SIDIS at small x at next-to-leading order: Gluon contribution

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We calculate the contribution of gluons to single inclusive hadron production at next-to-leading order (NLO) accuracy in deep inelastic scattering (DIS) at small x using the color glass condensate formalism. It is shown that the only divergence present is the standard collinear divergence, which is absorbed into scale evolution of quark-hadron fragmentation function. Our calculations are performed at finite N_c and we provide general finite N_c expressions for the structure of Wilson lines appearing in inclusive dihadron and single hadron production cross sections. We also comment on how one can obtain rapidity distribution of hadron multiplicities from our results.

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I. INTRODUCTION

Deep inelastic scattering (DIS) is the cleanest environment in which to probe internal partonic structure of hadrons and nuclei [1,2]. Due to the experimentally observed fast rise of gluons at small Bjorken x one expects that a high energy hadron or nucleus is a maximum occupancy state of predominantly gluons [3,4]. This high density gluonic state is referred to as the color glass condensate (CGC) [5–7] and is expected to give the dominant contribution to scattering cross sections at high energy. While there have been tantalizing hints of this saturated state at HERA, RHIC and the LHC [8–23,23–54] its presence is far from being firmly established (see [55–59] for reviews). The proposed Electron Ion Collider (EIC) will enable us to look for gluon saturation in various channels with unprecedented precision and accuracy. Among the most promising processes in which gluon saturation is expected to play a dominant role are single and double inclusive jet and hadron production. Measurements of transverse momenta, angular correlations and rapidity dependence of these processes will shed light on dynamics of gluons saturation and its energy dependence and enable a detailed comparison with theoretical expectations based on CGC formalism [60–69]. To improve the theoretical accuracy of CGC-based predictions there has been significant progress made in calculating next to leading order (NLO) and beyond eikonal corrections to observable which are sensitive to gluon saturation [70–130].

In [131] we calculated the next-to-leading order corrections to single inclusive hadron production in DIS (SIDIS) at small x when either quark or antiquark in the final state hadronizes. Here we continue our work toward a complete NLO calculation of SIDIS by considering hadronization of the produced gluon.[1] This channel is not present in a leading order calculations of SIDIS and appears only as part of the NLO corrections so that it is important to understand its role and contribution. Furthermore we include the full N_c dependence rather than making the large N_c approximation.

In the small x limit of DIS the virtual photon (transverse or longitudinal) splits into a quark antiquark pair (a dipole), which then multiply scatters from the target hadron or nucleus. To leading order (LO) accuracy the double inclusive production cross section can be written as

$$
\begin{aligned}
&\frac{d\sigma^{\gamma^* p/A\to q\bar{q}X}}{d^2\mathbf{p}\, d^2\mathbf{q}\, dy_1\, dy_2}\\
&=\frac{e^2Q^2N_c}{(2\pi)^7}\delta(1-z_1-z_2)(z_1z_2)^2\\
&\quad\times\int d^8\mathbf{x}[S_{122'1'}-S_{12}-S_{1'2'}+1]e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}}e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}}\\
&\quad\times\bigg\{4z_1z_2K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2'}|Q_1)\\
&\quad+(z_1^2+z_2^2)\frac{\mathbf{x}_{12}\cdot\mathbf{x}_{1'2'}}{|\mathbf{x}_{12}||\mathbf{x}_{1'2'}|}K_1(|\mathbf{x}_{12}|Q_1)K_1(|\mathbf{x}_{1'2'}|Q_1)\bigg\}\qquad(1)
\end{aligned}
$$

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[1] While this manuscript was being prepared for publication we were informed of a similar work [132] which however focuses on single inclusive jet production in DIS. We thank F. Salazar for bringing their work to our attention.

where the first and second terms in the curly bracket above correspond to the contribution of the longitudinal and transverse polarizations of the virtual photon. The incoming virtual photon (right moving) has momentum l^μ with $l^2 = -Q^2$ and we have set the transverse momentum of the photon to zero without any loss of generality. Furthermore p^μ (q^μ) is the momentum of the outgoing quark (antiquark) while z_1 (z_2) is its longitudinal momentum fraction relative to the photon. Also, $\mathbf{x}_1$ ($\mathbf{x}_2$) is the transverse coordinate of the quark (antiquark), and primed coordinates are used in the conjugate amplitude. Quark and antiquark rapidities y_1 and y_2 are related to their momentum fractions z_1 and z_2 via $\mathrm{d}y_i = \mathrm{d}z_i/z_i$.

