Restricted quantum focusing

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Quantum focusing is a powerful conjecture that plays a key role in the current proofs of many well-known quantum gravity theorems, including various consistency conditions, and causality constraints in AdS/CFT. We conjecture a (weaker) *restricted* quantum focusing, which we argue is sufficient to derive all known essential implications of quantum focusing. Subject to a technical assumption, we prove this conjecture in braneworld semiclassical gravity theories that are holographically dual to Einstein gravity in a higher-dimensional anti–de Sitter spacetime.

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I. INTRODUCTION

Spacetime is emergent in quantum gravity: at length scales much larger than the Planck length, an approximate semiclassical description emerges where local quantum fields propagate on a smooth spacetime manifold.

Despite its approximate nature, semiclassical gravity quantifies and explains deep quantum gravity concepts in simple geometric terms. The generalized entropy is central to this story. Let B be a partial Cauchy slice, such that ∂B is a smooth codimension-two spacelike submanifold. The generalized entropy of B is defined as [1–3]

$$S_{\mathrm{gen}}(B) = \frac{A(\partial B)}{4G_d} + S(B) + \cdots, \tag{1}$$

where $S(B)$ denotes the von Neumann entropy of the bulk fields in B and the ellipsis denotes subleading contributions to the generalized entropy from higher-curvature corrections to Einstein gravity [4].

The importance of the generalized entropy becomes particularly evident in the context of AdS/CFT [5]. Any conformal field theory (CFT) subsystem is dual to a quantum extremal region B, i.e., a stationary point of the generalized entropy functional [6,7], with $S_{\mathrm{gen}}(B)$ equal to the boundary subsystem's von Neumann entropy.[1] This has, for example, led to a derivation of the Page curve [9–11], extending even beyond AdS/CFT [12,13]. Furthermore, quantum extremal regions dictate salient features of the holographic bulk-to-boundary map, resulting, for instance, in concrete proposals for its computational complexity [14]. This has important consequences for the reconstruction of the black hole interior [14,15] and has further sharpened some proposed resolutions to the firewall paradox in evaporating black holes [16–18]. In addition, the generalized entropy outside black hole apparent horizons[2] has been identified with a coarse-grained entropy, giving the generalized second law of such horizons a statistical explanation [19–21].

The quantum focusing conjecture (QFC) [22], the quantum analogue of the classical focusing theorem, is a powerful constraint in semiclassical gravity whose implications are at the heart of the above discoveries' consistency. For example, the QFC is a crucial assumption in various existence proofs of quantum extremal regions [14,23,24] and their compatibility with causality on the boundary CFT [23,25]. The QFC also implies the quantum Bousso bound, quantum singularity theorems [26,27], the generalized second law of causal horizons and holographic screens [28], and the quantum null energy condition [22,29–31].

Despite its prominent role in semiclassical gravity and holography, quantum focusing remains without a general proof. The goal of this paper is to (partly) fill this gap. In Sec. II we conjecture a condition weaker than the QFC, which is sufficient to replace it in the aforementioned applications. Section III includes a proof of this and another relevant constraint in braneworld semiclassical gravity theories that are holographically dual to Einstein gravity in an asymptotically (locally) Anti–de Sitter (AdS) spacetime (henceforth referred to as braneworld gravity). We conclude in Sec. IV with a discussion of some related ideas and future directions.

[1]This is a special case of a more complicated story [8]. However, these complications can be ignored in a very large class of states.

[2]More accurately, quantum minimar surfaces [19–21].

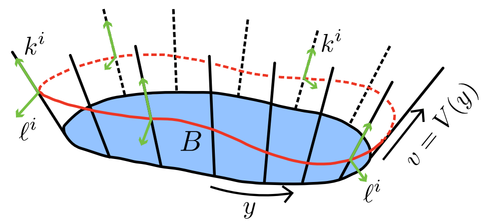

FIG. 1. Given a partial Cauchy slice B, we define the null hypersurface $N^+(B) = (\dot{J}^+(B) - B) \cup \partial B$ whose generators are depicted with straight lines with tangent vectors k^i. A future Cauchy slice Σ_V intersects $N^+(B)$ at $v = V(y)$. On any surface $v = V(y)$, we can define a null vector field ℓ^i orthogonal to it, which is outward and past-directed. The quantum expansions $\Theta_{(k)}$ and $\Theta_{(\ell)}$ of B_V are given by the rates of change of $S_{\rm gen}(B_V)$, per unit transverse area, per unit affine length, as the region is deformed locally at ∂B_V along the k^i and ℓ^i directions, respectively.

II. RESTRICTED QFC

We begin by defining some relevant objects. Let $J^+(B)$ be the causal future of B and $N^+(B) = (\dot{J}^+(B) - B) \cup \partial B$ a null hypersurface with affine generators k^i $(= \partial_v)$. Now, let Σ_V be a Cauchy slice nowhere to the past of ∂B that intersects $N^+(B)$ at $v = V(y) \geq 0$ ($v = 0$ on ∂B, and y denotes transverse coordinates on ∂B which label the generators) and let $B_V = \Sigma_V \cap J^+(B)$ (see Fig. 1).

Let $V_\lambda(y)$ be a one-parameter family of non-negative functions that satisfy $\partial_\lambda V_\lambda(y) \geq 0$. The QFC states that [28]

$$\partial_\lambda \Theta_{(k)}(V_\lambda; y) \leq 0 \quad \text{for all } y, \tag{2}$$

where

$$\Theta_{(k)}(V; y) = \frac{4G_d}{\sqrt{h_V}} \frac{\delta S_{\rm gen}(B_V)}{\delta V(y)} \tag{3}$$

is called the quantum expansion [28] of B_V at $y \in \partial B_V$ and h_V is the determinant of the induced metric on ∂B_V.

It is easy to see that the QFC implies

$$\Theta_{(k)}(0; y) \leq 0, \ \text{for } y \in \Gamma \subseteq \partial B \overset{V|_{\partial B - \Gamma} = 0}{\Longrightarrow} \Theta_{(k)}(V; y) \leq 0, \tag{4}$$

where $V|_{\partial B - \Gamma} = 0$ means that V is zero on all generators emanating from $\partial B - \Gamma$.

