_{12} - v_{11})$ and the abbreviation $\nu = 1 - v_{11}/(2v_{12})$. In the massless limit $\omega_{13}^{\pm}$ approaches v_{12} respectively 0, and ν goes to 1.

Again we have found a form of the ε-expansion, where the asymptotic behavior for $v_{11} \to 0$ is explicit. As for the one-denominator integral, we have a finite part and a part proportional to $v_{11}^{-\varepsilon}$. Note that Eq. (16) trivially reduces to Eq. (15) for $v_{11} = 0$, something that could not be easily seen from Eq. (11).

C. Asymptotic form of the double-massive two-denominator integral

Finally, we consider the double-massive two-denominator master integral,

$$I_{1,1}^{(2)}(v_{12}, v_{11}, v_{22}; \varepsilon) = \int \mathrm{d}\Omega_k \frac{1}{v_1 \cdot k v_2 \cdot k}. \tag{17}$$

We have already discussed the divergent behavior of its ε-expansion in the introduction, see Eq. (2).

Using two-point splitting, the double-massive integral can be expressed as a sum of single-massive integrals [5]. For the double-massive integral, we have to consider the cases of one or both masses approaching zero. To treat both limits together we will employ a splitting that treats v_1 and v_2 symmetrically and directly extracts the double massless limit.

We define two auxiliary vectors $v_3 = (1 - \lambda)v_1 + \lambda v_2$ and $v_4 = \mu v_1 + (1 - \mu)v_2$. Employing the two-point splitting lemma (6) first on $\Delta_k(v_1, v_2)$ inserting v_3 and subsequently on $\Delta_k(v_2, v_3)$ inserting v_4, we obtain the splitting

$$\Delta_k(v_1, v_2) = \lambda\Delta_k(v_1, v_3) + \mu\Delta_k(v_2, v_4) + (1 - \lambda - \mu)\Delta(v_3, v_4). \tag{18}$$

To make v_3 and v_4 massless as well as coinciding with v_1 respectively v_2 in the respective massless limit, we choose

$$\lambda = \frac{v_{12} - v_{11} - \sqrt{X}}{2v_{12} - v_{11} - v_{22}}, \qquad \mu = \frac{v_{12} - v_{22} - \sqrt{X}}{2v_{12} - v_{11} - v_{22}}, \tag{19}$$

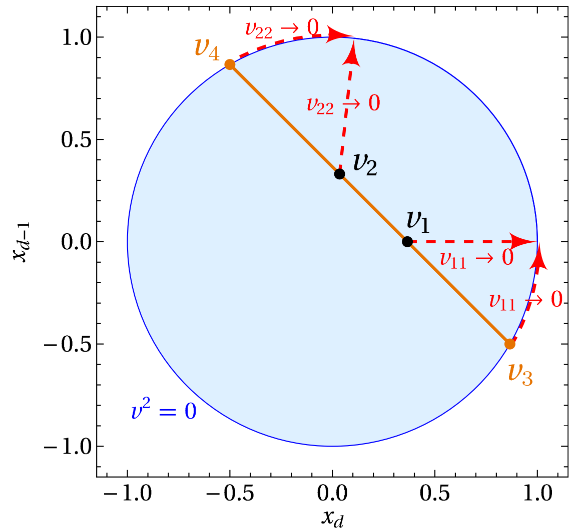

FIG. 3. Sketch illustrating the splitting of the double-massive two-denominator integral in Eq. (18). The figure shows the slice $x_0 = 1$ of Minkowski space, the blue circle indicates the intersection with the light cone where $v^2 = 0$.

with $X = v_{12}^2 - v_{11}v_{22}$. The scalar products of the auxiliary vectors are $v_{33} = v_{44} = 0$, $v_{13} = -\lambda\sqrt{X}$, $v_{24} = -\mu\sqrt{X}$, and $v_{34} = 2X/(2v_{12} - v_{11} - v_{22})$. Note that both v_{13} and v_{24} vanish in the respective massless limits. A graphical illustration of the splitting is given in Fig. 3.