We further define and use the notation

$$Q_i = Q\sqrt{z_i(1-z_i)}, \qquad \mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j,$$
$$\mathrm{d}^8\mathbf{x} = \mathrm{d}^2\mathbf{x}_1\,\mathrm{d}^2\mathbf{x}_2\mathrm{d}^2\mathbf{x}_{1'}\,\mathrm{d}^2\mathbf{x}_{2'}. \tag{2}$$

The production cross section is a convolution of the probability for a photon to split into a quark at transverse position $\mathbf{x}_1$ and an antiquark at position $\mathbf{x}_2$ represented by the Bessel functions, with multiple scatterings of the quark antiquark pair on the target encoded in the dipoles S_{ij} and quadrupoles S_{ijkl}. To calculate the single inclusive production cross section we need to integrate over one of the final state partons which we choose to be antiquark so that quark is produced. The case when antiquark is produced is identical so that our results can be multiplied by a factor of 2 to include antiquark production. Integrating over the produced antiquark phase space sets $\mathbf{x}_{2'} = \mathbf{x}_2$ and gives

$$\begin{aligned}
&\frac{\mathrm{d}\sigma^{\gamma^* p/A \to q(\mathbf{p},y_1)X}}{\mathrm{d}^2\mathbf{p}\,\mathrm{d}y_1} \\
&= \frac{e^2Q^2N_c}{(2\pi)^5}\int \mathrm{d}z_2\delta(1-z_1-z_2)(z_1^2 z_2) \\
&\times \int \mathrm{d}^6\mathbf{x}[S_{11'} - S_{12} - S_{1'2} + 1]e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} \\
&\times \bigg\{4z_1z_2K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2}|Q_1) \\
&+ (z_1^2 + z_2^2)\frac{\mathbf{x}_{12}\cdot\mathbf{x}_{1'2}}{|\mathbf{x}_{12}||\mathbf{x}_{1'2}|}K_1(|\mathbf{x}_{12}|Q_1)K_1(|\mathbf{x}_{1'2}|Q_1)\bigg\}.
\end{aligned} \tag{3}$$

II. SINGLE INCLUSIVE GLUON PRODUCTION

To calculate contribution of gluons to SIDIS we start with the next-to-leading (NLO) corrections to dihadron production in [104]. In principle there are both real and virtual contributions, however only real diagrams containing a gluon in the final state contribute so therefore we will focus on the real corrections here. The diagrams corresponding to real corrections are shown in Fig. 1.

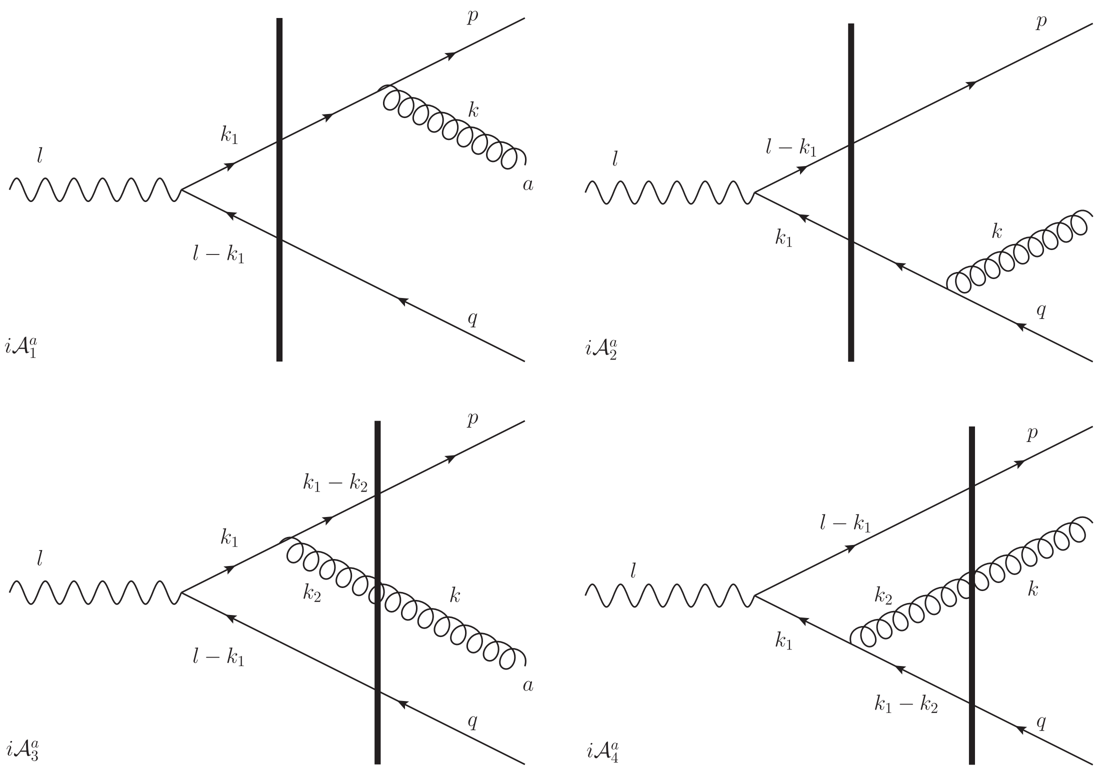

FIG. 1. The real corrections $i\mathcal{A}_1^a, \ldots, i\mathcal{A}_4^a$. The arrows on Fermion lines indicate Fermion number flow, whereas all momenta flow to the right. The thick solid line indicates multiple scatterings from the target.