Interestingly, Eq. (4) is all that is required of the QFC in the applications mentioned in the Introduction. A lookalike condition, unrelated to the QFC, that is also crucial to the aforementioned applications is[3]

$$\Theta_{(\ell)}(0; y) \leq 0 \overset{V(y) = 0}{\Longrightarrow} \Theta_{(\ell)}(V; y) \leq 0, \tag{5}$$

where for any region B_V we define ℓ^i as the past-outward-directed vector field orthogonal to ∂B_V. Then, $\Theta_{(\ell)}(V; y)$ is defined in obvious analogy with Eq. (3) (see Fig. 1).

Throughout our discussion so far, we can interchange $J^+(B)$ with $D^+(B)$ (the future domain of dependence of B). That is, we can consider inward deformations of B along future-directed null geodesics orthogonal to ∂B. Then, k^i would be future-inward directed and ℓ^i would be past-inward directed. Together, conditions (4) and (5), along with their inward and time-reversed versions, imply all of the applications mentioned in (the fourth paragraph of) the Introduction.[4] It is therefore highly desirable to prove them.

Here, we conjecture a *restricted* QFC, which states that

$$\Theta_{(k)}(V_\lambda; y) = 0 \Rightarrow \partial_\lambda \Theta_{(k)}(V_\lambda; y) \leq 0. \tag{6}$$

Even though the restricted QFC is weaker than the QFC, it is sufficient to derive (4). To see this, pick any V_λ such that $V_0(y) = 0$ and $V_1(y) = V(y)$, which further satisfies the property that for each y, $\Theta_{(k)}(V_\lambda; y)$ is a differentiable function of λ. We expect that all physical states allow such a choice.[5] Then, a violation of (4) at some transverse point y implies that there exists a λ such that $\Theta_{(k)}(V_\lambda, y) = 0$, but $\partial_\lambda \Theta_{(k)}(V_\lambda, y) > 0$. Therefore, (6) implies (4).

Similarly, the following inequality implies (5):

$$\Theta_{(\ell)}(V_\lambda; y) = 0 \overset{\partial_\lambda V_\lambda(y) = 0}{\Longrightarrow} \partial_\lambda \Theta_{(\ell)}(V_\lambda; y) \leq 0. \tag{7}$$

The rest of the paper is mainly devoted to proving conditions (6) and (7) in braneworld gravity.

III. PROOF OF RESTRICTED QUANTUM FOCUSING IN BRANEWORLD GRAVITY

We introduce the braneworld setup briefly in Sec. III A, reviewing the salient points of the construction for our purposes, before delving into the proofs of conditions (6)

[3]This condition involves a variation of the von Neumann entropy of B under null deformations of ∂B at *different* points. This can be rewritten as an expression involving the von Neumann entropy of three subsystems, which by the strong subadditivity of the von Neumann entropy acquires a sign [28,32]. But Eq. (5) also involves a contact term contribution from the Dong entropy piece of the generalized entropy [4].

[4]It is a straightforward exercise to show this in most cases which we leave to the interested and/or skeptical reader. Technically, an additional, often overlooked (and independent of the QFC) assumption is involved: the loss of generators along N^+, which happens generically due to caustics and self-intersections, cannot increase the value of $S_{\rm gen}$. Separately, to arrive at the quantum null energy condition, one needs to approach a classically stationary point y on ∂B through a family of surfaces that satisfy $\Theta_{(k)} = 0$ at y in the $G \to 0$ limit.

[5]The reader might object that, for example, in shock-wave geometries with a delta-function energy source this is not the case. However, such delta-function divergences only make sense as a distribution, and a physically reasonable state needs to involve a proper smearing of such delta functions which would then allow a differentiable choice.

and (7) in Sec. III B. For much more elaborate discussions of braneworld holography setups, see Refs. [33–44].

A. Brane setup

In the standard AdS/CFT setup, to compute the CFT_d partition function holographically one considers a cutoff surface at $z = \epsilon$, where z is the Fefferman-Graham (FG) radial coordinate of AdS_{d+1}, and computes the bulk action including the Gibbons-Hawking-York terms. Then, as ϵ is sent to zero, appropriate counterterms are added to cancel divergences.

One way to think about braneworld holography is to instead consider a (physical) cutoff surface at a finite distance with a metric that is free to fluctuate. The previously divergent contributions (no longer removed by counterterms) may now be interpreted as induced gravity on the brane (e.g., as in the Randall-Sundrum model [33]) which is coupled to a strongly interacting holographic CFT. This is a semiclassical gravity theory on the brane that is holographically dual to a higher-dimensional classical (Einstein gravity) theory.

Explicitly, consider the bulk action

$$I_{\text{total}} = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x\sqrt{-\bar{g}}\left(\bar{R} + \frac{d(d-1)}{L^2}\right) + \frac{1}{8\pi G_{d+1}} \int_{\text{brane}} d^d x\sqrt{-g}(K - T), \tag{8}$$

where G_{d+1} and L denote the bulk Newton constant and the AdS radius, respectively, $\bar{R}$ denotes the bulk Ricci scalar, and $K = g^{ij}K_{ij}$ is the trace of the brane extrinsic curvature tensor K_{ij}. There also exists a brane-tension term (proportional to T) which can fine-tune the brane location. The intrinsic metric on the brane is free to fluctuate, resulting in the equations of motion

$$K_{ij} = (K - T)g_{ij}. \tag{9}$$

A practical way to find explicit brane solutions like this is to start with an asymptotically locally AdS_{d+1} solution in FG coordinates,

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \tilde{g}_{ij}(\tilde{x}, z)d\tilde{x}^i d\tilde{x}^j), \tag{10}$$

with the condition (always achievable by an appropriate rescaling of z) that the smallest length scale on $\tilde{g}_{ij}(\tilde{x}, z = 0)$, denoted by L_0, satisfies $L_0 \gg L$. Then, the FG expansion remains valid at $z \sim L$ and it is easy to check that, with $T = (d-1)/L$, the brane will be located at

$$z = L + O(L/L_0), \tag{11}$$

where the subleading corrections are $\tilde{x}$ dependent in general. Importantly, by using the FG expansion one can check that the induced gravity on the brane is Einstein gravity plus higher-curvature corrections that are suppressed by powers of L [38]:

$$I_{\text{brane}} = \frac{1}{16\pi G_d} \int d^d x\sqrt{-g}\big(R + O(L^2R^2)\big) + I_{\text{CFT}}, \tag{12}$$

where x denotes brane coordinates, $O(L^2R^2)$ schematically denotes higher-derivative corrections, and I_{CFT} denotes the (nonlocal) action of the holographic CFT. Here,

$$G_d \sim \frac{G_{d+1}}{L}. \tag{13}$$

Combined with $L^{d-1}/G_{d+1} \sim c$, where c denotes the CFT's effective number of degrees of freedom, this gives

$$cG_d \sim L^{d-2}. \tag{14}$$

Therefore, L is the scale of the breakdown of the semiclassical expansion on the brane.

