Primary observables for gauge boson collider signals

Cristien Arzate, Spencer Chang, and Gabriel Jacobo
Department of Physics and Institute for Fundamental Science, University of Oregon, Eugene, Oregon 97403, USA

(Received 20 December 2023; accepted 29 February 2024; published 29 April 2024)

In this paper, we determine a basis for the on-shell four-point amplitudes $VVVV$ for Standard Model gauge bosons $V = W^{\pm}, Z, \gamma, g$. Following previous work, this completes the analysis of three- and four-point amplitudes for the Standard Model and could be used for model-independent searches for beyond the Standard Model physics at colliders. Our results include a Lagrangian parametrization for the "primary" amplitudes, where including additional derivatives leads to the Mandelstam "descendant" amplitudes, and upper bounds on the coupling constants from imposing unitarity. We also perform an estimate for the sensitivity for new Z decays at the High-Luminosity LHC, finding that $Z \to \gamma\bar{\ell}\ell$ could be searched for, but that other decay modes, like $Z \to (\gamma\gamma\gamma, \gamma gg)$, are too small to be discovered after imposing unitarity constraints.

DOI: 10.1103/PhysRevD.109.075046

I. INTRODUCTION

Recently, there has been enormous progress in determining the general structure of on-shell amplitudes for the Standard Model [1–7]. In addition to being of interest theoretically, these results could enable broad model-independent searches for beyond the Standard Model (BSM) physics, without relying on the standard effective field theory (EFT) parametrizations [i.e., Standard Model EFT (SMEFT) [8,9] and Higgs EFT [10]]. Indeed, amplitudes may be a better way to connect experiment and theory, given the direct connection to experimental analyses and since amplitudes do not require the EFT assumptions of power counting and do not suffer from ambiguous basis issues of Lagrangian operators.

In Refs. [6,7], the full structure of on-shell three- and four-point amplitudes involving the Higgs and top quark were determined. This leaves four-point gauge boson amplitudes as the remaining ones to be analyzed. In this paper, we complete this study for both massive ($W^{\pm}, Z$) and massless (γ, g) gauge boson interactions. At the High-Luminosity LHC (HL-LHC) and future colliders (e.g., e^+e^- colliders at the Z pole), we stand to increase our sample of W/Z particles by orders of magnitude. Our study of the amplitudes then allows us to consider if there are interesting amplitudes for three-body decays like $Z \to \gamma\gamma\gamma$ beyond those considered in the past.

This work confirms and extends existing work on parametrizing these amplitudes, so we will now describe the overlapping work. The three-point amplitudes that we list in Eqs. (1) and (2) were derived in [2]. Of the four-point amplitudes we studied, the amplitudes for $Zggg$ were derived in [1] and the amplitudes for $ffVh$ were derived in [2]. Four-point primary operators up to dimension eight have been derived in operator form in [4] and in spinor structure form in [5].

The rest of this paper is organized as follows: Section II describes what amplitudes we will explore and how to determine independent amplitudes. Section III discusses the Hilbert series results for our gauge boson operators. In Sec. IV, we discuss some relevant phenomenological issues, such as unitarity bounds on coupling strengths and also rough estimates for Z decays at the HL-LHC. Section V is the main body of results, where we list the operators for the primary amplitudes. In Sec. VI, we estimate which Z decay amplitudes are interesting for exploration at the HL-LHC. Finally, in Sec. VII, we conclude.

II. FINDING INDEPENDENT AMPLITUDES/ COUPLINGS FOR ELECTROWEAK GAUGE BOSONS

To find the most general on-shell amplitudes for gauge bosons, we impose invariance under $\mathrm{SU}(3)_c \times \mathrm{U}(1)_{em}$ and Lorentz symmetry. For three- and four-point interactions, this gives the following list:

$$\begin{aligned} &\text{3pt: } \bar{f}fV, hhV, hVV, VVV \\ &\text{4pt: } hhhV, \bar{f}fhV, \bar{f}fVV, hhVV, hVVV, VVVV, \end{aligned} \quad (1)$$

where f is a fermion, h is a Higgs boson, and V is any gauge boson. To fully characterize these four-point interactions, we also need additional three-point interactions for exchange diagrams, which add

$$\text{3pt additional: } hhh, \bar{f}fh. \tag{2}$$

Of these couplings, the three- and four-point couplings, except for $VVVV$, have been determined (e.g., [7,11]), so in this paper this leaves the following four-point couplings to analyze:

$$\begin{aligned} VVVV\colon\ & WWWW, WWZZ, ZZZZ, WWZ\gamma, ZZZ\gamma, \\ & WW\gamma\gamma, WWgg, ZZ\gamma\gamma, ZZgg, Z\gamma gg, Z\gamma\gamma\gamma, \\ & Zggg, \gamma\gamma\gamma\gamma, \gamma\gamma gg, \gamma ggg, gggg. \end{aligned} \tag{3}$$

In [7,11], an approach for determining a basis for independent operators for three- and four-point on-shell amplitudes was developed and explained in detail. Here, we will briefly summarize the three-step process of (1) enumerating an overcomplete basis of amplitudes, (2) determining the independent primary amplitudes, and (3) checking the result against a Hilbert series calculation. For those interested in the details, please refer to the discussion in [7,11].

For step 1, we use the fact that local on-shell four-point gauge boson amplitudes are Lorentz invariants involving gauge boson polarization and momenta contracted with the metric or the Levi-Civita tensor. For processes with massless gauge bosons, we use the associated field strength tensors to maintain gauge invariance and satisfy the Ward identity. We distinguish amplitudes that have no factors of Mandelstam variables from those with such factors. We refer to the former as primary amplitudes and the latter as descendant amplitudes, following the terminology of [7,11]. Note that our construction of amplitudes parallels that of [2,3] and that our primary amplitudes are equivalent to what were called stripped contact terms in [3]. By allowing for arbitrary Mandelstam factors in the descendants [12], we parametrize the most general on-shell four-point amplitudes. However, for on-shell three-point amplitudes, all Mandelstam invariants can be written in terms of the particle masses, so there are only primary amplitudes. A process with amplitude $\mathcal{M}$ can then be written as a linear sum $\mathcal{M} = \sum_a C_a \mathcal{M}_a$ of these parametrized amplitudes $\mathcal{M}_a$. Each $\mathcal{M}_a$ will have an associated Lagrangian operator $\mathcal{O}_a$ with mass dimension $d_{\mathcal{O}}$ and a dimensionless coupling c_a. The associated Lagrangian terms can then be written as

$$\mathcal{L}_{\text{amp}} = \sum_a \frac{c_a}{v^{d_{\mathcal{O}-4}}} \mathcal{O}_a. \tag{4}$$

The factors of Higgs vacuum expectation value (VEV) in the denominator are chosen such that the coupling remains dimensionless at any given mass dimension.

For the four-point amplitudes we consider here, there are commonly two or more identical bosons. In such cases, the amplitudes need to be symmetric under crossing exchange of the identical bosons. Assume we have the process $p_1(\epsilon_1) + p_2(\epsilon_2) \to p_3(\epsilon_3) + p_4(\epsilon_4)$, where the incoming particles are identical (note that, for the moment, we are not assuming any gauge charges for them), with the amplitude $\mathcal{M}(12;34)$, where $i = 1\ldots4$ is shorthand for p_i, ϵ_i. We can then form symmetric and antisymmetric combinations under 1,2 exchange,

$$\mathcal{M}_{\pm}(12;34) = \frac{1}{2}\left[\mathcal{M}(12;34) \pm \mathcal{M}(21;34)\right], \tag{5}$$

and we can use these to construct general amplitudes with Mandelstam invariants. Since under $1 \leftrightarrow 2$, s is invariant and $t \leftrightarrow u$, the most general $1 \leftrightarrow 2$ symmetric amplitude is

$$\begin{aligned} \mathcal{M}_{12}(12;34) = \mathcal{M}_{12}(21;34) = F\big(s,(t-u)^2\big)\mathcal{M}_+(12;34) \\ + (t-u)G\big(s,(t-u)^2\big)\mathcal{M}_-(12;34), \end{aligned} \tag{6}$$

where the polynomial functions F and G are $1 \leftrightarrow 2$ exchange symmetric.

For the case of three identical bosons, the amplitudes should first be symmetrized for the first two bosons and then that result should be symmetrized with respect to exchanges with the third particle. The result of this yields

$$\begin{aligned} \mathcal{M}_{123}(12;34) = H\big(s,(t-u)^2\big)\mathcal{M}_{12}(12;34) \\ + H\big(t,(s-u)^2\big)\mathcal{M}_{12}(13;24) \\ + H\big(u,(t-s)^2\big)\mathcal{M}_{12}(32;14), \end{aligned} \tag{7}$$

where exchange of incoming and outgoing particles has a minus sign and complex conjugation, e.g., $1 \leftrightarrow 3$ takes $p_1 \to -p_3, \epsilon_1 \to \epsilon_3^*$ and $p_3 \to -p_1, \epsilon_3 \to \epsilon_1^*$. The arguments of H have been chosen such that, under $i \leftrightarrow j$ exchange, with $i, j = 1, 2, 3$, the sum of the three terms is invariant, i.e., $\mathcal{M}_{123}(12;34)$ is invariant under the permutations of 1, 2, 3 in the argument of the function. This follows since under $1 \leftrightarrow 3$ exchange, $s \leftrightarrow u$ and under $2 \leftrightarrow 3$ exchange, $s \leftrightarrow t$. Finally, for the case of four identical bosons, the previous result needs to be symmetrized with respect to the fourth particle. This requires replacing $\mathcal{M}_{12}(12;34)$ with $\mathcal{M}_{12,34}(12;34) = \frac{1}{2}[\mathcal{M}_{12}(12;34) + \mathcal{M}_{12}(12;43)]$.

For step 2, we take this overcomplete basis of amplitudes and find the independent ones. To do this, we will work in increasing mass dimension for $\mathcal{O}_a$. In the center-of-mass frame for the $1+2 \to 3+4$, the weighted amplitude $E^n_{\text{c.m.}}\mathcal{M}$ is a polynomial of the kinematic variables $E_{\text{c.m.}}$, $\cos\theta_{\text{c.m.}}, \sin\theta_{\text{c.m.}}, |\boldsymbol{p}_{\text{initial}}|, |\boldsymbol{p}_{\text{final}}|$.[13] To proceed, we simplify the amplitude by replacing even powers of $\sin\theta_{\text{c.m.}}$, $|\boldsymbol{p}_{\text{initial}}|, |\boldsymbol{p}_{\text{final}}|$ with their solution in terms of $\cos\theta_{\text{c.m.}}$ and $E_{\text{c.m.}}$ leading to the general amplitude

$$E^n_{\text{c.m.}}\mathcal{M} = P + Q\sin\theta_{\text{c.m.}} + R|\boldsymbol{p}_{\text{initial}}| + S|\boldsymbol{p}_{\text{final}}| + T\sin\theta_{\text{c.m.}}|\boldsymbol{p}_{\text{initial}}| + U\sin\theta_{\text{c.m.}}|\boldsymbol{p}_{\text{final}}| + V|\boldsymbol{p}_{\text{initial}}||\boldsymbol{p}_{\text{final}}| + W\sin\theta_{\text{c.m.}}|\boldsymbol{p}_{\text{initial}}||\boldsymbol{p}_{\text{final}}|, \tag{8}$$

where P, Q, R, S, T, U, V, W are polynomials in $\cos\theta_{\text{c.m.}}$, $E_{\text{c.m.}}$. As argued in [11], for a redundancy to occur, i.e., $\mathcal{M} = 0$, one needs each of the $P, \ldots, W$ polynomials to independently vanish. Since the coefficients in the polynomials depend on the couplings c_a, one can use a numerical singular value decomposition to find how many redundancies there are and, by process of elimination, find an independent set of couplings c_a with corresponding amplitudes $\mathcal{M}_a$.

The third step provides a complementary constraint on the number of independent amplitudes. To do so, we determine the number of independent Lagrangian operators using the Hilbert series, which counts the number of operators at each mass dimension [14–20]. Since there is one-to-one correspondence between nonredundant operators $\mathcal{O}_a$ and amplitudes $\mathcal{M}_a$ (e.g., [1]), this counting can be used to check the numerical analysis. More will be said about the specifics of the Hilbert series in the next section.

III. HILBERT SERIES

The Hilbert series is a tool that provides the number of gauge invariant independent operators in a given EFT [14,16–21]. The Hilbert series counts the number of independent operators while taking into account symmetry constraints, equations of motion, and redundancies due to integration by parts. In Eq. (9), we list the Hilbert series for four-point interactions involving only electroweak gauge bosons (the Hilbert series for the other three- and four-point operators aforementioned can be found in [7,11]):

$$\begin{aligned}
H_{WWWW} &= \frac{2q^4+16q^6+22q^8+7q^{10}-2q^{12}}{(1-q^2)(1-q^4)}, \qquad H_{WWZZ} = \frac{2q^4+27q^6+40q^8+14q^{10}-2q^{12}}{(1-q^2)(1-q^4)},\\
H_{ZZZZ} &= \frac{q^4+4q^6+8q^8+11q^{10}+5q^{12}-2q^{14}}{(1-q^4)(1-q^6)}, \qquad H_{WWZ\gamma} = \frac{22q^6+34q^8+(2-4)q^{10}}{(1-q^2)^2},\\
H_{ZZZ\gamma} &= \frac{4q^6+14q^8+22q^{10}+12q^{12}+(4-2)q^{14}}{(1-q^4)(1-q^6)}, \qquad H_{WW\gamma\gamma} = H_{WWgg} = \frac{3q^6+19q^8+14q^{10}+(2-2)q^{12}}{(1-q^2)(1-q^4)},\\
H_{ZZ\gamma\gamma} &= H_{ZZgg} = \frac{3q^6+13q^8+7q^{10}-2q^{12}}{(1-q^2)(1-q^4)}, \qquad H_{Z\gamma gg} = \frac{12q^8+12q^{10}+(2-2)q^{12}}{(1-q^2)(1-q^4)},\\
H_{Z\gamma\gamma\gamma} &= \frac{4q^8+10q^{10}+8q^{12}+(4-2)q^{14}}{(1-q^4)(1-q^6)}, \qquad H_{Zggg} = \frac{6q^8+18q^{10}+16q^{12}+(8-2)q^{14}+2q^{16}}{(1-q^4)(1-q^6)},\\
H_{\gamma\gamma\gamma\gamma} &= \frac{3q^8+5q^{10}+q^{12}-2q^{14}}{(1-q^4)(1-q^6)}, \qquad H_{\gamma\gamma gg} = \frac{7q^8+5q^{10}-2q^{12}}{(1-q^2)(1-q^4)},\\
H_{\gamma ggg} &= \frac{4q^8+12q^{10}+8q^{12}+(6-2)q^{14}+4q^{16}}{(1-q^4)(1-q^6)}, \qquad H_{gggg} = \frac{9q^8+14q^{10}+16q^{12}+(9-2)q^{14}+(2-4)q^{16}}{(1-q^4)(1-q^6)}.
\end{aligned} \tag{9}$$

The correct way to interpret these Hilbert series is to take the exponent of each q to be the mass dimension and the corresponding coefficients to be the number of independent operators minus the number of redundancies at that mass dimension. When evaluating the Hilbert series, one cannot tell if there is such a cancellation. Only by looking at the independent amplitudes can we resolve this ambiguity; so using those results we have written out terms with a cancellation explicitly, e.g., $(6-2)q^{14}$ in the $H_{\gamma ggg}$ shows that there are six new primaries and two redundancies appearing at dimension 14. The appearance of these negative terms in the coefficients means that descendants of primary operators at a lower mass dimension become redundant to operators at the corresponding mass dimension of the negative term and that the higher-dimensional operators and their descendants should be discarded from the set of independent operators.

To illustrate this in a specific example, take the Hilbert series for $WWWW$. The numerator looks like

$$2q^4 + 16q^6 + 22q^8 + 7q^{10} - 2q^{12}. \tag{10}$$

So in terms of primary operators, at dimension four there are two operators, at dimension six there are 16 operators, at dimension eight there are 22 operators, at dimension ten there are seven operators, and at dimension 12 there are two redundancies. Now, to see the descendant structure of each operator, the denominator should be Taylor expanded. For example, the negative term turns out to be

$$H_{WWWW} \supset -\frac{2q^{12}}{(1-q^2)(1-q^4)} = -2q^{12}(1+q^2+q^4+\cdots)(1+q^4+q^8+\cdots). \quad (11)$$

This says that, if we consider the channel $W^+W^+ \to W^-W^-$, then there are two redundant operators at dimension 12 that have descendant structures of the form $s^n(t-u)^{2m}\mathcal{O}$. The reason that they follow that specific structure is because of the exchange symmetries that the operators have to obey, namely, a symmetry under $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. We will later find out that the redundant operators are descendants of two dimension-eight operators, allowing us to rewrite the negative term, along with the positive dimension-eight term, as

$$\begin{aligned} H_{WWWW} &\supset \frac{22q^8 - 2q^{12}}{(1-q^2)(1-q^4)} = \frac{20q^8 + 2q^8(1-q^4)}{(1-q^2)(1-q^4)} \\ &= \frac{20q^8}{(1-q^2)(1-q^4)} + \frac{2q^8}{(1-q^2)} \\ &= 20q^8(1+q^2+q^4+\cdots)(1+q^4+q^8+\cdots) \\ &\quad + 2q^8(1+q^2+q^4+\cdots). \end{aligned} \quad (12)$$

From this we see that the correct interpretation of the independent operators is that two dimension-eight primary operators have a descendant structure of $s^n\mathcal{O}$, while the other 20 dimension-eight primaries have a descendant structure of $s^n(t-u)^{2m}\mathcal{O}$. This means that, for the former two primary operators, we can throw out their descendants of the form $s^n(t-u)^{2m}\mathcal{O}$ for $n \geq 0$ and $m \geq 1$ and still have an independent set of operators. Again, this Hilbert series interpretation must be checked with the amplitudes to confirm this explanation. As another example for what the denominators mean, consider the denominator for the Hilbert series for $\gamma\gamma\gamma\gamma$,

$$\frac{1}{(1-q^4)(1-q^6)} = (1+q^4+q^8+\cdots) \times (1+q^6+q^{12}+\cdots). \quad (13)$$

The first set of parentheses says that there are powers of a four-dimensional function of Mandelstam variables and the second set of parentheses says that are powers of a six-dimensional function of Mandelstam variables. Because the $\gamma\gamma\gamma\gamma$ interaction should have exchange symmetries between all pairs of particles, the descendant structure should have the form $(s^2+t^2+u^2)^n(stu)^m$, in agreement with the dimensional analysis.

As mentioned earlier, some of the coefficients in the Hilbert series are written as a positive integer minus a negative integer. For example, this occurs in the Hilbert series for the $ZZZ\gamma$ interaction at dimension 14 as shown by the term $(4-2)q^{14}$. When evaluating the Hilbert series, this coefficient would be 2, but in this case, by studying the amplitudes, we find that there are four new primary operators and two redundancies at mass dimension 14, so we write the coefficient in this way to make this explicit. This also means that, for a given interaction, at mass dimensions higher than what we have explored, there could be terms with coefficients of zero, not because there are no primary operators present, but because there are the same number of redundancies as primaries at that dimension. An example of this happening occurs for the $Z\gamma gg$ interaction at mass dimension 12, which we have written the term as $(2-2)q^{12}$. Therefore, it is not guaranteed that we have enumerated all possible primary operators, since there can be cancellations with the redundancies. However, because those would appear at very high mass dimension, they are phenomenologically unimportant and so do not warrant much concern. This possibility is the reason why we have analyzed operators up to at least the first mass dimension that has a zero coefficient in the numerator and up to dimension 16 for operators of $Zggg$, γggg, and $gggg$ interactions.

IV. PHENOMENOLOGY

A. Unitarity

As in [7,11], we use unitarity constraints to place upper bounds on the couplings of the operators we have enumerated. We know that the Standard Model (SM) does not violate tree-level unitarity at high energies (e.g., [22,23]); therefore, a deviation from the SM will violate it at some high energy scale E_{max}. Our constraints will depend upon this scale and we roughly expect E_{max} above a TeV to be consistent with current LHC analyses, but values lower than a TeV to possibly be in tension. To compute the bounds, we follow the same techniques developed by [24–26] (see also [27]).

For each operator, we create a schematic SMEFT realization of it in order to compute our unitarity bounds. To illustrate this, we turn to the $WWWW$ interaction where there is a dimension-six primary operator $iW^{+\mu}\tilde{W}^{+}_{\nu\rho}D_\mu W^{-\nu}W^{-\rho} + \mathrm{H.c.}$ To realize this operator in SMEFT, one needs at least four Higgs doublets. The nonfield strength W's come from covariant derivatives acting on the doublets, leading to four covariant derivatives, which by integration by parts can act on just three of the doublets. The dual field strength tensor $\tilde{W}^a_{\mu\nu}$ needs to be contracted with the SU(2) generators T^a. This leads to a SMEFT operator, which we simplify into a schematic form,

$$iH^\dagger T^a D^\mu H \tilde{W}^a_{\nu\rho} D^\nu_\mu H^\dagger D^\rho H \to D^4 H^4 \tilde{W}^a_{\mu\nu}. \quad (14)$$

Primary operators that have either zero or one field strength tensor have a SMEFT operator with at least four Higgs doublets by the following argument where we try to use only two Higgs doublets. If there are no field strength tensors, the SMEFT operator has the schematic form $D^n(H^\dagger H)$ and, by integration by parts, can always be reduced to factors of $D^\mu D_\mu$ acting on one of the Higgs doublets. Using the equations of motion, each $D^\mu D_\mu$ can be removed and replaced with expressions that involve no gauge bosons. By iterating, we see that this does not realize the four gauge boson amplitude we wanted, thus we need operators with four Higgs doublets, like $D^n(H^\dagger H H^\dagger H)$. Since there are four Higgs doublets, it is not possible to write all Lorentz invariants in terms of $D^\mu D_\mu$'s, so the above argument cannot be applied. A similar argument works for operators with one field strength, $D^n(H^\dagger W_{\mu\nu} H)$, since derivative pairs on the three fields can also be moved on to individual fields. From Lorentz invariance, these are either of the form (1) $W^{\mu\nu}(D_\mu D_\nu H) \propto W^{\mu\nu} W_{\mu\nu} H$, and thus actually have two field strengths, or (2) $(D^\mu D_\mu H)$ or $(D^\mu D_\mu W_{\alpha\beta})$, which can be reduced by equations of motion to expressions with fewer covariant derivatives or more than one field strength, respectively. For operators with two or three field strength tensors, these arguments no longer work and we only need two Higgs doublets instead of four for the SMEFT operator. Finally, if we have four field strength tensors, then we do not need any Higgs doublets for the SMEFT operator.

