Scalar cosmological perturbations from quantum entanglement within Lorentzian quantum gravity

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We derive the dynamics of (isotropic) scalar perturbations from the mean-field hydrodynamics of full Lorentzian quantum gravity, as described by a two-sector (timelike and spacelike) Barrett-Crane group field theory model. The rich causal structure of this model allows us to consistently implement in the quantum theory the causal properties of a physical Lorentzian reference frame composed of four minimally coupled, massless, and free scalar fields. Using this frame, we are able to effectively construct relational observables that are used to recover macroscopic cosmological quantities. In particular, small isotropic scalar inhomogeneities emerge as a result of (relational) nearest-neighbor two-body entanglement between degrees of freedom of the underlying quantum gravity theory. The dynamical equations we obtain for geometric and matter perturbations show remarkable agreement with those of classical general relativity for sub-Planckian modes. Quantum gravity effects produce important deviations from the classical general relativistic dynamics for trans-Planckian modes, which we show to be associated to subhorizon scales in the physical reference frame we are employing.

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I. INTRODUCTION

The prevailing ΛCDM (Λ cold dark matter) paradigm [1] gives a remarkably accurate description of the large-scale structure of our Universe in terms of a homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) background spacetime, complemented by small, inhomogeneous perturbations thereof. This nearly homogeneous and isotropic geometry is sourced by a cosmic fluid permeating the Universe. Within the ΛCDM paradigm, this fluid is composed of standard and nonstandard matter, in the form of a cosmological constant Λ and cold dark matter.

However, despite its successes, this model falls short of providing a complete physical and conceptual clarification of critical open questions about our Universe touching for instance on the fate of the initial big bang singularity, the origin of cosmic structure, the enigmatic nature of dark energy and dark matter, and the issue of the Hubble tension. Quantum gravity (QG) may help shed some light on these unanswered questions. On the other hand, cosmology, with its precision observations, presents itself as an incredibly promising testing ground for quantum gravity theories. For these reasons, extracting cosmological physics from QG is a key step to make substantial progress in both QG and theoretical cosmology.

Nevertheless, extracting cosmology from full QG is a formidable challenge, especially for approaches with "pregeometric" degrees of freedom that differ significantly from the continuum fields of standard cosmology. Thus, two main difficulties arise in this endeavor: (i) The transition from the microscopic realm of QG cosmological physics requires an appropriate definition of a semiclassical and continuum limit [2]. This possibly involves a coarse-graining procedure where macroscopic degrees of freedom are identified as effectively emerging from the underlying QG entities [3]. (ii) Inherent to background independent approaches to QG is the absence of the conventional spacetime manifold structure. Consequently, notions of

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time evolution and spatial localization can only be understood in a relational sense [4–8].

Tensorial group field theories (TGFTs) [9–12] constitute a promising framework that offers a versatile tool set to tackle both of the aforementioned challenges. Combinatorially, TGFTs can be seen as a higher-dimensional generalization of matrix models [13] and are closely related to tensor models [14–17]. TGFT models which are enriched by quantum geometric degrees of freedom (encoded in group theoretic data) are called group field theories (GFTs) [9,18] and can be seen as quantum and statistical field theories of spacetime defined on a group manifold. Owing to their quantum geometric interpretation, GFT models can be related to many other QG approaches, such as loop quantum gravity (LQG) [19–21], spin foam models [22,23], simplicial gravity [24–28], or dynamical triangulations [29–32].

It is generally expected that the continuum limit of TGFTs manifests itself in a phase of coarse-grained collective behavior [33–36]. Although extremely challenging, substantial progress in exploring the phase diagram of TGFTs has been accomplished by adapting powerful tools from local statistical and quantum field theories; see for instance [12,37–45]. Primary examples of these techniques are the functional renormalization group (FRG) methodology [46–49] and Landau-Ginzburg mean-field theory [48,50,51]. Of particular relevance for this article, the Landau-Ginzburg method has been applied recently to quantum geometric TGFT models with Lorentzian signature, finding a remarkable robustness of the mean-field approximation [36,52].

Inspired by the physics of Bose-Einstein condensates [53], the GFT condensate cosmology program [34,54–57] models this mean-field description of GFTs in terms of coherent peaked states (CPSs) subject to the classical GFT equation of motion [57–63]. In this way, this program has successfully reproduced the Friedmann dynamics for spatially homogeneous and isotropic flat geometries at late relational times, while at early relational times, the initial big bang singularity of classical cosmology is replaced by a big bounce [57,61–63]. Of particular relevance to our present work, all of these results, initially obtained using an Engle-Pereira-Rovelli-Livine (EPRL)-like GFT model [27,64], have been independently reproduced through an extended formulation of the Barrett-Crane (BC) model [65]. This observation hints at a potential universal behavior of different microscopic GFT models after coarse graining in the continuum limit [36].

Scalar cosmological perturbations in the GFT condensate cosmology framework have been studied in [66–69], and most notably in [70] where the dynamics of general relativity (GR) in the super-horizon limit of large perturbation wavelength are reproduced. For non-negligible wave vectors k, qualitative deviations from classical results arise in the spatial derivative term of the perturbation equations. Building on the interpretations for this mismatch proposed in [70], here, we suggest that it ultimately originates from (i) an insufficient coupling between the physical reference frame and the causal structure of the underlying geometry and (ii) a lack of quantum gravitational correlations generating the macroscopic inhomogeneities.

In this paper, we address these issues by making use of the richer causal structures available in the extended Barrett-Crane GFT model. We derive the dynamics of scalar cosmological perturbations which are remarkably closer to those of GR and thereby significantly improve the results of [70]. To accomplish this, we first establish a connection between the causal character of the quantum geometry and that of the clock and rods, thus making the Lorentzian interpretation of the physical frame manifest. Building on this interplay, we introduce perturbed coherent peaked states that capture the collective behavior of spacelike and timelike tetrahedra. At the background level, these states reproduce the homogeneous cosmological dynamics obtained in previous studies. Perturbations on the other hand are encoded in the two-body quantum entanglement within and between the spacelike and timelike sector. This procedure differs significantly from the purely spacelike perturbations of [70] and can be seen as out-of-condensate perturbations. Finally, we extract and study the dynamics of cosmological perturbations from the mean-field quantum dynamics associated to such perturbed CPS, and we compare our results to predictions of classical GR. We refer the reader who is interested only in the cosmological dynamics and not its derivations directly to Sec. IV E.

The paper is organized as follows: We begin by setting up the complete BC model with spacelike and timelike tetrahedra in Sec. II A, introducing in particular the two-sector Fock space. In Sec. II B, we couple four reference and one matter field to the model, proposing a restriction to couple clocks and rods to the causality of the underlying geometry. Within the causally extended setting, CPS and their relational dynamics are introduced in Sec. III, first at the background and then at the level of first-order perturbations. In Secs. IV A–IV C, we study the expectation values of relevant operators with respect to the condensate state and derive dynamical equations thereof in an effective relational fashion. Solutions of these equations are analyzed in Sec. IV E and compared to solutions of the classical perturbation equations from GR.

II. COMPLETE BARRETT-CRANE MODEL: SPACELIKE AND TIMELIKE TETRAHEDRA

After a brief discussion on the status of causality in GFTs, we introduce the Barrett-Crane TGFT model with spacelike and timelike tetrahedra in Sec. II A. Thereafter, the coupling of matter and reference fields to this model are discussed in Sec. II B. In particular, we present the possibility to relate the causal nature of clocks and rods to the causal structure of the underlying quantum geometry.

A. Spacelike and timelike tetrahedra

As introduced in Sec. I, GFTs are quantum and statistical field theories, defined on a group manifold G, endowed with a quantum geometric interpretation. One-particle excitations of a d-dimensional simplicial GFT are interpreted as $(d-1)$-dimensional simplices which, under nonlocal interactions, form d-dimensional simplices that collectively build up spacetime. A choice of group G, dimension d, and GFT-action S_{GFT} defines a specific GFT model, the partition function of which can be perturbatively expanded in terms of Feynman amplitudes dual to simplicial pseudomanifolds [11,12,14]. Depending on the representation employed, these amplitudes can be associated to either spin foam or simplicial gravity amplitudes.

The basic prerequisite for this work is a quantum geometric GFT model with an accessible causal structure, the importance of which is highlighted in approaches like causal dynamical triangulations [30–32] or causal set theory [71]. The issue of properly encoding microcausality within GFTs, spin foam models, and LQG has been scarcely studied (see, e.g., Refs. [72–77]) and only recently aroused interest once again. This has been triggered by studies on the asymptotics [78–83] of the Conrady-Hnybida extension [84,85] of the EPRL spin foam model [23,86] which includes spacelike and timelike tetrahedra to encode Lorentzian quantum geometries. In addition, in the context of effective spin foams [87–89], the path integral for Lorentzian quantum gravity has been studied, shedding light on causality violating configurations [90–92]. Finally, forming the basis of the present work, a completion of the Lorentzian Barrett-Crane GFT and spin foam model [72,73,93] with $d=4$ and $G=\mathrm{SL}(2,\mathbb{C})$ has been developed in [17] which also includes timelike and lightlike tetrahedra.