For a general discussion, it is more convenient to consider Riemann normal coordinates in a neighborhood of the brane:

$$ds^2 = dn^2 + g_{ij}(n, x)dx^i dx^j, \tag{15}$$

where the brane is located at $n = 0$. In these coordinates, the brane equation of motion (9) at $n = 0$ gives

$$\partial_n g_{ij} = -\frac{2}{L} g_{ij}. \tag{16}$$

Now, consider a partial Cauchy slice B of the brane spacetime. We have

$$S_{\text{gen}}(B) = \frac{A(\bar{X}(B))}{4G_{d+1}}, \tag{17}$$

where $A(\bar{X}(B))$ denotes the area of the minimal area bulk extremal surface $\bar{X}(B)$ homologous to B [34,35,38–41,45]. We may view Eq. (17) as a definition of $S_{\text{gen}}(B)$ for our purposes, though it must be possible to derive it from an independent definition of $S_{\text{gen}}(B)$. Note that the homology condition here does not necessarily mean $\partial\bar{X} = \partial B$. In general, $\partial B \subset \partial\bar{X}$, where some connected components of $\partial\bar{X}$ may end with Neumann boundary conditions on a brane (see Fig. 2).

A powerful feature of the brane-world scenario is that merely bulk Einstein gravity induces Einstein gravity plus higher derivative corrections to all orders (in cG_d) on the brane. This is a very convenient setup to study quantities like the generalized entropy and conditions like the restricted QFC which make sense to all orders in the semiclassical expansion (controlled by L). Of course, the bulk theory receives both quantum and stringy corrections (discussed in Sec. IV A), which on the brane are

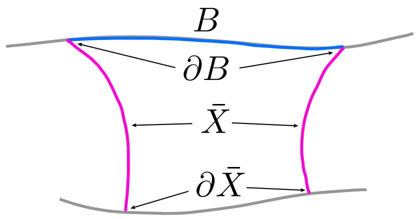

FIG. 2. An arbitrary region B on the (top) brane is shown with a minimal area extremal surface $\bar{X}$ homologous to it. While some connected components of $\bar{X}$ end on ∂B, in general there may be others that end on another brane (satisfying a Neumann boundary condition). In such cases, we can glue the solution to itself across the lower brane, reducing to a scenario with only a Dirichlet boundary condition.

interpreted as $1/c$ and inverse-coupling corrections, respectively.

Before going on, we comment on connections to previous works. In [38], Eq. (17) was expanded in small L where it was shown to reproduce the Bekenstein-Hawking entropy for the region B plus quantum corrections and local extrinsic curvature terms on ∂B [4]. Furthermore, our setup is close in spirit to the work of [46], which in the standard AdS/CFT setup used the Hubeny-Rangamani-Takayanagi formula [47,48] to prove the quantum null energy condition in holographic CFTs (see also later work [49,50]). The brane setup here is of course different in that it is a gravitational theory. But, in addition, there are two major technical differences with [46]. First, contrary to [46], we do not analyze the extremal surface $\bar{X}$ in a "near boundary/brane" expansion. The treatment is fully nonperturbative in that regard, enabling us to draw conclusions that hold to all orders in the brane semiclassical expansion parameter L. Furthermore, in [46] a crucial inequality was derived from "entanglement wedge nesting," proven earlier only in the context of standard AdS/CFT [51]. We therefore use another, more direct technique here.

B. Proof

Let $\bar{X}^\mu(n, y^a)$ specify embedding coordinates for $\bar{X}$, where μ denotes bulk coordinates and y^a is an extension of the coordinates on ∂B to the bulk. We work in a gauge where $\bar{X}^n = n$. Therefore, $\bar{X}^\mu = (n, \bar{X}^i(n, y^a))$ such that $\bar{X}^i(n = 0, y^a)$ is the embedding coordinates of the ∂B on the brane. For simplicity, we can pick coordinates on the brane such that $\bar{X}^i(n = 0, y^a) = y^a \delta_{ia}$ (no summation). Let $\bar{H}_{\alpha\beta}(n, y^a)$ denote the induced metric on $\bar{X}^\mu$.[6] Here α and β are either n or y^a, the coordinates on $\bar{X}$. Then,

$$A[\bar{X}] = \int dn dy^a \sqrt{\bar{H}}, \tag{18}$$

where $\bar{H} = \det(\bar{H}_{\alpha\beta})$. By taking a functional derivative of this area subject to the null deformation of B in the k^i or ℓ^i directions, we can compute the corresponding quantum expansions,

$$\Theta_{(k)}(B; y) = -\frac{k_\mu t^\mu}{\ell_S}\bigg|_{\partial B}, \tag{19}$$

$$\Theta_{(\ell)}(B; y) = -\frac{\ell_\mu t^\mu}{\ell_S}\bigg|_{\partial B}, \tag{20}$$

where B in the argument of Θ means evaluating it at $V = 0$. Further, k^μ and ℓ^μ are the push-forwards of k^i and ℓ^i, t^μ is the unique unit-normalized tangent vector of $\bar{X}$ orthogonal to ∂B, and $\ell_S = G_{d+1}/G_d$, the effective short-distance cutoff of the local gravitational theory on the brane.[7] Note that by definition $\ell_S \sim L$, and ℓ_S is simply introduced for convenience. Therefore, we take appropriate gauge-invariant length scales associated with the background spacetime, state, and region B to be much larger than ℓ_S to respect the semiclassical regime. Note that since $\bar{X}$ is extremal, the only contribution to Eqs. (19) and (20) comes from the subset of $\partial\bar{X}$ with Dirichlet boundary conditions, i.e. ∂B.