To calculate the coupling constraints, we need to estimate two quantities: the scattering amplitude and phase space factor for the highest and lowest particle multiplicity of an interaction process. By using these two quantities, we can put upper bounds on the coupling strengths of operators. Note that because we are calculating approximate bounds we only care about factors of v and neglect $O(1)$ factors like $\sqrt{2}$, g, g', $\sin\theta_W$, and $\cos\theta_W$. At high energy our amplitudes will then be of the form

$$\mathcal{M}(\phi_1 \cdots \phi_k \to \phi_{k+1} \cdots \phi_n) \sim c_a \frac{E^m}{v^{m+n-4}}, \qquad (15)$$

where E is the total energy, v is the Higgs VEV, k is the total number of incoming particles, and n is the total number of particles. The allowed values of n are determined by the SMEFT operator and can be varied by setting Higgs doublets to their VEV. To obtain m, we count the energy scaling of the various quantities in the schematic SMEFT operator. The scaling behavior of various quantities are

$$D \sim E, \qquad B_{\mu\nu} \sim E, \qquad W^a_{\mu\nu} \sim E, \qquad G^b_{\mu\nu} \sim E. \qquad (16)$$

The value of m is just the total energy scaling of the operator and the power of v is given by dimensional analysis. Next, the unitarity bound on the amplitude depends on the phase space factor of the initial and final states,

$$\mathcal{M}(\phi_1 \cdots \phi_k \to \phi_{k+1} \cdots \phi_n) \lesssim \frac{1}{\sqrt{\Phi_k(E)\Phi_{n-k}(E)}}, \qquad (17)$$

where $\Phi_k(E)$ is the total phase space for k particles with center-of-mass energy E. We follow [11] and work in the massless limit where, approximately,

$$\Phi_k(E) \sim \frac{1}{8\pi}\left(\frac{E}{4\pi}\right)^{2k-4}. \qquad (18)$$

Since unitarity is violated at some energy $E_{\max}$, we get that the unitarity bound on the couplings is

$$c_a \lesssim 2(4\pi)^{n-3}\left(\frac{v}{E_{\max}}\right)^{m+n-4}. \qquad (19)$$

Now, recall that the interaction of the operator $iW^{+\mu}\tilde{W}^+_{\nu\rho}D_\mu W^{-\nu}W^{-\rho}$ + H.c. has a schematic SMEFT form of $D^4H^4\tilde{W}^a_{\mu\nu}$. We now evaluate the amplitudes at high energy by using the equivalence theorem, using the Nambu-Goldstone bosons in H for longitudinal W's and Z's. We find that the best bounds come from using the derivative part of the covariant derivatives and not the transverse gauge bosons. Thus, for the schematic operator $D^4H^4\tilde{W}^a_{\mu\nu}$, we can have the five-point process $\phi\phi \to \phi\phi W_T$ or the four-point process $\phi\phi \to \phi W_T$, where ϕ is a Nambu-Goldstone boson. Under our approximation, these have amplitudes $\mathcal{M}(\phi\phi \to \phi\phi W_T) \sim c_a \frac{E^5}{v^6}$ and $\mathcal{M}(\phi\phi \to \phi W_T) \sim c_a \frac{E^5}{v^5}$. For the first case $n = 5$, $m = 5$, and for the second case $n = 4$, $m = 5$. Respectively, these lead to the bounds $c_a \lesssim \frac{0.07}{E^6_{\text{TeV}}}$ and $c_a \lesssim \frac{0.02}{E^5_{\text{TeV}}}$, where $E_{\text{TeV}} \equiv E_{\max}/\text{TeV}$. Thus, the higher multiplicity amplitude is more stringent for higher E_{TeV} and the lower multiplicity is more stringent for lower E_{TeV}, where they cross at $E_{\max} = 4\pi v$ for the bound $c_a \lesssim 2(4\pi)^{1-m}$. In this way, we proceed to calculate unitarity bounds on coupling strengths for all enumerated operators.

B. Electroweak gauge boson decays

In this subsection, we give estimates for modifications to electroweak gauge boson decays. Two-body decays of the Z are quite well covered, as $Z \to (\gamma\gamma, gg)$ are forbidden by the Landau-Yang theorem and $Z \to \bar{f}f$ were studied at LEP1 for vector and axial couplings [28]. This leaves only the dipole couplings to fermions, which interfere with the Standard Model with a rate proportional to the fermion mass [29]. We will now discuss the on-shell three-body decay modes of Z bosons that are allowed by the SM, which are

$$Z \to (\gamma\gamma\gamma, \gamma gg, ggg, \bar{f}f\gamma, \bar{f}fg). \qquad (20)$$

These decay modes occur in the Standard Model at higher order, so there can be interference with the new amplitudes.

We ignore the masses of the fermions, so that the mass of the Z is the only relevant energy scale. Then we can approximate the new decay amplitudes as

$$\mathcal{M}_{\mathcal{O}}(Z \to 3) \simeq \frac{c_{\mathcal{O}}}{v^{d_{\mathcal{O}}-4}} m_Z^{d_{\mathcal{O}}-4}, \tag{21}$$

where v is the Higgs VEV, $c_{\mathcal{O}}$ are couplings, and m_Z is the mass of the Z boson. If the SM amplitude is larger than the BSM one, then interference between the SM and BSM amplitudes forms the most significant contribution to the total decay amplitude. Making the same approximations as in [7], we estimate that the branching ratios including interference are

$$\mathrm{BR}(Z \to 3)_{\mathrm{BSM}} \approx \frac{m_Z}{512\pi^3\Gamma_Z} |\mathcal{M}(Z \to 3)_{\mathrm{SM}} + \mathcal{M}(Z \to 3)_{\mathrm{BSM}}|^2, \tag{22}$$

where we have approximated both the SM and BSM amplitudes as constants.

V. INDEPENDENT AMPLITUDES FOR ELECTROWEAK GAUGE BOSONS

In this section, we discuss the independent primary operators for $VVVV$ interaction amplitudes. We will also check the number of operators and redundancies with the Hilbert series for each interaction. In the second column of Tables I–XXI, we list the operators for the primary amplitudes. In addition, we give the amplitude's CP transformation, the dimension of the operator, the schematic form of a SMEFT operator realization, and the unitarity bounds on the coupling strength. An example of how these unitarity bounds are calculated can be found in Sec. IV A. Primary operators and their descendants are $\mathrm{SU}(3)_c \times \mathrm{U}(1)_{em}$ invariant so the covariant derivatives only involve the gluon and photon, whereas the covariant derivatives for the SMEFT operators are $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ invariant. Finally, for operators that have nontrivial SU(3) contractions, we will add a column to specify this. In the following, we discuss each interaction's table(s) in detail and describe how the amplitudes and their redundancies agree with the Hilbert series in Eq. (9).

Tables I and II list the primary operators for $W^+W^+W^-W^-$ interactions up to dimension ten. To analyze the amplitude, we assume the process $W_1^+W_2^+ \to W_3^+W_4^+$. In terms of primary operators, the Hilbert series predicts that there should be 2 dimension-four operators, 16 dimension-six operators, 22 dimension-eight operators, 7 dimension-ten operators, and 2 redundancies appearing at dimension 12. From our amplitude enumeration procedure, we find agreement with the Hilbert series and find that the dimension-12 descendant operators $s^2\mathcal{O}_{25}^{WWWW}$ and $s^2\mathcal{O}_{35}^{WWWW}$ are redundant, with $s = (p_{W_1^+} + p_{W_2^+})^2$. Because descendants of redundant operators continue to be redundant, the operators $s^n(t-u)^{2m}\mathcal{O}_{25}^{WWWW}$ and $s^n(t-u)^{2m}\mathcal{O}_{35}^{WWWW}$, for $n \geq 2$, $m \geq 0$ should be removed in order to form a set of independent operators. Remember that, since $t = (p_{W_1^+} - p_{W_3^+})^2$ and $u = (p_{W_1^+} - p_{W_4^+})^2$,

TABLE I. Primary dimension-four and -six operators for $W^+W^+W^-W^-$ interactions.

i	$\mathcal{O}_i^{WWWW}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
1	$W^{+\mu}W_\mu^+W^{-\nu}W_\nu^-$	$+$	4	D^4H^4	$\frac{0.09}{E_{\mathrm{TeV}}^4}$
2	$W^{+\mu}W^{+\nu}W_\mu^-W_\nu^-$	$+$			
3	$D^\rho W^{+\mu}W^{+\nu}\left(W_\mu^-\overleftrightarrow{D}_\rho W_\nu^-\right) + \mathrm{H.c.}$	$+$	6	D^6H^4	$\frac{0.006}{E_{\mathrm{TeV}}^6}$
4	$D_\nu W^{+\mu}D_\mu W^{+\nu}W^{-\rho}W_\rho^- + \mathrm{H.c.}$	$+$			
5	$W^{+\mu}W^{+\nu}W^{-\rho}D_{\mu\nu}W_\rho^- + \mathrm{H.c.}$	$+$			
6	$D_\nu W^{+\mu}D_\mu W^{+\rho}W^{-\nu}W_\rho^- + \mathrm{H.c.}$	$+$			
7	$W^{+\mu}W^{+\nu}D_{\mu\rho}W_\nu^-W^{-\rho} + \mathrm{H.c.}$	$+$			
8	$D_\rho W^{+\mu}W^{+\nu}D_\mu W^{-\rho}W_\nu^-$	$+$			
9	$i\varepsilon^{\mu\nu\rho\sigma}D_\mu W^{+\alpha}W_\nu^+D_\alpha W_\rho^-W_\sigma^- + \mathrm{H.c.}$	$+$		εD^6H^4	
10	$iD_\nu W^{+\mu}D_\mu W^{+\nu}W^{-\rho}W_\rho^- + \mathrm{H.c.}$	$-$		D^6H^4	
11	$iW^{+\mu}W^{+\nu}W^{-\rho}D_{\mu\nu}W_\rho^- + \mathrm{H.c.}$	$-$			
12	$iD_\nu W^{+\mu}D_\mu W^{+\rho}W^{-\nu}W_\rho^- + \mathrm{H.c.}$	$-$			
13	$\varepsilon^{\mu\nu\rho\sigma}D_\alpha W_\mu^+W_\nu^+\left(W_\rho^-\overleftrightarrow{D}^\alpha W_\sigma^-\right) + \mathrm{H.c.}$	$-$		εD^6H^4	
14	$\varepsilon^{\mu\nu\rho\sigma}D_\mu W^{+\alpha}W_\nu^+D_\alpha W_\rho^-W_\sigma^- + \mathrm{H.c.}$	$-$			
15	$iW^{+\mu}\tilde{W}_{\nu\rho}^+D_\mu W^{-\nu}W^{-\rho} + \mathrm{H.c.}$	$+$	6	$D^4H^4\tilde{W}_{\mu\nu}^a$	$\frac{0.02}{E_{\mathrm{TeV}}^5}, \frac{0.07}{E_{\mathrm{TeV}}^6}$
16	$iD^\mu W^{+\nu}W_\nu^+\tilde{W}_{\mu\rho}^-W^{-\rho} + \mathrm{H.c.}$	$+$			
17	$W^{+\mu}\tilde{W}_{\nu\rho}^+D_\mu W^{-\nu}W^{-\rho} + \mathrm{H.c.}$	$-$			
18	$D^\mu W^{+\nu}W_\nu^+\tilde{W}_{\mu\rho}^-W^{-\rho} + \mathrm{H.c.}$	$-$			

TABLE II. Primary dimension-eight and -ten operators for the $W^+W^+W^-W^-$ interaction. There are two redundancies that appear at dimension 12 such that, in order to form a set of independent operators, $s^n(t-u)^{2m}\mathcal{O}_{25}^{WWWW}$ and $s^n(t-u)^{2m}\mathcal{O}_{35}^{WWWW}$, with $n\geq 2$, $m\geq 0$, should be omitted.

i	$\mathcal{O}_i^{WWWW}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
19	$D_\sigma W^{+\mu} D_\mu W^{+\nu}\left(D_\nu W^{-\rho}\overleftrightarrow{D}^{\sigma} W^-_\rho\right)+\text{H.c.}$	+			
20	$D_{\rho\sigma} W^{+\mu} D_\mu W^{+\nu}\left(W^{-\rho}\overleftrightarrow{D}^{\sigma} W^-_\nu\right)+\text{H.c.}$	+			
21	$D_{\rho\sigma} W^+_\mu W^+_\nu\left(D^\mu W^{-\rho}\overleftrightarrow{D}^{\sigma} W^{-\nu}\right)+\text{H.c.}$	+		D^8H^4	
22	$D_{\nu\rho\sigma} W^{+\mu} D_\mu W^{+\nu} W^{-\rho} W^{-\sigma}+\text{H.c.}$	+			
23	$D_{\nu\rho} W^{+\mu} D_{\mu\sigma} W^{+\nu} W^{-\rho} W^{-\sigma}+\text{H.c.}$	+			
24	$D_{\rho} W^{+\mu} D_{\mu\sigma} W^{+\nu} D_\nu W^{-\rho} W^{-\sigma}+\text{H.c.}$	+			
25	$D_{\rho\sigma} W^{+\mu} W^{+\nu} D_{\mu\nu} W^{-\rho} W^{-\sigma}+\text{H.c.}$	+			
26	$\varepsilon^{\mu\nu\rho\sigma} D_{\mu\alpha} W^{+\beta} D_\beta W^+_\nu\left(iW^-_\rho\overleftrightarrow{D}^{\alpha} W^-_\sigma\right)+\text{H.c.}$	+		εD^8H^4	
27	$\varepsilon^{\mu\nu\rho\sigma} D_{\mu\alpha} W^{+\beta} D_\nu W^+_\beta\left(iW^-_\rho\overleftrightarrow{D}^{\alpha} W^-_\sigma\right)+\text{H.c.}$	+	8		$\frac{3\times10^{-4}}{E_{\text{TeV}}^8}$
28	$D_{\rho\sigma} W^{+\mu} D_\mu W^{+\nu}\left(iW^{-\rho}\overleftrightarrow{D}^{\sigma} W^-_\nu\right)+\text{H.c.}$	−			
29	$D_\sigma W^{+\mu} D_\mu W^{+\nu}\left(iD_\nu W^{-\rho}\overleftrightarrow{D}^{\sigma} W^-_\rho\right)+\text{H.c.}$	−		D^8H^4	
30	$iD_{\nu\rho\sigma} W^{+\mu} D_\mu W^{+\nu} W^{-\rho} W^{-\sigma}+\text{H.c.}$	−			
31	$\varepsilon^{\mu\nu\rho\sigma} D_{\mu\alpha} W^{+\beta} D_\beta W^+_\nu\left(W^-_\rho\overleftrightarrow{D}^{\alpha} W^-_\sigma\right)+\text{H.c.}$	−			
32	$\varepsilon^{\mu\nu\rho\sigma} D_{\mu\alpha} W^{+\beta} D_\nu W^+_\beta\left(W^-_\rho\overleftrightarrow{D}^{\alpha} W^-_\sigma\right)+\text{H.c.}$	−			
33	$\varepsilon^{\mu\nu\rho\sigma} D_{\mu\alpha} W^{+\beta} W^+_\nu\left(D_\rho W^-_\beta\overleftrightarrow{D}^{\alpha} W^-_\sigma\right)+\text{H.c.}$	−		εD^8H^4	
34	$\varepsilon^{\mu\nu\rho\sigma} D_\mu W^{+\alpha} D_{\nu\alpha} W^{+\beta} D_\beta W^-_\rho W^-_\sigma+\text{H.c.}$	−			
35	$\varepsilon^{\mu\nu\rho\sigma} D_{\mu\beta} W^{+\alpha} W^+_\nu D_{\rho\alpha} W^{-\beta} W^-_\sigma$	−			
36	$D_\sigma W^{+\mu} D_\mu W^+_{\nu\rho}\left(W^{-\rho}\overleftrightarrow{D}^{\sigma} W^{-\nu}\right)+\text{H.c.}$	+		$D^6H^4W^a_{\mu\nu}$	
37	$D_\sigma W^{+\mu} D_\mu \tilde{W}^+_{\nu\rho}\left(iW^{-\nu}\overleftrightarrow{D}^{\sigma} W^{-\rho}\right)+\text{H.c.}$	+			
38	$iD^{\nu\rho} W^{+\mu} D_\mu W^+_\nu \tilde{W}^-_{\rho\sigma} W^{-\sigma}+\text{H.c.}$	+	8	$D^6H^4\tilde{W}^a_{\mu\nu}$	$\frac{0.001}{E_{\text{TeV}}^7}$, $\frac{0.004}{E_{\text{TeV}}^8}$
39	$iD^{\rho} W^{+\mu} W^{+\nu} D_{\mu\nu} \tilde{W}^-_{\rho\sigma} W^{-\sigma}+\text{H.c.}$	+			
40	$D^{\rho} W^{+\mu} W^{+\nu} D_{\mu\nu} \tilde{W}^-_{\rho\sigma} W^{-\sigma}+\text{H.c.}$	−			
41	$D_{\nu\rho\alpha} W^{+\mu} D_{\mu\sigma} W^{+\nu}\left(W^{-\rho}\overleftrightarrow{D}^{\alpha} W^{-\sigma}\right)+\text{H.c.}$	+		$D^{10}H^4$	
42	$D_{\rho\sigma\alpha} W^{+\mu} D_\mu W^{+\nu}\left(D_\nu W^{-\rho}\overleftrightarrow{D}^{\alpha} W^{-\sigma}\right)+\text{H.c.}$	+			
43	$\varepsilon^{\mu\nu\rho\sigma} D_{\mu\beta\tau} W^{+\alpha} D_{\nu\alpha} W^{+\beta}\left(iW^-_\rho\overleftrightarrow{D}^{\tau} W^-_\sigma\right)+\text{H.c.}$	+		$\varepsilon D^{10}H^4$	
44	$\varepsilon^{\mu\nu\rho\sigma} D_{\mu\tau} W^{+\alpha} D_\nu W^{+\beta}\left(iD_{\alpha\beta} W^-_\rho\overleftrightarrow{D}^{\tau} W^-_\sigma\right)+\text{H.c.}$	+	10		$\frac{2\times10^{-5}}{E_{\text{TeV}}^{10}}$
45	$D_{\nu\rho\alpha} W^{+\mu} D_{\mu\sigma} W^{+\nu}\left(iW^{-\rho}\overleftrightarrow{D}^{\alpha} W^{-\sigma}\right)+\text{H.c.}$	−		$D^{10}H^4$	
46	$D_{\rho\sigma\alpha} W^{+\mu} D_\mu W^{+\nu}\left(iD_\nu W^{-\rho}\overleftrightarrow{D}^{\alpha} W^{-\sigma}\right)+\text{H.c.}$	−			
47	$\varepsilon^{\mu\nu\rho\sigma} D_{\mu\beta\tau} W^{+\alpha} D_{\nu\alpha} W^{+\beta}\left(W^-_\rho\overleftrightarrow{D}^{\tau} W^-_\sigma\right)+\text{H.c.}$	−		$\varepsilon D^{10}H^4$	

TABLE III. Primary dimension-four and -six operators for the ZZW^+W^- interaction.

i	$\mathcal{O}_i^{ZZWW}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
1	$Z^\mu Z_\mu W^{+\nu} W^-_\nu$	+	4	D^4H^4	$\frac{0.09}{E^4_{\text{TeV}}}$
2	$Z^\mu Z^\nu W^+_\mu W^-_\nu$	+			
3	$\partial^\rho Z^\mu Z^\nu \left(W^+_\mu \overleftrightarrow{D}_\rho W^-_\nu + \text{H.c.}\right)$	+	6	D^6H^4	$\frac{0.006}{E^6_{\text{TeV}}}$
4	$\partial_\nu Z^\mu \partial_\mu Z^\nu W^{+\rho} W^-_\rho$	+			
5	$Z^\mu Z^\nu \left(W^{+\rho} D_{\mu\nu} W^-_\rho + \text{H.c.}\right)$	+			
6	$\partial_\rho Z^\mu \partial_\mu Z^\nu \left(W^{+\rho} W^-_\nu + \text{H.c.}\right)$	+			
7	$Z^\mu \partial_{\mu\rho} Z^\nu \left(W^{+\rho} W^-_\nu + \text{H.c.}\right)$	+			
8	$\partial^\rho Z^\mu Z^\nu \left(W^+_\nu D_\mu W^-_\rho + \text{H.c.}\right)$	+			
9	$Z^\mu \partial^\rho Z^\nu \left(W^+_\nu D_\mu W^-_\rho + \text{H.c.}\right)$	+			
10	$\partial_{\nu\rho} Z^\mu Z_\mu (W^{+\nu} W^{-\rho})$	+			
11	$\partial_\nu Z^\mu \partial_\rho Z_\mu (W^{+\nu} W^{-\rho} + \text{H.c.})$	+			
12	$i\varepsilon^{\mu\nu\rho\sigma} \partial_\mu Z^\alpha \partial_\alpha Z_\nu W^+_\rho W^-_\sigma$	+		εD^6H^4	
13	$\varepsilon^{\mu\nu\rho\sigma} \partial^\alpha Z_\mu Z_\nu \left(iW^+_\rho D_\sigma W^-_\alpha + \text{H.c.}\right)$	+			
14	$\varepsilon^{\mu\nu\rho\sigma} \partial_\mu Z^\alpha Z_\nu \left(iW^+_\rho D_\sigma W^-_\alpha + \text{H.c.}\right)$	+			
15	$Z^\mu \partial_\mu Z^\nu \left(iW^{+\rho} D_\nu W^-_\rho + \text{H.c.}\right)$	−		D^6H^4	
16	$\partial^\mu Z^\nu \partial_\nu Z^\rho \left(iW^+_\mu W^-_\rho + \text{H.c.}\right)$	−			
17	$\partial^\mu Z^\nu Z^\rho \left(iW^+_\rho D_\nu W^-_\mu + \text{H.c.}\right)$	−			
18	$Z^\mu \partial^\rho Z^\nu \left(iW^+_\nu D_\mu W^-_\rho + \text{H.c.}\right)$	−			
19	$\varepsilon^{\mu\nu\rho\sigma} \partial_\alpha Z_\mu Z_\nu \left(W^+_\rho \overleftrightarrow{D}^\alpha W^-_\sigma + \text{H.c.}\right)$	−		εD^6H^4	
20	$\varepsilon^{\mu\nu\rho\sigma} \partial_\mu Z^\alpha Z_\nu \left(W^+_\rho D_\alpha W^-_\sigma + \text{H.c.}\right)$	−			
21	$iZ^\alpha \partial_\alpha \tilde{Z}^{\mu\nu} W^+_\mu W^-_\nu$	+	6	$D^4H^4\tilde{W}^a_{\mu\nu}$	$\frac{0.02}{E^5_{\text{TeV}}}$, $\frac{0.07}{E^6_{\text{TeV}}}$
22	$Z^\mu \tilde{Z}^{\nu\rho} \left(iW^+_\nu D_\rho W^-_\mu + \text{H.c.}\right)$	+			
23	$Z^\mu Z^\nu \left(iW^{+\rho} D_\mu \tilde{W}^-_{\nu\rho} + \text{H.c.}\right)$	+			
24	$Z^\mu \partial_\mu Z^{\nu\rho} \left(iW^+_\nu W^-_\rho + \text{H.c.}\right)$	−		$D^4H^4W^a_{\mu\nu}$	
25	$\partial^\rho \tilde{Z}^{\mu\nu} Z_\mu \left(W^+_\rho W^-_\nu + \text{H.c.}\right)$	−		$D^4H^4\tilde{W}^a_{\mu\nu}$	
26	$Z^\mu \tilde{Z}^{\nu\rho} \left(W^+_\nu D_\mu W^-_\rho + \text{H.c.}\right)$	−			
27	$\partial_\nu Z^\mu \tilde{Z}^{\nu\rho} \left(W^+_\mu W^-_\rho + \text{H.c.}\right)$	−			
28	$Z^\mu \partial_\mu Z_\nu \left(W^+_\rho \tilde{W}^{-\nu\rho} + \text{H.c.}\right)$	−			
29	$\partial_\mu Z^\rho Z_\rho \left(W^+_\nu \tilde{W}^{-\mu\nu} + \text{H.c.}\right)$	−			

s and $(t-u)^2$ are the Mandelstam invariants that respect the interchanges $W_1^+ \leftrightarrow W_2^+$ and $W_3^+ \leftrightarrow W_4^+$.

In Tables III–V we list the primary operators for the ZZW^+W^- interactions up to dimension ten after considering the process $ZZ \to W^+W^-$. The Hilbert series states that for these there should be 2 dimension-four operators, 27 dimension-six operators, 40 dimension-eight operators, 14 dimension-ten operators, and 2 redundancies that appear at dimension 12. Our findings are in agreement with the Hilbert series. The redundancies and their descendants are given by the operators $s^n(t-u)^{2m}\mathcal{O}_{39}^{ZZWW}$ and $s^n(t-u)^{2m}\mathcal{O}_{54}^{ZZWW}$, for $n \geq 2$ and $m \geq 0$. To form a set independent operators, they should all be omitted.

We list the primary operators for $ZZZZ$ interactions up to dimension 12 in Table VI. We achieve full agreement with the Hilbert series, from which we expect that there should be 1 dimension-four operator, 4 dimension-six operators, 8 dimension-eight operators, 11 dimension-ten operators, 5 dimension-12 operators, and 2 redundancies appearing at dimension 14. Operators $x^n y^m \mathcal{O}_{22}^{ZZZZ}$ and $x^n y^m \mathcal{O}_{23}^{ZZZZ}$ for $n \geq 1$, $m \geq 0$ are redundant, where $x = s^2 + t^2 + u^2$ and $y = stu$, and should not be included in a set of independent operators.