1. The Lorentzian Barrett-Crane GFT model

Restricting to spacelike tetrahedra and to homogeneous and isotropic condensates, it has been shown in [65] that at late times flat Friedmann dynamics emerge for the volume of the Universe. In this work, we go beyond this setting and include, as a minimal extension, also timelike tetrahedra while excluding lightlike tetrahedra from the outset (assumption DS1). Indeed, as we will argue below, timelike tetrahedra are necessary to properly couple the reference fields according to the signature of the quantum geometric building blocks (see Sec. II B for more details). As the results of Sec. IV show, this restricted set of causal configurations is already sufficient to yield GR-like cosmological perturbations.

Within the extension to spacelike and timelike tetrahedra, the group fields are functions $\varphi(g_v, X_\alpha)$ with $\alpha\in\{+,-\}$ assigning a spacelike, respectively a timelike signature. The four group elements $g_v=(g_1,g_2,g_3,g_4)$ are elements of $\mathrm{SL}(2,\mathbb{C})$, and X_α is a normal vector with the according signature. More precisely, X_α is an element of the homogeneous space $\mathrm{SL}(2,\mathbb{C})/\mathrm{U}^{(\alpha)}$, where $\mathrm{U}^{(+)}=\mathrm{SU}(2)$ and $\mathrm{U}^{(-)}=\mathrm{SU}(1,1)$ are the stabilizer subgroups of the respective normal vectors

$$X_+=(1,0,0,0),\qquad X_-=(0,0,0,1). \tag{2.1}$$

The fields $\varphi_\pm$ exhibit two defining symmetries,

$$\varphi(g_v,X_\alpha)=\varphi(g_v h^{-1}, h\cdot X_\alpha),\qquad \forall\ h\in\mathrm{SL}(2,\mathbb{C}), \tag{2.2}$$

$$\varphi(g_v,X_\alpha)=\varphi(g_v u_v, X_\alpha),\qquad \forall\ u_1,\ldots,u_4\in\mathrm{U}_{X_\alpha}, \tag{2.3}$$

referred to as closure and simplicity constraints, respectively, or both together as geometricity constraints. U_{X_α} denotes the stabilizer subgroup of $\mathrm{SL}(2,\mathbb{C})$ with respect to the normal vector X_α, which is isomorphic to $\mathrm{U}^{(\alpha)}$. Based on the ideas of [26,27], extending the domain $\mathrm{SL}(2,\mathbb{C})^4$ by the normal vector allows one to impose the constraints in a covariant and commuting fashion. Consequently, X_α is only considered as an auxiliary nondynamical variable that does not carry intrinsic geometric information.

Following the introduction of the fields φ, the model is then defined by its action $S[\varphi,\bar\varphi]$, which decomposes into a kinetic part,

$$K[\varphi,\bar\varphi]=\sum_\alpha\int_{\mathrm{SL}(2,\mathbb{C})^8}\mathrm{d}g_v\mathrm{d}g_w \times\int_{\mathrm{SL}(2,\mathbb{C})/\mathrm{U}^{(\alpha)}}\mathrm{d}X_\alpha\bar\varphi(g_v,X_\alpha)\mathcal{K}_\alpha(g_v,g_w)\varphi(g_w,X_\alpha), \tag{2.4}$$

with kernels $\mathcal{K}_\alpha$ and an interaction part $V[\varphi,\bar\varphi]$, the latter of which is explicitly given in [17,65]. *A priori*, the interaction V does incorporate all possible simplicial interactions composed of altogether five tetrahedra of spacelike and timelike signature.

For explicit computations as well as to connect to the spin foam formalism [22,23], the spin representation of the group field is a crucial tool that is going to be utilized heavily in this work. Following the constructions of [17], the expansion of $\varphi(g_v,X_\alpha)$ in terms of unitary irreducible $\mathrm{SL}(2,\mathbb{C})$-representation labels $(\rho,\nu)\in\mathbb{R}\times\mathbb{N}/2$ of the principal series is given by

$$\varphi(g_v,X_+)=\int\mathrm{d}\rho_v\sum_{j_v m_v l_v n_v}\varphi^{\rho_v}_{+,j_v m_v}\prod_{i=1}^4\rho_i^2 D^{(\rho_i,0)}_{j_i m_i l_i m_i}(g_i g_X)\bar{\mathcal{I}}^{\rho_i,+}_{l_i m_i}, \tag{2.5}$$

$$\varphi(g_v,X_-)=\int\mathrm{d}\rho_v\sum_{\nu_v}\sum_{j_v m_v l_v n_v}\varphi^{\rho_v\nu_v}_{-,j_v m_v} \times\prod_{i=1}^4(\rho_i^2\delta_{\nu_i,0}+\nu_i^2\delta(\rho_i)\chi_{\nu_i})D^{(\rho_i,\nu_i)}_{j_i m_i l_i n_i}(g_i g_X)\bar{\mathcal{I}}^{\rho_i\nu_i,-}_{l_i n_i}. \tag{2.6}$$

The continuous label $\rho \in \mathbb{R}$ is associated to spacelike faces, irrespective of the containing tetrahedron being spacelike or timelike. $\nu \in 2\mathbb{N}^+$ is associated to timelike faces, which are necessarily contained in timelike tetrahedra. These conditions follow from the simplicity constraint of the Barrett-Crane model [17,65], derived in the framework of integral geometry [94]. $\mathcal{I}^\pm$ are invariant symbols that ensure the constraints of Eqs. (2.2) and (2.3) which, upon integration over the normal vector, yield generalized Barrett-Crane intertwiners $B^{\rho_v \nu_v, \alpha}_{l_v n_v}$ defined in [17].

2. Extended Fock space

The extension of the GFT model to include also timelike tetrahedra necessitates the definition of an extended Fock space structure. That is because the individual field operators $\hat{\varphi}(g_v, X_+)$ and $\hat{\varphi}(g_v, X_-)$ are defined on different domains and act on different Fock spaces. These two sectors, denoted by $\mathcal{F}_+$ and $\mathcal{F}_-$, respectively, are defined as

$$\mathcal{F}_\pm \coloneqq \bigoplus_{N=0}^{\infty} \mathrm{sym}(\mathcal{H}_\pm^{(1)} \otimes \ldots \otimes \mathcal{H}_\pm^{(N)}), \tag{2.7}$$

where the one-particle Hilbert spaces for spacelike and timelike tetrahedra are given by

$$\mathcal{H}_+ \coloneqq L^2(\mathrm{SL}(2,\mathbb{C})^4 \times \mathrm{H}^3/\sim_+), \tag{2.8}$$

(see also [65]) and

$$\mathcal{H}_- \coloneqq L^2(\mathrm{SL}(2,\mathbb{C})^4 \times \mathrm{H}^{1,2}/\sim_-). \tag{2.9}$$

Here, $\sim_\pm$ denotes the imposition of geometricity constraints with respect to a timelike and spacelike normal, respectively. These normal vectors are elements of the two-sheeted and one-sheeted hyperboloids, $X_+ \in \mathrm{H}^3$ and $X_- \in \mathrm{H}^{1,2}$.

The total Fock space $\mathcal{F}$ of the theory is constructed as the tensor product of $\mathcal{F}_+$ and $\mathcal{F}_-$, i.e.,

$$\mathcal{F} \coloneqq \mathcal{F}_+ \otimes \mathcal{F}_- = \bigoplus_{N_{\mathrm{tot}}}^{\infty} \bigoplus_{N+M=N_{\mathrm{tot}}} \mathrm{sym}(\mathcal{H}_+^{\otimes N}) \otimes \mathrm{sym}(\mathcal{H}_-^{\otimes M}). \tag{2.10}$$

As usual in quantum field theory, the linear structure of the individual and total Fock spaces is strictly only possible when interactions, as introduced for the complete BC model in [17], are neglected, as assumed hereafter. For a discussion of this matter for the single-sector Fock space, we refer to [58]. Creation and annihilation operators of spacelike and timelike tetrahedra, abbreviated as $\hat{\varphi}_\pm^\dagger$ and $\hat{\varphi}_\pm$, respectively, are defined in terms of the creation and annihilation operators of the respective sectors. We frequently suppress the trivial action on the opposite sector, e.g., $\hat{\varphi}_+ \equiv \hat{\varphi}_+ \otimes \mathbb{1}_-$. Following the usual commutation rules, the operators $\hat{\varphi}_\pm$ satisfy the algebra

$$[\hat{\varphi}_\pm, \hat{\varphi}_\pm^\dagger] = \mathbb{1}_\pm, \qquad [\hat{\varphi}_\pm, \hat{\varphi}_\pm] = [\hat{\varphi}_\pm^\dagger, \hat{\varphi}_\pm^\dagger] = 0, \tag{2.11}$$

where $\mathbb{1}_\pm$ is the identity on $\mathcal{F}_\pm$ respecting closure and simplicity constraints. Notice that, by construction, operators of different sectors mutually commute

$$[\varphi_\pm, \varphi_\mp^\dagger] = [\hat{\varphi}_\pm, \hat{\varphi}_\mp] = [\hat{\varphi}_\pm^\dagger, \hat{\varphi}_\mp^\dagger] = 0. \tag{2.12}$$

The vacuum state $|\emptyset\rangle$ of the total Fock space is naturally defined as the state which is annihilated by both, $\hat{\varphi}_+$ and $\hat{\varphi}_-$. It therefore corresponds to the tensor product of the respective vacuum states, i.e., $|\emptyset\rangle = |\emptyset\rangle_+ \otimes |\emptyset\rangle_-$.