Upon integration of Eq. (18) we obtain,

$$\begin{aligned} I_{1,1}^{(2)}(v_{12}, v_{11}, v_{22}; \varepsilon) = \frac{\pi}{\sqrt{X}}[&v_{34}I_{1,1}^{(0)}(v_{34}; \varepsilon) \\ &- v_{13}I_{1,1}^{(1)}(v_{13}, v_{11}; \varepsilon) \\ &- v_{24}I_{1,1}^{(1)}(v_{24}, v_{22}; \varepsilon)]. \end{aligned} \tag{20}$$

This identity splits the double-massive integral into two single-massive integrals and a massless integral. For these we can use the ε-expansions from Eqs. (11) and (15), resulting in the representation of the double-massive integral,

$$\begin{aligned} I_{1,1}^{(2)}(v_{12}, v_{11}, v_{22}; \varepsilon) = \frac{\pi}{\sqrt{X}}\Bigg\{&2\left(\frac{v_{34}}{2}\right)^{-\varepsilon}\left[-\frac{1}{\varepsilon} - \varepsilon \mathrm{Li}_2\left(1 - \frac{v_{34}}{2}\right) + \varepsilon^2 f_0(v_{34}) + \mathcal{O}(\varepsilon^3)\right] \\ &- v_{11}^{-\varepsilon}\left(\frac{v_{11}}{v_{13}}\right)^{2\varepsilon}\left[-\frac{1}{\varepsilon} - 2\varepsilon(\mathrm{Li}_2(\omega_{13}^+) + \mathrm{Li}_2(\omega_{13}^-)) + \varepsilon^2 f_1(\omega_{13}^+, \omega_{13}^-) + \mathcal{O}(\varepsilon^3)\right] \\ &- v_{22}^{-\varepsilon}\left(\frac{v_{22}}{v_{24}}\right)^{2\varepsilon}\left[-\frac{1}{\varepsilon} - 2\varepsilon(\mathrm{Li}_2(\omega_{24}^+) + \mathrm{Li}_2(\omega_{24}^-)) + \varepsilon^2 f_1(\omega_{24}^+, \omega_{24}^-) + \mathcal{O}(\varepsilon^3)\right]\Bigg\}, \end{aligned} \tag{21}$$

where $\omega_{ij}^{\pm} = 1 - v_{ij}/(1 \pm \sqrt{1 - v_{ii}})$. The kinematic variables X, v_{13}, v_{24}, and v_{34} are defined in the text above; they all depend on v_{12}, v_{11}, and v_{22}. Importantly $v_{11}/v_{13} \to 2$ for $v_{11} \to 0$ and analogously $v_{22}/v_{23} \to 2$ for $v_{22} \to 0$.

The asymptotics of the double-massive integral is manifest in Eq. (21), we have a part constant in both massless limits, a part proportional to $v_{11}^{-\varepsilon}$, and a part proportional to $v_{22}^{-\varepsilon}$. Note that the $1/\varepsilon$ poles cancel between the parts if we expand in ε for finite v_{11} and v_{22}. For $v_{22}=0$ we immediately recover Eq. (16).

The full expressions for the functions $f_{0,1}$ parametrizing the order ε^2 parts of the massless respectively single-massive two-denominator integral can be found in the Appendix. The order ε^2 is included here, since applying the expansion (4) for both v_{11} and v_{22} may result in a $1/\varepsilon^2$ pole. In the limit $v_{11}\to 0$, we have $\omega_{13}^{+}\to 1$ and $\omega_{13}^{-}\to 0$. Analogously, in the limit $v_{22}\to 0$, it is $\omega_{24}^{+}\to 1$ and $\omega_{24}^{-}\to 0$. Hence, a double massless pole term $\delta(v_{11})\delta(v_{22})/\varepsilon^2$ will receive a contribution from the ε^2 coefficient function in the double massless limit. The specific value required for f_1 is $f_1(1,0)=-2\zeta_3$, where ζ_3 denotes Apéry's constant $\zeta_3=\sum_{n=1}^{\infty}1/n^3$.

If one is interested in only a single massless limit, say $v_{11}\to 0$ while v_{22} stays finite, we may expand $v_{22}^{-\varepsilon}$ allowing for some explicit simplifications of logarithms. In this case, we find the representation

$$I_{1,1}^{(2)}(v_{12},v_{11},v_{22};\varepsilon)=\frac{\pi}{\sqrt{X}}\Bigg\{-\frac{1}{\varepsilon}-2\log(2\nu)+\log\left(\frac{2v_{12}(v_{12}+\sqrt{X})}{v_{22}}-v_{11}\right)+2\varepsilon\Bigg(\mathrm{Li}_2(\omega_{24}^{+})+\mathrm{Li}_2(\omega_{24}^{-})+\mathrm{Li}_2\left(1-\frac{2v_{12}-v_{11}-v_{22}}{X}\right)+\frac{1}{4}\log^2\left(\frac{v_{22}}{v_{24}^2}\right)\Bigg)+\mathcal{O}(\varepsilon^2)+v_{11}^{-\varepsilon}\left[\frac{1}{\varepsilon}+2\log(2\nu)+2\varepsilon(\mathrm{Li}_2(\omega_{13}^{+})+\mathrm{Li}_2(\omega_{13}^{-})+\log^2(2\nu))+\mathcal{O}(\varepsilon^2)\right]\Bigg\},\tag{22}$$