The full NLO expressions for dihadron production are derived in [104], in order to calculate contribution of gluons one needs to "undo" integration of the final state gluon phase space and then integrate out the final state quark and antiquark so that only the produced gluon is left in the final state. As the original expressions in [104] are very long and not illuminating we will just show the results here,

$$\frac{\mathrm{d}\sigma^L_{1\times 1}}{\mathrm{d}^2\mathbf{k}\,\mathrm{d}y} = 4\frac{e^2Q^2g^2N_c}{(2\pi)^6}C_F\int \mathrm{d}z_1\,\mathrm{d}z_2\delta(1-z_1-z_2-z)z_2^2(1-z_2)^2[z_1^2+(1-z_2)^2] \times \int \mathrm{d}^6\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}_{1'1}} K_0(|\mathbf{x}_{12}|Q_2)K_0(|\mathbf{x}_{1'2}|Q_2)[S_{11'}-S_{12}-S_{1'2}+1]\int\frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}\frac{e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}}}{[z\mathbf{p}-z_1\mathbf{k}]^2} \tag{4}$$

$$\frac{\mathrm{d}\sigma^L_{2\times 2}}{\mathrm{d}^2\mathbf{k}\,\mathrm{d}y} = 4\frac{e^2Q^2g^2N_c}{(2\pi)^6}C_F\int \mathrm{d}z_1\,\mathrm{d}z_2\delta(1-z_1-z_2-z)z_1^2(1-z_1)^2[z_2^2+(1-z_1)^2] \times \int \mathrm{d}^6\mathbf{x} e^{i\mathbf{k}\cdot\mathbf{x}_{2'2}} K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{12'}|Q_1)[S_{22'}-S_{12}-S_{12'}+1]\int\frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\frac{e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}}}{[z\mathbf{q}-z_2\mathbf{k}]^2} \tag{5}$$

$$\frac{\mathrm{d}\sigma^L_{1\times 2}}{\mathrm{d}^2\mathbf{k}\mathrm{d}y} = 4\frac{e^2Q^2g^2N_c}{(2\pi)^6}\int \mathrm{d}z_1\mathrm{d}z_2\delta(1-z_1-z_2-z)\left[\frac{z_1(1-z_1)z_2(1-z_2)}{z}\right]\left[\frac{z_1(1-z_1)+z_2(1-z_2)}{z}\right] \times\frac{1}{(2\pi)^2}\int \mathrm{d}^8\mathbf{x}\frac{\mathbf{x}_{1'1}\cdot\mathbf{x}_{2'2}}{\mathbf{x}_{1'1}^2\mathbf{x}_{2'2}^2}K_0(|\mathbf{x}_{12}|Q_2)K_0(|\mathbf{x}_{1'2'}|Q_1)e^{i\mathbf{k}\cdot[\mathbf{x}_{2'1}+\frac{z_1}{z}\mathbf{x}_{1'1}+\frac{z_2}{z}\mathbf{x}_{2'2}]} \times\left[\frac{N_c}{2}S_{12}S_{1'2'}+C_F[1-S_{12}-S_{2'1'}]-\frac{1}{2N_c}S_{122'1'}\right] \tag{6}$$

$$\frac{\mathrm{d}\sigma^L_{3\times 3}}{\mathrm{d}^2\mathbf{k}\mathrm{d}y} = 4\frac{e^2Q^2g^2N_c}{(2\pi)^6}\int \mathrm{d}z_1\,\mathrm{d}z_2\delta(1-z_1-z_2-z)z_2^2[z_1^2+(1-z_2)^2] \times\frac{1}{(2\pi)^2}\int \mathrm{d}^8\mathbf{x}\frac{\mathbf{x}_{31}\cdot\mathbf{x}_{3'1}}{\mathbf{x}_{31}^2\mathbf{x}_{3'1}^2}[K_0(QX)K_0(QX')]_{1'=1,2'=2}e^{i\mathbf{k}\cdot\mathbf{x}_{3'3}} \times\left\{\frac{N_c}{2}[S_{33'}S_{33'}-S_{13}S_{23}-S_{13'}S_{23'}]+C_F-\frac{1}{2N_c}[1-2S_{12}]\right\}. \tag{7}$$

$$\frac{\mathrm{d}\sigma^L_{4\times 4}}{\mathrm{d}^2\mathbf{k}\mathrm{d}y} = 4\frac{e^2Q^2g^2N_c}{(2\pi)^6}\int \mathrm{d}z_1\,\mathrm{d}z_2\delta(1-z_1-z_2-z)z_1^2[z_2^2+(1-z_1)^2] \times\frac{1}{(2\pi)^2}\int \mathrm{d}^8\mathbf{x}\frac{\mathbf{x}_{32}\cdot\mathbf{x}_{3'2}}{\mathbf{x}_{32}^2\mathbf{x}_{3'2}^2}[K_0(QX)K_0(QX')]_{1'=1,2'=2}\,e^{i\mathbf{k}\cdot\mathbf{x}_{3'3}} \times\left\{\frac{N_c}{2}[S_{33'}S_{33'}-S_{13}S_{23}-S_{13'}S_{23'}]+C_F-\frac{1}{2N_c}[1-2S_{12}]\right\}. \tag{8}$$