In our gauge, we have $\bar{H}_{na}|_{\partial B} = 0$, a condition that we can preserve in a neighborhood of ∂B on $\bar{X}$ by defining the extension of the y^a coordinates into the bulk appropriately. We also have

$$\bar{H}_{nn}(n, y) = 1 + g_{ij}\partial_n \bar{X}^i \partial_n \bar{X}^j. \tag{21}$$

To make sure that $\bar{X}$ is a spacelike surface, we need $\bar{H}_{nn} > 0$. Using Eq. (19), one can write $\partial_n \bar{X}^i$ in terms of the quantum expansions of B,

$$\partial_n \bar{X}^i|_{n=0} = -\frac{\ell_S \Theta_{(\ell)}}{\sqrt{1 - 2\ell_S^2 \Theta_{(k)}\Theta_{(\ell)}}} k^i - \frac{\ell_S \Theta_{(k)}}{\sqrt{1 - 2\ell_S^2 \Theta_{(k)}\Theta_{(\ell)}}} \ell^i, \tag{22}$$

where k^i and ℓ^i are normalized such that $k^i \ell_i = 1$. Then,

$$H_{nn}|_{n=0} > 0 \Rightarrow 2\ell_S^2 \Theta_{(k)}\Theta_{(\ell)} < 1. \tag{23}$$

In fact, we expect (and henceforth assume) from the validity of the semiclassical expansion that

[6] To sum up the notation, μ and ν are bulk indices, i and j are brane indices, α and β denote indices along $\bar{X}$, and a denotes indices on ∂B. So, e.g., $\mu = \{n, i\}$ and $\alpha = \{n, a\}$.

[7] More specifically, this scale is the species scale of the brane semiclassical theory. At this scale, the equations for the metric remain classical, though they are no longer local. This is analogous to physics at the string scale. See Ref. [52].

$$|\Theta_{(k)}\Theta_{(\ell)}| \ll \ell_S^{-2}. \tag{24}$$

This makes sense because $\Theta_{(k)}\Theta_{(\ell)}$ is a coordinate-invariant quantity related to the brane region and state. For example, if we take B to be the ball of radius R in Minkowski space, this condition is equivalent to $R \gg \ell_S$, which is clearly required for the validity of the semiclassical analysis. From now on, we add the condition (24) to the list of other curvature invariants that satisfy the semiclassical condition.

Let $V_{\lambda=0}(y) = 0$. Without loss of generality, we focus on the conditions (6) and (7) when evaluated at $\lambda = 0$. In order to compute $\partial_\lambda \Theta_{(k)}(V_\lambda)|_{\lambda=0}$ and $\partial_\lambda \Theta_{(\ell)}(V_\lambda)|_{\lambda=0}$, we need to calculate the response of the extremal surface $\bar{X}(B)$ to an infinitesimal deformation of B at ∂B in the k^i direction. A deformation of $\bar{X}$ can be specified by a deformation vector field $\alpha \bar{k}^\mu + \beta \bar{\ell}^\mu$ in the normal bundle of $\bar{X}$, where $\bar{k}^\mu$ and $\bar{\ell}^\mu$ are null vector fields orthogonal to $\bar{X}$ (which we normalize with $\bar{k}^\mu \bar{\ell}_\mu = 1$) and α and β are scalar functions on $\bar{X}$. To deform $\bar{X}$, we can then follow (by a fixed affine parameter λ) geodesics fired from $\bar{X}$ along $\alpha \bar{k}^\mu + \beta \bar{\ell}^\mu$. After some computation from Eq. (19), we get

$$\partial_\lambda \Theta_{(k)}(V_\lambda; y) = \frac{1}{\ell_S (\bar{H}_{nn})^{\frac{1}{2}}} \left(-k_i \partial_n (\alpha \bar{k}^i + \beta \bar{\ell}^i)|_{n=0} + \ell_S^3 (\bar{H}_{nn})^{\frac{3}{2}} \Theta_{(k)} \partial_\lambda (\Theta_{(k)}\Theta_{(\ell)}) \right). \tag{25}$$

At ∂B, the deformation of $\bar{X}$ projected onto the brane needs to satisfy the following condition:

$$(\alpha \bar{k}^i + \beta \bar{\ell}^i)|_{\partial B} = (\partial_\lambda V_\lambda) k^i, \tag{26}$$

where $\bar{k}^i$ and $\bar{\ell}^i$ are the projections of $\bar{k}^\mu$ and $\bar{\ell}^\mu$ onto the brane. In general, $\bar{k}^i$ ($\bar{\ell}^i$) is different from k^i (ℓ^i). See Fig. 3.

Using the definitions of $\bar{k}^\mu$ and $\bar{\ell}^\mu$, it is possible to derive

$$\bar{k}^i|_{\partial B} = k^i + \frac{-1 + \ell_S^2 \Theta_{(k)}\Theta_{(\ell)} + \sqrt{1 - 2\ell_S^2 \Theta_{(k)}\Theta_{(\ell)}}}{\ell_S^2 \Theta_{(\ell)}^2} \ell^i, \tag{27}$$

$$\bar{\ell}^i|_{\partial B} = \frac{1 - \ell_S^2 \Theta_{(k)}\Theta_{(\ell)} + \sqrt{1 - 2\ell_S^2 \Theta_{(k)}\Theta_{(\ell)}}}{2} \ell^i - \frac{\ell_S^2 \Theta_{(\ell)}^2}{2} k^i. \tag{28}$$

From this, we can derive

$$\alpha|_{\partial B} = \frac{1 - \ell_S^2 \Theta_{(k)}\Theta_{(\ell)} + \sqrt{1 - 2\ell_S^2 \Theta_{(k)}\Theta_{(\ell)}}}{2\sqrt{1 - 2\ell_S^2 \Theta_{(k)}\Theta_{(\ell)}}} \partial_\lambda V_\lambda, \tag{29}$$

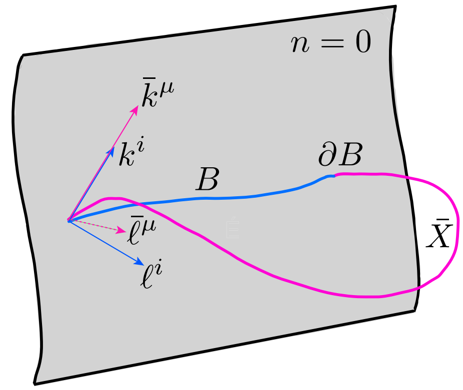

FIG. 3. B is a subregion on the brane (located at $n = 0$). The generalized entropy of B is computed by the Bekenstein-Hawking entropy of the minimal area bulk extremal surface $\bar{X}$ homologous to B. The null vector fields k^i and ℓ^i orthogonal to ∂B and the null vector fields $\bar{k}^\mu$ and $\bar{\ell}^\mu$ orthogonal to $\partial \bar{X}$ are depicted. k^i and $\bar{k}^\mu$ align at a point $y \in \partial B$ where $\Theta_{(k)}(B; y) = 0$.