In Tables VII and VIII, by using the process $W^+W^- \to Z\gamma$, we enumerate the primary operators for the $W^+W^-Z\gamma$ interaction up to dimension ten. We agree

TABLE IV. Primary dimension-eight operators for the ZZW^+W^- interaction.

i	$\mathcal{O}_i^{ZZWW}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
30	$\partial^\sigma Z^\mu \partial_\mu Z^\nu \left(W^{+\rho} \overleftrightarrow{D}_\sigma D_\nu W^-_\rho + \text{H.c.}\right)$	+	8	D^8H^4	$\frac{3\times10^{-4}}{E_{\text{TeV}}^8}$
31	$\partial_{\rho\sigma} Z^\mu \partial_\mu Z^\nu \left(W^{+\rho} \overleftrightarrow{D}^\sigma W^-_\nu + \text{H.c.}\right)$	+			
32	$\partial^\sigma Z^\mu \partial^\rho Z^\nu \left(W^+_\mu \overleftrightarrow{D}_\sigma D_\nu W^-_\rho + \text{H.c.}\right)$	+			
33	$\partial^{\rho\sigma} Z^\mu Z^\nu \left(W^+_\mu \overleftrightarrow{D}_\sigma D_\nu W^-_\rho + \text{H.c.}\right)$	+			
34	$\partial^{\nu\sigma} Z^\mu \partial^\rho Z_\mu \left(W^+_\nu \overleftrightarrow{D}_\sigma W^-_\rho + \text{H.c.}\right)$	+			
35	$\partial_{\nu\rho\sigma} Z^\mu \partial_\mu Z^\nu (W^{+\rho} W^{-\sigma})$	+			
36	$\partial_{\nu\rho} Z^\mu \partial_{\mu\sigma} Z^\nu (W^{+\rho} W^{-\sigma})$	+			
37	$\partial_\sigma Z^\mu \partial_{\mu\rho} Z^\nu \left(W^{+\rho} D_\nu W^{-\sigma} + \text{H.c.}\right)$	+			
38	$\partial_{\rho\sigma} Z^\mu Z^\nu \left(W^{+\rho} D_{\mu\nu} W^{-\sigma} + \text{H.c.}\right)$	+			
39	$\partial_\sigma Z^\mu \partial_\rho Z^\nu \left(W^{+\rho} D_{\mu\nu} W^{-\sigma} + \text{H.c.}\right)$	+			
40	$\varepsilon^{\mu\nu\rho\sigma} \partial_{\mu\alpha} Z^\beta Z_\nu \left(iW^+_\rho \overleftrightarrow{D}^\alpha D_\beta W^-_\sigma + \text{H.c.}\right)$	+		εD^8H^4	
41	$\varepsilon^{\mu\nu\rho\sigma} \partial_{\mu\beta} Z^\alpha \partial_\alpha Z_\nu \left(iW^+_\rho D_\sigma W^{-\beta} + \text{H.c.}\right)$	+			
42	$\partial_\sigma Z^\mu \partial_{\mu\rho} Z^\nu \left(iW^{+\rho} \overleftrightarrow{D}^\sigma W^-_\nu + \text{H.c.}\right)$	−		D^8H^4	
43	$\partial^\sigma Z^\mu \partial^\rho Z^\nu \left(iW^+_\mu \overleftrightarrow{D}_\sigma D_\nu W^-_\rho + \text{H.c.}\right)$	−			
44	$\partial^{\rho\sigma} Z^\mu Z^\nu \left(iW^+_\mu \overleftrightarrow{D}_\sigma D_\nu W^-_\rho + \text{H.c.}\right)$	−			
45	$\partial^{\rho\sigma} Z^\mu \partial_\mu Z^\nu \left(iW^+_\nu \overleftrightarrow{D}_\sigma W^-_\rho + \text{H.c.}\right)$	−			
46	$\partial^{\nu\rho\sigma} Z^\mu Z_\mu \left(iW^+_\nu \overleftrightarrow{D}_\sigma W^-_\rho + \text{H.c.}\right)$	−			
47	$\partial^{\rho\sigma} Z^\mu \partial_\mu Z^\nu \left(iW^+_\rho D_\nu W^-_\sigma + \text{H.c.}\right)$	−			
48	$\partial^\sigma Z^\mu \partial_{\mu\rho} Z^\nu \left(iW^{+\rho} D_\nu W^-_\sigma + \text{H.c.}\right)$	−			
49	$Z^\mu \partial_{\mu\rho\sigma} Z^\nu \left(iW^{+\rho} D_\nu W^{-\sigma} + \text{H.c.}\right)$	−			
50	$\varepsilon^{\mu\nu\rho\sigma} \partial_{\mu\alpha} Z^\beta \partial_\beta Z_\nu \left(W^+_\rho \overleftrightarrow{D}^\alpha W^-_\sigma\right)$	−		εD^8H^4	
51	$\varepsilon^{\mu\nu\rho\sigma} \partial_{\mu\alpha} Z^\beta \partial_\nu Z_\beta \left(W^+_\rho \overleftrightarrow{D}^\alpha W^-_\sigma\right)$	−			
52	$\varepsilon^{\mu\nu\rho\sigma} \partial_\alpha Z_\mu \partial_\nu Z^\beta \left(W^+_\rho \overleftrightarrow{D}^\alpha D_\sigma W^-_\beta + \text{H.c.}\right)$	−			
53	$\varepsilon^{\mu\nu\rho\sigma} \partial_\mu Z^\alpha \partial_{\nu\alpha} Z^\beta \left(W^+_\rho D_\beta W^-_\sigma + \text{H.c.}\right)$	−			
54	$\varepsilon^{\mu\nu\rho\sigma} \partial_{\mu\beta} Z^\alpha Z_\nu \left(W^+_\rho D_{\sigma\alpha} W^{-\beta} + \text{H.c.}\right)$	−			

(Table continued)

TABLE IV. *(Continued)*

i	$\mathcal{O}_i^{ZZWW}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
55	$\partial^\sigma Z^\mu \partial_\mu Z^{\nu\rho}\left(W_\nu^+ \overleftrightarrow{D}_\sigma W_\rho^-\right)$	$+$		$D^6H^4W^a_{\mu\nu}$	
56	$\partial_{\alpha\beta}\tilde{Z}^{\mu\nu}Z_\mu\left(iW^{+\beta}\overleftrightarrow{D}^\alpha W_\nu^- + \text{H.c.}\right)$	$+$			
57	$\partial_\alpha Z^\beta \tilde{Z}^{\mu\nu}\left(iW_\mu^+\overleftrightarrow{D}^\alpha D_\beta W_\nu^- + \text{H.c.}\right)$	$+$			
58	$\partial_{\mu\alpha}Z^\beta \tilde{Z}^{\mu\nu}\left(iW_\beta^+\overleftrightarrow{D}^\alpha W_\nu^- + \text{H.c.}\right)$	$+$			
59	$\partial_{\mu\beta}Z^\alpha \tilde{Z}^{\mu\nu}\left(iW_\nu^+ D_\alpha W^{-\beta} + \text{H.c.}\right)$	$+$			
60	$\partial_\alpha Z^\beta \partial_\beta Z_\mu\left(iW_\nu^+\overleftrightarrow{D}^\alpha \tilde{W}^{-\mu\nu} + \text{H.c.}\right)$	$+$			
61	$\partial_{\mu\alpha}Z^\beta Z_\beta\left(iW_\nu^+\overleftrightarrow{D}^\alpha \tilde{W}^{-\mu\nu} + \text{H.c.}\right)$	$+$			
62	$\partial_\mu Z^\alpha \partial_\alpha Z^\beta\left(iW_\nu^+ D_\beta \tilde{W}^{-\mu\nu} + \text{H.c.}\right)$	$+$	8	$D^6H^4\tilde{W}^a_{\mu\nu}$	$\frac{0.001}{E_{\text{TeV}}^7}, \frac{0.004}{E_{\text{TeV}}^8}$
63	$Z^\alpha \partial_{\mu\alpha}Z^\beta\left(iW_\nu^+ D_\beta \tilde{W}^{-\mu\nu} + \text{H.c.}\right)$	$+$			
64	$\partial^\alpha \tilde{Z}^{\mu\nu}Z^\beta\left(W_\mu^+\overleftrightarrow{D}_\alpha D_\nu W_\beta^- + \text{H.c.}\right)$	$-$			
65	$\partial^\alpha Z^\rho \partial_\rho \tilde{Z}^{\mu\nu}\left(W_\mu^+\overleftrightarrow{D}_\alpha W_\nu^-\right)$	$-$			
66	$\partial_{\nu\rho}Z^\mu \partial_\mu \tilde{Z}^{\nu\sigma}\left(W^{+\rho}W_\sigma^- + \text{H.c.}\right)$	$-$			
67	$\partial^\alpha Z^\mu Z^\beta\left(W^{+\nu}\overleftrightarrow{D}_\alpha D_\beta \tilde{W}_{\mu\nu}^- + \text{H.c.}\right)$	$-$			
68	$\partial_{\mu\beta}Z^\alpha \partial_\alpha Z^\beta\left(W_\nu^+\tilde{W}^{-\mu\nu} + \text{H.c.}\right)$	$-$			
69	$\partial_\mu Z^\alpha Z^\beta\left(W_\nu^+ D_{\alpha\beta}\tilde{W}^{-\mu\nu} + \text{H.c.}\right)$	$-$			

with the Hilbert series that there are 22 dimension-six operators, 34 dimension-eight operators, 2 dimension-ten operators, and 4 redundancies that appear at dimension ten. These redundancies and their descendants correspond to operators $s^n t^m \mathcal{O}_{44}^{WW\gamma Z}$, $s^n t^m \mathcal{O}_{49}^{WW\gamma Z}$, $s^n t^m \mathcal{O}_{54}^{WW\gamma Z}$, and $s^n t^m \mathcal{O}_{55}^{WW\gamma Z}$, with $n \geq 1$, $m \geq 0$. To form a complete set of independent operators they should be removed.

In Tables IX and X we list the primary operators for the $ZZZ\gamma$ interaction up to dimension 14. We obtain full agreement with the Hilbert series. There are 4 dimension-six operators, 14 dimension-eight operators, 22 dimension-ten operators, 12 dimension-12 operators, 4 dimension-14 operators, and 2 redundancies that appear at dimension 14. To form a set of independent operators, $x^n y^m \mathcal{O}_{28}^{ZZZ\gamma}$ and $x^n y^m \mathcal{O}_{29}^{ZZZ\gamma}$, with $x = s^2 + t^2 + u^2$, $y = stu$, $n \geq 1$, and $m \geq 0$, should be omitted.

The primary operators for the $W^+W^-\gamma\gamma$ interaction up to dimension 12 are listed in Table XI and they agree with the expectations from the Hilbert series. There are 3 dimension-six operators, 19 dimension-eight operators, 14 dimension-ten operators, 2 dimension-12 operators, and 2 redundancies that show up at dimension 12. Note that, in this case, the coefficient for q^{12} exactly cancels between the two operators and two redundancies. The following operators and their descendants should be removed to maintain an independent set of operators: $s^n(t-u)^{2m}\mathcal{O}_9^{WW\gamma\gamma}$ and $s^n(t-u)^{2m}\mathcal{O}_{17}^{WW\gamma\gamma}$, with $n \geq 0$, and $m \geq 1$. The primary operators for the W^+W^-gg interaction up to dimension 12 can be obtained by replacing $F_{\mu\nu} \to G^A_{\mu\nu}$ and contracting the SU(3) indices with δ_{AB}. The redundancies of the $W^+W^-\gamma\gamma$ interaction apply to the corresponding operators of the W^+W^-gg interaction.

Primary operators for the $ZZ\gamma\gamma$ interaction up to dimension ten are tabulated in Table XII. Our results agree with the Hilbert series, from which we expect there to be 3 dimension-six operators, 13 dimension-eight operators, 7 dimension-ten operators, and 2 redundancies appearing at dimension 12. The operators and descendants $s^n(t-u)^{2m}\mathcal{O}_9^{ZZ\gamma\gamma}$ and $s^n(t-u)^{2m}\mathcal{O}_{12}^{ZZ\gamma\gamma}$, with $n \geq 0$ and $m \geq 1$, should be omitted in order to form a set of independent operators. The primary operators for the $ZZgg$ interaction up to dimension 12 can be obtained by making the replacement $F_{\mu\nu} \to G^A_{\mu\nu}$ and contracting the SU(3) indices with δ_{AB}. The redundancies of the $ZZ\gamma\gamma$ interaction apply to the corresponding operators of the $ZZgg$ interaction.

Table XIII lists the primary operators for the $Z\gamma gg$ interaction up to dimension 12. Agreeing with the Hilbert series, we find 12 dimension-eight operators, 12 dimension-ten operators, 2 dimension-12 operators, and

TABLE V. Primary dimension-ten operators for the ZZW^+W^- interaction. Two redundancies, which are descendants of dimension-eight operators, appear at dimension 12. To form an independent set of operators, $s^n(t-u)^{2m}\mathcal{O}_{39}^{ZZWW}$ and $s^n(t-u)^{2m}\mathcal{O}_{54}^{ZZWW}$, with $n \geq 2$, $m \geq 0$, should be omitted.

i	$\mathcal{O}_i^{ZZWW}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
70	$\partial_{\nu\rho\alpha}Z^\mu \partial_{\mu\sigma}Z^\nu \left(W^{+\rho}\overleftrightarrow{D}^{\alpha}W^{-\sigma} + \text{H.c.}\right)$	$+$	10	$D^{10}H^4$	$\frac{2\times10^{-5}}{E_{\text{TeV}}^{10}}$
71	$\partial_{\nu\alpha}Z^\mu \partial_{\rho\sigma}Z^\nu \left(iW^{+\rho}\overleftrightarrow{D}^{\alpha}D_\mu W^{-\sigma} + \text{H.c.}\right)$	$+$			
72	$\partial_{\nu\rho\sigma\alpha}Z^\mu Z^\nu \left(W^{+\rho}\overleftrightarrow{D}^{\alpha}D_\mu W^{-\sigma} + \text{H.c.}\right)$	$+$			
73	$\partial_{\rho\alpha}Z^\mu \partial_\sigma Z^\nu \left(W^{+\rho}\overleftrightarrow{D}^{\alpha}D_{\mu\nu} W^{-\sigma} + \text{H.c.}\right)$	$+$			
74	$\varepsilon^{\mu\nu\rho\sigma}\partial_{\mu\tau}Z^\alpha \partial_{\nu\alpha}Z^\beta \left(iW^+_\rho\overleftrightarrow{D}^{\tau}D_\beta W^-_\sigma + \text{H.c.}\right)$	$+$		$\varepsilon D^{10}H^4$	
75	$\varepsilon^{\mu\nu\rho\sigma}\partial_{\mu\beta\tau}Z^\alpha Z_\nu \left(iW^+_\rho\overleftrightarrow{D}^{\tau}D_{\sigma\alpha} W^{-\beta}\right)$	$+$			
76	$\partial_{\nu\rho\sigma\alpha}Z^\mu \partial_\mu Z^\nu \left(iW^{+\rho}\overleftrightarrow{D}^{\alpha}W^{-\sigma}\right)$	$-$		$D^{10}H^4$	
77	$\partial_{\sigma\alpha}Z^\mu \partial_{\mu\rho}Z^\nu \left(iW^{+\rho}\overleftrightarrow{D}^{\alpha}D_\nu W^{-\sigma} + \text{H.c.}\right)$	$-$			
78	$\partial_{\rho\sigma\alpha}Z^\mu Z^\nu \left(iW^{+\rho}\overleftrightarrow{D}^{\alpha}D_{\mu\nu} W^{-\sigma} + \text{H.c.}\right)$	$-$			
79	$\varepsilon^{\mu\nu\rho\sigma}\partial_{\mu\beta\tau}Z^\alpha \partial_{\nu\alpha}Z^\beta \left(W^+_\rho\overleftrightarrow{D}^{\tau}W^-_\sigma\right)$	$-$		$\varepsilon D^{10}H^4$	
80	$\varepsilon^{\mu\nu\rho\sigma}\partial_{\mu\tau}Z^\alpha \partial_\nu Z^\beta \left(W^+_\rho\overleftrightarrow{D}^{\tau}D_{\alpha\beta} W^-_\sigma + \text{H.c.}\right)$	$-$			
81	$\partial_{\nu\rho\alpha}Z^\mu \partial_\mu Z^\nu \left(iW^+_\sigma\overleftrightarrow{D}^{\alpha}\tilde{W}^{-\rho\sigma} + \text{H.c.}\right)$	$+$	10	$D^8H^4\tilde{W}^a_{\mu\nu}$	$\frac{8\times10^{-5}}{E_{\text{TeV}}^{9}}, \frac{3\times10^{-4}}{E_{\text{TeV}}^{10}}$
82	$\partial_{\rho\alpha}Z^\mu Z^\nu \left(iW^+_\sigma\overleftrightarrow{D}^{\alpha}D_{\mu\nu}\tilde{W}^{-\rho\sigma} + \text{H.c.}\right)$	$+$			
83	$\partial_{\nu\rho\alpha}Z^\mu Z^\nu \left(W^+_\sigma\overleftrightarrow{D}^{\alpha}D_\mu\tilde{W}^{-\rho\sigma} + \text{H.c.}\right)$	$-$			

TABLE VI. Primary operators for the $ZZZZ$ interaction up to dimension 12. Two redundancies, which are descendants of dimension-ten operators, appear at dimension 14. To form an independent set of operators, $x^n y^m \mathcal{O}_{22}^{ZZZZ}$ and $x^n y^m \mathcal{O}_{23}^{ZZZZ}$, where $x = s^2 + t^2 + u^2$ and $y = stu$, with $n \geq 1$ and $m \geq 0$, should be removed.

i	$\mathcal{O}_i^{ZZZZ}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
1	$Z^\mu Z_\mu Z^\nu Z_\nu$	$+$	4	D^4H^4	$\frac{0.09}{E_{\text{TeV}}^4}$
2	$\partial^\rho Z^\mu Z^\nu \left(Z_\mu \overleftrightarrow{\partial}_\rho Z_\nu\right)$	$+$	6	D^6H^4	$\frac{0.006}{E_{\text{TeV}}^6}$
3	$\partial_\nu Z^\mu \partial_\mu Z^\nu Z^\rho Z_\rho$	$+$			
4	$Z^\mu Z^\nu \partial_{\mu\nu} Z^\rho Z_\rho$	$+$			
5	$\tilde{Z}^{\mu\nu} Z^\rho \partial_\rho Z_\mu Z_\nu$	$-$	6	$D^4H^4\tilde{W}^a_{\mu\nu}$	$\frac{0.02}{E_{\text{TeV}}^5}, \frac{0.07}{E_{\text{TeV}}^6}$
6	$\partial_\sigma Z^\mu \partial_\mu Z^\rho \left(\partial_\rho Z^\nu \overleftrightarrow{\partial}^{\sigma} Z_\nu\right)$	$+$	8	D^8H^4	$\frac{3\times10^{-4}}{E_{\text{TeV}}^8}$
7	$\partial_{\nu\sigma} Z^\mu \partial_\mu Z^\rho \left(Z^\nu \overleftrightarrow{\partial}^{\sigma} Z_\rho\right)$	$+$			
8	$\partial_{\nu\rho} Z^\mu \partial_{\mu\sigma} Z^\nu Z^\rho Z^\sigma$	$+$			
9	$\partial^{\rho\sigma} Z^\mu \partial_\sigma Z^\nu \left(Z_\mu \overleftrightarrow{\partial}_\rho Z_\nu\right)$	$+$			
10	$\varepsilon^{\mu\nu\rho\sigma}\partial_{\alpha\sigma} Z^\beta \partial_\beta Z_\mu \left(Z_\nu \overleftrightarrow{\partial}^{\alpha} Z_\rho\right)$	$-$		εD^8H^4	
11	$\varepsilon^{\mu\nu\rho\sigma}\partial_{\alpha\rho} Z^\beta \partial_\sigma Z_\beta \left(Z_\mu \overleftrightarrow{\partial}^{\alpha} Z_\nu\right)$	$-$			

(Table continued)

TABLE VI. *(Continued)*

i	$\mathcal{O}_i^{ZZZZ}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
12	$\partial^\sigma Z^\mu \partial_\mu Z^{\nu\rho}\left(Z_\nu \overleftrightarrow{\partial}_\sigma Z_\rho\right)$	$+$	8	$D^6H^4W^a_{\mu\nu}$	$\frac{0.001}{E^7_{\rm TeV}}, \frac{0.004}{E^8_{\rm TeV}}$
13	$\partial_{\mu\beta} Z^\alpha \partial_\alpha Z^\beta \tilde{Z}^{\mu\nu} Z_\nu$	$-$		$D^6H^4\tilde{W}^a_{\mu\nu}$	
14	$\partial_{\nu\rho\alpha} Z^\mu \partial_{\mu\sigma} Z^\nu \left(Z^\rho \overleftrightarrow{\partial}^\alpha Z^\sigma\right)$	$+$	10	$D^{10}H^4$	$\frac{2\times10^{-5}}{E^{10}_{\rm TeV}}$
15	$\partial_{\rho\sigma\alpha} Z^\mu \partial_\mu Z^\nu \left(\partial_\nu Z^\rho \overleftrightarrow{\partial}^\alpha Z^\sigma\right)$	$+$			
16	$\partial_\alpha Z^\mu \partial_{\mu\rho\sigma} Z^\nu \left(\partial_\nu Z^\rho \overleftrightarrow{\partial}^\alpha Z^\sigma\right)$	$+$			
17	$\partial^{\alpha\beta} Z^\mu \partial_{\mu\alpha} Z^\nu \left(\partial_\nu Z^\rho \overleftrightarrow{\partial}_\beta Z_\rho\right)$	$+$			
18	$\partial^{\alpha\rho\sigma} Z^\mu \partial_{\mu\alpha} Z^\nu \left(Z_\rho \overleftrightarrow{\partial}_\sigma Z_\nu\right)$	$+$			
19	$\varepsilon^{\mu\nu\rho\sigma}\partial_{\mu\beta\tau} Z^\alpha \partial_{\nu\alpha} Z^\beta \left(Z_\rho \overleftrightarrow{\partial}^\tau Z_\sigma\right)$	$-$		$\varepsilon D^{10}H^4$	
20	$\varepsilon^{\mu\nu\rho\sigma}\partial_{\mu\tau} Z^\alpha \partial_\nu Z^\beta \left(\partial_{\alpha\beta} Z_\rho \overleftrightarrow{\partial}^\tau Z_\sigma\right)$	$-$			
21	$\varepsilon^{\mu\nu\rho\sigma}\partial^{\alpha\tau}_\sigma Z^\beta \partial_{\beta\tau} Z_\mu \left(Z_\nu \overleftrightarrow{\partial}_\alpha Z_\rho\right)$	$-$			
22	$\varepsilon^{\mu\nu\rho\sigma}\partial^{\alpha\tau}_\rho Z^\beta \partial_{\sigma\tau} Z_\beta \left(Z_\mu \overleftrightarrow{\partial}_\alpha Z_\nu\right)$	$-$			
23	$\partial^{\alpha\sigma} Z^\mu \partial_{\mu\alpha} Z^{\nu\rho}\left(Z_\rho \overleftrightarrow{\partial}_\sigma Z_\nu\right)$	$+$	10	$D^8H^4W^a_{\mu\nu}$	$\frac{8\times10^{-5}}{E^9_{\rm TeV}}, \frac{3\times10^{-4}}{E^{10}_{\rm TeV}}$
24	$\partial_\tau Z^\alpha \partial_{\mu\alpha} Z^\beta \left(\partial_\beta \tilde{Z}^{\mu\sigma} \overleftrightarrow{\partial}^\tau Z_\sigma\right)$	$-$		$D^8H^4\tilde{W}^a_{\mu\nu}$	
25	$\partial^{\nu\rho\alpha\beta} Z^\mu \partial_{\mu\sigma\beta} Z_\nu \left(Z_\rho \overleftrightarrow{\partial}_\alpha Z^\sigma\right)$	$+$	12	$D^{12}H^4$	$\frac{1\times10^{-6}}{E^{12}_{\rm TeV}}$
26	$\partial^{\rho\sigma\alpha\beta} Z^\mu \partial_{\mu\beta} Z^\nu \left(\partial_\nu Z_\rho \overleftrightarrow{\partial}_\alpha Z_\sigma\right)$	$+$			
27	$\partial^{\alpha\beta} Z^\mu \partial_{\mu\rho\sigma\beta} Z^\nu \left(\partial_\nu Z^\rho \overleftrightarrow{\partial}_\alpha Z^\sigma\right)$	$+$			
28	$\varepsilon^{\mu\nu\rho\sigma}\partial^{\beta\tau\pi}_\mu Z^\alpha \partial_{\nu\alpha\pi} Z_\beta \left(Z_\rho \overleftrightarrow{\partial}_\tau Z_\sigma\right)$	$-$		$\varepsilon D^{12}H^4$	
29	$\varepsilon^{\mu\nu\rho\sigma}\partial^{\alpha\tau\pi}_\rho Z^\beta \partial_{\sigma\tau\pi} Z_\beta \left(Z_\mu \overleftrightarrow{\partial}_\alpha Z_\nu\right)$	$-$			

2 redundancies at dimension 12. To form a set of independent operators, the operators and descendants $s^n(t-u)^{2m}\mathcal{O}_4^{Z\gamma gg}$, $s^n(t-u)^{2m}\mathcal{O}_9^{Z\gamma gg}$, with $n \geq 0$, $m \geq 1$, should be removed.