$$\frac{\mathrm{d}\sigma^L_{3\times 4}}{\mathrm{d}^2\mathbf{k}\,\mathrm{d}y} = -4\frac{e^2Q^2g^2N_c}{(2\pi)^6}\int \mathrm{d}z_1\mathrm{d}z_2\delta(1-z_1-z_2-z)z_1z_2[z_1(1-z_1)+z_2(1-z_2)] \times\frac{1}{(2\pi)^2}\int \mathrm{d}^8\mathbf{x}\frac{\mathbf{x}_{31}\cdot\mathbf{x}_{3'2}}{\mathbf{x}_{31}^2\,\mathbf{x}_{3'2}^2}[K_0(QX)K_0(QX')]_{1'=1,2'=2}e^{i\mathbf{k}\cdot\mathbf{x}_{3'3}} \times\left\{\frac{N_c}{2}[S_{33'}S_{33'}-S_{13}S_{23}-S_{13'}S_{23'}]+C_F-\frac{1}{2N_c}[1-2S_{12}]\right\}. \tag{9}$$

$$\frac{\mathrm{d}\sigma^L_{1\times 3}}{\mathrm{d}^2\mathbf{k}\,\mathrm{d}y} = 4\frac{e^2Q^2g^2N_c}{(2\pi)^6}\int \mathrm{d}z_1\,\mathrm{d}z_2\delta(1-z_1-z_2-z)\frac{z_2}{z}z_2(1-z_2)[z_1^2+(1-z_2)^2] \times\frac{1}{(2\pi)^2}\int \mathrm{d}^8\mathbf{x}\frac{\mathbf{x}_{1'1}\cdot\mathbf{x}_{3'1'}}{\mathbf{x}_{1'1}^2\mathbf{x}_{3'1'}^2}[K_0(|\mathbf{x}_{12}|Q_2)K_0(QX')]_{2'=2}e^{i\mathbf{k}\cdot[\mathbf{x}_{3'1}+\frac{z_1}{z}\mathbf{x}_{1'1}]} \times\left\{\frac{N_c}{2}[S_{3'1'}S_{13'}-S_{3'1'}S_{23'}]+C_F[1-S_{12}]-\frac{1}{2N_c}[S_{11'}-S_{21'}]\right\}. \tag{10}$$

$$\frac{\mathrm{d}\sigma^L_{1\times 4}}{\mathrm{d}^2\mathbf{k}\,\mathrm{d}y} = -4\frac{e^2Q^2g^2N_c}{(2\pi)^6}\int \mathrm{d}z_1\mathrm{d}z_2\delta(1-z_1-z_2-z)\frac{z_1}{z}z_2(1-z_2)[z_1(1-z_1)+z_2(1-z_2)]$$
$$\times\frac{1}{(2\pi)^2}\int \mathrm{d}^8\mathbf{x}\frac{\mathbf{x}_{1'1}\cdot\mathbf{x}_{3'2}}{\mathbf{x}^2_{1'1}\mathbf{x}^2_{3'2}}[K_0(|\mathbf{x}_{12}|Q_2)K_0(QX')]_{2'=2}e^{i\mathbf{k}\cdot[\mathbf{x}_{3'1}+\frac{z_1}{z}\mathbf{x}_{1'1}]}$$
$$\times\left\{\frac{N_c}{2}[S_{3'1'}S_{13'}-S_{3'1'}S_{23'}]+C_F[1-S_{12}]-\frac{1}{2N_c}[S_{11'}-S_{21'}]\right\}. \tag{11}$$

$$\frac{\mathrm{d}\sigma^L_{2\times 3}}{\mathrm{d}^2\mathbf{k}\mathrm{d}y} = -4\frac{e^2Q^2g^2N_c}{(2\pi)^6}\int \mathrm{d}z_1\mathrm{d}z_2\delta(1-z_1-z_2-z)\frac{z_2}{z}z_1(1-z_1)[z_1(1-z_1)+z_2(1-z_2)]$$
$$\times\frac{1}{(2\pi)^2}\int \mathrm{d}^8\mathbf{x}\frac{\mathbf{x}_{2'2}\cdot\mathbf{x}_{3'1}}{\mathbf{x}^2_{2'2}\mathbf{x}^2_{3'1}}[K_0(|\mathbf{x}_{12}|Q_1)K_0(QX')]_{1'=1}e^{i\mathbf{k}\cdot[\mathbf{x}_{3'2}+\frac{z_2}{z}\mathbf{x}_{2'2}]}$$
$$\times\left\{\frac{N_c}{2}[S_{3'2'}S_{23'}-S_{3'2'}S_{13'}]+C_F[1-S_{12}]-\frac{1}{2N_c}[S_{22'}-S_{2'1}]\right\}. \tag{12}$$