$$\beta|_{\partial B} = \frac{1 - \ell_S^2 \Theta_{(k)}\Theta_{(\ell)} - \sqrt{1 - 2\ell_S^2 \Theta_{(k)}\Theta_{(\ell)}}}{\ell_S^2 \Theta_{(\ell)}^2 \sqrt{1 - 2\ell_S^2 \Theta_{(k)}\Theta_{(\ell)}}} \partial_\lambda V_\lambda. \tag{30}$$

We can simplify the above expressions using only $\ell_S^2 \Theta_{(k)}\Theta_{(\ell)} \ll 1$:

$$\bar{k}^i|_{\partial B} = k^i - \frac{\ell_S^2 \Theta_{(k)}^2}{2} \ell^i + \cdots, \tag{31}$$

$$\bar{\ell}^i|_{\partial B} = \ell^i - \frac{\ell_S^2 \Theta_{(\ell)}^2}{2} k^i + \cdots, \tag{32}$$

and

$$\alpha|_{\partial B} = \partial_\lambda V_\lambda + \cdots, \tag{33}$$

$$\beta|_{\partial B} = \frac{\ell_S^2 \Theta_{(k)}^2}{2} \partial_\lambda V_\lambda + \cdots. \tag{34}$$

As an important side note, it does not make sense to demand that the absolute values of $\ell_S \Theta_{(k)}$ and $\ell_S \Theta_{(\ell)}$ are small because their values can change under a simultaneous rescaling of k^i and ℓ^i. In other words, these dimensionless quantities are coordinate dependent.[8]

As functions on $\bar{X}$, α and β are constrained by the fact that the deformation of $\bar{X}$ needs to take it to a nearby extremal surface. Deriving this constraint is a straightforward exercise (see, e.g., [53]). The result is

[8]For example, for an evaporating black hole in infalling Eddington-Finkelstein coordinates, it is possible to make them arbitrarily large [10,11].

$$\begin{pmatrix} \hat{D}_+ & -\bar{\varsigma}^2_{(\bar{\ell})} - 8\pi G\bar{T}_{\mu\nu}\bar{\ell}^\mu\bar{\ell}^\nu \\ -\bar{\varsigma}^2_{(\bar{k})} - 8\pi G\bar{T}_{\mu\nu}\bar{k}^\mu\bar{k}^\nu & \hat{D}_- \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0, \tag{35}$$

where $\bar{\varsigma}^2_{(k)}$ and $\bar{\varsigma}^2_{(\ell)}$ denote shear-squared terms on $\bar{X}$, and

$$\hat{D}_\pm = -\overline{\nabla}^2 \mp 2\chi^\alpha\overline{\nabla}_\alpha - \left(\bar{\chi}^\alpha\bar{\chi}_\alpha \pm \overline{\nabla}_\alpha\bar{\chi}^\alpha + \bar{G}_{\mu\nu}\bar{k}^\mu\bar{\ell}^\nu - \frac{\bar{r}}{2}\right), \tag{36}$$

where $\overline{\nabla}_\alpha$ is the covariant derivative on $\bar{X}^\mu$, $\bar{\chi}_\alpha = \bar{\ell}^\mu\overline{\nabla}_\alpha\bar{k}_\mu$, $\bar{G}_{\mu\nu}$ is the bulk Einstein tensor, and $\bar{r}$ is the intrinsic Ricci scalar on $\bar{X}$.

The matrix in Eq. (35) is a particular linear operator acting on pairs of scalar functions on $\bar{X}$. The result is a "cooperative elliptic system," which has in particular been studied in [54] and was first discussed in the context of the standard AdS/CFT correspondence in [53]. We use an important theorem in these works, a special case of which (adapted to our needs) we state here:

Theorem 1. Consider a fully coupled cooperative elliptic system, i.e., a system of linear differential equations

$$\begin{pmatrix} \hat{L}_1 & f \\ g & \hat{L}_2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0, \tag{37}$$

where A and B are functions on an open domain U of $\mathbb{R}^n$, f and g are nonpositive functions, and L_i (for $i = 1$ or 2) are elliptic operators,

$$\hat{L}_i = (H_i)^{\alpha\beta}\partial_\alpha\partial_\beta + (b_i)^\alpha\partial_\alpha + c_i, \tag{38}$$

where $(H_i)^{\alpha\beta}$ are positive-definite matrices for each i. Now, suppose Eq. (37) has a supersolution (A^+, B^+), i.e.,

$$A^+|_U \geq 0, \tag{39}$$

$$B^+|_U \geq 0, \tag{40}$$

$$(\hat{L}_1 A^+ + f B^+)|_U \geq 0, \tag{41}$$

$$(\hat{L}_2 B^+ + g A^+)|_U \geq 0, \tag{42}$$

and either A^+ or B^+ is nonzero somewhere on ∂U, or either (41) or (42) is not saturated somewhere in U. Then, for *any* (sufficiently smooth) solution (A, B) to Eq. (37), either

$$\begin{cases} A|_{\partial U} \geq 0 \\ B|_{\partial U} \geq 0 \end{cases} \Rightarrow \begin{cases} A|_U > 0, \\ B|_U > 0, \end{cases} \tag{43}$$

or

$$\begin{cases} A|_U = 0, \\ B|_U = 0. \end{cases}$$

In [53], Theorem 1 was applied to the extremal surface deviation (35) in the standard AdS/CFT context. We assume that the extension of this theorem from domains of $\mathbb{R}^n$ to a manifold like $\bar{X}$ is trivial. The bulk null energy condition implies[9]

$$\left(-\bar{\varsigma}^2_{(\bar{k})} - 8\pi G\bar{T}_{\mu\nu}\bar{k}^\mu\bar{k}^\nu\right)\Big|_{\bar{X}} \leq 0, \tag{44}$$

$$\left(-\bar{\varsigma}^2_{(\bar{\ell})} - 8\pi G\bar{T}_{\mu\nu}\bar{\ell}^\mu\bar{\ell}^\nu\right)\Big|_{\bar{X}} \leq 0. \tag{45}$$

In the highly nongeneric case where one of the above inequalities in saturated everywhere on $\bar{X}$, the analysis becomes trivial. Therefore, without losing anything, we restrict to the case where they are both nonsaturated somewhere on $\bar{X}$. Then, the only remaining step to make Theorem 1 nontrivially applicable is to demonstrate the existence of a supersolution. In the standard AdS/CFT context, this follows from the (classical) maximin prescription [51].