3. Operators

Building up on the Fock space structure we introduced, operators are in general defined as convolutions of kernels with creation and annihilation operators (see [20,57,62] for further details).[1] For the purposes of this work, we are particularly interested in one- and two-body operators on both sectors, i.e., the spacelike and timelike ones. The most important one-body operators are the α-number operator

$$\hat{N}_\alpha = \int \mathrm{d}g_v \mathrm{d}X_\alpha \hat{\varphi}^\dagger(g_v, X_\alpha)\hat{\varphi}(g_v, X_\alpha), \tag{2.13}$$

and the spatial three-volume operator

$$\hat{V} = \int \mathrm{d}g_v \mathrm{d}X_+ \hat{\varphi}^\dagger(g_v, X_+) V(g_v) \hat{\varphi}(g_v, X_+). \tag{2.14}$$

In spin representation, the kernel V of the volume operator is given in analogy to the eigenvalues of the LQG volume operator [95–98]. In case of isotropy of the representation labels $(\rho_i \equiv \rho)$ the kernel scales as $V \sim \rho^{3/2}$ [57,62,65].

A two-body operator $\hat{O}_{\alpha\beta}$ that describes a nontrivial correlation between the sectors α and β is generally given by

$$\hat{O}_{\alpha\beta} = \int \mathrm{d}g_v \mathrm{d}X_\alpha \mathrm{d}g_w \mathrm{d}X_\beta O(g_v, X_\alpha, g_w, X_\beta) \hat{\varphi}^\dagger \times (g_v, X_\alpha) \times \hat{\varphi}^\dagger(g_w, X_\beta), \tag{2.15}$$

where $\times$ symbolizes operator multiplication, "$\cdot$", if $\alpha = \beta$ or a tensor product, "$\otimes$", if $\alpha \neq \beta$. Notice that, $\hat{O}_{\alpha\beta}$ does not factorize in general, thus creating an entangled state when acting on a product state in $\mathcal{F}_+ \otimes \mathcal{F}_-$. In Sec. III B, we introduce three such operators, $\widehat{\delta\Phi}, \widehat{\delta\Psi}$, and $\widehat{\delta\Xi}$, which encode two-body quantum entanglement, constituting the source of cosmological perturbations that we later derive.

[1] If the group is noncompact and the field symmetries yield empty group integrations, a regularization procedure is required, for which we refer to [58,65] (assumption KS5).

B. Coupling scalar fields: Matter and physical Lorentzian reference frame

As mentioned in Sec. I, in background independent theories, physical observables are naturally understood as relational, localizing dynamical degrees of freedom with respect to other dynamical degrees of freedom. However, the implementation of a relational description in full general relativity is quite complicated, even more so for quantum gravity theories. This is especially true for approaches characterized by new fundamental pregeometric degrees of freedom, since relationality—as we usually understand it—is tightly related to the emergence of continuum notions [57]. In the context of classical homogeneous cosmology, the simplest implementation of the relational strategy involves a minimally coupled massless free (MCMF) scalar field, which serves as a relational clock [99,100]. If inhomogeneities are taken into account, 3 additional degrees of freedom, serving as relational rods, are required. This can be achieved by including three more MCMF "rod" scalar fields [100]. Another explicit example for such a physical reference frame is given by Brown-Kuchař dust [101–103], which has for instance been employed to define a fully relational cosmological perturbation theory [104,105].

In this work, following [70], we implement the relational strategy by using a physical Lorentzian reference frame composed of four MCMF scalar fields, serving as the dynamical clock and rods of our system. Furthermore, we introduce an additional MCMF "matter" scalar field which is assumed to dominate the field content of the emergent cosmology (assumption DS5). Following the strategy of [62,106,107], the scalar fields are coupled to the GFT in such a way that the Feynman amplitudes correspond to simplicial gravity path integrals with a minimally coupled scalar field placed on dual vertices of the triangulation and propagating along dual edges (see DS2). As shown in detail in Refs. [62,106], this is realized by extending the domain of the group field by the scalar field values,

$$\varphi(g_v, X_\alpha) \to \varphi(g_v, X_\alpha, \chi^\mu, \phi), \qquad (2.16)$$

where $\chi^\mu = (\chi^0, \boldsymbol{\chi})$ are the four reference fields with χ^0 being the clock and $\boldsymbol{\chi}$ being the three rods, and where ϕ denotes the additional "matter" scalar field. We use $\mu, \nu, \ldots \in \{0, 1, 2, 3\}$ as indices for the reference fields as well as for harmonic coordinates, discussed in Appendix C. Lower Latin letters $i, j, \ldots \in \{1, 2, 3\}$ denote hereby spatial components. For other than harmonic coordinates, spacetime indices are given by lower Latin letters $a, b, \ldots \in \{0, 1, 2, 3\}$.

The two kinetic kernels $\mathcal{K}_\alpha$, entering Eq. (2.4), are extended to

$$\mathcal{K}_\pm(g_v, g_w) \to \mathcal{K}_\pm(g_v, g_w, (\chi^\mu - \chi'^\mu)^2, (\phi - \phi')^2), \qquad (2.17)$$

respecting the translation and reflection invariance of the classical actions for the fields χ^μ and ϕ. Importantly, it is the kinetic kernels which encode the propagation of the scalar fields along the simplicial complex. Since the scalar field is understood to be constant on a single simplex, the five group fields entering the vertex action $V[\varphi, \bar{\varphi}]$ carry the same scalar field value. In this sense, the interactions are local with respect to the scalar field degrees of freedom [65,106].

Notice that the Fock space structure introduced in Sec. II A naturally extends to the case where such scalar fields are present, such that the nonzero commutation relations of Eq. (2.11) are given by

$$[\hat{\varphi}_\pm(\chi^\mu, \phi), \hat{\varphi}^\dagger_\pm(\chi'^\mu, \phi')] = \mathbb{1}_\pm \delta^{(4)}(\chi^\mu - \chi'^\mu)\delta(\phi - \phi'). \qquad (2.18)$$

Also, operators now include an integration over the full domain including the scalar field values. For instance, the α-number operator is now defined as

$$\hat{N}_\alpha = \int \mathrm{d}g_v\, \mathrm{d}X_\alpha\, \mathrm{d}^4\chi\, \mathrm{d}\phi\, \hat{\varphi}^\dagger(g_v, X_\alpha, \chi^\mu, \phi)\hat{\varphi}(g_v, X_\alpha, \chi^\mu, \phi). \qquad (2.19)$$

For further details, we refer to [57,65].

1. Clock and rods

In GR, perturbation equations clearly distinguish between derivatives with respect to (clock) time and (rods) space. As we can see explicitly from, e.g., Eq. (C17), it is not just a matter of signature: when the physical frame is made of four minimally coupled, massless, and free scalar fields, their harmonic behavior imposes a different relative weight of a^4 between (relational) space and time derivatives. Since symmetries on field space of the classical action are naturally reflected at the level of the kernels of the GFT action [62], one would naively expect that, by imposing Lorentz symmetry at the level of the frame field action, one would recover effective equations which at least show the right signature of temporal and spatial derivative terms. However, as shown in [70], this seems not to be the case: The signature of effective equations turns out to be independent on the symmetries of the frame action, and to be fixed essentially only by the parameters of the CPSs. On top of that, in the effective perturbation equations derived in [70], temporal and spatial derivatives enter with the same weight. As emphasized already in [70], this property of the effective equations is the source of a crucial mismatch with GR when perturbations momenta are $k > 0$.

Both of the above results seem to suggest a difficulty in distinguishing between rods and clock at the level of the underlying GFT. This difficulty, however, may be solved by carefully coupling the frame according to the causal structure of the underlying geometry. To do so, in this work we take advantage of the extended structure of the present model, which, crucially, includes both spacelike and timelike tetrahedra.

In the next two following paragraphs, we restrict our attention to the reference fields, since we do not intend to impose the same causality conditions on the matter field ϕ.

2. Classical and discrete perspective

In the continuum, the action of the four reference fields is given by

$$S[\chi] = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\sum_{\mu=0}^{3} g^{ab}\partial_a\chi^\mu\partial_b\chi^\mu, \qquad (2.20)$$

and we enforce our assumption on the signature of clock and rods in terms of the conditions

$$g(\partial\chi^0, \partial\chi^0) < 0, \qquad (2.21a)$$

$$g(\partial\chi^i, \partial\chi^i) > 0, \qquad (2.21b)$$

meaning that the clock has a timelike gradient and rods have a spacelike gradient. Notice that here we are not imposing a Lorentz symmetry (on field space) of the action. Indeed, as already argued in [70], only this choice for the action guarantees the appropriate sign for the energy density of the four fields. We note two points here: First, the conditions on the gradients are not implied by the Klein-Gordon equation, but constitute indeed an additional physical requirement. Second, despite the point-particle intuition, a massless scalar field does not necessarily have a lightlike gradient. One of the simplest counter examples is that of a massless scalar field in a homogeneous background that can be used as a clock, therefore having a timelike gradient.[2]

Introducing a discretization of the continuum scalar field theory on a two-complex Γ, where the fields are placed on dual vertices $v\in\Gamma$, the action can be written as

$$S_\Gamma[\chi] = \frac{1}{2}\sum_{(vv')\in\Gamma} V^{*(4)}_{(vv')}\sum_{\mu=0}^{3}\left(\frac{\chi^\mu_v - \chi^\mu_{v'}}{l_{(vv')}}\right)^2, \qquad (2.22)$$

with $V^{*(4)}_{(vv')}$ being the Voronoi four volume dual to the dual edge (vv') and with $l_{(vv')}$ being the dual edge length [108]. The sum over (vv') denotes the sum over all dual edges which already suggests that, after quantization, the scalar field dynamics are going to be encoded in the edge amplitudes, i.e., in the kinetic kernels of the GFT. Notice, that the discretization of a continuum field theory is always accompanied by ambiguities in the construction of discrete derivatives and dual geometric quantities [108].