The primary operators for the $Z\gamma\gamma\gamma$ interaction up to dimension 14 are enumerated in Table XIV. The Hilbert series predicts that there are 4 dimension-eight operators, 10 dimension-ten operators, 8 dimension-12 operators, 4 dimension-14 operators, and 2 redundancies that appear at dimension 14. Our results are in agreement with this prediction. In order to form a set of independent operators, the operators and descendants $x^n y^m \mathcal{O}_9^{Z\gamma\gamma\gamma}$ and $x^n y^m \mathcal{O}_{13}^{Z\gamma\gamma\gamma}$, with $x = s^2 + t^2 + u^2$, $y = stu$, $n \geq 1$, and $m \geq 0$, should be omitted. Symmetric (in any $g \leftrightarrow g$ particle exchange of their kinematic variables) primary operators for the $Zggg$ interaction up to dimension 14 can be obtained by making the replacement $F_{\mu\nu} \to G^A_{\mu\nu}$ and contracting SU(3) indices with the fully symmetric structure constant tensor d_{ABC}. The redundancies of the $Z\gamma\gamma\gamma$ interaction apply to the corresponding operators of the $Zggg$ interaction.

In Table XV, we list the antisymmetric (in any $g \leftrightarrow g$ particle exchange of kinematics) primary operators for the $Zggg$ interaction up to dimension 16. Note that the SU(3) indices of the gluon field strengths $G^A_{\mu\nu}$ are suppressed in our notation and taken to be contracted with the fully antisymmetric structure constant tensor f_{ABC}, so that under the combined color and kinematic exchange, the gluons obey Bose-Einstein statistics. When taken with the symmetric $Zggg$ operators obtained from modifications to operators in Table XIV, which was discussed in the last paragraph, our results agree with the Hilbert series. We find that there are 6 dimension-eight operators, 18 dimension-ten operators, 16 dimension-12 operators, 8 dimension-14 operators, 2 dimension-16 operators, and 2 redundancies that appear at dimension 14. These redundancies come from operators involving the fully symmetric structure

TABLE VII. Primary dimension-six operators for the $W^+W^-Z\gamma$ interaction.

i	$\mathcal{O}_i^{WWZ\gamma}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
1	$\left(W^{+\mu}D_\nu W^-_\mu + \text{H.c.}\right)F^{\nu\rho}Z_\rho$	+			
2	$\left(W^+_\mu D_\rho W^-_\nu + \text{H.c.}\right)F^{\rho\mu}Z^\nu$	+			
3	$\left(W^{+\mu}D_\mu W^-_\nu + \text{H.c.}\right)F^{\nu\rho}Z_\rho$	+		$D^4H^4B^{\mu\nu}$	
4	$\left(W^{+\mu}W^{-\nu} + \text{H.c.}\right)\partial_\nu F_{\mu\rho}Z^\rho$	+			
5	$\left(W^+_\mu D_\rho W^-_\nu + \text{H.c.}\right)F^{\mu\nu}Z^\rho$	+			
6	$\left(iW^{+\mu}D_\mu W^-_\nu + \text{H.c.}\right)\tilde{F}^{\nu\rho}Z_\rho$	+		$D^4H^4\tilde{B}^{\mu\nu}$	
7	$\left(iW^{+\mu}D_\nu W^-_\mu + \text{H.c.}\right)\tilde{F}_{\nu\rho}Z_\rho$	+			
8	$\varepsilon^{\mu\nu\rho\sigma}\left(iW^+_\mu D^\alpha W^-_\nu + \text{H.c.}\right)F_{\rho\alpha}Z_\sigma$	+		$\varepsilon D^4H^4B^{\mu\nu}$	
9	$\varepsilon^{\mu\nu\rho\sigma}\left(iW^+_\mu D_\nu W^{-\alpha} + \text{H.c.}\right)F_{\rho\alpha}Z_\sigma$	+			
10	$\left(iW^{+\mu}\overset{\leftrightarrow}{D}^\rho W^-_\mu\right)F_{\rho\sigma}Z^\sigma$	−	6		$\frac{0.02}{E_{\text{TeV}}^5}$, $\frac{0.07}{E_{\text{TeV}}^6}$
11	$\left(iW^{+\mu}D^\rho W^{-\nu} + \text{H.c.}\right)F_{\mu\rho}Z_\nu$	−		$D^4H^4B^{\mu\nu}$	
12	$\left(iW^{+\mu}W^{-\nu} + \text{H.c.}\right)\partial_\nu F_{\mu\rho}Z^\rho$	−			
13	$\left(iW^{+\mu}D_\mu W^{-\nu} + \text{H.c.}\right)F_{\nu\rho}Z^\rho$	−			
14	$\left(W^{+\mu}D_\mu W^-_\nu + \text{H.c.}\right)\tilde{F}^{\nu\rho}Z_\rho$	−			
15	$\left(W^{+\mu}W^-_\nu + \text{H.c.}\right)\partial_\mu\tilde{F}^{\nu\rho}Z_\rho$	−		$D^4H^4\tilde{B}^{\mu\nu}$	
16	$\left(W^{+\mu}D_\nu W^-_\mu + \text{H.c.}\right)\tilde{F}^{\nu\rho}Z_\rho$	−			
17	$\varepsilon^{\mu\nu\rho\sigma}\left(W^+_\mu D^\alpha W^-_\nu + \text{H.c.}\right)F_{\rho\alpha}Z_\sigma$	−		$\varepsilon D^4H^4B^{\mu\nu}$	
18	$\varepsilon^{\mu\nu\rho\sigma}\left(W^+_\mu D_\nu W^{-\alpha} + \text{H.c.}\right)F_{\rho\alpha}Z_\sigma$	−			
19	$\left(W^+_\mu W^-_{\nu\rho} + \text{H.c.}\right)F^{\nu\rho}Z^\mu$	+		$D^2H^2W^a_{\mu\nu}B^{\mu\nu}$	
20	$\left(iW^{+\mu}\tilde{W}^{-\nu\rho} + \text{H.c.}\right)F_{\nu\mu}Z_\rho$	+	6	$D^2H^2\tilde{W}^a_{\mu\nu}B^{\mu\nu}$	$\frac{0.09}{E_{\text{TeV}}^4}$
21	$\left(iW^{+\mu}W^{-\nu\rho} + \text{H.c.}\right)F_{\nu\rho}Z_\mu$	−		$D^2H^2W^a_{\mu\nu}B^{\mu\nu}$	
22	$\left(W^{+\mu}\tilde{W}^{-\nu\rho} + \text{H.c.}\right)F_{\nu\mu}Z_\rho$	−		$D^2H^2\tilde{W}^a_{\mu\nu}B^{\mu\nu}$	

constant tensor d_{ABC} and can be obtained from Table XIV. They, along with their descendants, should be removed to form a set of independent operators.

In Table XVI, we have a list of the primary operators for the $\gamma\gamma\gamma\gamma$ interaction up to dimension 12. The Hilbert series predicts 3 dimension-eight operators, 5 dimension-ten operators, 1 dimension-12 operator, and 2 redundancies at dimension 14, which are the results that we find. We can form a set of independent operators by removing the descendants $x^n y^m \mathcal{O}_6^{\gamma\gamma\gamma\gamma}$ and $x^n y^m \mathcal{O}_7^{\gamma\gamma\gamma\gamma}$, with $n \geq 1$ and $m \geq 0$, $x = s^2 + t^2 + u^2$, $y = stu$ which are redundant.

In Table XVII, we can find the primary operators for the $\gamma\gamma gg$ interaction up to dimension ten. There are seven dimension-eight operators and five dimension-ten operators. This result is consistent with the corresponding Hilbert series. At dimension 12, there are two redundancies such that we should omit the descendants $s^n(t-u)^{2m}\mathcal{O}_4^{\gamma\gamma gg}$ and $s^{m+1}(t-u)^{2n}\mathcal{O}_6^{\gamma\gamma gg}$, with $n \geq 0$ and $m \geq 1$, in order to have a list of independent operators.

In Tables XVIII and XIX, we enumerate a set of primary operators for the γggg interaction up to dimension 16. We find 4 dimension-eight operators, 12 dimension-ten operators, 8 dimension-12 operators, 6 dimension-14 operators, 4 dimension-16 operators, and 2 redundancies at dimension 14, agreeing with the Hilbert series. We should remove the descendant operators $x^n y^m \mathcal{O}_8^{\gamma ggg}$, $x^n y^m \mathcal{O}_{16}^{\gamma ggg}$, with $x = s^2 + t^2 + u^2$, $y = stu$, $n \geq 1$, and $m \geq 0$ to exclude redundancies. In this case, we are including in the tables the fully symmetric and antisymmetric structure constants d_{ABC} and f_{ABC}, respectively, which implicitly contract the SU(3) indices of the $G^A_{\mu\nu}$'s.

Four-gluon scattering proceeds as in Fig. 1. An example Lagrangian operator for four-gluon interactions is $f(T^A, T^B, T^C, T^C) G^A_{\mu\nu} G^{B\nu\rho} G^C_{\rho\sigma} G^{D\sigma\mu}$, where we have chosen a structure for the contraction of the Lorentz indices and f represents a configuration of the trace of the generators. In this case, we have two ways to express f (up to trivial permutations) to keep the amplitude invariant: $\mathrm{Tr}(T^A T^B)\mathrm{Tr}(T^C T^D)$ and $\mathrm{Tr}(T^A T^B T^C T^D)$. Considering that the gluons are identical, this gives two different possible amplitudes,

$$\begin{aligned} M_{\mathrm{Tr}^2}(1A2B;3C4D) = {} & \mathrm{Tr}(T^A T^B)\mathrm{Tr}(T^C T^D) M_{\mathrm{Tr}^2}(12;34) \\ & + \mathrm{Tr}(T^A T^C)\mathrm{Tr}(T^B T^D) M_{\mathrm{Tr}^2}(13;24) \\ & + \mathrm{Tr}(T^A T^D)\mathrm{Tr}(T^B T^C) M_{\mathrm{Tr}^2}(14;23), \end{aligned} \tag{23}$$

TABLE VIII. Primary dimension-eight and -ten operators for the $W^+W^-Z\gamma$ interaction. At dimension ten there are four redundancies, which are descendants of dimension-eight operators. In order to form a set of independent operators, $s^n t^m \mathcal{O}_{44}^{WW\gamma Z}$, $s^n t^m \mathcal{O}_{49}^{WW\gamma Z}$, $s^n t^m \mathcal{O}_{54}^{WW\gamma Z}$, and $s^n t^m \mathcal{O}_{55}^{WW\gamma Z}$, with $n \geq 1$, $m \geq 0$, should be omitted.

i	$\mathcal{O}_i^{WWZ\gamma}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
23	$\left(D_\nu W^{+\mu} D_{\mu\rho} W^{-\nu} + \mathrm{H.c.}\right) F^{\rho\sigma} Z_\sigma$	+			
24	$\left(D^\rho W^{+\mu} D_\mu W^{-\nu} + \mathrm{H.c.}\right) \partial_\nu F_{\rho\sigma} Z^\sigma$	+			
25	$\left(W^+_\mu D^{\mu\rho} W^{-\nu} + \mathrm{H.c.}\right) \partial_\nu F_{\rho\sigma} Z^\sigma$	+			
26	$\left(W^{+\mu} D^\rho W^{-\nu} + \mathrm{H.c.}\right) \partial_{\mu\nu} F_{\rho\sigma} Z^\sigma$	+		$D^6 H^4 B^{\mu\nu}$	
27	$\left(D_\sigma W^{+\mu} D_{\mu\rho} W^{-\nu} + \mathrm{H.c.}\right) F_{\rho\sigma} Z_\nu$	+			
28	$\left(D_{\rho\sigma} W^{+\mu} D_\mu W^-_\nu + \mathrm{H.c.}\right) F^{\nu\rho} Z^\sigma$	+			
29	$\left(W^{+\mu} D^{\rho\sigma} W^{-\nu} + \mathrm{H.c.}\right) \partial_\nu F_{\mu\rho} Z_\sigma$	+			
30	$\left(D_{\nu\sigma} W^{+\mu} D_\rho W^-_\mu + \mathrm{H.c.}\right) F^{\nu\rho} Z^\sigma$	+			
31	$\varepsilon^{\mu\nu\rho\sigma} \left(i D^\alpha_\mu W^{+\beta} D_\beta W^-_\nu + \mathrm{H.c.}\right) F_{\rho\alpha} Z_\sigma$	+			
32	$\varepsilon^{\mu\nu\rho\sigma} \left(i D_\mu W^+_\beta D^{\alpha\beta} W^-_\nu + \mathrm{H.c.}\right) F_{\rho\alpha} Z_\sigma$	+		$\varepsilon D^6 H^4 B^{\mu\nu}$	
33	$\varepsilon^{\mu\nu\rho\sigma} \left(i W^+_\mu D^\alpha_\nu W^{-\beta} + \mathrm{H.c.}\right) \partial_\beta F_{\rho\alpha} Z_\sigma$	+	8		$\frac{0.001}{E^7_{\mathrm{TeV}}}, \frac{0.004}{E^8_{\mathrm{TeV}}}$
34	$\left(i D_\nu W^{+\mu} D_{\mu\rho} W^{-\nu} + \mathrm{H.c.}\right) F^{\rho\sigma} Z_\sigma$	−			
35	$\left(i D^\rho W^{+\mu} D_\mu W^{-\nu} + \mathrm{H.c.}\right) \partial_\nu F_{\rho\sigma} Z^\sigma$	−			
36	$\left(i W^+_\mu D^{\mu\rho} W^{-\nu} + \mathrm{H.c.}\right) \partial_\nu F_{\rho\sigma} Z^\sigma$	−			
37	$\left(i W^{+\mu} D^\rho W^{-\nu} + \mathrm{H.c.}\right) \partial_{\mu\nu} F_{\rho\sigma} Z^\sigma$	−		$D^6 H^4 B^{\mu\nu}$	
38	$\left(i D_\sigma W^{+\mu} D_{\mu\rho} W^{-\nu} + \mathrm{H.c.}\right) F^{\rho\sigma} Z_\nu$	−			
39	$\left(i D_{\rho\sigma} W^{+\mu} D_\mu W^-_\nu + \mathrm{H.c.}\right) F^{\nu\rho} Z^\sigma$	−			
40	$\left(i W^{+\mu} D^{\rho\sigma} W^{-\nu} + \mathrm{H.c.}\right) \partial_\nu F_{\mu\rho} Z_\sigma$	−			
41	$\varepsilon^{\mu\nu\rho\sigma} \left(D^\alpha_\mu W^{+\beta} D_\beta W^-_\nu + \mathrm{H.c.}\right) F_{\rho\alpha} Z_\sigma$	−			
42	$\varepsilon^{\mu\nu\rho\sigma} \left(D_\mu W^+_\beta D^{\alpha\beta} W^-_\nu + \mathrm{H.c.}\right) F_{\rho\alpha} Z_\sigma$	−		$\varepsilon D^6 H^4 B^{\mu\nu}$	
43	$\varepsilon^{\mu\nu\rho\sigma} \left(W^+_\mu D^\alpha_\nu W^{-\beta} + \mathrm{H.c.}\right) \partial_\beta F_{\rho\alpha} Z_\sigma$	−			

(Table continued)

TABLE VIII. *(Continued)*

i	$\mathcal{O}_i^{WWZ\gamma}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
44	$\left(D_\sigma W^{+\mu} D_\mu W^-_{\nu\rho} + \text{H.c.}\right) F^{\nu\rho} Z^\sigma$	$+$	8	$D^4H^2W^a_{\mu\nu}B^{\mu\nu}$	$\frac{0.006}{E^6_{\text{TeV}}}$
45	$\left(iD^\mu W^{+\nu} D_\nu \tilde{W}^{-\rho\sigma} + \text{H.c.}\right) F_{\rho\mu} Z_\sigma$	$+$		$D^4H^2\tilde{W}^a_{\mu\nu}B^{\mu\nu}$	
46	$\left(iW^+_\nu D^{\nu\mu} \tilde{W}^{-\rho\sigma} + \text{H.c.}\right) F_{\rho\nu} Z_\sigma$	$+$			
47	$\left(iD^\rho W^{+\mu} \tilde{W}^{-\nu\sigma} + \text{H.c.}\right) \partial_\mu F_{\nu\rho} Z_\sigma$	$+$			
48	$\left(iW^{+\mu} D^\rho \tilde{W}^{-\nu\sigma} + \text{H.c.}\right) \partial_\mu F_{\nu\rho} Z_\sigma$	$+$			
49	$\left(iD_\sigma W^{+\mu} D_\mu W^-_{\nu\rho} + \text{H.c.}\right) F^{\nu\rho} Z^\sigma$	$-$		$D^4H^2W^a_{\mu\nu}B^{\mu\nu}$	
50	$\left(D^\mu W^{+\nu} D_\nu \tilde{W}^{-\rho\sigma} + \text{H.c.}\right) F_{\rho\mu} Z_\sigma$	$-$		$D^4H^2\tilde{W}^a_{\mu\nu}B^{\mu\nu}$	
51	$\left(W^+_\nu D^{\nu\mu} \tilde{W}^{-\rho\sigma} + \text{H.c.}\right) F_{\rho\nu} Z_\sigma$	$-$			
52	$\left(W^{+\mu} D^\rho \tilde{W}^{-\nu\sigma} + \text{H.c.}\right) \partial_\mu F_{\nu\rho} Z_\sigma$	$-$			
53	$\left(D^\rho W^{+\mu} \tilde{W}^{-\nu\sigma} + \text{H.c.}\right) \partial_\mu F_{\nu\rho} Z_\sigma$	$-$			
54	$\left(D_\nu W^+_\mu D_\rho \tilde{W}^{-\nu\sigma} + \text{H.c.}\right) F^{\mu\rho} Z_\sigma$	$-$			
55	$\left(iD_\nu W^+_{\mu\rho} \tilde{W}^{-\nu\sigma} + \text{H.c.}\right) F^{\mu\rho} Z_\sigma$	$+$	8	$D^2H^2W^a_{\mu\nu}\tilde{W}^a_{\mu\nu}B^{\mu\nu}$	$\frac{0.02}{E^5_{\text{TeV}}}, \frac{0.07}{E^6_{\text{TeV}}}$
56	$\left(D_\nu W^+_{\mu\rho} \tilde{W}^{-\nu\sigma} + \text{H.c.}\right) F^{\mu\rho} Z_\sigma$	$-$			
57	$\left(D_{\nu\rho\sigma} W^{+\mu} D_{\mu\alpha} W^{-\nu} + \text{H.c.}\right) F^{\rho\alpha} Z^\sigma$	$+$	10	$D^8H^4B^{\mu\nu}$	$\frac{8\times10^{-5}}{E^9_{\text{TeV}}}, \frac{3\times10^{-4}}{E^{10}_{\text{TeV}}}$
58	$\left(D_{\nu\rho} W^{+\mu} D_{\mu\alpha} \tilde{W}^{-\nu\sigma} + \text{H.c.}\right) F^{\rho\alpha} Z_\sigma$	$-$	10	$D^6H^2\tilde{W}^a_{\mu\nu}B^{\mu\nu}$	$\frac{3\times10^{-4}}{E^8_{\text{TeV}}}$

TABLE IX. Primary dimension-six, -eight, and -ten operators for the $ZZZ\gamma$ interaction.

i	$\mathcal{O}_i^{ZZZ\gamma}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
1	$Z^\mu \partial_\nu Z_\mu Z_\rho F^{\nu\rho}$	$+$	6	$D^4H^4B^{\mu\nu}$	$\frac{0.02}{E^5_{\text{TeV}}}, \frac{0.07}{E^6_{\text{TeV}}}$
2	$Z^\mu \partial_\mu Z_\nu Z_\rho F^{\nu\rho}$	$+$			
3	$Z^\mu \partial_\mu Z_\nu Z_\rho \tilde{F}^{\nu\rho}$	$-$		$D^4H^4\tilde{B}^{\mu\nu}$	
4	$Z^\mu \tilde{Z}^{\nu\rho} Z_\rho F_{\nu\mu}$	$-$	6	$D^2H^2\tilde{W}^a_{\mu\nu}B^{\mu\nu}$	$\frac{0.09}{E^4_{\text{TeV}}}$
5	$\partial_\nu Z^\mu \partial_{\mu\rho} Z^\nu Z_\sigma F^{\rho\sigma}$	$+$	8	$D^6H^4B^{\mu\nu}$	$\frac{0.001}{E^7_{\text{TeV}}}, \frac{0.004}{E^8_{\text{TeV}}}$
6	$\partial^\rho Z^\mu \partial_\mu Z^\nu Z^\sigma \partial_\nu F_{\rho\sigma}$	$+$			
7	$Z_\mu \partial^{\mu\rho} Z^\nu Z^\sigma \partial_\nu F_{\rho\sigma}$	$+$			
8	$Z^\mu \partial^\rho Z^\nu Z^\sigma \partial_{\mu\nu} F_{\rho\sigma}$	$+$			
9	$\partial_\sigma Z^\mu \partial_{\mu\rho} Z^\nu Z_\nu F^{\rho\sigma}$	$+$			
10	$\partial^\sigma Z^\mu \partial_{\nu\sigma} Z_\mu Z_\rho F^{\nu\rho}$	$+$			
11	$\partial^\sigma Z_\mu \partial_{\rho\sigma} Z_\nu Z^\nu F^{\rho\mu}$	$+$			
12	$\partial^\sigma Z^\mu \partial_{\nu\sigma} Z_\mu Z_\rho \tilde{F}^{\nu\rho}$	$-$		$D^6H^4\tilde{B}^{\mu\nu}$	
13	$\varepsilon^{\mu\nu\rho\sigma} \partial^\alpha_\mu Z^\beta \partial_\beta Z_\nu Z_\sigma F_{\rho\alpha}$	$-$		$\varepsilon D^6H^4B^{\mu\nu}$	
14	$\varepsilon^{\mu\nu\rho\sigma} \partial_\mu Z_\beta \partial^{\alpha\beta} Z_\nu Z_\sigma F_{\rho\alpha}$	$-$			

(Table continued)