This brings us to our main technical assumption: in our setup, where the bulk is cut off by a brane, we henceforth *assume* that such a supersolution exists. We leave a proof of this assumption to future work, but we comment here on why we believe this is a mild assumption. It is possible to prove that the matrix operator in Eq. (35) has a real eigenvalue (called the principal eigenvalue) that is equal to or smaller than the real part of all of its other eigenvalues, and whose corresponding eigenvector is a pair of positive functions on $\bar{X}$ [55]. The central assumption here would then follow if this eigenvalue is positive. In the standard AdS/CFT setup, the positivity of this eigenvalue is a simple consequence of the (classical) maximin prescription. Now, from Eq. (11), we expect that for an $\bar{X}$ anchored to the brane, this eigenvalue is only perturbatively (in L/L_0) different from that of standard AdS/CFT, therefore maintaining its positive sign.

Last, if there exist connected components of $\partial\bar{X}$ satisfying Neumann boundary conditions on some brane, we can "double up" the solution by gluing across the brane, which would then reduce the boundary conditions of $\bar{X}$ to purely Dirichlet ones (see Fig. 2).

Assuming the existence of a supersolution, it follows that

$$\begin{cases} \alpha|_{\partial B} \geq 0 \\ \beta|_{\partial B} \geq 0 \end{cases} \Rightarrow \begin{cases} \alpha|_{\bar{X}} \geq 0, \\ \beta|_{\bar{X}} \geq 0. \end{cases} \tag{46}$$

[9]Alternatively, we can simply assume the (classical) restricted focusing in the bulk.

Armed with (46), we can now prove our main results, the conditions (6) and (7). First, since $\partial_\lambda V_\lambda \geq 0$, Eqs. (33) and (34) imply the lhs of (46). Furthermore, $\Theta_{(k)} = 0$ simplifies Eq. (25) in the following way:

$$\Theta_{(k)}(B;y) = 0$$
$$\Rightarrow \partial_\lambda \Theta_{(k)}(V_\lambda;y)|_{\lambda=0} = -\frac{\alpha k_i \partial_n \bar{k}^i + \partial_n \beta}{\ell_S}\Big|_{n=0}. \quad (47)$$

By Eq. (30), $\beta(n=0,y)=0$. Then, Eq. (46) implies that

$$\partial_n \beta(n,y)|_{n=0} \geq 0. \quad (48)$$

To make contact with $k_i \bar{\ell}^i$, we first use $\bar{k}^\mu \bar{k}_\mu = 0$:

$$(\bar{k}^n)^2 + g_{ij}\bar{k}^i \bar{k}^j = 0, \quad (49)$$

where $\bar{k}^n$ is the component of $\bar{k}^\mu$ orthogonal to the brane. Note that $\Theta_{(k)}(B;y)=0$ implies $\bar{k}^i(n=0,y)=k^i$, which in turn implies $\bar{k}^n(n=0,y)=0$. Taking an n derivative results in

$$g_{ij}k^i \partial_n \bar{k}^j(n,y)|_{n=0} = 0, \quad (50)$$

where we made use of the brane equations of motion $\partial_n g_{ij}|_{n=0} \propto g_{ij}$, $\bar{k}^i(n=0,y)=k^i$, and $\bar{k}_n \partial_n \bar{k}^n(n,y)|_{n=0} = 0$. The last condition requires showing that $\partial_n \bar{k}^n$ is convergent enough as $n \to 0$ which we have relegated to the Appendix. Putting everything together,

$$\Theta_{(k)}(B;y) = 0$$
$$\Rightarrow \partial_\lambda \Theta_{(k)}(V_\lambda;y)|_{\lambda=0} = -\frac{\partial_n \beta(n,y)|_{n=0}}{\ell_S} \leq 0. \quad (51)$$

This concludes the proof of (6). Let us emphasize the role played by the condition $\Theta_{(k)} = 0$ in the restricted QFC. Through Eq. (30), it implies $\beta(n=0,y)=0$, which by Eq. (46) leads to $\partial_n \beta(n,y)|_{n=0} \geq 0$, something crucial in deriving the bound in Eq. (51). Without it, we were unable to prove a bound on $\partial_\lambda \Theta_{(k)}(V_\lambda)$. However, in Sec. IV B we discuss a concrete sense in which $\beta(n,y) \geq 0$ "approximately" bounds how positive $\partial_\lambda \Theta_{(k)}(V_\lambda;y)$ can get when $\Theta_{(k)}(V_\lambda;y) \neq 0$. Note also that the derivation of Eq. (51) did not rely on truncating the small-ℓ_S expansion anywhere, and therefore the result is expected to be true to all orders in the ℓ_S expansion.[10]

[10]In fact, hinging on the existence of the extremal surface $\bar{X}$, the proof holds nonperturbatively.

For the condition (7), we have

$$\Theta_{(\ell)}(B;y) = 0$$
$$\overset{\partial_\lambda V_\lambda(y)=0}{\Rightarrow} \partial_\lambda \Theta_{(\ell)}(V_\lambda;y)|_{\lambda=0} = -\frac{\partial_n \alpha(n,y)|_{n=0}}{\ell_S} \leq 0. \quad (52)$$

The inequality in (52) follows from

$$\partial_\lambda V_\lambda(y) = 0 \Rightarrow \alpha(n=0,y) = 0 \Rightarrow \partial_n \alpha(n,y)|_{n=0} \geq 0, \quad (53)$$

where the second implication follows from the condition (46).