[2]The authors thank S. Gielen and D. Oriti for a clarifying exchange on this matter.

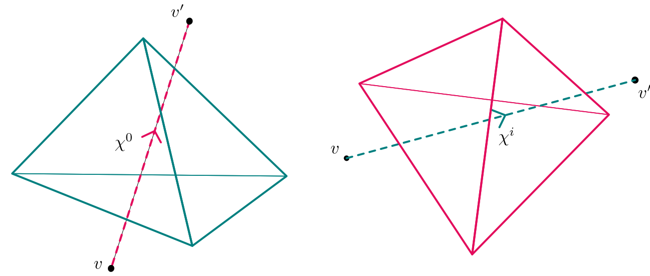

FIG. 1. Left: a spacelike tetrahedron (teal) and its corresponding timelike dual edge (pink), connecting the dual vertices v and v'. Following the restriction in Eq. (2.25a), the clock χ^0 only propagates along timelike dual edges. Right: a timelike tetrahedron (pink) and its corresponding spacelike dual edge (teal). In this case, Eq. (2.25b) imposes that rods χ^i only propagate along spacelike dual edges.

A necessary condition for a discretization to be viable is that it exhibits the correct continuum limit [109].

Given the Lorentzian structure of the original continuum manifold, Γ is the dual complex of a Lorentzian discretization. This implies that edges (vv') which are dual to spacelike and timelike tetrahedra are timelike and spacelike, respectively, for which a visual intuition is given in Fig. 1. Notice that (i) the propagation of the scalar field is only sensitive to the signature of the dual edges and (ii) lightlike dual edges are excluded as a consequence of lightlike tetrahedra being excluded from the outset. Based on this, the discrete scalar field action in Eq. (2.22) can then be split into spacelike and timelike dual edges

$$S_\Gamma[\chi] = S^+_\Gamma[\chi] + S^-_\Gamma[\chi], \qquad (2.23)$$

defined by

$$S_\Gamma[\chi] = \frac{1}{2}\sum_{\mu=0}^{3}\left[\sum_{(vv')\mathrm{t.l.}} w_{vv'}(\chi^\mu_v - \chi^\mu_{v'})^2 + \sum_{(vv')\mathrm{s.l.}} w_{vv'}(\chi^\mu_v - \chi^\mu_{v'})^2\right], \qquad (2.24)$$

where $w_{vv'}$ denotes the geometric coefficients $V^{*(4)}_{(vv')}/l^2_{(vv')}$. Clearly, both types of reference fields propagate *a priori* on both types of dual edges. In analogy to the continuum equations (2.21), we propose to align the causal character of the reference frame with that of geometry by introducing the conditions

$$\chi^0_v - \chi^0_{v'} = 0, \quad \text{for } (vv') \text{ spacelike}, \qquad (2.25a)$$

$$\chi^i_v - \chi^i_{v'} = 0, \quad \text{for } (vv') \text{ timelike}. \qquad (2.25b)$$

Formulated geometrically, the clock propagates along timelike dual edges, and rods propagate along spacelike dual edges, as depicted in Fig. 1.[3] As a result, the discrete scalar field action splits into a clock and a rod part, associated to the signature of the respective dual edges. In the following, we discuss the realization of the conditions (2.25) at the level of the GFT coupling.

3. Restriction of kinetic kernels

Proceeding in parallel to [106], the scalar field coupling is obtained by considering the simplicial gravity path integral on a given complex Γ for the coupled gravity-matter system. As geometric quantities, dual edge lengths, and tetrahedron volumes can be rewritten in terms of bivector variables $B \in \mathfrak{sl}(2,\mathbb{C})$. Then, the GFT model which generates these amplitudes is derived, showing that the details of propagation are encoded in the kinetic kernel, while the GFT interaction is local with respect to the scalar fields. In particular, the details of the discretized geometric quantities $L^*_{vv'}$ and $V^{(3)}_{vv'}$ are encoded in the kinetic kernels, which we keep implicitly defined for the rest of this work.

As a result of the assignment of clocks to timelike dual edges and rods to spacelike dual edges above, the kinetic kernels $\mathcal{K}_\pm$ have a restricted dependence, given by

$$\mathcal{K}_+(g_v, g_w, (\chi - \chi')^2) = \mathcal{K}_+(g_v, g_w, (\chi^0 - \chi^{0\prime})^2), \tag{2.26a}$$

$$\mathcal{K}_-(g_v, g_w, (\chi - \chi')^2) = \mathcal{K}_-(g_v, g_w, |\boldsymbol{\chi} - \boldsymbol{\chi}'|^2). \tag{2.26b}$$

This structure of the kinetic kernels corresponds to a strong imposition of the classical discrete conditions in Eq. (2.25). Clearly, a weaker imposition (e.g., via a Gaussian) including quantum fluctuations around the classical behavior is also possible. For the rest of this work, we assume Eqs. (2.26) to hold (assumption DS2). Notice again, that the matter field ϕ is *a priori* not affected by this restriction.

III. COHERENT PEAKED STATES AND PERTURBATIONS

A crucial ingredient for the extraction of cosmological physics is the identification of states that can be associated with continuum, classical physics. Since GFTs are many-body quantum field theories of atoms of spacetime, by analogy with condensed matter systems, one would naturally expect these states to exhibit some form of collective behavior. The simplest form of such collective behavior is captured by coherent (or condensate) states. Importantly, strong evidence has recently been provided for the existence of such a condensate phase in quantum geometric TGFTs [36,52]. One-body condensates of spacelike tetrahedra (whose condensate wave function encodes the macroscopic physics of the system) have indeed been used to derive an effective cosmological dynamics that exhibits a resolution of the big bang into a big bounce [57,62,65] and offers intriguing phenomenological implications such as dynamical isotropization [110], emergent inflation [111,112], or a late-time de Sitter phase [113]. Motivated by these encouraging results and given the need to introduce timelike tetrahedra to improve the frame coupling, we propose to describe the background component of our collective states as the tensor product of a spacelike and a timelike one-body condensate

$$|\sigma; x^0\rangle \otimes |\tau; x^0, \boldsymbol{x}\rangle = \mathcal{N}_\sigma \mathcal{N}_\tau e^{\hat{\sigma}\otimes\mathbb{1}+\mathbb{1}\otimes\hat{\tau}}|\emptyset\rangle, \tag{3.1}$$

where the spacelike and timelike condensate states (whose details are discussed in Secs. III A 1 and III A 2) are denoted as $|\sigma; x^0\rangle$ and $|\tau; x^0, \boldsymbol{x}\rangle$ respectively. At the right-hand side of the above equation, we have rewritten the two condensate states as exponentials of the one-body operators $\hat{\sigma}$ and $\hat{\tau}$, respectively acting on the vacuum of $\mathcal{F}_+ \otimes \mathcal{F}_-$. Finally, $\mathcal{N}_\sigma$ and $\mathcal{N}_\tau$ are normalization factors. We emphasize that the timelike condensate state $|\tau; x^0, \boldsymbol{x}\rangle$ is part of the background since the spatial peaking on $\boldsymbol{x}$ only enters the peaking function and not the reduced condensate wave function $\tilde{\tau}$, as discussed in detail down below.

Given the above choice of background states, the most natural way to describe cosmological inhomogeneities would seem to be to include small inhomogeneous perturbations of the one-body condensate wave functions (since this is where the macroscopic physics of the system is encoded). However, as it will become clear in Sec. III C 1, the mean-field dynamics of the two one-body condensate wave functions turn out to be completely decoupled (at least in the regime of negligible interactions that we will consider below). Therefore, this choice would produce results that are effectively equivalent to those found in [70], and thus eventually lead to the puzzling indistinguishably between clocks and rods discussed in the previous section and to the consequent mismatch with GR.

For this reason, and following the intriguing idea that nontrivial geometries are associated with entanglement of quantum gravity degrees of freedom, in this paper we propose an alternative description of cosmological inhomogeneities in terms of correlations of the underlying GFT quanta. Since, however, cosmological inhomogeneities are a macroscopic phenomenon from the QG perspective, the perturbations describing them should be realized in a collective manner at the level of the GFT. Therefore, we consider states of the form

$$|\Delta; x^0, \boldsymbol{x}\rangle = \mathcal{N}_\Delta \exp\big(\hat{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\hat{\tau} + \widehat{\delta\Phi}\otimes\mathbb{1} + \widehat{\delta\Psi} + \mathbb{1}\otimes\widehat{\delta\Xi}\big)|\emptyset\rangle, \tag{3.2}$$

[3]Notice that these conditions enforce the gradients of clocks and rods to have only temporal, respectively spatial, entries, which is a stronger condition than requiring the signature to be timelike or spacelike.

where the perturbations are encoded in the operators $\widehat{\delta\Phi}$, $\widehat{\delta\Psi}$, and $\widehat{\delta\Xi}$ (see also KS1). In general, these can be a combination of n-body operators, each of which encoding n-body correlations within and in between the spacelike and timelike sectors. However, in the following, we will restrict to the simplest nontrivial case, i.e., two-body operators. Also, as the perturbations are assumed to be small, the final form of states we will employ are a linearized version of Eq. (3.2). Notice that in the picture we propose, we do not consider perturbations at the level of quantum geometric operators but rather at the level of the states, chosen appropriately to describe GR-like perturbations.