TABLE IX. *(Continued)*

i	$\mathcal{O}_i^{ZZZ\gamma}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
15	$\partial^\mu Z^\nu \partial_\nu \tilde{Z}^{\rho\sigma} Z_\sigma F_{\rho\mu}$	$-$			
16	$Z_\mu \partial^{\mu\nu} \tilde{Z}^{\rho\sigma} Z_\sigma F_{\rho\nu}$	$-$	8	$D^4 H^2 \tilde{W}^a_{\mu\nu} B^{\mu\nu}$	$\frac{0.006}{E^6_{\text{TeV}}}$
17	$\partial_\nu Z_\mu \partial_\rho \tilde{Z}^{\nu\sigma} Z_\sigma F^{\mu\rho}$	$-$			
18	$\partial_\nu Z_{\mu\rho} \tilde{Z}^{\nu\sigma} Z_\sigma F^{\mu\rho}$	$-$	8	$D^2 H^2 W^a_{\mu\nu} \tilde{W}^a_{\mu\nu} B^{\mu\nu}$	$\frac{0.02}{E^5_{\text{TeV}}}, \frac{0.07}{E^6_{\text{TeV}}}$
19	$\partial_{\nu\rho\sigma} Z^\mu \partial_{\mu\alpha} Z^\nu Z^\sigma F^{\rho\alpha}$	$+$			
20	$\partial^\alpha_\nu Z^\mu \partial_{\mu\rho\alpha} Z^\nu Z_\sigma F^{\rho\sigma}$	$+$			
21	$\partial^{\rho\alpha} Z^\mu \partial_{\mu\alpha} Z^\nu Z^\sigma \partial_\nu F_{\rho\sigma}$	$+$			
22	$\partial_\alpha Z_\mu \partial^{\mu\rho\alpha} Z^\nu Z^\sigma \partial_\nu F_{\rho\sigma}$	$+$			
23	$\partial_\alpha Z^\mu \partial^{\rho\alpha} Z^\nu Z^\sigma \partial_{\mu\nu} F_{\rho\sigma}$	$+$		$D^8 H^4 B^{\mu\nu}$	
24	$\partial^\alpha_\sigma Z^\mu \partial_{\mu\rho\alpha} Z^\nu Z_\nu F^{\rho\sigma}$	$+$			
25	$\partial_{\rho\sigma\alpha} Z^\mu \partial^\alpha_\mu Z_\nu Z^\sigma F^{\nu\rho}$	$+$			
26	$\partial_\alpha Z^\mu \partial^{\rho\sigma\alpha} Z^\nu Z_\sigma \partial_\nu F_{\mu\rho}$	$+$	10		$\frac{8\times10^{-5}}{E^9_{\text{TeV}}}, \frac{3\times10^{-4}}{E^{10}_{\text{TeV}}}$
27	$\partial_{\nu\sigma\alpha} Z^\mu \partial^\alpha_\rho Z_\mu Z^\sigma F^{\nu\rho}$	$+$			
28	$\partial^{\sigma\alpha} Z^\mu \partial_{\nu\sigma\alpha} Z_\mu Z_\rho F^{\nu\rho}$	$+$			
29	$\partial^{\rho\alpha} Z^\mu \partial_{\nu\rho\alpha} Z_\mu Z_\sigma \tilde{F}^{\nu\sigma}$	$-$		$D^8 H^4 \tilde{B}^{\mu\nu}$	
30	$\varepsilon^{\mu\nu\rho\sigma} \partial^{\alpha\tau}_\mu Z^\beta \partial_{\beta\tau} Z_\nu Z_\sigma F_{\rho\alpha}$	$-$			
31	$\varepsilon^{\mu\nu\rho\sigma} \partial_{\mu\tau} Z_\beta \partial^{\alpha\beta\tau} Z_\nu Z_\sigma F_{\rho\alpha}$	$-$		$\varepsilon D^8 H^4 B^{\mu\nu}$	
32	$\varepsilon^{\mu\nu\rho\sigma} \partial_\tau Z_\mu \partial^{\alpha\tau}_\nu Z^\beta Z_\sigma \partial_\beta F_{\rho\alpha}$	$-$			
33	$\partial^\alpha_\sigma Z^\mu \partial_{\mu\alpha} Z_{\nu\rho} Z^\sigma F^{\nu\rho}$	$+$		$D^6 H^2 W^a_{\mu\nu} B^{\mu\nu}$	
34	$\partial_{\nu\rho} Z^\mu \partial_{\mu\alpha} \tilde{Z}^{\nu\sigma} Z_\sigma F^{\rho\alpha}$	$-$			
35	$\partial^{\mu\alpha} Z^\nu \partial_{\nu\alpha} \tilde{Z}^{\rho\sigma} Z_\sigma F_{\rho\mu}$	$-$			
36	$\partial_\alpha Z_\mu \partial^{\mu\nu\alpha} \tilde{Z}^{\rho\sigma} Z_\sigma F_{\rho\nu}$	$-$	10		$\frac{3\times10^{-4}}{E^8_{\text{TeV}}}$
37	$\partial^{\rho\alpha} Z^\mu \partial_\alpha \tilde{Z}^{\nu\sigma} Z_\sigma \partial_\mu F_{\nu\rho}$	$-$		$D^6 H^4 \tilde{W}^a_{\mu\nu} B^{\mu\nu}$	
38	$\partial_\alpha Z^\mu \partial^{\rho\alpha} \tilde{Z}^{\nu\sigma} Z_\sigma \partial_\mu F_{\nu\rho}$	$-$			
39	$\partial^\alpha_\nu Z_\mu \partial_{\rho\alpha} \tilde{Z}^{\nu\sigma} Z_\sigma F^{\mu\rho}$	$-$			
40	$\partial_{\nu\alpha} Z_{\mu\rho} \partial^\alpha \tilde{Z}^{\nu\sigma} Z_\sigma F^{\mu\rho}$	$-$	10	$D^4 H^2 W^a_{\mu\nu} \tilde{W}^a_{\mu\nu} B^{\mu\nu}$	$\frac{0.001}{E^7_{\text{TeV}}}, \frac{0.004}{E^8_{\text{TeV}}}$

TABLE X. Primary dimension-12 and -14 operators for the $ZZZ\gamma$ interaction. At dimension 14, two descendant operators become redundant. To form a set of independent operators, $x^n y^m \mathcal{O}_{28}^{ZZZ\gamma}$ and $x^n y^m \mathcal{O}_{29}^{ZZZ\gamma}$, with $x = s^2 + t^2 + u^2$, $y = stu$, $n \geq 1$, and $m \geq 0$, should be omitted.

i	$\mathcal{O}_i^{ZZZ\gamma}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
41	$\partial_{\nu\rho\sigma\beta} Z^\mu \partial^\beta_{\mu\alpha} Z^\nu Z^\sigma F^{\rho\alpha}$	$+$			
42	$\partial^{\alpha\beta}_\nu Z^\mu \partial_{\mu\rho\alpha\beta} Z^\nu Z_\sigma F^{\rho\sigma}$	$+$			
43	$\partial^{\rho\alpha\beta} Z^\mu \partial_{\mu\alpha\beta} Z^\nu Z^\sigma \partial_\nu F_{\rho\sigma}$	$+$		$D^{10} H^4 B^{\mu\nu}$	
44	$\partial_{\alpha\beta} Z_\mu \partial^{\mu\rho\alpha\beta} Z^\nu Z^\sigma \partial_\nu F_{\rho\sigma}$	$+$			
45	$\partial_{\alpha\beta} Z^\mu \partial^{\rho\alpha\beta} Z^\nu Z^\sigma \partial_{\mu\nu} F_{\rho\sigma}$	$+$	12		$\frac{5\times10^{-6}}{E^{11}_{\text{TeV}}}, \frac{2\times10^{-5}}{E^{12}_{\text{TeV}}}$
46	$\partial_{\nu\sigma}{}^{\alpha\beta} Z^\mu \partial_{\rho\alpha\beta} Z_\mu Z^\sigma F^{\nu\rho}$	$+$			
47	$\varepsilon^{\mu\nu\rho\sigma} \partial_\mu{}^{\alpha\tau\pi} Z^\beta \partial_{\beta\tau\pi} Z_\nu Z_\sigma F_{\rho\alpha}$	$-$			
48	$\varepsilon^{\mu\nu\rho\sigma} \partial_{\mu\tau\pi} Z_\beta \partial^{\alpha\beta\tau\pi} Z_\nu Z_\sigma F_{\rho\alpha}$	$-$		$\varepsilon D^{10} H^4 B^{\mu\nu}$	
49	$\varepsilon^{\mu\nu\rho\sigma} \partial_{\tau\pi} Z_\mu \partial_\nu{}^{\alpha\tau\pi} Z^\beta Z_\sigma \partial_\beta F_{\rho\alpha}$	$-$			

(Table continued)

TABLE X. *(Continued)*

i	$\mathcal{O}_i^{ZZZ\gamma}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
50	$\partial_{\nu\rho}{}^{\beta}Z^{\mu}\partial_{\mu\alpha\beta}\tilde{Z}^{\nu\sigma}Z_{\sigma}F^{\rho\alpha}$	$-$	12	$D^8H^2\tilde{W}^a_{\mu\nu}B^{\mu\nu}$	$\frac{2\times10^{-5}}{E_{\text{TeV}}^{10}}$
51	$\partial^{\mu\alpha\beta}Z^{\nu}\partial_{\nu\alpha\beta}\tilde{Z}^{\rho\sigma}Z_{\sigma}F_{\rho\mu}$	$-$			
52	$\partial_{\nu\alpha\beta}Z_{\mu\rho}\partial^{\alpha\beta}\tilde{Z}^{\nu\sigma}Z_{\sigma}F^{\mu\rho}$	$-$	12	$D^6H^2\tilde{W}^a_{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	$\frac{8\times10^{-5}}{E_{\text{TeV}}^{9}}, \frac{3\times10^{-4}}{E_{\text{TeV}}^{10}}$
53	$\partial_{\nu\rho\sigma\beta\tau}Z^{\mu}\partial_{\mu\alpha}{}^{\beta\tau}Z^{\nu}Z^{\sigma}F^{\rho\alpha}$	$+$	14	$D^{12}H^4B^{\mu\nu}$	$\frac{3\times10^{-7}}{E_{\text{TeV}}^{13}}, \frac{9\times10^{-7}}{E_{\text{TeV}}^{14}}$
54	$\left(\partial_{\nu\alpha\beta}Z^{\mu}\overset{\leftrightarrow}{\partial}{}^{\tau}\partial_{\mu\rho}{}^{\alpha\beta}Z^{\nu}\right)Z_{\sigma}\partial_{\tau}F^{\rho\sigma}$	$+$			
55	$\varepsilon^{\mu\nu\rho\sigma}\left(\partial_{\mu}{}^{\alpha\tau\pi}Z^{\beta}\overset{\leftrightarrow}{\partial}{}^{\delta}\partial_{\beta\tau\pi}Z_{\nu}\right)Z_{\sigma}\partial_{\delta}F_{\rho\alpha}$	$-$		$\varepsilon D^{12}H^4B^{\mu\nu}$	
56	$\partial_{\nu\rho\beta\tau}Z^{\mu}\partial_{\mu\alpha}{}^{\beta\tau}\tilde{Z}^{\nu\sigma}Z_{\sigma}F^{\rho\alpha}$	$-$	14	$D^{10}H^2\tilde{W}^a_{\mu\nu}B^{\mu\nu}$	$\frac{1\times10^{-6}}{E_{\text{TeV}}^{12}}$

TABLE XI. Operators up to dimension 12 for the $W^+W^-\gamma\gamma$ interaction. At dimension 12, there are two redundancies such that, in order to form a set of independent operators, $s^n(t-u)^{2m}\mathcal{O}_9^{WW\gamma\gamma}$ and $s^n(t-u)^{2m}\mathcal{O}_{17}^{WW\gamma\gamma}$, with $n\geq 0$ and $m\geq 1$, should be omitted. Operators for W^+W^-gg interactions can be obtained by replacing $F_{\mu\nu}$'s with $G^A_{\mu\nu}$'s contracted with δ_{AB}. The same redundancies apply to the corresponding operators.

i	$\mathcal{O}_i^{WW\gamma\gamma}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
1	$W^{+\mu}W^-_{\mu}F^{\nu\rho}F_{\nu\rho}$	$+$	6	$D^2H^2B^{\mu\nu}B^{\mu\nu}$	$\frac{0.09}{E_{\text{TeV}}^{4}}$
2	$W^{+\mu}W^-_{\nu}F_{\mu\rho}F^{\nu\rho}$	$+$			
3	$\left(W^{+\mu}W^-_{\nu}+\text{H.c.}\right)F_{\mu\rho}\tilde{F}^{\nu\rho}$	$-$		$D^2H^2B^{\mu\nu}\tilde{B}^{\mu\nu}$	
4	$W^{+\mu}W^{-\nu}F^{\rho\sigma}\partial_{\mu\nu}F_{\rho\sigma}$	$+$	8	$D^4H^2B^{\mu\nu}B^{\mu\nu}$	$\frac{0.006}{E_{\text{TeV}}^{6}}$
5	$W^{+\mu}W^{-\nu}\partial_{\mu}F^{\rho\sigma}\partial_{\nu}F_{\rho\sigma}$	$+$			
6	$\left(W^{+\mu}D_{\sigma}W^{-\nu}+\text{H.c.}\right)F^{\rho\sigma}\partial_{\mu}F_{\nu\rho}$	$+$			
7	$\left(W^{+\mu}D^{\sigma}W^-_{\nu}+\text{H.c.}\right)F^{\nu\rho}\partial_{\mu}F_{\rho\sigma}$	$+$			
8	$\left(W^{+\mu}D_{\rho}{}^{\sigma}W^-_{\mu}+\text{H.c.}\right)F^{\nu\rho}F_{\nu\sigma}$	$+$			
9	$\left(W^+_{\mu}\overset{\leftrightarrow}{D}{}^{\sigma}W^{-\nu}+\text{H.c.}\right)\partial_{\sigma}F^{\mu\rho}F_{\rho\nu}$	$+$			
10	$\left(iW^+_{\mu}D^{\rho}W^{-\nu}+\text{H.c.}\right)\tilde{F}^{\mu\sigma}\partial_{\nu}F_{\sigma\rho}$	$+$		$D^4H^2B^{\mu\nu}\tilde{B}^{\mu\nu}$	
11	$\varepsilon^{\mu\nu\rho\sigma}\left(iW^+_{\mu}D_{\nu}W^{-\alpha}+\text{H.c.}\right)F_{\rho}{}^{\beta}\partial_{\alpha}F_{\sigma\beta}$	$+$		$\varepsilon D^4H^2B^{\mu\nu}B^{\mu\nu}$	
12	$\left(iW^{+\mu}D_{\sigma}W^{-\nu}+\text{H.c.}\right)F^{\rho\sigma}\partial_{\mu}F_{\nu\rho}$	$-$		$D^4H^2B^{\mu\nu}B^{\mu\nu}$	
13	$\left(iW^{+\mu}D^{\sigma}W^-_{\nu}+\text{H.c.}\right)F^{\nu\rho}\partial_{\mu}F_{\rho\sigma}$	$-$			
14	$\left(W^{+\mu}\overset{\leftrightarrow}{D}{}^{\sigma}W^-_{\nu}+\text{H.c.}\right)\partial_{\sigma}F_{\mu\rho}\tilde{F}^{\nu\rho}$	$-$		$D^4H^2B^{\mu\nu}\tilde{B}^{\mu\nu}$	
15	$\left(W^+_{\mu}W^{-\nu}+\text{H.c.}\right)\partial_{\nu}F_{\rho\sigma}\partial^{\sigma}\tilde{F}^{\mu\rho}$	$-$			
16	$\varepsilon^{\mu\nu\rho\sigma}\left(W^+_{\mu}D_{\beta}W^-_{\nu}+\text{H.c.}\right)F_{\rho\alpha}\partial_{\sigma}F^{\alpha\beta}$	$-$		$\varepsilon D^4H^2B^{\mu\nu}B^{\mu\nu}$	
17	$\varepsilon^{\mu\nu\rho\sigma}\left(W^+_{\mu}\overset{\leftrightarrow}{D}{}^{\beta}W^-_{\nu}+\text{H.c.}\right)\partial_{\beta}F^{\alpha}{}_{\rho}F_{\alpha\sigma}$	$-$			

(Table continued)

TABLE XI. *(Continued)*

i	$\mathcal{O}_i^{WW\gamma\gamma}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
18	$\left(W^{+\mu}D^{\sigma}W^{-}_{\nu\rho}+\text{H.c.}\right)F^{\nu\rho}F_{\sigma\mu}$	+	8	$D^2H^2W^a_{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	$\frac{0.02}{E^5_{\text{TeV}}},\frac{0.07}{E^6_{\text{TeV}}}$
19	$\left(iW^{+}_{\mu}D_{\rho}\tilde{W}^{-\mu\sigma}+\text{H.c.}\right)F^{\nu\rho}F_{\sigma\nu}$	+		$D^2H^2\tilde{W}^a_{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	
20	$\left(W^{+}_{\mu}D_{\rho}W^{-}_{\nu\sigma}+\text{H.c.}\right)\tilde{F}^{\mu\rho}F^{\nu\sigma}$	−		$D^2H^2W^a_{\mu\nu}B^{\mu\nu}\tilde{B}^{\mu\nu}$	
21	$\left(W^{+}_{\mu}\tilde{W}^{-\mu\nu}+\text{H.c.}\right)F^{\rho\sigma}\partial_{\sigma}F_{\nu\rho}$	−		$D^2H^2\tilde{W}^a_{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	
22	$\left(W^{+}_{\mu}D_{\sigma}\tilde{W}^{-\mu\nu}+\text{H.c.}\right)F^{\rho\sigma}F_{\nu\rho}$	−			
23	$\left(W^{+\mu}\overleftrightarrow{D}^{\alpha}W^{-\nu}+\text{H.c.}\right)\partial_{\mu\alpha}F^{\rho\sigma}\partial_{\nu}F_{\rho\sigma}$	+	10	$D^6H^2B^{\mu\nu}B^{\mu\nu}$	$\frac{3\times10^{-4}}{E^8_{\text{TeV}}}$
24	$\left(W^{+\mu}\overleftrightarrow{D}^{\alpha}D_{\sigma}W^{-\nu}+\text{H.c.}\right)\partial_{\alpha}F^{\rho\sigma}\partial_{\mu}F_{\nu\rho}$	+			
25	$\varepsilon^{\mu\nu\rho\sigma}\left(iW^{+}_{\mu}\overleftrightarrow{D}^{\tau}D_{\beta}W^{-}_{\nu}+\text{H.c.}\right)\partial_{\tau}F_{\rho\alpha}\partial_{\sigma}F^{\alpha\beta}$	+		$\varepsilon D^6H^2B^{\mu\nu}B^{\mu\nu}$	
26	$(iW^{+\mu}\overleftrightarrow{D}^{\alpha}W^{-\nu}+\text{H.c.})\partial_{\alpha}F^{\rho\sigma}\partial_{\mu\nu}F_{\rho\sigma}$	−		$D^6H^2B^{\mu\nu}B^{\mu\nu}$	
27	$\left(iW^{+\mu}\overleftrightarrow{D}^{\alpha}D_{\sigma}W^{-\nu}+\text{H.c.}\right)\partial_{\alpha}F^{\rho\sigma}\partial_{\mu}F_{\nu\rho}$	−			
28	$\left(iW^{+\mu}\overleftrightarrow{D}_{\alpha}D^{\rho}W^{-\nu}+\text{H.c.}\right)\partial^{\alpha}F_{\nu\sigma}\partial^{\sigma}F_{\mu\rho}$	−			
29	$\varepsilon^{\mu\nu\rho\sigma}\left(W^{+}_{\mu}\overleftrightarrow{D}_{\tau}W^{-}_{\nu}\right)\partial^{\alpha\tau}F_{\rho\beta}\partial^{\beta}F_{\sigma\alpha}$	−		$\varepsilon D^6H^2B^{\mu\nu}B^{\mu\nu}$	
30	$\varepsilon^{\mu\nu\rho\sigma}(W^{+}_{\mu}\overleftrightarrow{D}^{\tau}D^{\alpha\beta}W^{-}_{\nu}+\text{H.c.})\partial_{\tau}F_{\rho\beta}F_{\sigma\alpha}$	−			
31	$\left(W^{+}_{\mu}\overleftrightarrow{D}_{\alpha}D_{\sigma}W^{-}_{\nu\rho}+\text{H.c.}\right)\partial^{\alpha}F^{\nu\rho}F^{\sigma\mu}$	+	10	$D^4H^2W^a_{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	$\frac{0.001}{E^7_{\text{TeV}}},\frac{0.004}{E^8_{\text{TeV}}}$
32	$\left(W^{+\mu}D_{\sigma}W^{-}_{\alpha\nu}+\text{H.c.}\right)F^{\rho\sigma}\partial_{\mu\rho}F^{\alpha\nu}$	+			
33	$\left(iW^{+}_{\mu}\overleftrightarrow{D}_{\alpha}D_{\rho}W^{-}_{\nu\sigma}+\text{H.c.}\right)\partial^{\alpha}F^{\nu\sigma}\tilde{F}^{\mu\rho}$	+		$D^4H^2W^a_{\mu\nu}B^{\mu\nu}\tilde{B}^{\mu\nu}$	
34	$\left(iW^{+}_{\mu}\overleftrightarrow{D}_{\alpha}\tilde{W}^{-\mu\nu}+\text{H.c.}\right)\partial^{\alpha}F^{\rho\sigma}\partial_{\sigma}F_{\nu\rho}$	+		$D^4H^2\tilde{W}^a_{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	
35	$\left(iW^{+}_{\mu}\overleftrightarrow{D}_{\alpha}D_{\sigma}\tilde{W}^{-\mu\nu}+\text{H.c.}\right)\partial^{\alpha}F_{\nu\rho}F^{\rho\sigma}$	+			
36	$\left(W^{+\mu}D^{\rho}{}_{\alpha}\tilde{W}^{-}_{\mu\nu}+\text{H.c.}\right)F^{\sigma\alpha}\partial^{\nu}F_{\rho\sigma}$	−			
37	$\left(iW^{+\mu}\overleftrightarrow{D}^{\beta}D_{\rho}{}^{\sigma}W^{-\nu}+\text{H.c.}\right)\partial_{\beta}F^{\alpha\rho}\partial_{\mu\nu}F_{\alpha\sigma}$	−	12	$D^8H^2B^{\mu\nu}B^{\mu\nu}$	$\frac{2\times10^{-5}}{E^{10}_{\text{TeV}}}$
38	$\left(iW^{+}_{\mu}\overleftrightarrow{D}^{\beta}D^{\sigma}{}_{\alpha}\tilde{W}^{-\mu\nu}+\text{H.c.}\right)\partial_{\beta}F_{\rho\sigma}\partial_{\nu}F^{\rho\alpha}$	+	12	$D^6H^2\tilde{W}^a_{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	$\frac{8\times10^{-5}}{E^9_{\text{TeV}}},\frac{3\times10^{-4}}{E^{10}_{\text{TeV}}}$

$$\begin{aligned}M_{\text{Tr}}(1A2B;3C4D) = {} & \text{Tr}(T^AT^BT^CT^D)M_{\text{Tr}}(12;34)+\text{Tr}(T^AT^BT^DT^C)M_{\text{Tr}}(12;43)\\ & +\text{Tr}(T^AT^CT^BT^D)M_{\text{Tr}}(13;24)+\text{Tr}(T^AT^CT^DT^B)M_{\text{Tr}}(13;42)\\ & +\text{Tr}(T^AT^DT^BT^C)M_{\text{Tr}}(14;23)+\text{Tr}(T^AT^DT^CT^B)M_{\text{Tr}}(14;32), \end{aligned} \tag{24}$$

where the right-hand side factorizes the amplitude into color factors and subamplitudes that only depend on kinematics and polarizations. Given the structure of the operators, the subamplitudes have the following identities under exchange of kinematics, $M_{\text{Tr}^2}(12;34)=M_{\text{Tr}^2}(21;34)=M_{\text{Tr}^2}(12;43)=M_{\text{Tr}^2}(21;43)=M_{\text{Tr}^2}(34;12)=M_{\text{Tr}^2}(43;12)=M_{\text{Tr}^2}(34;21)=M_{\text{Tr}^2}(43;21)$ and $M_{\text{Tr}}(12;34)=M_{\text{Tr}}(23;41)=M_{\text{Tr}}(34;12)=M({}_{\text{Tr}}41;23)$. By forming

TABLE XII. Operators up to dimension ten for the $ZZ\gamma\gamma$ interaction. At dimension 12, there are two redundancies such that, in order to form a set of independent operators, $s^n(t-u)^{2m}\mathcal{O}_9^{ZZ\gamma\gamma}$ and $s^n(t-u)^{2m}\mathcal{O}_{12}^{ZZ\gamma\gamma}$, with $n \geq 0$ and $m \geq 1$, should be omitted. Operators for $ZZgg$ interactions can be obtained by replacing $F_{\mu\nu}$'s with $G^A_{\mu\nu}$'s contracted with δ_{AB}. The same redundancies apply to the corresponding operators.