$$\frac{\mathrm{d}\sigma^L_{2\times 4}}{\mathrm{d}^2\mathbf{k}\,\mathrm{d}y} = 4\frac{e^2Q^2g^2N_c}{(2\pi)^6}\int \mathrm{d}z_1\,\mathrm{d}z_2\delta(1-z_1-z_2-z)\frac{z_1}{z}z_1(1-z_1)[(1-z_1)^2+z_2^2]$$
$$\times\frac{1}{(2\pi)^2}\int \mathrm{d}^8\mathbf{x}\frac{\mathbf{x}_{2'2}\cdot\mathbf{x}_{3'2'}}{\mathbf{x}^2_{2'2}\mathbf{x}^2_{3'2'}}[K_0(|\mathbf{x}_{12}|Q_1)K_0(QX')]_{1'=1}e^{i\mathbf{k}\cdot[\mathbf{x}_{3'2}+\frac{z_2}{z}\mathbf{x}_{2'2}]}$$
$$\times\left\{\frac{N_c}{2}[S_{2'3'}S_{23'}-S_{3'2'}S_{13'}]+C_F[1-S_{12}]-\frac{1}{2N_c}[S_{22'}-S_{2'1}]\right\}, \tag{13}$$

where $\mathbf{k}$ and y are the transverse momentum and rapidity of the produced gluon with $\mathrm{d}y = \frac{\mathrm{d}z}{z}$ and z is the fraction of the photon momentum carried by the radiated gluon $z \equiv \frac{k^+}{l^+}$. We also use the following notation for coordinate dependence of some of the Bessel functions

$$X = \sqrt{z_1z_2\mathbf{x}^2_{12}+z_1z\mathbf{x}^2_{13}+z_2z\mathbf{x}^2_{23}} \tag{14}$$

Furthermore, X' that appears in some of the terms is the same as X above but with $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \to \mathbf{x}_{1'}, \mathbf{x}_{2'}, \mathbf{x}_{3'}$. Expressions in Eqs. (4)–(13) are our main result. Nevertheless they contain divergences which must be regulated before they can give meaningful results. Finally, it is worth noticing presence of a quadrupole ($S_{122'1'}$) in (6) which is however suppressed at large N_c.

III. DIVERGENCES

Divergences can arise due to quantum corrections to tree level cross sections. Possible divergences include ultraviolet (UV) divergences which happen when loop 4-momenta go to infinity $\mathbf{k}^\mu \to \infty$, soft divergences which arise when loop 4-momenta go to zero $\mathbf{k}^\mu \to 0$ (for massless theories), collinear divergences which arise when the angle between a radiated parton and its parent (after radiation) go to zero ($\theta \to 0$), and rapidity divergences when the radiated parton carries very small fraction z of its parents momentum ($z \to 0$).

It is straightforward to check that there are no UV or soft divergences in Eqs. (4)–(13). UV divergences may occur when transverse separations (in coordinate space) go to zero, however in this limit the radiation kernels are UV finite as can be seen for example in

$$\frac{\mathbf{x}_{2'2}\cdot\mathbf{x}_{3'1}}{\mathbf{x}^2_{2'2}\mathbf{x}^2_{3'1}} = -\frac{1}{2}\left[\frac{(\mathbf{x}_{2'2}-\mathbf{x}_{3'1})^2}{\mathbf{x}^2_{2'2}\mathbf{x}^2_{3'1}}-\frac{1}{\mathbf{x}^2_{2'2}}-\frac{1}{\mathbf{x}^2_{3'1}}\right] \tag{15}$$

as $\mathbf{x}_{2'} \leftrightarrow \mathbf{x}_2$, $\mathbf{x}_{3'} \leftrightarrow \mathbf{x}_1$. Possible soft divergences (when transverse coordinates go to infinity in coordinate space) vanish due to behavior of Bessel functions at large values of their arguments or by the rapidly oscillating phase factors. Furthermore as the gluon is produced at finite momentum there cannot be rapidity divergences. However, there is a collinear divergence present in some of the terms, namely when the multiple scatterings on the target happen before radiation of the gluon [Eqs. (4) and (5)]. This collinear divergence appears as a pole in

$$\frac{1}{[z\mathbf{p}-z_1\mathbf{k}]^2} \tag{16}$$

when integrating over the quark transverse momentum $\mathbf{p}$ (and similarly for antiquark). In principle one absorbs this collinear divergence into bare fragmentation function which makes it scale dependent. This was discussed in great detail in [104] for contributions of quark-quark splitting P_{qq} to the evolution of quark-hadron fragmentation function. It is worth remembering DGLAP evolution equation for scale dependence of quark-hadron fragmentation function is a matrix equation which mixes quark-hadron and gluon-hadron fragmentation functions. This is symbolically written as