Finally, in Sec. III C, we derive and discuss the effective relational dynamics of the linearized perturbed states.

A. Coherent peaked states for spacelike and timelike tetrahedra

As discussed above, our working assumption is that the spacelike and timelike sector of the background structure separate. Since the total Fock space $\mathcal{F}$, introduced in Sec. II A, is given in terms of a tensor product of $\mathcal{F}_+$ and $\mathcal{F}_-$, the states of the background are therefore product states.

1. Spacelike CPS

On the spacelike Fock space $\mathcal{F}_+$, we introduce the coherent peaked state

$$|\sigma_{\epsilon^+,\pi_0^+};x^0\rangle = \mathcal{N}_\sigma \exp\left(\int \mathrm{d}g_v\, \mathrm{d}X_+\, \mathrm{d}^4\chi\, \mathrm{d}\phi\, \sigma_{\epsilon^+,\pi_0^+;x^0}(g_v,X_+,\chi^0,\phi)\hat{\varphi}^\dagger(g_v,X_+,\chi^\mu,\phi)\right)|\emptyset\rangle, \tag{3.3}$$

which is assumed to be normalized via the factor $\mathcal{N}_\sigma$ [57,65]. The key ingredient is the condensate wave function $\sigma_{\epsilon^+,\pi_0^+;x^0}$, with x^0 being the reference field value on which the state is peaked and with ϵ^+ and π_0^+ characterizing the peaking properties (see [57,63] for further details on the formalism of coherent peaked states). It can be understood as a mean field, since it is the expectation value of the spacelike group field operator

$$\langle\sigma_{\epsilon^+,\pi_0^+};x^0|\hat{\varphi}(g_v,X_+,\chi^\mu,\phi)|\sigma_{\epsilon^+,\pi_0^+};x^0\rangle = \sigma_{\epsilon^+,\pi_0^+;x^0}(g_v,X_+,\chi^0,\phi). \tag{3.4}$$

The condensate wave function factorizes as

$$\sigma_{\epsilon^+,\pi_0^+;x^0}(g_v,X_+,\chi^0,\phi) = \eta_{\epsilon^+}(\chi^0-x^0;\pi_0^+)\tilde{\sigma}(g_v,X_+,\chi^0,\phi), \tag{3.5}$$

wherein

$$\eta_{\epsilon^+}(\chi^0-x^0;\pi_0^+) = \mathcal{N}_{\epsilon^+}\exp\left(-\frac{(\chi^0-x^0)^2}{2\epsilon^+}\right)\mathrm{e}^{i\pi_0^+(\chi^0-x^0)} \tag{3.6}$$

encodes the Gaussian peaking on the reference field value x^0 together with a phase factor that ensures finiteness of the reference field momenta [57,65] (see also KS3). $\mathcal{N}_\epsilon^+$ is a normalization factor of the Gaussian function. The remaining geometric information is carried by the reduced condensate wave function $\tilde{\sigma}(g_v,X_+,\chi^0,\phi)$.

2. Choice of spacelike peaking

We have chosen a particular peaking for the spacelike condensate wave function as well as a specific dependence on the components of the reference fields χ^μ. Recall that the group field $\varphi(g_v,X_+,\chi^\mu,\phi)$ does depend on all components of the reference fields, even if the kinetic kernel is restricted according to Eq. (2.26a). However, since we want the background condensates to be associated to perfectly homogeneous geometries, the condensate wave function is peaked only on a chosen clock value, and the reduced condensate wave function is a function of the clock only (assumption KS3).

3. Symmetries of $\tilde{\sigma}$

Importantly, the reduced condensate wave function $\tilde{\sigma}$ satisfies additional symmetries besides that of the group field, given in Eqs. (2.2) and (2.3). First, notice that the embedding of a single tetrahedron in Minkowski space, as dictated by the group field, is not a gauge-invariant information. In fact, we argue that the embedding information should be realized in a relational fashion, which is ensured by the clock peaking. As argued in [65], so-called adjoint covariance

$$\tilde{\sigma}(g_v,X_+,\chi^0,\phi) = \tilde{\sigma}(hg_vh^{-1},h\cdot X_+,\chi^0,\phi), \quad \forall\ h\in\mathrm{SL}(2,\mathbb{C}), \tag{3.7}$$

can be understood as averaging over the embedding of a single tetrahedron, therefore eliminating this gauge-variant information. The resulting domain of the reduced condensate wave function carries the correct number of degrees of freedom, corresponding to the homogeneous spatial metric at a relational instance of time, i.e., it is diffeomorphic to minisuperspace [57,59,65]. We refer to this property as *relational homogeneity* [57].[4] The remaining two conditions are most clearly seen in spin representation. *A priori*, the four spacelike faces of a tetrahedron, labelled by ρ_i, take different values. Imposing however that the tetrahedra are equilateral, which is often referred to as an isotropy condition [62,65], we fix for the remainder all ρ_i to be equal (assumption KS2).[5] Furthermore, the $\mathrm{SL}(2,\mathbb{C})$-intertwiner labels arising from Eq. (3.7) are fixed [65]. To simplify matters even further, we assume that the condensate is dominated by a single spin label ρ, as justified by [65,110,114] (assumption DC2). As a result of all of these additional conditions, the reduced condensate wave function has a spin representation given by

$$\tilde{\sigma}_\rho(\chi^0,\phi) \equiv \tilde{\sigma}(\chi^0,\phi), \tag{3.8}$$

where we suppress the fixed label ρ in the notation for the remainder. Following this introduction of coherent states for the spacelike background, we elaborate in the following on the timelike background.

4. Timelike CPS

Following the arguments of the introduction of this section, we assume the timelike background to be described by a condensate which, as it turns out in Sec. IV, proves sufficient to capture GR-like perturbations. Following this idea, we denote the condensate state on $\mathcal{F}_-$ as

$$|\tau_{\epsilon^-,\pi_0^-,\delta,\pi_x};x^0,\boldsymbol{x}\rangle = \mathcal{N}_\tau \exp\left(\int \mathrm{d}g_v \mathrm{d}X_- \mathrm{d}^4\chi \mathrm{d}\phi\, \tau_{\epsilon^-,\pi_0^-,\delta,\pi_x;x^0,\boldsymbol{x}}(g_v,X_-,\chi^\mu,\phi)\hat{\varphi}^\dagger(g_v,X_-,\chi^\mu,\phi)\right)|\emptyset\rangle, \tag{3.9}$$

which is now an eigenstate of the timelike group field operator, again normalized by the factor $\mathcal{N}_\tau$. Similar to the spacelike case, the timelike condensate wave function $\tau_{\epsilon^-,\pi_0^-,\delta,\pi_x;x^0,\boldsymbol{x}}$ factorizes according to

$$\tau_{\epsilon^-,\pi_0^-,\delta,\pi_x;x^0,\boldsymbol{x}}(g_v,X_-,\chi^\mu,\phi) = \eta_{\epsilon^-}(\chi^0 - x^0,\pi_0^-)\eta_\delta(|\boldsymbol{\chi}-\boldsymbol{x}|,\pi_x)\tilde{\tau}(g_v,X_-,\chi^0,\phi), \tag{3.10}$$

where $\tilde{\tau}(g_v,X_-,\chi^0,\phi)$ is the timelike reduced condensate wave function. Besides a clock peaking, the timelike condensate is also peaked on the rod variables $\boldsymbol{\chi}$ via

$$\eta_\delta(|\boldsymbol{\chi}-\boldsymbol{x}|,\pi_x) = \mathcal{N}_\delta \exp\left(-\frac{|\boldsymbol{\chi}-\boldsymbol{x}|^2}{2\delta}\right)\mathrm{e}^{i\pi_x|\boldsymbol{\chi}-\boldsymbol{x}|}. \tag{3.11}$$

We have chosen an isotropic peaking of the rod variables with the same parameters δ and π_x for every spatial direction, following the strategy of [70] (see also KS2 and KS3).

5. Choice of timelike peaking

Since the timelike condensate is associated to the background structure, the reduced condensate wave function $\tilde{\tau}(g_v,\chi^0,\phi,X_-)$ only depends on the relational clock χ^0. A peaking on rod variables is added for the timelike condensate to associate spatial derivatives to the timelike sector, as we discuss in more detail in Sec. III C 2 (see also assumption KS3).

[4]Previously, the equivalence of minisuperspace and the domain of the condensate wave function had been established using only the geometric variables [59,62,65]. In this work, however, we advocate for a relational notion of such an equivalence.

[5]Notice that while this notion of isotropy seems natural from a geometric point of view, a different restriction onto the reduced condensate wave function has been explored [110]. However, at the level of the background dynamics this produces physically equivalent results which is line with naive universality arguments.