i	$\mathcal{O}_i^{ZZ\gamma\gamma}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
1	$Z^\mu Z_\mu F^{\nu\rho} F_{\nu\rho}$	+	6	$D^2H^2B^{\mu\nu}B^{\mu\nu}$	$\frac{0.09}{E_{\mathrm{TeV}}^4}$
2	$Z^\mu Z_\nu F_{\mu\rho} F^{\nu\rho}$	+			
3	$Z^\mu Z_\nu F_{\mu\rho} \tilde{F}^{\nu\rho}$	−		$D^2H^2B^{\mu\nu}\tilde{B}^{\mu\nu}$	
4	$Z^\mu Z^\nu F^{\rho\sigma} \partial_{\mu\nu} F_{\rho\sigma}$	+	8	$D^4H^2B^{\mu\nu}B^{\mu\nu}$	$\frac{0.006}{E_{\mathrm{TeV}}^6}$
5	$Z^\mu Z^\nu \partial_\mu F^{\rho\sigma} \partial_\nu F_{\rho\sigma}$	+			
6	$Z^\mu \partial_\sigma Z^\nu F^{\rho\sigma} \partial_\mu F_{\nu\rho}$	+			
7	$Z^\mu \partial^\sigma Z_\nu F^{\nu\rho} \partial_\mu F_{\rho\sigma}$	+			
8	$Z^\mu \partial_\rho{}^\sigma Z_\mu F^{\nu\rho} F_{\nu\sigma}$	+			
9	$\left(Z_\mu \overleftrightarrow{\partial}^\sigma Z^\nu\right) \partial_\sigma F^{\mu\rho} F_{\rho\nu}$	+			
10	$Z_\mu Z^\nu \partial_\nu F_{\rho\sigma} \partial^\sigma \tilde{F}^{\mu\rho}$	−		$D^4H^2B^{\mu\nu}\tilde{B}^{\mu\nu}$	
11	$\epsilon^{\mu\nu\rho\sigma} Z_\mu \partial_\beta Z_\nu F_{\rho\alpha} \partial_\sigma F^{\alpha\beta}$	−		$\epsilon D^4H^2B^{\mu\nu}B^{\mu\nu}$	
12	$\epsilon^{\mu\nu\rho\sigma}\left(Z_\mu \overleftrightarrow{\partial}^\beta Z_\nu\right) \partial_\beta F^\alpha{}_\rho F_{\alpha\sigma}$	−			
13	$Z^\mu \partial^\sigma Z_{\nu\rho} F^{\nu\rho} F_{\sigma\mu}$	+	8	$D^2H^2B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	$\frac{0.02}{E_{\mathrm{TeV}}^5}, \frac{0.07}{E_{\mathrm{TeV}}^6}$
14	$Z_\mu \partial_\rho Z_{\nu\sigma} \tilde{F}^{\mu\rho} F^{\nu\sigma}$	−		$D^2H^2B^{\mu\nu}B^{\mu\nu}\tilde{B}^{\mu\nu}$	
15	$Z_\mu \tilde{Z}^{\mu\nu} F^{\rho\sigma} \partial_\sigma F_{\nu\rho}$	−			
16	$Z_\mu \partial_\sigma \tilde{Z}^{\mu\nu} F^{\rho\sigma} F_{\nu\rho}$	−			
17	$\left(Z^\mu \overleftrightarrow{\partial}^\alpha Z^\nu\right) \partial_{\mu\alpha} F^{\rho\sigma} \partial_\nu F_{\rho\sigma}$	+	10	$D^6H^2B^{\mu\nu}B^{\mu\nu}$	$\frac{3\times10^{-4}}{E_{\mathrm{TeV}}^8}$
18	$\left(Z^\mu \overleftrightarrow{\partial}^\alpha \partial_\sigma Z^\nu\right) \partial_\alpha F^{\rho\sigma} \partial_\mu F_{\nu\rho}$	+			
19	$\epsilon^{\mu\nu\rho\sigma}\left(Z_\mu \overleftrightarrow{\partial}_\tau Z_\nu\right) \partial^{\alpha\tau} F_{\rho\beta} \partial^\beta F_{\sigma\alpha}$	−		$\epsilon D^6H^2B^{\mu\nu}B^{\mu\nu}$	
20	$\epsilon^{\mu\nu\rho\sigma}\left(Z_\mu \overleftrightarrow{\partial}^\tau \partial^{\alpha\beta} Z_\nu\right) \partial_\tau F_{\rho\beta} F_{\sigma\alpha}$	−			
21	$Z^\mu \partial_\rho Z^{\nu\alpha} F^{\rho\sigma} \partial_{\mu\sigma} F_{\nu\alpha}$	+	10	$D^4H^2B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	$\frac{0.001}{E_{\mathrm{TeV}}^7}, \frac{0.004}{E_{\mathrm{TeV}}^8}$
22	$\left(Z_\mu \overleftrightarrow{\partial}_\alpha \partial_\sigma Z_{\nu\rho}\right) \partial^\alpha F^{\nu\rho} F^{\sigma\mu}$	+			
23	$Z_\mu \partial_\rho{}^\alpha \tilde{Z}^{\mu\nu} F_{\sigma\alpha} \partial_\nu F^{\rho\sigma}$	−		$D^4H^2B^{\mu\nu}B^{\mu\nu}\tilde{B}^{\mu\nu}$	

candidate subamplitudes with the correct symmetries under the exchange of kinematics, we can find the independent $M_{\mathrm{Tr}^2}(12;34)$ and $M_{\mathrm{Tr}}(12;34)$. These will, respectively, lead to an independent set of $M_{\mathrm{Tr}^2}(1A2B;3C4D)$ and $M_{\mathrm{Tr}}(1A2B;3C4D)$. However, it is possible that there will still be redundancies between the two types of amplitudes $M_{\mathrm{Tr}^2}(1A2B;3C4D)$ and $M_{\mathrm{Tr}}(1A2B;3C4D)$.

In the following, we show that, if there is a redundancy between the subamplitudes of the two types, then there will be a redundancy among the full amplitudes after putting in the color factors. To see this, assume that there is a redundancy between two subamplitudes $\hat{M}(12;34) = M_{\mathrm{Tr}^2}(12;34) = M_{\mathrm{Tr}}(12;34)$. Now $\hat{M}$ is invariant under both of the permutation symmetries of the two subamplitudes and one can show that this means $\hat{M}$ is invariant under arbitrary permutations of the four particles. Then we can show that

$$
\begin{aligned}
M_{\mathrm{Tr}^2}(1A2B;3C4D) &= (\mathrm{Tr}(T^AT^B)\mathrm{Tr}(T^CT^D) \\
&\quad + \mathrm{Tr}(T^AT^C)\mathrm{Tr}(T^BT^D) \\
&\quad + \mathrm{Tr}(T^AT^D)\mathrm{Tr}(T^BT^C))\hat{M}(12;34) \\
&= (\mathrm{Tr}(T^AT^BT^CT^D) + \mathrm{Tr}(T^AT^BT^DT^C) \\
&\quad + \mathrm{Tr}(T^AT^CT^BT^D) + \mathrm{Tr}(T^AT^CT^DT^B) \\
&\quad + \mathrm{Tr}(T^AT^DT^BT^C) \\
&\quad + \mathrm{Tr}(T^AT^DT^CT^B))\hat{M}(12;34) \\
&= M_{\mathrm{Tr}}(1A2B;3C4D), \qquad (25)
\end{aligned}
$$

TABLE XIII. Primary operators up to dimension 12 for the $Z\gamma gg$ interaction. There are two redundancies that both appear at dimension 12. To form a set of operators that are independent, $s^n(t-u)^{2m}\mathcal{O}_4^{Z\gamma gg}$, $s^n(t-u)^{2m}\mathcal{O}_9^{Z\gamma gg}$, with $n \geq 0$, $m \geq 1$, should be omitted.

i	$\mathcal{O}_i^{Z\gamma gg}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
1	$G^{\mu\nu}D_\sigma G_{\mu\rho}F_\nu{}^\rho Z^\sigma$	$+$	8	$D^2H^2G^{\mu\nu}G^{\mu\nu}B^{\mu\nu}$	$\frac{0.02}{E_{\text{TeV}}^5}, \frac{0.07}{E_{\text{TeV}}^6}$
2	$G^{\mu\nu}D_\sigma G_{\nu\rho}F^{\rho\sigma}Z_\mu$	$+$			
3	$G^{\mu\nu}D^\sigma G_{\mu\rho}F_{\nu\sigma}Z^\rho$	$+$			
4	$G^{\mu\nu}D_\rho G_{\mu\nu}F^{\rho\sigma}Z_\sigma$	$+$			
5	$G^{\mu\nu}G^{\rho\sigma}\partial_\rho F_{\mu\nu}Z_\sigma$	$+$			
6	$G^{\mu\nu}G_{\mu\rho}\partial^\rho F_{\nu\sigma}Z^\sigma$	$+$			
7	$D_\rho G_{\mu\nu}G^{\nu\rho}\tilde{F}^{\mu\sigma}Z_\sigma$	$-$		$D^2H^2G^{\mu\nu}G^{\mu\nu}\tilde{B}^{\mu\nu}$	
8	$G^{\mu\nu}\tilde{G}^{\rho\sigma}\partial_\rho F_{\mu\nu}Z_\sigma$	$-$		$D^2H^2G^{\mu\nu}\tilde{G}^{\mu\nu}B^{\mu\nu}$	
9	$G^{\mu\nu}D_\nu\tilde{G}^{\rho\sigma}F_{\mu\rho}Z_\sigma$	$-$			
10	$\varepsilon^{\mu\nu\rho\sigma}G_{\mu\alpha}D_\nu G^{\alpha\beta}F_{\rho\beta}Z_\sigma$	$-$		$\varepsilon D^2H^2G^{\mu\nu}G^{\mu\nu}B^{\mu\nu}$	
11	$\varepsilon^{\mu\nu\rho\sigma}G_{\mu\alpha}G^{\alpha\beta}\partial_\nu F_{\rho\beta}Z_\sigma$	$-$			
12	$\varepsilon^{\mu\nu\rho\sigma}G_{\mu\alpha}D^\beta G_\nu{}^\alpha F_{\rho\beta}Z_\sigma$	$-$			
13	$G^{\mu\nu}D_{\mu\alpha}G^{\rho\sigma}\partial_\nu F_{\rho\sigma}Z^\alpha$	$+$	10	$D^4H^2G^{\mu\nu}G^{\mu\nu}B^{\mu\nu}$	$\frac{0.001}{E_{\text{TeV}}^7}, \frac{0.004}{E_{\text{TeV}}^8}$
14	$D_\rho G^{\mu\nu}D_{\sigma\alpha}G_{\mu\nu}F^{\rho\sigma}Z^\alpha$	$+$			
15	$\left(G^{\mu\nu}\overleftrightarrow{D}^\alpha D_\sigma G_{\mu\rho}\right)\partial_\alpha F_\nu{}^\rho Z^\sigma$	$+$			
16	$\left(G^{\mu\nu}\overleftrightarrow{D}^\alpha D_\nu G^{\rho\sigma}\right)\partial_\alpha F_{\mu\rho}Z_\sigma$	$+$			
17	$\left(G^{\mu\nu}\overleftrightarrow{D}^\alpha D_\sigma G_{\nu\rho}\right)\partial_\alpha F^{\rho\sigma}Z_\mu$	$+$			
18	$\left(G^{\mu\nu}\overleftrightarrow{D}^\alpha G^{\rho\sigma}\right)\partial_{\nu\alpha}F_{\mu\rho}Z_\sigma$	$+$			
19	$G^{\mu\nu}D_\rho G_{\mu\alpha}\partial_\nu{}^\alpha\tilde{F}^{\rho\sigma}Z_\sigma$	$-$		$D^4H^2G^{\mu\nu}G^{\mu\nu}\tilde{B}^{\mu\nu}$	
20	$\left(G^{\mu\nu}\overleftrightarrow{D}^\alpha D_\nu G_{\mu\rho}\right)\partial_\alpha\tilde{F}^{\rho\sigma}Z_\sigma$	$-$			
21	$\left(G_{\mu\nu}\overleftrightarrow{D}^\alpha G^{\nu\rho}\right)\partial_{\rho\alpha}\tilde{F}^{\mu\sigma}Z_\sigma$	$-$			
22	$D^\alpha G_{\mu\nu}D^{\nu\rho}\tilde{G}^{\mu\sigma}F_{\rho\alpha}Z_\sigma$	$-$		$D^4H^2G^{\mu\nu}\tilde{G}^{\mu\nu}B^{\mu\nu}$	
23	$\varepsilon^{\mu\nu\rho\sigma}\left(G_{\mu\alpha}\overleftrightarrow{D}^\tau D_\nu G^{\alpha\beta}\right)\partial_\tau F_{\rho\beta}Z_\sigma$	$-$		$\varepsilon D^4H^2G^{\mu\nu}G^{\mu\nu}B^{\mu\nu}$	
24	$\varepsilon^{\mu\nu\rho\sigma}\left(G_{\mu\alpha}\overleftrightarrow{D}_\tau G_{\nu\beta}\right)\partial^{\beta\tau}F_\rho{}^\alpha Z_\sigma$	$-$			
25	$\left(G^{\mu\nu}\overleftrightarrow{D}_\beta D^{\sigma\alpha}G_{\mu\rho}\right)\partial^{\rho\beta}F_{\nu\sigma}Z_\alpha$	$+$	12	$D^6H^2G^{\mu\nu}G^{\mu\nu}B^{\mu\nu}$	$\frac{8\times10^{-5}}{E_{\text{TeV}}^9}, \frac{3\times10^{-4}}{E_{\text{TeV}}^{10}}$
26	$\left(G^{\mu\nu}\overleftrightarrow{D}_\beta D_\alpha G_{\mu\rho}\right)\partial_\nu{}^{\rho\beta}\tilde{F}^{\alpha\sigma}Z_\sigma$	$-$		$D^6H^2G^{\mu\nu}G^{\mu\nu}\tilde{B}^{\mu\nu}$	

which shows that the full amplitudes are also redundant. In the second equality, we have used the group theory identity for SU(3),

$$\sum_{\text{six distinct perms}} \mathrm{Tr}(T^AT^BT^CT^D) = \sum_{\text{three distinct perms}} \mathrm{Tr}(T^AT^B)\mathrm{Tr}(T^CT^D) \qquad (26)$$

with the distinct permutations given in (23) and (24). This implies that it is enough to look for the independent subamplitudes of both types to characterize the independent full amplitudes.

For simplicity, we adopt the notation $\mathrm{Tr}(T^2)\mathrm{Tr}(T^2)$ and $\mathrm{Tr}(T^4)$ to represent $\mathrm{Tr}(T^AT^B)\mathrm{Tr}(T^CT^D)$ and $\mathrm{Tr}(T^AT^BT^CT^D)$, respectively. In Tables XX and XXI, we show a set of primary operators for $gggg$ interactions, up to dimension 16. We find 9 dimension-eight operators, 14 dimension-ten operators, 16 dimension-12 operators, 9 dimension-14 operators, 2 dimension-16 operators, 2 redundancies at dimension 14, and 4 redundancies at dimension 16, which agrees with the corresponding Hilbert series. The redundancies are the operators and

TABLE XIV. Primary operators up to dimension 14 for the $Z\gamma\gamma\gamma$ interaction. At dimension 14, two operators become redundant to operators at dimension ten. To form a set of independent operators, $x^n y^m \mathcal{O}_9^{Z\gamma\gamma\gamma}$ and $x^n y^m \mathcal{O}_{13}^{Z\gamma\gamma\gamma}$, with $x = s^2 + t^2 + u^2$, $y = stu$, $n \geq 1$, and $m \geq 0$, should be omitted. Replacing all $F_{\mu\nu}$'s with $G^A_{\mu\nu}$'s contracted with d_{ABC} yields the symmetric (in exchange of gluon kinematics) primary operators up to dimension 14 for the $Zggg$ interaction. The same redundancies apply to the corresponding operators.

i	$\mathcal{O}_i^{Z\gamma\gamma\gamma}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
1	$F^{\mu\nu}\partial_\sigma F_{\nu\rho}F^{\rho\sigma}Z_\mu$	$+$	8	$D^2H^2B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	$\frac{0.02}{E_{\text{TeV}}^5}, \frac{0.07}{E_{\text{TeV}}^6}$
2	$F^{\mu\nu}\partial^\sigma F_{\mu\rho}F_{\nu\sigma}Z^\rho$	$+$			
3	$\partial_\rho F_{\mu\nu}F^{\nu\rho}\tilde{F}^{\sigma\mu}Z_\sigma$	$-$		$D^2H^2B^{\mu\nu}B^{\mu\nu}\tilde{B}^{\mu\nu}$	
4	$\varepsilon^{\mu\nu\rho\sigma}F_{\mu\alpha}F^{\alpha\beta}\partial_\nu F_{\beta\rho}Z_\sigma$	$-$		$\varepsilon D^2H^2B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	
5	$\partial_{\rho\alpha}F^{\mu\nu}\partial_\sigma F_{\mu\nu}F^{\rho\sigma}Z^\alpha$	$+$	10	$D^4H^2B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	$\frac{0.001}{E_{\text{TeV}}^7}, \frac{0.004}{E_{\text{TeV}}^8}$
6	$\partial_{\sigma\alpha}F^{\mu\nu}\partial^\alpha F_{\nu\rho}F^\rho{}_\mu Z^\sigma$	$+$			
7	$\partial^\alpha F^{\mu\nu}\partial_{\sigma\alpha}F_{\nu\rho}F^{\rho\sigma}Z_\mu$	$+$			
8	$\partial_\alpha F^{\mu\nu}\partial^{\sigma\alpha}F_{\mu\rho}F_{\nu\sigma}Z^\rho$	$+$			
9	$\partial^\alpha F^{\mu\nu}\partial_\alpha F^{\rho\sigma}\partial_\rho F_{\mu\nu}Z_\sigma$	$+$			
10	$F^{\mu\nu}\partial_\rho F_{\mu\sigma}\partial_\nu{}^\sigma\tilde{F}^{\rho\alpha}Z_\alpha$	$-$		$D^4H^2B^{\mu\nu}B^{\mu\nu}\tilde{B}^{\mu\nu}$	
11	$\partial_\rho F_{\mu\nu}\partial_\sigma{}^\nu\tilde{F}^{\mu\alpha}F^{\rho\sigma}Z_\alpha$	$-$			
12	$\partial_{\rho\alpha}F_{\mu\nu}\partial^\alpha F^{\nu\rho}\tilde{F}^{\mu\sigma}Z_\sigma$	$-$			
13	$\varepsilon^{\mu\nu\rho\sigma}\partial^\tau F_{\mu\alpha}\partial_{\nu\tau}F^{\alpha\beta}F_{\rho\beta}Z_\sigma$	$-$		$\varepsilon D^4H^2B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	
14	$\varepsilon^{\mu\nu\rho\sigma}\partial^\tau F_{\mu\alpha}\partial_\tau F^{\alpha\beta}\partial_\nu F_{\rho\beta}Z_\sigma$	$-$			
15	$\partial_{\rho\alpha\beta}F^{\mu\nu}\partial_\sigma{}^\beta F_{\mu\nu}F^{\rho\sigma}Z^\alpha$	$+$	12	$D^6H^2B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	$\frac{8\times10^{-5}}{E_{\text{TeV}}^9}, \frac{3\times10^{-4}}{E_{\text{TeV}}^{10}}$
16	$\partial_{\sigma\alpha\beta}F^{\mu\nu}\partial^{\alpha\beta}F_{\nu\rho}F^\rho{}_\mu Z^\sigma$	$+$			
17	$\partial^{\alpha\beta}F^{\mu\nu}\partial_{\sigma\alpha\beta}F_{\nu\rho}F^{\rho\sigma}Z_\mu$	$+$			
18	$\left(F^{\mu\nu}\overleftrightarrow{\partial}_\beta\partial^{\alpha\sigma}F_{\mu\rho}\right)\partial^{\rho\beta}F_{\nu\sigma}Z_\alpha$	$+$			
19	$\partial^\beta F^{\mu\nu}\partial_{\rho\beta}F_{\mu\sigma}\partial_\nu{}^\sigma\tilde{F}^{\rho\alpha}Z_\alpha$	$-$		$D^6H^2B^{\mu\nu}B^{\mu\nu}\tilde{B}^{\mu\nu}$	
20	$\partial_\rho{}^\beta F_{\mu\nu}\partial_{\sigma\beta}{}^\nu\tilde{F}^{\mu\alpha}F^{\rho\sigma}Z_\alpha$	$-$			
21	$\partial_{\rho\alpha\beta}F_{\mu\nu}\partial^{\alpha\beta}F^{\nu\rho}\tilde{F}^{\mu\sigma}Z_\sigma$	$-$			
22	$\left(F^{\mu\nu}\overleftrightarrow{\partial}_\beta\partial_\rho F_{\mu\alpha}\right)\partial^{\alpha\beta}{}_\nu\tilde{F}^{\sigma\rho}Z_\sigma$	$-$			
23	$\partial^{\beta\tau}F^{\mu\nu}\partial_{\mu\alpha\beta\tau}F^{\rho\sigma}\partial_\nu F_{\rho\sigma}Z^\alpha$	$+$	14	$D^8H^2B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	$\frac{5\times10^{-6}}{E_{\text{TeV}}^{11}}, \frac{2\times10^{-5}}{E_{\text{TeV}}^{12}}$
24	$\partial_{\rho\alpha\beta\tau}F^{\mu\nu}\partial_\sigma{}^{\beta\tau}F_{\mu\nu}F^{\rho\sigma}Z^\alpha$	$+$			
25	$\partial^{\beta\tau}F^{\mu\nu}\partial_{\rho\beta\tau}F_{\mu\sigma}\partial_\nu{}^\sigma\tilde{F}^{\rho\alpha}Z_\alpha$	$-$		$D^8H^2B^{\mu\nu}B^{\mu\nu}\tilde{B}^{\mu\nu}$	
26	$\partial_\rho{}^{\beta\tau}F_{\mu\nu}\partial_{\sigma\beta\tau}{}^\nu\tilde{F}^{\mu\alpha}F^{\rho\sigma}Z_\alpha$	$-$			

descendants given by $x^n y^m \mathcal{O}_4^{gggg}$, $x^n y^m \mathcal{O}_9^{gggg}$, $x^n y^m \mathcal{O}_{12}^{gggg}$, $x^n y^m \mathcal{O}_{14}^{gggg}$, $x^n y^m \mathcal{O}_{19}^{gggg}$, $x^n y^m \mathcal{O}_{20}^{gggg}$, with $x = s^2 + t^2 + u^2$, $y = stu$, $n \geq 0$, and $m \geq 1$, and should be removed if we want a set of independent operators.

VI. DECAYS OF THE Z BOSON

Now that we have the amplitudes and unitarity bounds, we can continue the analysis of modifications to the decays of the Z boson, taking into account the upper bounds on coupling strengths from the tables. First, we start with the decays $Z \to \bar{f}f(\gamma, g)$. Such decays occur in the Standard Model through radiation of the gauge boson off of the fermions. Such emissions are collinear enhanced, so there could be hope that the contact amplitudes in Ref. [7] could be distinguished in differential distributions. In this part of phase space, we estimate $\text{BR}_{\text{SM}}(Z \to \gamma\bar{f}f) \approx \frac{\alpha}{4\pi}\text{BR}_{\text{SM}}(Z \to \bar{f}f)$. Assuming just the irreducible background from the Standard Model in the channel $Z \to \gamma\bar{\mu}\mu$, we find that to get a 1σ fluctuation at the HL-LHC with $\sim 6 \times 10^9$ Z bosons would require unitarity violation at about 5 TeV for the dimension-six and dimension-seven amplitudes, which have two field strengths. Of course, reducible backgrounds will reduce this estimate, but this suggests that for reasonably high unitarity violating scales, this could be searched for at the LHC. For $Z \to \bar{f}fg$, the fermions would need to be a quark-antiquark pair, for which the b quark would be the most promising.