$$\frac{\partial}{\partial \log Q^2}\begin{pmatrix} D_{h/q} \\ D_{h/g}\end{pmatrix} \sim \begin{pmatrix} P_{qq} & P_{gq} \\ P_{qg} & P_{gg}\end{pmatrix}\begin{pmatrix} D_{h/q} \\ D_{h/g}\end{pmatrix} \quad (17)$$

where in this work we are including the contribution of P_{gq} channel to evolution of the quark-hadron fragmentation function. To do this we define a bare gluon-hadron fragmentation function $D^{(0)}_{h/g}(z_h)$ and convolute the partonic production cross section with this fragmentation function to describe hadronization of the parton. To be specific we focus on Eq. (4); we first shift $z_1 \longrightarrow z_1 - z$ and then $\mathbf{p} \longrightarrow \mathbf{p} - \mathbf{k}$ to get

$$k^+ \frac{\mathrm{d}\sigma_{1\times 1}}{\mathrm{d}^2\mathbf{k}\,\mathrm{d}k^+} = 4\frac{e^2Q^2g^2N_c}{(2\pi)^6}C_F\int \mathrm{d}^2\mathbf{x}_1\,\mathrm{d}^2\mathbf{x}_{1'}\,\mathrm{d}^2\mathbf{x}_2[S_{11'} - S_{12} - S_{1'2} + 1]\int_z^1 \mathrm{d}z_1 K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2}|Q_1) \times \frac{z_1^2}{z^2}z_1^2(1-z_1)^2\left[1 + \left(1 - \frac{z}{z_1}\right)^2\right]\int \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}\frac{e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}}}{[\mathbf{p} - \frac{z_1}{z}\mathbf{k}]^2} \quad (18)$$

To go from the partonic to hadronic production cross section we convolute the parton level cross section with the bare gluon-hadron fragmentation function $D^{(0)}_{h/g}(z_h)$

$$k_h^+ \frac{\mathrm{d}\sigma_{1\times 1}}{\mathrm{d}^2\mathbf{k}_h\,\mathrm{d}k_h^+} \equiv \int_0^1 \frac{\mathrm{d}z_h}{z_h^2}D^{(0)}_{h/g}(z_h)k^+\frac{\mathrm{d}\sigma_{1\times 1}}{\mathrm{d}^2\mathbf{k}\,\mathrm{d}k^+} \quad (19)$$

where $\mathbf{k}_h, k_h^+$ are now the produced hadron momenta and $z_h \equiv \frac{k_h^\mu}{k^\mu}$. We further define a new variable $\xi \equiv \frac{z}{z_1}$ in terms of which the hadron production cross section can be written as

$$k_h^+ \frac{\mathrm{d}\sigma_{1\times 1}}{\mathrm{d}^2\mathbf{k}_h\,\mathrm{d}k_h^+} = 4\frac{e^2Q^2g^2N_c}{(2\pi)^6}C_F\int \mathrm{d}^2\mathbf{x}_1\,\mathrm{d}^2\mathbf{x}_{1'}\,\mathrm{d}^2\mathbf{x}_2\,[S_{11'} - S_{12} - S_{1'2} + 1]\int_0^1\frac{\mathrm{d}z_h}{z_h^2}D^{(0)}_{h/g}(z_h) \times \int_z^1\frac{\mathrm{d}\xi}{\xi^2}\left(\frac{z}{\xi}\right)^3\left(1 - \frac{z}{\xi}\right)^2\frac{[1 + (1-\xi)^2]}{\xi}K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2}|Q_1)\int\frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}\frac{e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}}}{[\mathbf{p} - \frac{1}{\xi}\mathbf{k}]^2} \quad (20)$$

note that we now have $z \equiv \frac{k^+}{l^+} = \frac{k_h^+}{z_h l^+}$ and that $Q_1^2 = Q^2 z_1(1-z_1) = Q^2\frac{z}{\xi}(1 - \frac{z}{\xi})$. The integration over $\mathbf{p}$ is divergent and must be regulated. Here we use dimensional regularization which gives

$$\int\frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}\frac{e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}}}{[\mathbf{p} - \frac{1}{\xi}\mathbf{k}]^2} = \frac{1}{2\pi}\left[\frac{1}{\varepsilon} - \log(\pi e^{\gamma_E}\mu|\mathbf{x}_{1'1}|)\right]e^{i\frac{\mathbf{k}_h}{\xi z_h}\cdot\mathbf{x}_{1'1}} \quad (21)$$