In the remainder of this section, we discuss two additional inequalities that follow from the strong subadditivity of the von Neumann entropy (see footnote 3 and [28,32]). These are

$$\frac{\delta\Theta_{(k)}(V;y)}{\delta V(y')}\Big|_{y\neq y'} \leq 0, \quad (54)$$

$$\frac{\delta\Theta_{(\ell)}(V;y)}{\delta V(y')}\Big|_{y\neq y'} \leq 0. \quad (55)$$

Deriving these conditions is a nice consistency check. This can be done by choosing $V_\lambda = \lambda \delta^{d-2}(y-y')$. Then, for $y \neq y'$,

$$\frac{\delta\Theta_{(k)}(V;y)}{\delta V(y')}\Big|_{V=0} = \frac{-1}{\ell_S}\Big(\partial_n \beta(n,y)|_{n=0} + \frac{\ell_S^2 \Theta_{(k)}(B;y)^2}{2}\partial_n \alpha(n,y)|_{n=0}\Big), \quad (56)$$

$$\frac{\delta\Theta_{(\ell)}(V;y)}{\delta V(y')}\Big|_{V=0} = \frac{-1}{\ell_S}\Big(\partial_n \alpha(n,y)|_{n=0} + \frac{\ell_S^2 \Theta_{(\ell)}(B;y)^2}{2}\partial_n \beta(n,y)|_{n=0}\Big), \quad (57)$$

where we have dropped terms suppressed by $\ell_S^2 \Theta_k \Theta_\ell$, which are not relevant since they multiply either $\partial_n \beta$ or $\partial_n \alpha$ in the expressions above. Now, the condition (46) implies that for $y \neq y'$,

$$\partial_n \alpha(n,y)|_{n=0} \geq 0, \quad (58)$$

$$\partial_n \beta(n,y)|_{n=0} \geq 0, \quad (59)$$

resulting in the desired signs (54) and (55).

IV. DISCUSSION

The following ideas will be explored and expanded on in forthcoming work.

A. Bulk quantum and higher-curvature corrections

Even though Eq. (17) already includes all perturbative-in-G_d corrections, it receives additional *bulk* quantum, i.e., $O(G_{d+1})$, and higher-curvature corrections, i.e., $O(\delta)$, where δ is the small scale suppressing the higher-curvature terms in the bulk gravity action. Studying these corrections (which are $1/c$ and inverse coupling corrections from the brane perspective) is very important since it will elucidate whether restricted QFC (or at least its proof here) is an accident of a leading-order analysis of the braneworld or something that holds more generally. We comment on how one could extend the proof of restricted QFC to include these corrections, leaving a thorough analysis to future work. Following the quantum extremal surface prescription [56], we assume that the exact formula, i.e., to all orders in bulk perturbation theory, for the brane generalized entropy is given by

$$S_{\text{gen}}(B) = S_{\text{gen}}(\bar{H}(B)), \tag{60}$$

where $\bar{H}(B)$ is the homology slice of the quantum extremal surface $\bar{X}(B)$ homologous to B with the smallest bulk generalized entropy. Here the bulk generalized entropy is given by

$$S^{\text{bulk}}_{\text{gen}}(\bar{H}(B)) = Q(\bar{X}(B)) + S^{\text{bulk}}(\bar{H}(B)) + \cdots, \tag{61}$$

where $Q(\bar{X})$ is the Dong entropy functional [4]

$$Q(\bar{X}) = \frac{A(\bar{X})}{4G_{d+1}} + O(\lambda) \tag{62}$$

and $S^{\text{bulk}}(\bar{H}(B))$ denotes the bulk von Neumann entropy in $H(B)$.

To prove (6) and (7), these perturbative corrections only matter if Eq. (48) is saturated at leading order. For the condition (6), saturation implies

$$\partial_n \beta(n,y)|_{n=0} = 0. \tag{63}$$

A generalization of the Hopf lemma [57] then implies the very stringent condition that

$$\beta|_{\bar{X}} = 0. \tag{64}$$

That is, at leading order a small null deformation of ∂B in the null direction generates a null deformation *everywhere* on $\bar{X}$. Inspecting the extremal surface deviation (35), this also implies that to leading order (in δ or G_{d+1}) $\bar{X}$ lies on a locally stationary horizon. This simplifies the analysis greatly. The next-to-leading-order corrections can be solved for explicitly by Eq. (35). A possibility is that $\partial_\lambda \Theta_{(k)}(V_\lambda; y)$ reduces at next-to-leading-order to integrated bulk restricted QFC. Higher-order corrections will not be important if the saturation of the integrated bulk restricted QFC is only possible to all orders in the bulk δ or G_{d+1} expansions. A similar argument can be made for the condition (7).

One could also consider a generalization of our setup where additional intrinsic brane curvature terms (beyond the pure tension term) are added directly to the brane action [58]. If such terms are perturbatively small, i.e., they only cause small changes to the coefficients of the brane gravity derivative expansion, Eq. (12), then it is possible that the treatment discussed earlier in this subsection would suffice to generalize the proofs of restricted QFC. If such corrections are large though, we do not know how to use our method to derive the restricted QFC. One possibility is that such theories are pathological. This possibility was discussed in [59], where the authors emphasized that nontension terms added to the brane lead to a brane null geodesic not being a bulk null geodesic (since K_{ij} will no longer be proportional to g_{ij}), therefore violating the "brane causality condition," i.e., there will be bulk causal curves connecting points on the brane that are spacelike separated on the brane's causal structure.

B. Approximate QFC

By Taylor expanding β near $n = 0$, we get [modulo O(1) factors in the coefficients]

$$\beta(n,y) \sim \ell_S^2 \Theta_{(k)}^2 - n\ell_S(\partial_\lambda \Theta_{(k)} + \Theta_{(k)}^2) + n^2 \partial_n^2 \beta(n,y)|_{n=0} + O(n^3). \tag{65}$$

From Eq. (35), we have that $\partial_n^2 \beta(n,y)|_{n=0} \sim \ell_S^{-2}$. Therefore, if at some value of n the first two terms become equal while their absolute values are much larger than the third- and higher-order terms, then $\beta(n,y) \geq 0$ would be violated. It is easy to check that this leads to an "approximate quantum focusing" condition[11]

$$\partial_\lambda \Theta_{(k)} \lesssim (\partial_\lambda V_\lambda) \Theta_{(k)}^2. \tag{66}$$

This bound becomes a sharp statement when there exists a perturbative parameter ϵ in the problem and $\partial_\lambda \Theta_{(k)}$ and $\Theta_{(k)}$ acquire ϵ expansions. Then, in the $\epsilon \to 0$ limit, (66) states that the leading lhs term, if it is of lower order in ϵ than the leading rhs term, is nonpositive.