TABLE XV. Antisymmetric (in exchange of gluon kinematics) primary operators up to dimension 16 for the $Zggg$ interaction. Note that SU(3) indices are suppressed, where the $G^A_{\mu\nu}$'s are contracted with the fully antisymmetric structure constant tensor f_{ABC}.

i	$\mathcal{O}_i^{Zggg,A}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
1	$G^{\mu\nu} D_\sigma G_{\nu\rho} G^{\rho\sigma} Z_\mu$	+	8	$D^2 H^2 G^{\mu\nu} G^{\mu\nu} G^{\mu\nu}$	$\frac{0.02}{E_{\mathrm{TeV}}^5}, \frac{0.07}{E_{\mathrm{TeV}}^6}$
2	$G^{\mu\nu} D_\nu G_{\mu\rho} \tilde{G}^{\rho\sigma} Z_\sigma$	−		$D^2 H^2 G^{\mu\nu} G^{\mu\nu} \tilde{G}^{\mu\nu}$	
3	$D^\alpha G^{\mu\nu} D_{\sigma\alpha} G_{\mu\rho} G_\nu{}^\rho Z^\sigma$	+	10	$D^4 H^2 G^{\mu\nu} G^{\mu\nu} G^{\mu\nu}$	$\frac{0.001}{E_{\mathrm{TeV}}^7}, \frac{0.004}{E_{\mathrm{TeV}}^8}$
4	$D^\alpha G^{\mu\nu} D_{\nu\alpha} G_{\rho\sigma} G_\mu{}^\rho Z^\sigma$	+			
5	$D^\alpha G^{\mu\nu} D_{\sigma\alpha} G_{\nu\rho} G^{\rho\sigma} Z_\mu$	+			
6	$G^{\mu\nu} D^\sigma{}_\alpha G_{\mu\rho} D^\rho G_{\nu\sigma} Z^\alpha$	+			
7	$D^\alpha G^{\mu\nu} D_{\nu\alpha} G_{\mu\rho} \tilde{G}^{\rho\sigma} Z_\sigma$	−		$D^4 H^2 G^{\mu\nu} G^{\mu\nu} \tilde{G}^{\mu\nu}$	
8	$D^\alpha G_{\mu\nu} D_\alpha G^{\nu\rho} D_\rho \tilde{G}^{\mu\sigma} Z_\sigma$	−			
9	$G^{\mu\nu} D_\alpha G_{\mu\rho} D_\nu{}^\rho \tilde{G}^{\sigma\alpha} Z_\sigma$	−			
10	$\varepsilon^{\mu\nu\rho\sigma} D^\tau G_{\mu\alpha} D_{\nu\tau} G^{\alpha\beta} G_{\rho\beta} Z_\sigma$	−		$\varepsilon D^4 H^2 G^{\mu\nu} G^{\mu\nu} G^{\mu\nu}$	
11	$D^{\alpha\beta} G^{\mu\nu} D_{\nu\alpha\beta} G_{\rho\sigma} G_\mu{}^\rho Z^\sigma$	+	12	$D^6 H^2 G^{\mu\nu} G^{\mu\nu} G^{\mu\nu}$	$\frac{8\times10^{-5}}{E_{\mathrm{TeV}}^9}, \frac{3\times10^{-4}}{E_{\mathrm{TeV}}^{10}}$
12	$D^\beta G^{\mu\nu} D^\sigma{}_{\alpha\beta} G_{\mu\rho} D^\rho G_{\nu\sigma} Z^\alpha$	+			
13	$\left(G^{\mu\nu} \overleftrightarrow{D}^\beta D_{\mu\alpha} G^{\rho\sigma}\right) D_{\nu\beta} G_{\rho\sigma} Z^\alpha$	+			
14	$\left(D_{\rho\alpha} G^{\mu\nu} \overleftrightarrow{D}^\beta D_\sigma G_{\mu\nu}\right) D_\beta G^{\rho\sigma} Z^\alpha$	+			
15	$D^{\alpha\beta} G^{\mu\nu} D_{\nu\alpha\beta} G_{\mu\rho} \tilde{G}^{\rho\sigma} Z_\sigma$	−		$D^6 H^2 G^{\mu\nu} G^{\mu\nu} \tilde{G}^{\mu\nu}$	
16	$D^\beta G^{\mu\nu} D_{\alpha\beta} G_{\mu\rho} D_\nu{}^\rho \tilde{G}^{\sigma\alpha} Z_\sigma$	−			
17	$(G^{\mu\nu} \overleftrightarrow{D}^\beta D_\rho G_{\mu\alpha}) D_{\nu\beta}{}^\alpha \tilde{G}^{\rho\sigma} Z_\sigma$	−			
18	$\left(D_\rho G^{\mu\nu} \overleftrightarrow{D}^\beta D_{\nu\alpha} \tilde{G}_{\mu\sigma}\right) D_\beta G^{\rho\alpha} Z^\sigma$	−			
19	$D^{\beta\tau} G^{\mu\nu} D^\sigma{}_{\alpha\beta\tau} G_{\mu\rho} D^\rho G_{\nu\sigma} Z^\alpha$	+	14	$D^8 H^2 G^{\mu\nu} G^{\mu\nu} G^{\mu\nu}$	$\frac{5\times10^{-6}}{E_{\mathrm{TeV}}^{11}}, \frac{2\times10^{-5}}{E_{\mathrm{TeV}}^{12}}$
20	$\left(D^\tau G^{\mu\nu} \overleftrightarrow{D}^\beta D_{\mu\alpha\tau} G^{\rho\sigma}\right) D_{\nu\beta} G_{\rho\sigma} Z^\alpha$	+			
21	$D^{\beta\tau} G^{\mu\nu} D_{\alpha\beta\tau} G_{\mu\rho} D_\nu{}^\rho \tilde{G}^{\sigma\alpha} Z_\sigma$	−		$D^8 H^2 G^{\mu\nu} G^{\mu\nu} \tilde{G}^{\mu\nu}$	
22	$\left(D^\tau G^{\mu\nu} \overleftrightarrow{D}^\beta D_{\rho\tau} G_{\mu\alpha}\right) D_{\nu\beta}{}^\alpha \tilde{G}^{\rho\sigma} Z_\sigma$	−			
23	$\left(D^{\tau\pi} G^{\mu\nu} \overleftrightarrow{D}^\beta D_{\mu\alpha\tau\pi} G^{\rho\sigma}\right) D_{\nu\beta} G_{\rho\sigma} Z^\alpha$	+	16	$D^{10} H^2 G^{\mu\nu} G^{\mu\nu} G^{\mu\nu}$	$\frac{3\times10^{-7}}{E_{\mathrm{TeV}}^{13}}, \frac{9\times10^{-7}}{E_{\mathrm{TeV}}^{14}}$
24	$\left(D^{\tau\pi} G^{\mu\nu} \overleftrightarrow{D}^\beta D_{\rho\tau\pi} G_{\mu\alpha}\right) D_{\nu\beta}{}^\alpha \tilde{G}^{\rho\sigma} Z_\sigma$	−		$D^{10} H^2 G^{\mu\nu} G^{\mu\nu} \tilde{G}^{\mu\nu}$	

However, these would have substantial QCD backgrounds, so our optimistic analysis would be entirely too unrealistic.

We now turn to Z decays into three gauge bosons. The $Z \to 3\gamma$ decay is allowed in the SM, but is a higher-order process that is only possible through at least a single W boson or fermion loop [30–37], leading to a predicted branching ratio of $\sim 5 \times 10^{-10}$ [37,38]. Most recently, an ATLAS search found the bound $\mathrm{BR}(Z \to 3\gamma) < 2.2 \times 10^{-6}$ [39]. In this analysis, the background consists of a irreducible part with three or more prompt photons and a reducible component of a combination of photons and electrons or hadronic jets misidentified as photons. With the HL-LHC luminosity of $\sim 3000\ \mathrm{fb}^{-1}$ [40,41], this bound should naively improve to about $\mathrm{BR}(Z \to 3\gamma) \lesssim 3 \times 10^{-7}$. This is much larger than the SM prediction, so we need the BSM amplitude to dominate, which we estimate requires unitarity violation at 500 (330) GeV for dimension-eight (ten) operators. Such low unitarity violating scales would require new physics to hide itself from LHC searches and so disfavors the possibility of observing this at the HL-LHC without substantial improvements in reducing reducible backgrounds. In fact, if we impose our unitarity bounds to be violated at a TeV, this predicts $\mathrm{BR}(Z \to 3\gamma) \lesssim 2 \times 10^{-9}$ for dimension-eight amplitudes, which is extremely challenging to observe at a hadron collider. However, one could hope that a future lepton collider producing 10^{12} Z's might have more controllable backgrounds that could reach these small branching ratios.

TABLE XVI. Operators up to dimension 12 for the $\gamma\gamma\gamma\gamma$ interaction. At dimension 14, there are two redundancies such that, in order to have a set of independent operators, $x^n y^m \mathcal{O}_6^{\gamma\gamma\gamma\gamma}$ and $x^n y^m \mathcal{O}_7^{\gamma\gamma\gamma\gamma}$, with $x = s^2 + t^2 + u^2$, $y = stu$, $n \geq 1$, and $m \geq 0$, should be omitted.

i	$\mathcal{O}_i^{\gamma\gamma\gamma\gamma}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
1	$F^{\mu\nu}F_{\mu\rho}F_{\nu\sigma}F^{\sigma\rho}$	+		$B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	
2	$F^{\mu\nu}F_{\rho\sigma}F_{\mu\nu}F^{\rho\sigma}$	+	8		$\frac{0.1}{E_{\text{TeV}}^4}$
3	$F^{\nu\sigma}F_{\sigma\rho}\tilde{F}_{\mu\nu}F^{\rho\mu}$	−		$B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}\tilde{B}^{\mu\nu}$	
4	$\partial_\beta F^{\mu\nu}\partial_\alpha F_{\nu\rho}F^{\rho}{}_{\mu}F^{\alpha\beta}$	+			
5	$\partial_\alpha F^{\mu\nu}\partial_\sigma F_{\nu\rho}F^{\rho\sigma}F^{\alpha}{}_{\mu}$	+		$D^2B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	
6	$F^{\mu\nu}\partial_{\alpha\sigma}F_{\nu\rho}F^{\rho\sigma}F^{\alpha}{}_{\mu}$	+	10		$\frac{0.006}{E_{\text{TeV}}^6}$
7	$\partial^\nu F^{\sigma\mu}\partial_\alpha F_{\mu\nu}\tilde{F}_{\rho\sigma}F^{\alpha\rho}$	−		$\varepsilon D^2B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	
8	$\varepsilon^{\mu\nu\rho\sigma}\partial^\tau F_{\rho\alpha}F^{\alpha\beta}\partial_\sigma F_{\beta\mu}F_{\tau\nu}$	−			
9	$\partial_\alpha{}^\beta F^{\mu\nu}\partial_{\sigma\beta}F_{\nu\rho}F^{\rho\sigma}F^{\alpha}{}_{\mu}$	+	12	$D^4B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}B^{\mu\nu}$	$\frac{4\times10^{-4}}{E_{\text{TeV}}^8}$

TABLE XVII. Operators up to dimension ten for the $\gamma\gamma gg$ interaction. At dimension 12, there are two redundancies that should be omitted, along with their descendants, if we want to have a list of independent operators. The corresponding redundancies are given by $s^n(t-u)^{2m}\mathcal{O}_4^{\gamma\gamma gg}$ and $s^{m+1}(t-u)^{2n}\mathcal{O}_6^{\gamma\gamma gg}$, with $n \geq 0$ and $m \geq 1$.

i	$\mathcal{O}_i^{\gamma\gamma gg}$	CP	$d_{\mathcal{O}_i}$	SMEFT operator form	c Unitarity bound
1	$F^{\mu\nu}F_{\mu\nu}G^{\rho\sigma}G_{\rho\sigma}$	+			
2	$F^{\mu\nu}F^{\rho\sigma}G_{\mu\nu}G_{\rho\sigma}$	+		$B^{\mu\nu}B^{\mu\nu}G^{\mu\nu}G^{\mu\nu}$	
3	$F^{\mu\nu}F_{\nu\rho}G^{\rho\sigma}G_{\sigma\mu}$	+			
4	$F^{\mu\nu}F^{\rho\sigma}G_{\nu\rho}G_{\sigma\mu}$	+	8		$\frac{0.1}{E_{\text{TeV}}^4}$
5	$\tilde{F}^{\mu\nu}F_{\mu\nu}G^{\rho\sigma}G_{\rho\sigma}$	−		$B^{\mu\nu}\tilde{B}^{\mu\nu}G^{\mu\nu}G^{\mu\nu}$	
6	$\tilde{F}^{\mu\nu}F^{\rho\sigma}G_{\mu\nu}G_{\rho\sigma}$	−			
7	$F^{\mu\nu}F_{\mu\nu}\tilde{G}^{\rho\sigma}G_{\rho\sigma}$	−		$B^{\mu\nu}B^{\mu\nu}G^{\mu\nu}\tilde{G}^{\mu\nu}$	
8	$\partial^\alpha F^{\mu\nu}\partial^\sigma F_{\nu\rho}G^{\rho}{}_{\mu}G_{\sigma\alpha}$	+			
9	$F^{\mu\nu}\partial^\alpha{}_\sigma F_{\nu\rho}G^{\rho\sigma}G_{\alpha\mu}$	+		$D^2B^{\mu\nu}B^{\mu\nu}G^{\mu\nu}G^{\mu\nu}$	
10	$F^{\mu\nu}\partial^\alpha F^{\rho\sigma}D_\rho G_{\mu\nu}G_{\alpha\sigma}$	+	10		$\frac{0.006}{E_{\text{TeV}}^6}$
11	$\varepsilon^{\mu\nu\rho\sigma}F_{\rho\alpha}\partial^\tau{}_\sigma F^{\alpha\beta}G_{\beta\mu}G_{\tau\nu}$	−		$\varepsilon D^2B^{\mu\nu}B^{\mu\nu}G^{\mu\nu}G^{\mu\nu}$	
12	$\varepsilon^{\mu\nu\rho\sigma}\partial^\tau F_{\rho\alpha}F^{\alpha\beta}D_\sigma G_{\beta\mu}G_{\tau\nu}$	−			

TABLE XVIII. Primary dimension-eight and -ten operators for the γggg interaction. $G_{\mu\nu}^A$'s are contracted with the fully symmetric structure constant tensor d_{ABC} and the fully antisymmetric structure constant tensor f_{ABC}.

i	$\mathcal{O}_i^{\gamma ggg}$	CP	$d_{\mathcal{O}_i}$	SU(3)	SMEFT operator form	c Unitarity bound
1	$G^{\mu\nu}G_{\mu\rho}G_{\nu\sigma}F^{\rho\sigma}$	+			$G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}B^{\mu\nu}$	
2	$G^{\mu\nu}G_{\mu\nu}G_{\rho\sigma}F^{\rho\sigma}$	+				
3	$G^{\mu\nu}G_{\nu\rho}\tilde{G}_{\mu\sigma}F^{\rho\sigma}$	−	8	d_{ABC}	$G^{\mu\nu}G^{\mu\nu}\tilde{G}^{\mu\nu}B^{\mu\nu}$	$\frac{0.1}{E_{\text{TeV}}^4}$
4	$G^{\mu\nu}G_{\nu\rho}G^{\rho\sigma}\tilde{F}_{\sigma\mu}$	−			$G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}\tilde{B}^{\mu\nu}$	

(Table continued)

TABLE XVIII. *(Continued)*

i	$\mathcal{O}_i^{\gamma ggg}$	CP	$d_{\mathcal{O}_i}$	SU(3)	SMEFT operator form	c Unitarity bound
5	$D_\alpha G^{\mu\nu} D_\sigma G_{\nu\rho} G^\rho{}_\mu F^{\sigma\alpha}$	+				
6	$D^\alpha G^{\mu\nu} D_\sigma G_{\nu\rho} G^{\rho\sigma} F_{\alpha\mu}$	+		d_{ABC}		
7	$G^{\mu\nu} D_\sigma{}^\alpha G_{\nu\rho} G^{\rho\sigma} F_{\alpha\mu}$	+			$D^2 G^{\mu\nu} G^{\mu\nu} G^{\mu\nu} B^{\mu\nu}$	
8	$D_\alpha G^{\mu\nu} D^\sigma G_{\mu\rho} G_{\nu\sigma} F^{\alpha\rho}$	+				
9	$D_\alpha G^{\mu\nu} D_\nu G^{\rho\sigma} G_{\mu\rho} F^\alpha{}_\sigma$	+		f_{ABC}		
10	$D^\alpha G^{\mu\nu} G^{\rho\sigma} D_\nu G_{\rho\mu} F_{\alpha\sigma}$	+				
11	$D^{\rho\alpha} G^{\mu\nu} G_{\nu\rho} \tilde{G}_\mu{}^\sigma F_{\sigma\alpha}$	−	10	d_{ABC}		$\frac{0.006}{E_{\mathrm{TeV}}^6}$
12	$D_\rho G_{\mu\nu} D^\alpha G^{\nu\rho} \tilde{G}^{\mu\sigma} F_{\sigma\alpha}$	−			$D^2 G^{\mu\nu} G^{\mu\nu} \tilde{G}^{\mu\nu} B^{\mu\nu}$	
13	$D^\rho G^{\mu\nu} D_\alpha G_{\nu\rho} \tilde{G}_{\sigma\mu} F^{\alpha\sigma}$	−		f_{ABC}		
14	$G^{\mu\nu} D_\alpha G_{\nu\rho} D^\rho \tilde{G}_{\sigma\mu} F^{\alpha\sigma}$	−				
15	$\varepsilon^{\mu\nu\rho\sigma} D^\tau G_{\rho\alpha} G^{\alpha\beta} D_\sigma G_{\beta\mu} F_{\tau\nu}$	−		d_{ABC}	$\varepsilon D^4 G^{\mu\nu} G^{\mu\nu} G^{\mu\nu} B^{\mu\nu}$	
16	$\varepsilon^{\mu\nu\rho\sigma} G_{\rho\alpha} D^\tau G^{\alpha\beta} D_\sigma G_{\mu\beta} F_{\tau\nu}$	−				

TABLE XIX. Primary dimension-12, -14, and -16 operators for the γggg interaction. There are two redundancies that appear at dimension 14. We can form a set of independent operators by removing the operators and descendants, $x^n y^m \mathcal{O}_8^{\gamma ggg}$, $x^n y^m \mathcal{O}_{16}^{\gamma ggg}$, with $x = s^2 + t^2 + u^2$, $y = stu$, $n \geq 1$, and $m \geq 0$, which are redundant.

i	$\mathcal{O}_i^{\gamma ggg}$	CP	$d_{\mathcal{O}_i}$	SU(3)	SMEFT operator form	c Unitarity bound
17	$D^{\alpha\beta} G^{\mu\nu} D_{\sigma\beta} G_{\nu\rho} G^{\rho\sigma} F_{\alpha\mu}$	+		d_{ABC}		
18	$D_\beta G^{\mu\nu} D_\sigma{}^{\alpha\beta} G_{\nu\rho} G^{\rho\sigma} F_{\alpha\mu}$	+			$D^4 G^{\mu\nu} G^{\mu\nu} G^{\mu\nu} B^{\mu\nu}$	
19	$D_\beta G^{\mu\nu} D_\nu{}^{\alpha\beta} G^{\rho\sigma} G_{\rho\mu} F_{\alpha\sigma}$	+		f_{ABC}		
20	$D_{\rho\beta} G^{\mu\nu} D^{\alpha\beta} G_{\mu\nu} G^{\rho\sigma} F_{\sigma\alpha}$	+				
21	$D^{\rho\alpha\beta} G^{\mu\nu} D_\beta G_{\nu\rho} \tilde{G}_\mu{}^\sigma F_{\sigma\alpha}$	−	12	d_{ABC}		$\frac{4\times 10^{-4}}{E_{\mathrm{TeV}}^8}$
22	$D_{\rho\beta} G_{\mu\nu} D^{\alpha\beta} G^{\nu\rho} \tilde{G}^{\mu\sigma} F_{\sigma\alpha}$	−			$D^4 G^{\mu\nu} G^{\mu\nu} \tilde{G}^{\mu\nu} B^{\mu\nu}$	
23	$D^{\rho\beta} G^{\mu\nu} D_{\alpha\beta} G_{\nu\rho} \tilde{G}_{\sigma\mu} F^{\alpha\sigma}$	−		f_{ABC}		
24	$\varepsilon^{\mu\nu\rho\sigma} D^\pi G_{\rho\alpha} D^\tau{}_{\sigma\pi} G^{\alpha\beta} G_{\beta\mu} F_{\tau\nu}$	−			$\varepsilon D^4 G^{\mu\nu} G^{\mu\nu} G^{\mu\nu} B^{\mu\nu}$	
25	$D_{\beta\tau} G^{\mu\nu} D_\sigma{}^{\alpha\beta\tau} G_{\nu\rho} G^{\rho\sigma} F_{\alpha\mu}$	+		d_{ABC}	$D^6 G^{\mu\nu} G^{\mu\nu} G^{\mu\nu} B^{\mu\nu}$	
26	$D_{\alpha\beta\tau} G^{\mu\nu} D_\nu{}^{\beta\tau} G^{\rho\sigma} G_{\mu\rho} F^\alpha{}_\sigma$	+		f_{ABC}		
27	$D_{\beta\tau} G^{\mu\nu} D_\nu{}^{\alpha\beta\tau} G^{\rho\sigma} G_{\rho\mu} F_{\alpha\sigma}$	+	14			$\frac{2\times 10^{-5}}{E_{\mathrm{TeV}}^{10}}$
28	$D^{\rho\alpha\beta\tau} G^{\mu\nu} D_{\beta\tau} G_{\nu\rho} \tilde{G}_\mu{}^\sigma F_{\sigma\alpha}$	−		d_{ABC}	$D^6 G^{\mu\nu} G^{\mu\nu} \tilde{G}^{\mu\nu} B^{\mu\nu}$	
29	$D^{\rho\beta\tau} G^{\mu\nu} D_{\alpha\beta\tau} G_{\nu\rho} \tilde{G}_{\sigma\mu} F^{\alpha\sigma}$	−		f_{ABC}		
30	$\varepsilon^{\mu\nu\rho\sigma} D^{\pi\eta} G_{\rho\alpha} D^\tau{}_{\sigma\pi\eta} G^{\alpha\beta} G_{\beta\mu} F_{\tau\nu}$	−			$\varepsilon D^6 G^{\mu\nu} G^{\mu\nu} G^{\mu\nu} B^{\mu\nu}$	
31	$\left(D^{\tau\pi}{}_\alpha G^{\mu\nu} \overset{\leftrightarrow}{D}{}^\beta D_{\tau\pi\sigma} G_{\nu\rho}\right) D_\beta G^\rho{}_\mu F^{\sigma\alpha}$	+			$D^8 G^{\mu\nu} G^{\mu\nu} G^{\mu\nu} B^{\mu\nu}$	
32	$\left(D^{\tau\pi} G^{\mu\nu} \overset{\leftrightarrow}{D}{}^\beta D^\alpha{}_{\sigma\tau\pi} G_{\nu\rho}\right) D_\beta G^{\rho\sigma} F_{\alpha\mu}$	+	16	f_{ABC}		$\frac{10^{-6}}{E_{\mathrm{TeV}}^{12}}$
33	$\left(D^{\rho\alpha\tau\pi} G^{\mu\nu} \overset{\leftrightarrow}{D}{}^\beta D_{\tau\pi} G_{\nu\rho}\right) D_\beta \tilde{G}_\mu{}^\sigma F_{\sigma\alpha}$	−			$D^8 G^{\mu\nu} G^{\mu\nu} \tilde{G}^{\mu\nu} B^{\mu\nu}$	
34	$\varepsilon^{\mu\nu\rho\sigma} (D^{\tau\pi\eta} G_{\rho\alpha} \overset{\leftrightarrow}{D}{}^\xi D_{\pi\eta} G^{\alpha\beta}) D_{\sigma\xi} G_{\beta\mu} F_{\tau\nu}$	−			$\varepsilon D^8 G^{\mu\nu} G^{\mu\nu} G^{\mu\nu} B^{\mu\nu}$	

TABLE XX. Primary dimension-eight and -ten operators for the $gggg$ interaction. $G^A_{\mu\nu}$'s are contacted with trace factors, where $\mathrm{Tr}(T^2)\mathrm{Tr}(T^2)$ represents $\mathrm{Tr}(T^AT^B)\mathrm{Tr}(T^CT^D)$, and $\mathrm{Tr}(T^4)$ represents $\mathrm{Tr}(T^AT^BT^CT^D)$.