Further defining $z_h' = \xi z_h$ and then relabeling z_h' as z_h and multiplying by 2 to take into account contribution of antiquark radiation given by Eq. (5) leads to

$$k_h^+ \frac{\mathrm{d}\sigma_{(1\times 1 + 2\times 2)}}{\mathrm{d}^2\mathbf{k}_h\,\mathrm{d}k_h^+} = 8\frac{e^2Q^2N_c}{(2\pi)^5}\int \mathrm{d}^2\mathbf{x}_1\,\mathrm{d}^2\mathbf{x}_{1'}\,\mathrm{d}^2\mathbf{x}_2[S_{11'} - S_{12} - S_{1'2} + 1]\int_0^1\frac{\mathrm{d}z_h}{z_h^2}\left(\frac{k_h^+}{l^+}\frac{1}{z_h}\right)^3\left(1 - \frac{k_h^+}{l^+}\frac{1}{z_h}\right)^2 \times e^{i\frac{\mathbf{k}_h}{z_h}\cdot\mathbf{x}_{1'1}}K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2}|Q_1)\int_{z_h}^1\frac{\mathrm{d}\xi}{\xi}\frac{\alpha_s}{\pi}P_{gq}(\xi)\left[\frac{1}{\varepsilon} - \log(\pi e^{\gamma_E}\mu|\mathbf{x}_{1'1}|)\right]D^{(0)}_{h/g}\left(\frac{z_h}{\xi}\right) \quad (22)$$

where we now have $Q_1^2 = Q^2\frac{k_h^+}{z_h l^+}(1 - \frac{k_h^+}{z_h l^+})$ and the quark-gluon splitting function P_{gq} is defined as

$$P_{gq}(\xi) \equiv C_F\frac{1 + (1-\xi)^2}{\xi} \quad (23)$$

This is our result for Eqs. (4) and (5) which are the only terms containing divergences. We note that unlike the quark-quark splitting which requires a $+$ prescription to regulate the large ξ limit there is no need for such a prescription here in this channel. Such a prescription would have come from the would be virtual corrections which do not exist to this order in this channel.

At this point one can add the contribution of the LO term as well as the case of quark-quark splitting channel already considered in [131], the full result for terms involving collinear divergences is

$$k_h^+ \frac{\mathrm{d}\sigma}{\mathrm{d}^2\mathbf{k}_h\,\mathrm{d}k_h^+} = 8\,\frac{e^2Q^2N_c}{(2\pi)^5}\int \mathrm{d}^2\mathbf{x}_1\,\mathrm{d}^2\mathbf{x}_{1'}\,\mathrm{d}^2\mathbf{x}_2[S_{11'} - S_{12} - S_{1'2} + 1]\int_0^1 \frac{\mathrm{d}z_h}{z_h^2}\left(\frac{k_h^+}{l^+}\frac{1}{z_h}\right)^3\left(1 - \frac{k_h^+}{l^+}\frac{1}{z_h}\right)^2$$
$$\times\, e^{i\frac{\mathbf{k}_h}{z_h}\cdot\mathbf{x}_{1'1}} K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2}|Q_1)\int_{z_h}^1 \frac{\mathrm{d}\xi}{\xi}\left\{\left[\delta(1-\xi) + \frac{\alpha_s}{\pi}P_{qq}(\xi)\left[\frac{1}{\varepsilon} - \log(\pi e^{\gamma_E}\mu|\mathbf{x}_{1'1}|)\right]D_{h/q}^{(0)}\left(\frac{z_h}{\xi}\right)\right]\right.$$
$$\left. + \frac{\alpha_s}{\pi}P_{gq}(\xi)\left[\frac{1}{\varepsilon} - \log(\pi e^{\gamma_E}\mu|\mathbf{x}_{1'1}|)\right]D_{h/g}^{(0)}\left(\frac{z_h}{\xi}\right)\right\}. \tag{24}$$

Depending on the renormalization scheme (MS, $\overline{\mathrm{MS}}, \cdots$) it is standard to subtract the $1/\varepsilon$ divergence in addition to some/no finite terms ($\gamma_E, \cdots$) or to cancel them using counter terms and to define the renormalized parton-hadron fragmentation function $D(z_h, \mu^2)$ in terms of which the single inclusive hadron production cross section can be written as

$$k_h^+ \frac{\mathrm{d}\sigma^L_{SIDIS}}{\mathrm{d}^2\mathbf{k}_h\mathrm{d}k_h^+} = 8\,\frac{e^2Q^2N_c}{(2\pi)^5}\int \mathrm{d}^2\mathbf{x}_1\,\mathrm{d}^2\mathbf{x}_{1'}\,\mathrm{d}^2\mathbf{x}_2[S_{11'} - S_{12} - S_{1'2} + 1]\int_0^1 \frac{\mathrm{d}z_h}{z_h^2}\left(\frac{k_h^+}{l^+}\frac{1}{z_h}\right)^3\left(1 - \frac{k_h^+}{l^+}\frac{1}{z_h}\right)^2$$
$$\times\, e^{i\frac{\mathbf{k}_h}{z_h}\cdot\mathbf{x}_{1'1}} K_0(|\mathbf{x}_{12}|Q_1)K_0(|\mathbf{x}_{1'2}|Q_1)D_{h/q}(z_h, \mu^2) + \cdots. \tag{25}$$

where $D_{h/q}(z_h, \mu^2)$ now includes contribution of both quark-quark and quark-gluon splitting channels and $\cdots$ represents the finite terms given in Eqs. (6)–(13) for gluons (and analogous but different terms for quarks [131]).