6. Symmetries of $\tilde{\tau}$

In addition to the symmetries of $\hat{\varphi}(g_v,X_-,\chi^\mu,\phi)$, we introduce additional conditions to the timelike reduced condensate wave function $\tilde{\tau}$, similar to $\tilde{\sigma}$ above. Most importantly, these restrictions ensure that $\tilde{\tau}$ carries the correct degrees of freedom. First, $\tilde{\tau}$ also satisfies adjoint covariance

$$\tilde{\tau}(g_v,X_-,\chi^0,\phi) = \tilde{\tau}(hg_vh^{-1},h\cdot X_-,\chi^0,\phi), \quad \forall\ h\in\mathrm{SL}(2,\mathbb{C}), \tag{3.12}$$

with the resulting $\mathrm{SL}(2,\mathbb{C})$-intertwiner label in spin representation being fixed. As a result, the domain of $\tilde{\tau}$ corresponds to the metric degrees of freedom on a $(2+1)$-dimensional slice at a given instance of relational time. Therefore, the number of degrees of freedom of the spacelike and timelike condensates is the same, which is important for the later analysis. Since timelike tetrahedra admit an arbitrary mixture of spacelike and timelike faces, the reduced condensate wave function $\tilde{\tau}$ carries *a priori* all possible combinations of $(\rho,0)$ and $(0,\nu)$ labels, as Eq. (2.6) indicates. In the following, we are going to restrict to the case where the condensate wave function only carries spacelike faces and fix the corresponding label to the same label ρ as for $\tilde{\sigma}$ (assumptions KS2, KS4, and DC2). Besides a simplification of the dynamics, there are two further reasons to restrict to spacelike faces only. First, current developments in the Landau-Ginzburg mean-field analysis of the complete Barrett-Crane model [36,52,115] suggest that a condensate phase for timelike tetrahedra exists if the faces are all spacelike. Second, as detailed in [17], correlations between spacelike and timelike tetrahedra can only be mediated via spacelike faces. Hence, correlations between the spacelike and timelike sector, introduced below, are only possible if the faces carry the same signature.

As a result, the timelike reduced condensate wave function in spin representation is of the form

$$\tilde{\tau}_\rho(\chi^0,\phi) \equiv \tilde{\tau}(\chi^0,\phi), \tag{3.13}$$

where we again suppress the fixed label ρ in the remainder.

In summary, the background structure on the total Fock space is defined by the state

$$|\sigma_{\epsilon^+,\pi_0^+};x^0\rangle \otimes |\tau_{\epsilon^-,\pi_0^-,\delta,\pi_x};x^0,\boldsymbol{x}\rangle, \tag{3.14}$$

on which the group field operators act accordingly. The effective relational dynamics of this background state is computed in Sec. III C 1.

B. Perturbed coherent peaked states

Following the introduction of this section, inhomogeneities are encoded in the perturbed coherent peaked state of Eq. (3.2), with the three two-body operators $\widehat{\delta\Phi}\otimes\mathbb{1}_-$, $\widehat{\delta\Psi}$, and $\mathbb{1}_+\otimes\widehat{\delta\Xi}$ sourcing a quantum entanglement within and between the spacelike and timelike sectors. These three operators are defined in analogy to $\hat{O}_{++}$, $\hat{O}_{+-}$, and $\hat{O}_{--}$ in Eq. (2.15), respectively. The bilocal functions $\mathcal{O}_{\alpha\beta}$ are referred to as kernels.

A priori, the kernels $\delta\Phi$, $\delta\Psi$, and $\delta\Xi$ that define the three operators above are bilocal functions on the respective domains. On both copies of the domain, we impose the same restrictions as for the spacelike and timelike reduced condensate wave functions, respectively (see assumptions KS3, KS4, and DC2). As a result, the spin representation of the three kernels is explicitly given by

$$\delta\Phi(\chi^\mu,\phi,\chi'^\mu,\phi'),\quad \delta\Psi(\chi^\mu,\phi,\chi'^\mu,\phi')\ \text{and}\ \delta\Xi(\chi^\mu,\phi,\chi'^\mu,\phi'), \tag{3.15}$$

where we suppressed the dependence on the fixed $SL(2,\mathbb{C})$-representation label ρ in the notation.[6] Crucially, the three two-body operators do not factorize into one-body operators if the kernels do not factorize accordingly. As a result, acting with $\widehat{\delta\Phi}$, $\widehat{\delta\Psi}$, and $\widehat{\delta\Xi}$ on the tensor product of spacelike and timelike condensate creates an entangled state in the respective sectors.

Finally, since we are interested in small perturbations, we employ the linearized form of $|\Delta;x^0,\boldsymbol{x}\rangle$, given by

$$|\Delta;x^0,\boldsymbol{x}\rangle \approx \mathcal{N}_\Delta(\mathbb{1}+\widehat{\delta\Phi}+\widehat{\delta\Psi}+\widehat{\delta\Xi})|\sigma;x^0\rangle\otimes|\tau,x^0,\boldsymbol{x}\rangle. \tag{3.16}$$

This is the state that we are going to employ for computing the cosmological dynamics, including perturbations. In the following two sections, we show how to obtain effective relational dynamics as the expectation value of the GFT equations of motion and how to connect macroscopic quantities such as the three volume to the expectation value of quantum geometric operators.

C. Effective relational dynamics of perturbed CPS

Average relational dynamics of GFT condensates are, at a mean-field level, obtained by taking the expectation value of the GFT equations of motion with respect to the macroscopic state $|\Delta;x^0,\boldsymbol{x}\rangle$; see also DS3. Owing to the presence of two fields, corresponding to spacelike and timelike tetrahedra, there are two effective relational equations of motion, which are given by

$$\left\langle \Delta;x^0,\boldsymbol{x}\left|\frac{\delta S[\hat{\varphi},\hat{\varphi}^\dagger]}{\delta\hat{\varphi}(g_v,X_\alpha,x^\mu,\phi)}\right|\Delta;x^0,\boldsymbol{x}\right\rangle = 0, \tag{3.17}$$

for each signature $\alpha\in\{+,-\}$. Notice, that these equations correspond to the first of an infinite tower of Schwinger-Dyson equations [58,62]. Ultimately, the dynamics of the perturbed CPS $|\Delta;x^0\boldsymbol{x}\rangle$ will govern the dynamics of cosmological observables obtained as the expectation value of GFT operators with respect to $|\Delta;x^0\boldsymbol{x}\rangle$. However, as it requires particular care, we dedicate to this step Secs. IV A–IV C while focusing here only on the dynamics of the condensate.

For the remainder of this work, we assume negligible interactions (assumption DS4), which was shown in [62,65] to be a valid approximation at late but not very late times; see also [58] for a discussion. Notice, that the argument provided therein also applies to the timelike sector. One of the crucial consequences of this assumption is that higher orders of the Schwinger-Dyson equations reduce to powers of the lowest order equations (3.17), as we show in Appendix B. Hence, solutions of Eq. (3.17) solve also all higher orders. We comment on this matter in more detail in Sec. V. Notice that despite negligible interactions, the spacelike and timelike sectors get coupled via the spacelike-timelike quantum correlation $\delta\Psi$, as we show in detail in Sec. III C 2.

[6]Notice, that the functions $\delta\Phi$, $\delta\Psi$, and $\delta\Xi$ carry a dependence on the clock and on the rods. Together with the restriction in Eq. (3.27) below, where the two copies of reference fields are identified via a δ distribution, the domain of these kernels corresponds to the metric degrees of freedom at a relational spacetime point. Thus, the two-body correlations describe perturbations of the relational notion of homogeneity by a direct rod dependence.

Owing to the presence of perturbations, the two equations of motion can be separated into a zeroth-order background part and a first-order perturbation part, which we discuss separately in Secs. III C 1 and III C 2, respectively.

1. Background equations of motion

At background level, the two equations of motion in spin representation are given by

$$0 = \int \mathrm{d}\chi^0 \mathrm{d}\phi' \mathcal{K}_+(\chi^0, (\phi - \phi')^2)\sigma(\chi^0 + x^0, \phi'), \tag{3.18}$$

$$0 = \int \mathrm{d}^4\chi \mathrm{d}\phi' \mathcal{K}_-(\boldsymbol{\chi}, (\phi - \phi')^2)\tau(\chi^0 + x^0, \boldsymbol{\chi} + \boldsymbol{x}, \phi'), \tag{3.19}$$

where we note that due to spatial homogeneity, the background equation of motion on the spacelike sector would actually contain an empty integration over the rods χ^i which we henceforth regularize by introducing a fiducial cell of finite volume. For a further analysis, we perform a Fourier transform of the matter field variables $\phi \to \pi_\phi$. Following [70], we assume a peaking of both condensate wave functions on a fixed scalar field momentum p_ϕ, realized by a Gaussian peaking. Since the scalar field is minimally coupled, its canonical conjugate momentum is constant at the classical continuum level. We translate this idea to the present context by peaking on a fixed value p_ϕ (assumption KC1). Exploiting furthermore the peaking properties of σ and τ, defined in Eqs. (3.5) and (3.10), respectively, we obtain the dynamical equations for the reduced condensate wave functions $\tilde{\sigma}$ and $\tilde{\tau}$:

$$\partial_0^2\tilde{\sigma}(x^0, p_\phi) - 2i\tilde{\pi}_0^+\partial_0\tilde{\sigma}(x^0, p_\phi) - E_+^2(\pi_\phi)\tilde{\sigma}(x^0, p_\phi) = 0, \tag{3.20}$$

$$\partial_0^2\tilde{\tau}(x^0, p_\phi) - 2i\tilde{\pi}_0^-\partial_0\tilde{\tau}(x^0, p_\phi) - E_-^2\tilde{\tau}(x^0, p_\phi) = 0, \tag{3.21}$$

where the quantities $E_\pm$ and $\tilde{\pi}_0^\pm$ are defined in Appendix D 1, to which we refer for further details.