i	$\mathcal{O}_i^{gggg}$	CP	$d_{\mathcal{O}_i}$	SU(3) trace	SMEFT operator form	c Unitarity bound
1	$G^{A\mu\nu}G^{B}{}_{\mu\nu}G^{C\rho\sigma}G^{D}{}_{\rho\sigma}$	$+$	8	$\mathrm{Tr}(T^2)\mathrm{Tr}(T^2)$	$G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}$	$\frac{0.1}{E^4_{\mathrm{TeV}}}$
2	$G^{A\mu\nu}G^{B\rho\sigma}G^{C}{}_{\mu\nu}G^{D}{}_{\rho\sigma}$	$+$				
3	$G^{A\mu\nu}G^{B}{}_{\nu\rho}G^{C\rho\sigma}G^{D}{}_{\sigma\mu}$	$+$				
4	$G^{A\mu\nu}G^{B\rho\sigma}G^{C}{}_{\nu\rho}G^{D}{}_{\sigma\mu}$	$+$				
5	$G^{A\mu\nu}G^{B\rho\sigma}G^{C}{}_{\mu\nu}G^{D}{}_{\rho\sigma}$	$+$		$\mathrm{Tr}(T^4)$		
6	$G^{A\mu\nu}G^{B}{}_{\nu\rho}G^{C}{}_{\sigma\mu}G^{D\rho\sigma}$	$+$				
7	$G^{A\mu\nu}\tilde{G}^{B}{}_{\mu\nu}G^{C\rho\sigma}G^{D}{}_{\rho\sigma}$	$-$		$\mathrm{Tr}(T^2)\mathrm{Tr}(T^2)$	$G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}\tilde{G}^{\mu\nu}$	
8	$G^{A\mu\nu}G^{B\rho\sigma}\tilde{G}^{C}{}_{\mu\nu}G^{D}{}_{\rho\sigma}$	$-$		$\mathrm{Tr}(T^4)$		
9	$\tilde{G}^{A\mu\nu}G^{B}{}_{\nu\rho}G^{C\rho\sigma}G^{D}{}_{\sigma\mu}$	$-$				
10	$D^\alpha G^{A\mu\nu}D_\alpha G^{B}{}_{\mu\nu}G^{C\rho\sigma}G^{D}{}_{\rho\sigma}$	$+$	10	$\mathrm{Tr}(T^2)\mathrm{Tr}(T^2)$	$D^2G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}$	$\frac{0.006}{E^6_{\mathrm{TeV}}}$
11	$D^\alpha G^{A\mu\nu}D_\alpha G^{B\rho\sigma}G^{C}{}_{\mu\nu}G^{D}{}_{\rho\sigma}$	$+$				
12	$D^\alpha G^{A\mu\nu}D_\alpha G^{B}{}_{\nu\rho}G^{C\rho\sigma}G^{D}{}_{\sigma\mu}$	$+$				
13	$D^\alpha G^{A\mu\nu}D_\alpha G^{B\rho\sigma}G^{C}{}_{\nu\rho}G^{D}{}_{\sigma\mu}$	$+$				
14	$\left(G^{A\mu\nu}\overleftrightarrow{D}^\alpha G^{B\rho\sigma}\right)D_\alpha G^{C}{}_{\mu\nu}G^{D}{}_{\rho\sigma}$	$+$				
15	$D^\alpha G^{A\mu\nu}G^{B\rho\sigma}D_\alpha G^{C}{}_{\mu\nu}G^{D}{}_{\rho\sigma}$	$+$		$\mathrm{Tr}(T^4)$		
16	$D^\alpha G^{A\mu\nu}G^{B}{}_{\nu\rho}D_\alpha G^{C}{}_{\sigma\mu}G^{D\rho\sigma}$	$+$				
17	$\left(G^{A\mu\nu}\overleftrightarrow{D}^\alpha G^{C\rho\sigma}\right)D_\alpha G^{B}{}_{\mu\nu}G^{D}{}_{\rho\sigma}$	$+$				
18	$D_\alpha G^{A\mu\nu}D_\sigma G^{B}{}_{\nu\rho}G^{C\rho}{}_{\mu}G^{D\sigma\alpha}$	$+$				
19	$D^\alpha G^{A\mu\nu}D_\alpha\tilde{G}^{B}{}_{\mu\nu}G^{C\rho\sigma}G^{D}{}_{\rho\sigma}$	$-$		$\mathrm{Tr}(T^2)\mathrm{Tr}(T^2)$	$D^2G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}\tilde{G}^{\mu\nu}$	
20	$D^\alpha G^{A\mu\nu}G^{B\rho\sigma}D_\alpha\tilde{G}^{C}{}_{\mu\nu}G^{D}{}_{\rho\sigma}$	$-$		$\mathrm{Tr}(T^4)$		
21	$D^\alpha\tilde{G}^{A\mu\nu}G^{B}{}_{\nu\rho}D_\alpha G^{C\rho\sigma}G^{D}{}_{\sigma\mu}$	$-$				
22	$\varepsilon^{\mu\nu\rho\sigma}D^\tau G^{A}{}_{\rho\alpha}G^{B\alpha\beta}D_\sigma G^{C}{}_{\beta\mu}G^{D}{}_{\tau\nu}$	$-$		$\mathrm{Tr}(T^2)\mathrm{Tr}(T^2)$	$\varepsilon D^2G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}$	
23	$\varepsilon^{\mu\nu\rho\sigma}D^\tau G^{A}{}_{\rho\alpha}G^{B\alpha\beta}D_\sigma G^{C}{}_{\beta\mu}G^{D}{}_{\tau\nu}$	$-$		$\mathrm{Tr}(T^4)$		

TABLE XXI. Primary dimension-12, -14, and -16 operators for the $gggg$ interaction. There are two redundancies that appear at dimension 14 and four at dimension 16. To form a set of independent operators, the operators and descendants $x^n y^m \mathcal{O}_4^{gggg}$, $x^n y^m \mathcal{O}_9^{gggg}$, $x^n y^m \mathcal{O}_{12}^{gggg}$, $x^n y^m \mathcal{O}_{14}^{gggg}$, $x^n y^m \mathcal{O}_{19}^{gggg}$, $x^n y^m \mathcal{O}_{20}^{gggg}$, with $x = s^2 + t^2 + u^2$, $y = stu$, $n \geq 0$, and $m \geq 1$, should be omitted.

i	$\mathcal{O}_i^{gggg}$	CP	$d_{\mathcal{O}_i}$	SU(3) trace	SMEFT operator form	c Unitarity bound
24	$\left(G^{A\mu\nu}\overleftrightarrow{D}^{\alpha\beta}G^{B}{}_{\mu\nu}\right)D_{\alpha\beta}G^{C\rho\sigma}G^{D}{}_{\rho\sigma}$	+	12	$\mathrm{Tr}(T^2)\mathrm{Tr}(T^2)$	$D^4G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}$	$\frac{4\times10^{-4}}{E_{\mathrm{TeV}}^{8}}$
25	$\left(G^{A\mu\nu}\overleftrightarrow{D}^{\alpha\beta}G^{B\rho\sigma}\right)D_{\alpha\beta}G^{C}{}_{\mu\nu}G^{D}{}_{\rho\sigma}$	+				
26	$(G^{A\mu\nu}\overleftrightarrow{D}^{\alpha\beta}G^{B}{}_{\nu\rho})D_{\alpha\beta}G^{C\rho\sigma}G^{D}{}_{\sigma\mu}$	+				
27	$\left(D^{\beta}G^{A\mu\nu}\overleftrightarrow{D}^{\alpha}D_{\beta}G^{B\rho\sigma}\right)D_{\alpha}G^{C}{}_{\mu\nu}G^{D}{}_{\rho\sigma}$	+				
28	$\left(G^{A\mu\nu}\overleftrightarrow{D}^{\alpha\beta}G^{C}{}_{\mu\nu}\right)D_{\alpha\beta}G^{B\rho\sigma}G^{D}{}_{\rho\sigma}$	+		$\mathrm{Tr}(T^4)$		
29	$\left(G^{A\mu\nu}\overleftrightarrow{D}^{\alpha\beta}G^{C}{}_{\sigma\mu}\right)D_{\alpha\beta}G^{B}{}_{\nu\rho}G^{D\rho\sigma}$	+				
30	$\left(D^{\beta}G^{A\mu\nu}\overleftrightarrow{D}^{\alpha}D_{\beta}G^{C\rho\sigma}\right)D_{\alpha}G^{B}{}_{\mu\nu}G^{D}{}_{\rho\sigma}$	+				
31	$D_{\alpha\beta}G^{A\mu\nu}D_{\sigma}G^{B}{}_{\nu\rho}D^{\beta}G^{C\rho}{}_{\mu}G^{D\sigma\alpha}$	+				
32	$\left(D_{\alpha}G^{A\mu\nu}\overleftrightarrow{D}^{\beta}G^{C\rho}{}_{\mu}\right)D_{\sigma\beta}G^{B}{}_{\nu\rho}G^{D\sigma\alpha}$	+				
33	$D_{\alpha\beta}G^{A\mu\nu}D_{\sigma}{}^{\beta}G^{B}{}_{\nu\rho}G^{C\rho}{}_{\mu}G^{D\sigma\alpha}$	+				
34	$\left(G^{A\mu\nu}\overleftrightarrow{D}^{\alpha\beta}\tilde{G}^{B}{}_{\mu\nu}\right)D_{\alpha\beta}G^{C\rho\sigma}G^{D}{}_{\rho\sigma}$	−		$\mathrm{Tr}(T^2)\mathrm{Tr}(T^2)$	$D^4G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}\tilde{G}^{\mu\nu}$	
35	$\left(G^{A\mu\nu}\overleftrightarrow{D}^{\alpha\beta}\tilde{G}^{C}{}_{\mu\nu}\right)D_{\alpha\beta}G^{B\rho\sigma}G^{D}{}_{\rho\sigma}$	−		$\mathrm{Tr}(T^4)$		
36	$\left(D^{\rho}{}_{\alpha}G^{A\mu\nu}\overleftrightarrow{D}^{\beta}\tilde{G}^{C}{}_{\mu\sigma}\right)D_{\beta}G^{B}{}_{\nu\rho}G^{D\sigma\alpha}$	−				
37	$\varepsilon^{\mu\nu\rho\sigma}D^{\tau\pi}G^{A}{}_{\rho\alpha}D_{\pi}G^{B\alpha\beta}D_{\sigma}G^{C}{}_{\beta\mu}G^{D}{}_{\tau\nu}$	−		$\mathrm{Tr}(T^2)\mathrm{Tr}(T^2)$	$\varepsilon D^4G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}$	
38	$\varepsilon^{\mu\nu\rho\sigma}D^{\tau\pi}G^{A}{}_{\rho\alpha}G^{B\alpha\beta}D_{\sigma\pi}G^{C}{}_{\beta\mu}G^{D}{}_{\tau\nu}$	−		$\mathrm{Tr}(T^4)$		
39	$\varepsilon^{\mu\nu\rho\sigma}\left(D^{\tau}G^{A}{}_{\rho\alpha}\overleftrightarrow{D}^{\pi}D_{\sigma}G^{C}{}_{\beta\mu}\right)D_{\pi}G^{B\alpha\beta}G^{D}{}_{\tau\nu}$	−				
40	$\left(G^{A\mu\nu}\overleftrightarrow{D}^{\alpha\beta\tau}G^{B\rho\sigma}\right)D_{\alpha\beta\tau}G^{C}{}_{\mu\nu}G^{D}{}_{\rho\sigma}$	+	14	$\mathrm{Tr}(T^2)\mathrm{Tr}(T^2)$	$D^6G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}$	$\frac{2\times10^{-5}}{E_{\mathrm{TeV}}^{10}}$
41	$\left(G^{A\mu\nu}\overleftrightarrow{D}^{\alpha\beta\tau}G^{C\rho\sigma}\right)D_{\alpha\beta\tau}G^{B}{}_{\mu\nu}G^{D}{}_{\rho\sigma}$	+		$\mathrm{Tr}(T^4)$		
42	$\left(D_{\alpha}G^{A\mu\nu}\overleftrightarrow{D}^{\beta\tau}G^{C\rho}{}_{\mu}\right)D_{\sigma\beta\tau}G^{B}{}_{\nu\rho}G^{D\sigma\alpha}$	+				
43	$\left(D_{\alpha\tau}G^{A\mu\nu}\overleftrightarrow{D}^{\beta}D^{\tau}G^{C\rho}{}_{\mu}\right)D_{\sigma\beta}G^{B}{}_{\nu\rho}G^{D\sigma\alpha}$	+				
44	$\left(D_{\alpha}G^{A\mu\nu}\overleftrightarrow{D}^{\beta\tau}D_{\sigma}G^{B}{}_{\nu\rho}\right)D_{\beta\tau}G^{C\rho}{}_{\mu}G^{D\sigma\alpha}$	+				
45	$\left(D^{\rho}{}_{\alpha\tau}G^{A\mu\nu}\overleftrightarrow{D}^{\beta}D^{\tau}\tilde{G}^{C}{}_{\mu\sigma}\right)D_{\beta}G^{B}{}_{\nu\rho}G^{D\sigma\alpha}$	−		$\mathrm{Tr}(T^4)$	$D^6G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}\tilde{G}^{\mu\nu}$	
46	$\varepsilon^{\mu\nu\rho\sigma}\left(D^{\tau}G^{A}{}_{\rho\alpha}\overleftrightarrow{D}^{\pi\eta}G^{B\alpha\beta}\right)D_{\sigma\pi\eta}G^{C}{}_{\beta\mu}G^{D}{}_{\tau\nu}$	−		$\mathrm{Tr}(T^2)\mathrm{Tr}(T^2)$	$\varepsilon D^4G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}$	
47	$\varepsilon^{\mu\nu\rho\sigma}\left(D^{\tau}G^{A}{}_{\rho\alpha}\overleftrightarrow{D}^{\pi\eta}D_{\sigma}G^{C}{}_{\beta\mu}\right)D_{\pi\eta}G^{B\alpha\beta}G^{D}{}_{\tau\nu}$	−		$\mathrm{Tr}(T^4)$		
48	$\varepsilon^{\mu\nu\rho\sigma}\left(D^{\tau\eta}G^{A}{}_{\rho\alpha}\overleftrightarrow{D}^{\pi}D_{\sigma\eta}G^{C}{}_{\beta\mu}\right)D_{\pi}G^{B\alpha\beta}G^{D}{}_{\tau\nu}$	−				
49	$\left(D_{\alpha}G^{A\mu\nu}\overleftrightarrow{D}^{\beta\tau\pi}G^{C\rho}{}_{\mu}\right)D_{\sigma\beta\tau\pi}G^{B}{}_{\nu\rho}G^{D\sigma\alpha}$	+	16	$\mathrm{Tr}(T^4)$	$D^8G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}$	$\frac{10^{-6}}{E_{\mathrm{TeV}}^{12}}$
50	$\left(D^{\rho}{}_{\alpha}G^{A\mu\nu}\overleftrightarrow{D}^{\beta\tau\pi}\tilde{G}^{C}{}_{\mu\sigma}\right)D_{\beta\tau\pi}G^{B}{}_{\nu\rho}G^{D\sigma\alpha}$	−			$D^8G^{\mu\nu}G^{\mu\nu}G^{\mu\nu}\tilde{G}^{\mu\nu}$	

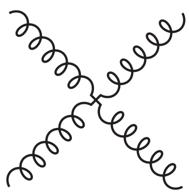

FIG. 1. Four-gluon interaction. Labels correspond to color indices, momenta, and polarization and we consider the process $(A,1)+(B,2)\to(C,3)+(D,4)$.

Next, we look at $Z\to\gamma gg$ and $Z\to ggg$ decays. The branching ratios are predicted to be $\sim 4.9\times10^{-6}$ [34,42,43] and $\sim 1.8\times10^{-5}$ [34,35,42,43], respectively. Since our unitarity bound estimates are the same for $Z\to(\gamma\gamma\gamma,\gamma gg,ggg)$, even if we optimistically assume that there is only the irreducible background from the Standard Model prediction and that unitarity is violated at a TeV, we find that there is at most a 2σ fluctuation for the HL-LHC run for the dimension-eight interactions. Since these would be likely to have substantial reducible backgrounds, this shows that these are unlikely to be observable. Such a conclusion was also reached in [34], which showed that measuring the associated $gg\to Zg$ interaction at a hadron collider was "rather remote" due to there being too much background, particularly from $q\bar{q}\to Zg$, $qg\to Zq$, and $\bar{q}g\to Z\bar{q}$ processes. They also concluded that attempts at measuring the coupling from observations of the $Z\to ggg$ process at a lepton collider would suffer too greatly from $Z\to q\bar{q}g$ background.

VII. CONCLUSIONS

This paper has determined the allowed on-shell amplitudes for four-point interactions of gauge bosons in the Standard Model. Following [6,7], this has completed the analysis of all three- and four-point interactions for the Standard Model content. For certain couplings, this required studying Lagrangian operators up to mass dimension 16, which demonstrates the efficacy of the numerical approach used in these papers.

The characterization of these amplitudes holds the promise of allowing the most general model-independent collider searches by studying the interactions of Standard Model particles. They also serve as a useful intermediary between experimental and theoretical analyses since, in comparison to EFT operators, they are interpretable and do not suffer from basis ambiguities. As an illustration of phenomenological study, we investigated the potential for discovering new physics in Z decays. In these estimates, we showed that $Z\to\gamma\bar{\ell}\ell$ is of interest at the HL-LHC, but other modes like $Z\to\gamma\gamma\gamma$ would require unitarity violation well below a TeV, which are likely in violation of direct search constraints.

Moving forward, it will be useful to perform realistic phenomenological studies at the HL-LHC and future colliders. Another direction is to use the Mandelstam descendants as a model for theoretical uncertainties. Finally, the utilization of on-shell amplitudes in realistic analyses will undoubtedly require solutions to practical challenges along the way. We and our collaborators are currently exploring such questions and hope that this work has motivated others to do the same.

Note added. Recently, an updated reference [44] for the Standard Model predictions for the Z branching ratios to massless gauge bosons was brought to our attention. These new results do not affect our conclusions.

ACKNOWLEDGMENTS

We would like to thank M. Luty for useful discussions. The work of C. A. and S. C. was supported in part by the U.S. Department of Energy under Award No. DE-SC0011640.

[1] Y. Shadmi and Y. Weiss, Effective field theory amplitudes the on-shell way: Scalar and vector couplings to gluons, J. High Energy Phys. 02 (2019) 165.
[2] G. Durieux, T. Kitahara, Y. Shadmi, and Y. Weiss, The electroweak effective field theory from on-shell amplitudes, J. High Energy Phys. 01 (2020) 119.
[3] G. Durieux, T. Kitahara, C. S. Machado, Y. Shadmi, and Y. Weiss, Constructing massive on-shell contact terms, J. High Energy Phys. 12 (2020) 175.
[4] Z.-Y. Dong, T. Ma, J. Shu, and Z.-Z. Zhou, The new formulation of Higgs effective field theory, J. High Energy Phys. 09 (2023) 101.
[5] H. Liu, T. Ma, Y. Shadmi, and M. Waterbury, An EFT hunter's guide to two-to-two scattering: HEFT and SMEFT on-shell amplitudes, J. High Energy Phys. 05 (2023) 241.
[6] S. Chang, M. Chen, D. Liu, and M. A. Luty, Primary observables for indirect searches at colliders, J. High Energy Phys. 07 (2023) 030.
[7] L. Bradshaw and S. Chang, Primary observables for top quark collider signals, Phys. Rev. D **108** (2023).
[8] W. Buchmuller and D. Wyler, Effective Lagrangian analysis of new interactions and flavor conservation, Nucl. Phys. **B268**, 621 (1986).

[9] B. Grzadkowski, M. Iskrzynski, M. Misiak, and J. Rosiek, Dimension-six terms in the standard model Lagrangian, J. High Energy Phys. 10 (2010) 085.

[10] F. Feruglio, The chiral approach to the electroweak interactions, Int. J. Mod. Phys. A **08**, 4937 (1993).

[11] S. Chang, M. Chen, D. Liu, and M. A. Luty, Primary observables for indirect searches at colliders, J. High Energy Phys. 07 (2023) 030.

[12] For processes with identical particles, these Mandelstam factors must be invariant under the relevant crossing symmetries, as we will discuss later.

[13] In some cases, there can be inverse factors of $E_{\mathrm{c.m.}}$, which we can get rid of by taking the power n to be large enough.

[14] L. Lehman and A. Martin, Hilbert series for constructing Lagrangians: Expanding the phenomenologist's toolbox, Phys. Rev. D **91** (2015).

[15] B. Henning, X. Lu, T. Melia, and H. Murayama, Hilbert series and operator bases with derivatives in effective field theories, Commun. Math. Phys. **347**, 363 (2015).

[16] L. Lehman and A. Martin, Low-derivative operators of the standard model effective field theory via Hilbert series methods, J. High Energy Phys. 02 (2016) 081.

[17] B. Henning, X. Lu, T. Melia, and H. Murayama, 2, 84, 30, 993, 560, 15456, 11962, 261485, ...: Higher dimension operators in the SM EFT, J. High Energy Phys. 08 (2017) 016; 09 (2019) 19(E).

[18] B. Henning, X. Lu, T. Melia, and H. Murayama, Operator bases, s-matrices, and their partition functions, J. High Energy Phys. 10 (2017) 199.

[19] L. Gráf, B. Henning, X. Lu, T. Melia, and H. Murayama, 2, 12, 117, 1959, 45171, 1170086, …: A Hilbert series for the QCD chiral Lagrangian, J. High Energy Phys. 01 (2021) 142.

[20] L. Gráf, B. Henning, X. Lu, T. Melia, and H. Murayama, Hilbert series, the Higgs mechanism, and HEFT, J. High Energy Phys. 02 (2023) 064.

[21] B. Henning, X. Lu, T. Melia, and H. Murayama, Hilbert series and operator bases with derivatives in effective field theories, Commun. Math. Phys. **347**, 363 (2016).

[22] J. M. Cornwall, D. N. Levin, and G. Tiktopoulos, Derivation of gauge invariance from high-energy unitarity bounds on the s matrix, Phys. Rev. D **10**, 1145 (1974).

[23] D. Liu and Z. Yin, Gauge invariance from on-shell massive amplitudes and tree-level unitarity, Phys. Rev. D **106**, 076003 (2022).

[24] S. Chang and M. A. Luty, The Higgs trilinear coupling and the scale of new physics, J. High Energy Phys. 03 (2020) 140.

[25] F. Abu-Ajamieh, S. Chang, M. Chen, and M. A. Luty, Higgs coupling measurements and the scale of new physics, J. High Energy Phys. 07 (2021) 056.

[26] F. Abu-Ajamieh, The scale of new physics from the Higgs couplings to $\gamma\gamma$ and γz, J. High Energy Phys. 06 (2022) 091.

[27] A. Falkowski and R. Rattazzi, Which EFT, J. High Energy Phys. 10 (2019) 255.

[28] S. Schael *et al.* (ALEPH, DELPHI, L3, OPAL, SLD, LEP Electroweak Working Group, SLD Electroweak Group, SLD Heavy Flavour Group Collaborations), Precision electroweak measurements on the Z resonance, Phys. Rep. **427**, 257 (2006).

[29] R. S. Gupta, A. Pomarol, and F. Riva, BSM primary effects, Phys. Rev. D **91**, 035001 (2015).

[30] M. Baillargeon and F. Boudjema, Contribution of the bosonic loops to the three-photon decay of the z, Phys. Lett. B **272**, 158 (1991).

[31] F.-x. Dong, X.-d. Jiang, and X.-j. Zhou, Partial decay width of the z into three photons via w loops, Phys. Rev. D **46**, 5074 (1992).

[32] M.-Z. Yang and X.-J. Zhou, The decay of the Z boson into three photons via W boson loops, Phys. Rev. D **52**, 5018 (1995).

[33] M. L. Laursen, K. O. Mikaelian, and M. A. Samuel, Z^0 decay into three gluons, Phys. Rev. D **23**, 2795 (1981).

[34] J. J. Van der Bij and E. W. N. Glover, Z boson production and decay via gluons, Nucl. Phys. **B313**, 237 (1989).

[35] W. Bernreuther, M. J. Duncan, and E. W. N. Glover, Rare Z decays, https://cds.cern.ch/record/199833.

[36] Z Physics at LEP-1. Workshop, edited by G. Altarelli, R. Kleiss, and C. Verzegnassi, *Standard Physics, CERN Yellow Reports: Conference Proceedings*, (CERN, Geneva, 1989), Vol. 1, 10.5170/CERN-1989-008-V-1.

[37] E. W. N. Glover and A. G. Morgan, Z boson decay into photons, Z. Phys. C **60**, 175 (1993).

[38] See [45] for a study of effective Lagrangians for the $Z\gamma\gamma\gamma$ interaction and [46] for a study where the decay is mediated by a scalar loop in a two Higgs doublet model using a minimally supersymmetric Standard Model.

[39] G. Aad *et al.* (ATLAS Collaboration), Search for new phenomena in events with at least three photons collected in pp collisions at $\sqrt{s} = 8$ TeV with the ATLAS detector, Eur. Phys. J. C **76**, 210 (2016).

[40] O. Brüning and L. Rossi, The high-luminosity Large Hadron Collider, Nat. Rev. Phys. **1**, 241 (2019).

[41] G. Arduini, R. Bruce, R. De Maria, M. Giovannozzi, G. Iadarola, J. Jowett, E. Métral, Y. Papaphilippou, and R. Tomás Garcia, Machine layout and performance, CERN Yellow Reports: Monographs (2020), https://e-publishing.cern.ch/index.php/CYRM/article/view/1154.

[42] M. L. Laursen, M. A. Samuel, G. B. Tupper, and A. Sen, Z^0 decay into two gluons and a photon for massive quarks, Phys. Rev. D **27**, 196 (1983).

[43] M. A. Pérez, G. Tavares-Velasco, and J. J. Toscano, New physics effects in rare Z decays, Int. J. Mod. Phys. A **19**, 159 (2004).

[44] D. d'Enterria and V. Dung Le, Rare and exclusive few-body decays of the Higgs, Z, W bosons, and the top quark, arXiv:2312.11211.

[45] M. Stöhr and J. Hořejší, Effective Lagrangians for z boson decay into photons, Phys. Rev. D **49**, 3775 (1994).

[46] H. König, Contribution of scalar loops to the three-photon decay of the z, Phys. Rev. D **50**, 602 (1994).