It would be interesting to explore nontrivial applications of (66). Here we provide one. In [60], it was found that in Einstein gravity plus higher-curvature corrections, classical focusing (of the Dong entropy functional [4]) is upheld on cross sections of a causal horizon that is a slight perturbation of a Killing horizon. This was shown by observing that $\partial_\lambda \Theta_{(k)} = -G_d (\partial_\lambda V_\lambda) T_{ij} k^i k^j + O(G_d^2)$, which is then nonpositive at $O(G_d)$ by the null energy condition.

[11] Douglas Stanford suggested a similar bound during a discussion about this work.

While this does not follow from the restricted QFC (since $\Theta_{(k)}$ is generally nonzero on such perturbed horizons), it does follow from (66): on the perturbed horizon, $\Theta_{(k)} = O(G_d)$, forcing any leading term in $\partial_\lambda \Theta_{(k)}$ lower than $O(G_d^2)$ to be nonpositive.

C. Does a QFC counterexample exist?

As discussed earlier, while restricted quantum focusing (6) has a natural proof in the braneworld scenario, it is not clear to us how to leverage the same technique to prove the original QFC (2). This begs the question of whether the QFC is true.

Here we discuss a setup where a QFC counterexample may be plausible. By Raychaudhuri's equation in Einstein gravity, we have

$$\Theta'_{(k)} = -\frac{\theta^2_{(k)}}{d-2} - \varsigma^2_{(k)} - 4G_d\big(2\pi\langle \hat{T}_{ij}\rangle k^i k^j - \hat{S}''_{(k)}\big), \tag{67}$$

where $\theta_{(k)}$ and $\varsigma^2_{(k)}$ are the classical expansion and the shear-squared of ∂B, respectively, $\langle \hat{T}_{ij}\rangle$ is the expectation value of the renormalized stress-energy tensor, and

$$\Theta'_{(k)}\delta^{d-2}(y-y') = \lim_{V\to\lambda\delta^{d-2}(y-y')} \partial_\lambda \Theta_{(k)}(V_\lambda; y), \tag{68}$$

$$\hat{S}''_{(k)}\delta^{d-2}(y-y') = \lim_{V\to\lambda\delta^{d-2}(y-y')} \partial_\lambda \left(\frac{1}{\sqrt{h_V}}\frac{\delta \hat{S}}{\delta V}\bigg|_{V_\lambda}\right), \tag{69}$$

where $\hat{S}$ denotes the renormalized von Neumann entropy of bulk fields. In [61,62], substantial evidence was provided that for interacting CFTs, at least when the domain of dependence of B is a Rindler wedge, we have

$$2\pi\langle \hat{T}_{ij}\rangle k^i k^j = \hat{S}''_{(k)}. \tag{70}$$

Now, let B be a ball in flat space. We then expect new terms in Eq. (70). In particular, by dimensional analysis we expect a term proportional to $\theta_{(k)}\hat{S}'_{(k)}$. Such a term does not have a definite sign and, when $\theta_{(k)}/S'_{(k)} = O(G_d)$, its sign may affect the sign of $\Theta'_{(k)}$.

Interestingly, when we instead consider the restricted QFC, where we have the additional constraint $\Theta_{(k)} = 0$, i.e.,

$$\Theta_{(k)} = \theta_{(k)} + 4G_d \hat{S}'_{(k)} = 0, \tag{71}$$

then $\theta_{(k)}\hat{S}'_{(k)}$ does acquire a definite sign, giving the restricted QFC a fighting chance. Examples like this will be explored in forthcoming work.

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APPENDIX: JUSTIFYING EQ. (50)

The bulk metric in a neighborhood of the brane is given by

$$ds^2 = dn^2 + g_{ij}(n,x)dx^i dx^j. \tag{A1}$$

The vector fields $\bar{k}^\mu$ are orthogonal to $\bar{X}$ and null, so, in particular,

$$\bar{k}^n + g_{ij}\bar{k}^i \partial_n \bar{X}^j = 0, \tag{A2}$$

$$(\bar{k}^n)^2 + g_{ij}\bar{k}^i\bar{k}^j = 0. \tag{A3}$$

Taking n derivatives of the above equations, we get

$$\lim_{n\to 0}(\partial_n \bar{k}^n + g_{ij}\partial_n\bar{k}^i\partial_n\bar{X}^j + g_{ij}\bar{k}^i\partial_n^2\bar{X}^j) = 0, \tag{A4}$$

$$\lim_{n\to 0}(\bar{k}^n\partial_n\bar{k}^n + g_{ij}\partial_n\bar{k}^i\bar{k}^j) = 0, \tag{A5}$$

where we used $\Theta_{(k)}(B;y) = 0$ and the brane equations of motion to simplify the first expression. By Eq. (22), a combination of the above equations gives

$$\lim_{n\to 0}\left[\partial_n\bar{k}^n\left(1 + \frac{\ell_S\Theta_\ell\bar{k}^n}{\sqrt{1-2\ell_S^2\Theta_{(k)}\Theta_{(\ell)}}}\right) + g_{ij}\bar{k}^i\partial_n^2\bar{X}^j\right] = 0. \tag{A6}$$

Since $\Theta_{(k)}(B;y)$ and $\Theta_{(\ell)}(B;y)$ are finite, this implies via Eq. (22) that $\partial_n\bar{X}^i|_{n=0}$ is finite. We now take advantage of the extremal surface equation:

$$\frac{1}{\sqrt{\bar{H}}}\partial_\alpha\big(\sqrt{\bar{H}}\bar{H}^{\alpha\beta}\partial_\beta\bar{X}^i\big) + \bar{H}^{\alpha\beta}\bar{\Gamma}^i_{kl}\partial_\alpha\bar{X}^k\partial_\beta\bar{X}^l = 0. \tag{A7}$$

This equation relates $g_{ij}\bar{k}^i\partial_n^2\bar{X}^j|_{n=0}$ to terms involving $\partial_n\bar{X}^i$. By the smoothness of ∂B, we expect $\Theta_{(k)}(B;y)$ and $\Theta_{(\ell)}(B;y)$ to be well behaved (e.g., finite and differentiable at y), which then forces $g_{ij}\bar{k}^i\partial_n^2\bar{X}^j|_{n=0}$ to be finite. Since $\bar{k}^n(n=0,y) = 0$, Eq. (A6) now implies the desired result that $\lim_{n\to 0}\bar{k}^n(n,y)\partial_n\bar{k}^n(n,y) = 0$.

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