Following the procedure of [57,62,70], the reduced condensate wave functions can be decomposed into a radial and angular part, denoted as $r_\alpha(x^0, p_\phi)$ and $\theta_\alpha(x^0, p_\phi)$, respectively. Splitting the resulting equations into a real and imaginary part, one obtains

$$r_\alpha''(x^0, p_\phi) - \frac{Q_\alpha^2(p_\phi)}{r_\alpha^3(x^0, p_\phi)} - \mu_\alpha^2(p_\phi)r_\alpha(x^0, p_\phi) = 0, \tag{3.22}$$

$$\theta_\alpha'(x^0, p_\phi) - \tilde{\pi}_0^\alpha - \frac{Q_\alpha(p_\phi)}{r_\alpha^2(x^0, p_\phi)} = 0, \tag{3.23}$$

where a prime denotes differentiation with respect to x^0, Q_α are integration constants and the μ_α are defined as $\mu_\alpha^2(p_\phi) \coloneqq E_\alpha^2(p_\phi) - (\tilde{\pi}_0^\alpha)^2$. As demonstrated in Appendix D 1, E_- and thus μ_- are actually independent of the peaked matter momentum p_ϕ.

2. Classical limit

As elaborated previously [57,62,63], the semiclassical limit of the condensate is obtained at late relational timescales where the moduli of the condensate wave functions r_α are dominant with respect to Q_α and μ_α but where interactions are still negligible (see also DC1). Furthermore, it has been shown in [110], that in this limit, expectation values of for instance the volume operator are sharply peaked, providing a highly nontrivial consistency check for the semiclassical interpretation. In this limit, the background equations of motion simplify significantly, yielding solutions

$$\tilde{\sigma}(x^0, p_\phi) = \tilde{\sigma}_0 \mathrm{e}^{(\mu_+ + i\tilde{\pi}_0^+)x^0}, \tag{3.24}$$

$$\tilde{\tau}(x^0, p_\phi) = \tilde{\tau}_0 \mathrm{e}^{(\mu_- + i\tilde{\pi}_0^-)x^0}, \tag{3.25}$$

where $\tilde{\sigma}_0$ and $\tilde{\tau}_0$ are determined by initial conditions. These two equations will be heavily employed to derive the dynamics of cosmological quantities at the background level such as the volume dynamics in Eq. (4.8), the dynamics of spacelike and timelike particle number in Eqs. (4.32) and (4.33), or the dynamics of the background scalar field in Eq. (4.52). Furthermore, the dynamics of the cosmological observables at first order in perturbations crucially depend on the background condensate solutions above.

This concludes the effective background equations of motion. In the next section, we compute the effective equations of motion at first order in perturbations.

3. Perturbed equations of motion

At first order in perturbations, there are two equations, one for the spacelike and one for the timelike sector. Since there are three distinct two-body correlations $\delta\Phi$, $\delta\Psi$, and $\delta\Xi$, there is a dynamical freedom for one of the variables if one assumes negligible interactions and works within a first-order perturbative framework. In the following, we utilize this freedom to relate the functions $\delta\Phi$ and $\delta\Psi$ via an arbitrary function $\mathrm{f}(\chi^\mu)$ that will be ultimately fixed by matching the perturbed volume to the classical quantity of GR. We provide a physical interpretation of this assumption in Sec. IV B in terms of the perturbations of the timelike number operator, δN_-.

4. Spacelike perturbed dynamics

Dynamics of the spacelike sector at first order of perturbations are governed by

$$0 = \int d^4\chi d\phi' \mathcal{K}_+((\chi^0)^2, (\phi - \phi')^2) \times \int d^4\chi' d\phi'' \big[\delta\Psi(\chi^\mu + x^\mu, \phi', \chi^{\mu\prime}, \phi'')\bar{\tau}(\chi^{\mu\prime}, \phi'') + \delta\Phi(\chi^\mu + x^\mu, \phi', \chi^{\mu\prime}, \phi'')\bar{\sigma}(\chi^{0\prime}, \phi'')\big]. \tag{3.26}$$

As a first simplification, we choose the bilocal kernel $\delta\Psi$ to depend only on one copy of relational frame data (assumption KC2), i.e.,

$$\delta\Psi(\chi^\mu, \pi_\phi, \chi^{\mu\prime}, \pi'_\phi) = \delta\Psi(\chi^\mu, \pi_\phi)\delta^{(4)}(\chi^\mu - \chi'^\mu)\delta(\pi_\phi - \pi'_\phi). \tag{3.27}$$

From a simplicial gravity perspective, locality with respect to the reference fields χ^μ corresponds to correlations only within the same four simplex, which can be compared to nearest-neighbor interactions in statistical spin systems. For the momenta of the matter field π_ϕ, the condition is interpreted as momentum conservation across tetrahedra of the same four simplex. Next, we exploit the dynamical freedom for one of the perturbation functions by imposing the relation (assumption DC3)

$$\delta\Phi(\chi^\mu, \pi_\phi) = \mathrm{f}(\chi^\mu)\delta\Psi(\chi^\mu, \pi_\phi), \tag{3.28}$$

with the complex valued function f defined as

$$\mathrm{f}(\chi^0, \boldsymbol{\chi}) = f(\chi^0) e^{i\theta_f(\chi^0)} |\eta_\delta(|\boldsymbol{\chi} - \boldsymbol{x}|)| e^{2i\pi_0^+\chi^0}. \tag{3.29}$$

Here, f and θ_f are real functions that only depend on the reference clock χ^0. In addition, we consider the following relations between the peaking parameters $\epsilon^\pm$ and $\pi_0^\pm$ of the different sectors:

$$\epsilon^+ = \epsilon^-, \qquad \pi_0^+ = -\pi_0^-, \tag{3.30}$$

also entering assumption KC3. Besides simplifying the spacelike equations of motion at first order in perturbations, we show in Sec. IV A that the expression of the perturbed three-volume δV takes a manageable form under Eq. (3.30).

Within this set of choices, the peaking properties of the spacelike and timelike condensate wave functions yield the following equation of motion for $\delta\Psi$:

$$0 = \partial_0^2(\delta\Psi(J_0\bar{\tilde{\tau}} + f e^{i\theta_f}\bar{\tilde{\sigma}})) - 2i\tilde{\pi}_0^+\partial_0(\delta\Psi(J_0\bar{\tilde{\tau}} + f e^{i\theta_f}\bar{\tilde{\sigma}})) - E_+^2\delta\Psi(J_0\bar{\tilde{\tau}} + f e^{i\theta_f}\bar{\tilde{\sigma}}) + \alpha\bar{\tilde{\tau}}\nabla_{\boldsymbol{x}}^2\delta\Psi, \tag{3.31}$$

for which a detailed derivation is provided in Appendix D 2. All of the functions above depend on the peaked value of the matter momentum, p_ϕ, but we suppress that dependence for notational clarity. As written out explicitly before, the two reduced condensate wave functions $\tilde{\sigma}$ and $\tilde{\tau}$ depend on the relational time x^0 since they are part of the background, while in contrast, $\delta\Psi(x^\mu, p_\phi)$ depends on all four reference field values. The remaining coefficients $E_+(p_\phi)$, $\tilde{\pi}_0^+$, J_0 and α are defined in Appendix D 2 and are entirely determined by peaking parameters and the peaked matter momentum p_ϕ.

Solving the background equations of motion and inserting them in the first-order perturbation equation, one obtains an equation for $\delta\Psi$ which is of the general form

$$0 = \delta\Psi'' + t_1(x^0, p_\phi)\delta\Psi' + t_0(x^0, p_\phi)\delta\Psi + s_2(x^0, p_\phi)\nabla_{\boldsymbol{x}}^2\delta\Psi, \tag{3.32}$$

with complex functions $t_i(x^0, p_\phi)$ and $s_2(x^0, p_\phi)$. The conditions on these coefficients to yield GR-like perturbation equations are discussed in Sec. IV.