It will be interesting to investigate the properties of these results in various limits; for example when the produced hadron transverse momentum $|\mathbf{k}|$ is much smaller than the photon virtuality Q one expects to encounter the so-called Sudakov logs ($\log\, Q^2/\mathbf{k}^2$) which could be large and hence would require resummation. This would further complicate the search for saturation effects as, depending on kinematics, they could be masked by Sudakov effects. A possible way around this would be to measure single inclusive hadrons with $\mathbf{k}^2 \sim Q^2$ to minimize the effect of Sudakov logs.

It is worth mentioning that if one further integrates over the produced gluon transverse momentum $\mathbf{k}$ one can obtain the rapidity dependence of parton multiplicity distribution $\frac{dN_g}{dy}$ [132]. This would then require a mechanism besides fragmentation functions to describe hadronization of the produced gluons. This may require some care specially for the collinear divergent terms in Eqs. (4) and (5). This is however beyond the score of this work and will be investigated later.

In summary we have calculated contribution of gluons to single inclusive hadron production in Deep Inelastic Scattering at small x and at finite N_c for longitudinal photon exchange. These contributions are part of the next-to-leading order corrections to SIDIS. Some of these contributions exhibit collinear divergences which are then absorbed into Leading Log evolution of fragmentations functions. The rest are finite and constitute part of NLO contributions to SIDIS. The case of transverse photon exchange is conceptually similar but algebraically more tedious and will be reported elsewhere.

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APPENDIX: STRUCTURE OF FINITE N_c CORRELATIONS

Here we have kept the N_c suppressed parts of NLO corrections in this work while in [104,131,133] we made the large N_c approximation. Rather than repeating the full results derived in those papers keeping N_c finite we show the finite N_c generalization of those expressions for the case of *dihadron* production. To get the finite N_c results for the case of quark hadronizing as done in [131] one would integrate over the final state phase space for antiquark and gluon which would then set some of these coordinates equal. Labeling of terms corresponds to the diagrams involved as before (virtual diagrams are shown in [104,131,133]). We also factor out an overall factor of N_c since the LO result also contains an explicit N_c. These expressions then can be used to replace the large N_c expressions in [104,131,133] easily.

$$\sigma_{1\times 1} = \sigma_{2\times 2} = N_c C_F\{S_{122'1'} - S_{12} - S_{2'1'} + 1\} \tag{A1}$$

$$\sigma_{1\times 2} = N_c\left\{\frac{N_c}{2}S_{12}S_{2'1'} + C_F[1 - S_{12} - S_{2'1'}] - \frac{1}{2N_c}S_{122'1'}\right\} \tag{A2}$$

$$\sigma_{3\times 3} = \sigma_{4\times 4} = \sigma_{3\times 4} = N_c\left\{\frac{N_c}{2}[S_{133'1'}S_{322'3'} - S_{13}S_{32} - S_{3'1'}S_{2'3'}] + C_F - \frac{1}{2N_c}[S_{122'1'} - S_{12} - S_{2'1'}]\right\} \tag{A3}$$

$$\sigma_{1\times 3} = \sigma_{1\times 4} = N_c\left\{\frac{N_c}{2}[S_{122'3'}S_{3'1'} - S_{3'1'}S_{2'3'}] + C_F[1 - S_{12}] - \frac{1}{2N_c}[S_{122'1'} - S_{2'1'}]\right\} \tag{A4}$$

$$\sigma_{2\times 3} = \sigma_{2\times 4} = N_c\left\{\frac{N_c}{2}[S_{123'1'}S_{2'3'} - S_{3'1'}S_{2'3'}] + C_F[1 - S_{12}] - \frac{1}{2N_c}[S_{122'1'} - S_{2'1'}]\right\} \tag{A5}$$

$$\sigma_{5\times LO} = \sigma_{7\times LO} = N_c\left\{\frac{N_c}{2}[S_{322'1'}S_{13} - S_{32}S_{13}] + C_F[1 - S_{2'1'}] - \frac{1}{2N_c}[S_{122'1'} - S_{12}]\right\} \tag{A6}$$

$$\sigma_{6\times LO} = \sigma_{8\times LO} = N_c\left\{\frac{N_c}{2}[S_{2'1'13}S_{32} - S_{13}S_{32}] + C_F[1 - S_{2'1'}] - \frac{1}{2N_c}[S_{122'1'} - S_{12}]\right\} \tag{A7}$$

$$\sigma_{9\times LO} = \sigma_{10\times LO} = \sigma_{11\times LO} = \sigma_{12\times LO} = \sigma_{14\times LO} = N_c C_F\{S_{122'1'} - S_{12} - S_{2'1'} + 1\} \tag{A8}$$

$$\sigma_{13\times LO} = N_c\left\{\frac{N_c}{2}[S_{2'1'}S_{12}] + C_F[1 - S_{2'1'} - S_{12}] - \frac{1}{2N_c}[S_{122'1'}]\right\} \tag{A9}$$

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