with the abbreviation $\nu=1-v_{11}/(2v_{14})$, where $v_{14}=\sqrt{X}(v_{12}-v_{11}+\sqrt{X})/(2v_{12}-v_{11}-v_{22})$. The asymptotic form for $v_{22}\to 0$ at finite v_{11} is the same upon interchanging $v_{11}\leftrightarrow v_{22}$.

III. CONCLUSION

We have established ε-expansions with manifest small-mass asymptotics for all massive angular master integrals with up to two denominators. The main results of this paper are the asymptotic expansions of the following:

(i) Massive integral $I_1^{(1)}$ in Eq. (12);
(ii) single-massive integral $I_{1,1}^{(1)}$ in Eq. (16);
(iii) double-massive integral $I_{1,1}^{(2)}$ in Eq. (21).

By means of recursion relations derived from IBP identities these results extend to all two-denominator angular integrals with integer coefficients. In the construction of the asymptotic expansion the two-point splitting lemma proved to be an immensely useful tool. It allowed for the extraction of the massless limits in terms of suitable massless angular integrals.

The presented method based on the two-point splitting lemma straightforwardly generalizes to angular integrals with more than two denominators. In these cases, splitting can be successively performed on pairs of denominators. However, for massive angular integrals with three and more linearly independent denominators, i.e., denominators that are not reducible to two-denominator integrals by partial fractioning, ε-expansions are not known in the literature at present. Once these become available, the analysis presented in this work can be extended.

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APPENDIX: ORDER ε^2 COEFFICIENTS OF THE DOUBLE-MASSIVE INTEGRAL

The explicit form of the order ε^2 coefficient functions f_0 and f_1 of the massless respectively single-massive two-denominator integrals appearing in Eq. (21) are

$$f_0(v)=\mathrm{Li}_3\left(1-\frac{2}{v}\right)-\mathrm{Li}_2\left(1-\frac{v}{2}\right)\log\frac{v}{2}-\frac{1}{6}\log^3\frac{v}{2}\tag{A1}$$

and

$$f_1(\omega^+,\omega^-)=2\mathrm{S}_{1,1,1}(\omega^+,\omega^-)-2\mathrm{Li}_3(\omega^+)+2\mathrm{Li}_3\left(\frac{\omega^+}{\omega^+-1}\right)-2\mathrm{Li}_2(\omega^+)\log(1-\omega^+)-\frac{1}{3}\log^3(1-\omega^+)+(\omega^+\leftrightarrow\omega^-),\tag{A2}$$

where $\mathrm{S}_{1,1,1}$ denotes the double Nielsen polylogarithm [5]

$$\mathrm{S}_{1,1,1}(x,y)=\int_0^1\frac{\mathrm{d}t}{t}\log(1-xt)\log(1-yt).\tag{A3}$$

This generalized polylogarithm of weight 3 and depth 2 can be expressed in terms of classical polylogarithms [38,39]. For $0 < y < x < 1$ it holds that,

$$\mathrm{S}_{1,1,1}(x,y) = -\mathrm{Li}_3(1-x) - \mathrm{Li}_3(1-y) + \mathrm{Li}_3\left(\frac{y}{x}\right) + \mathrm{Li}_3\left(\frac{1-x}{1-y}\right) - \mathrm{Li}_3\left(\frac{y(1-x)}{x(1-y)}\right) - \mathrm{Li}_2(y)\log(1-x) \\ + \mathrm{Li}_2(1-x)\log(1-y) + \mathrm{Li}_2\left(\frac{y}{x}\right)\log\left(\frac{1-x}{1-y}\right) - \frac{1}{2}\log^2(1-y)\log\left(\frac{y}{x}\right) + \frac{\pi^2}{6}\log(1-y) + \zeta_3. \tag{A4}$$

The specific value needed for the double massless limit is $\mathrm{S}_{1,1,1}(1,0) = 0$, which can be trivially read off from the integral representation (A3).

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