5. Timelike perturbed dynamics

As we will explicitly see in the next section, the dynamics of observables other than the spatial three volume, such as the matter field, its momentum, or the total number operator, are governed by the equations of motion of both sectors, spacelike and timelike. For this reason, in the following, we study the perturbed equations of motion on the timelike sector which, in spin representation, are given by

$$0 = \int d^4\chi d\phi' \mathcal{K}_-(|\boldsymbol{\chi}|^2, (\phi - \phi')^2) \times \int d^4\chi' d\phi'' \big[\delta\Psi(\chi^{0\prime}, \boldsymbol{\chi}', \phi', \chi^\mu + x^\mu, \phi'')\bar{\sigma}(\chi^{0\prime}, \phi'') + \delta\Xi(\chi^\mu + x^\mu, \phi', \chi^{0\prime}, \boldsymbol{\chi}', \phi'')\bar{\tau}(\chi^{\mu\prime}, \phi'')\big]. \tag{3.33}$$

Using the peaking properties of σ and τ, the locality condition in Eq. (3.27), the relation of peaking parameters in Eq. (3.30), as well as the classical background equations of motion in Eqs. (3.24) and (3.25), one obtains

$$0 = \bar{\tilde{\sigma}} \int d^3\chi \mathcal{K}_-(|\boldsymbol{\chi}|, p_\phi^2)\left[\partial_0^2\delta\Psi + 2\mu_+\partial_0\delta\Psi - \frac{\mathcal{K}_+^{(2)}}{\mathcal{K}_+^{(0)}}\delta\Psi\right] + \bar{\tilde{\tau}}\mathcal{K}_-^{(0)} J_{0,\boldsymbol{0}}[\partial_0^2\delta\Xi + 2\mu_-\partial_0\delta\Xi - \beta\delta\Xi + \gamma\nabla_{\boldsymbol{x}}^2\delta\Xi], \tag{3.34}$$

for which a derivation is given in Appendix D 2, including a definition of the parameters β and γ. Since the space dependence of the first term is integrated out, solutions $\delta\Xi$ need to be space independent, i.e., $\delta\Xi(x^\mu, p_\phi) \equiv \delta\Xi(x^0, p_\phi)$. Hence, the space derivative acting on $\delta\Xi$ vanishes, and the equation reduces to a second-order inhomogeneous ordinary differential equation. In particular, the pure time dependence of $\delta\Xi$ will have important consequences for the behavior of timelike particle number perturbations δN_-, discussed in detail in Sec. IV B.

IV. DYNAMICS OF OBSERVABLES

In the spirit of obtaining cosmology as a hydrodynamic limit of QG, classical cosmological quantities are associated with averages on the above condensate states of appropriate one-body observables defined within the GFT Fock space. Importantly, this can only hold under the assumption that quantum fluctuations of such observables on the states of interest are small. As emphasized in [63,116], this classicality requirement is automatically satisfied at late relational times. For this reason, in the following we will focus only on this regime.

Observables of interest for cosmological applications can be roughly divided into two categories: geometric observables (such as volume, area, curvature, etc.) and matter observables (such as the scalar field operators and their momenta). However, as one might expect, not all operators available in the quantum theory fall into these two categories. A particularly important example of "observables" that have no classical counterpart are number operators, i.e., operators that count the number of GFT (timelike and/or spacelike) quanta. In fact, the classical limit turns out to be associated with a large number of quanta in the condensate and is thus directly controlled by the above quantities.

More concretely, one can compute the expectation value of a second-quantized operator $\hat{\mathcal{O}}$ on the perturbed condensate states $|\Delta; x^0, \boldsymbol{x}\rangle$ by using the algebra of creation and annihilation operators. The result can be split generically as

$$\mathcal{O}(x^0, \boldsymbol{x}) \equiv \langle\Delta; x^0, \boldsymbol{x}|\hat{\mathcal{O}}|\Delta; x^0, \boldsymbol{x}\rangle = \bar{\mathcal{O}}(x^0) + \delta\mathcal{O}(x^0, \boldsymbol{x}), \tag{4.1}$$

where $\bar{\mathcal{O}}$ and $\delta\mathcal{O}$ are background and perturbed contributions, respectively. Note that, owing to the peaking properties of the states $|\Delta; x^0, \boldsymbol{x}\rangle$, the above expectation value is localized in relational space and time. In this sense, the quantities obtained are effective relational observables. As such, their dynamics should be compared, at least in an appropriate limit, with the dynamics of the corresponding classical cosmological relational observables. Since these are gauge-invariant extensions of gauge-fixed quantities, one could alternatively compare the dynamics of the above expectation values with the dynamics of the corresponding classical cosmological observables in harmonic gauge (since the physical frame used to localize quantities is in fact harmonic) [57,58,62,70], as derived in Appendix C.

In Secs. IV A–IV C, respectively, we will compute expectation values and dynamics of geometric, number, and matter operators. Then, in Sec. IV D we will use the results of the previous sections to derive the dynamics of an appropriate "curvature-like variable." Finally, in Sec. IV E, we will compare the effective GFT perturbation dynamics with the GR ones in harmonic gauge.

A. Volume operator and related geometric observables

Classically, the geometry of a flat, slightly inhomogeneous universe is characterized by the line element

$$\begin{aligned}\mathrm{d}s^2 = {}& -a^6(1+2A)\mathrm{d}t^2 + a^4\partial_i B\mathrm{d}t\mathrm{d}x^i \\ & + a^2((1-2\psi)\delta_{ij} + 2\partial_i\partial_j E)\mathrm{d}x^i\mathrm{d}x^j,\end{aligned} \tag{4.2}$$

where we have considered only scalar perturbations captured by the small functions A, B, ψ, and E. Note that, for the reasons explained above, Eq. (4.2) is written in harmonic gauge. At the background level, this means working in harmonic ($\bar{N}^2 = a^6$), rather than proper ($\bar{N}^2 = 1$) or conformal ($\bar{N}^2 = a^2$) time, while at the level of perturbations this forces the functions A, B, ψ, and E to satisfy the constraints (C13) (see Appendix C for more details).

Note that isotropic information on the spatial geometry is fully captured by the local three volume, $\sqrt{-g_{(3)}} = a^3(1 - 3\psi + \nabla^2 E)$. In this section, we will mainly focus on computing the expectation value of the corresponding three-volume GFT operator. Note that while this is clearly sufficient to characterize the full geometry of a homogeneous and isotropic universe [in particular, the Hubble parameter is $\mathscr{H} = \bar{V}'/(3\bar{V})$], it is a strong restriction at the perturbation level, as it only captures information related to a specific combination of ψ and E. To extract a full-fledged cosmological perturbation theory from GFTs, it is therefore imperative to construct more sophisticated geometric observables. This is an avenue of research that has been only tentatively explored [117,118]. We will return to this topic in Sec. V.

Using the choices on the perturbation functions $\delta\Phi$ and $\delta\Psi$ in Eqs. (3.27) and (3.28), respectively, the expectation value of $\hat{V}$ on the perturbed condensate state $|\Delta; x^0, \boldsymbol{x}\rangle$ is given by

$$\begin{aligned}\langle\Delta; x^0, \boldsymbol{x}|\hat{V}|\Delta; x^0, \boldsymbol{x}\rangle = {}& v\int \mathrm{d}^4\chi\mathrm{d}\pi_\phi\bar{\sigma}(\chi^0, \pi_\phi)\sigma(\chi^0, \pi_\phi) + 2v\mathfrak{Re}\left\{\int \mathrm{d}^4\chi\mathrm{d}\pi_\phi\delta\Psi(\chi^\mu, \pi_\phi)\bar{\sigma}(\chi^0, \pi_\phi)\bar{\tau}(\chi^\mu, \pi_\phi)\right\} \\ & + 2v\mathfrak{Re}\left\{\int \mathrm{d}^4\chi\mathrm{d}\pi_\phi\mathrm{f}(\chi^\mu)\delta\Psi(\chi^\mu, \pi_\phi)\bar{\sigma}(\chi^0, \pi_\phi)\bar{\sigma}(\chi^0, \pi_\phi)\right\},\end{aligned} \tag{4.3}$$

where v is similar to a volume eigenvalue [62,65], scaling as $\rho^{3/2}$. Applying Eq. (3.30) and exploiting the peaking properties of σ and τ, the volume expectation value becomes

$$\langle\Delta;x^0,\boldsymbol{x}|\hat{V}|\Delta;x^0,\boldsymbol{x}\rangle = v|\tilde{\sigma}(x^0,p_\phi)|^2\int \mathrm{d}^3\chi + 2v\mathfrak{Re}\left\{J_0\delta\Psi(x^\mu,p_\phi)\bar{\tilde{\sigma}}(x^0,p_\phi)\bar{\tilde{\tau}}(x^0,p_\phi)+\frac{J_2}{2}\bar{\tilde{\sigma}}(x^0,p_\phi)\bar{\tilde{\tau}}(x^0,p_\phi)\nabla^2_{\boldsymbol{x}}\delta\Psi(x^\mu,p_\phi)\right\}$$
$$+2v\mathfrak{Re}\{\delta\Psi(x^\mu,p_\phi)f(x^0)\mathrm{e}^{i\theta_f(x^0)}\bar{\tilde{\sigma}}(x^0,p_\phi)\bar{\tilde{\sigma}}(x^0,p_\phi)\}. \tag{4.4}$$

The first term, containing an empty rod integration (see assumption KS5), defines the background volume

$$\bar{V} = v|\tilde{\sigma}(x^0,p_\phi)|^2, \tag{4.5}$$

while the remaining two contributions make up the perturbations of the volume

$$\delta V(x^\mu,p_\phi) = 2v\mathfrak{Re}\{\delta\Psi(x^\mu,p_\phi)f(x^0)\mathrm{e}^{i\theta_f(x^0)}\bar{\tilde{\sigma}}(x^0,p_\phi)\bar{\tilde{\sigma}}(x^0,p_\phi)\}$$
$$+2v\mathfrak{Re}\left\{J_0\delta\Psi(x^\mu,p_\phi)\bar{\tilde{\sigma}}(x^0,p_\phi)\bar{\tilde{\tau}}(x^0,p_\phi)+\frac{J_2}{2}\bar{\tilde{\sigma}}(x^0,p_\phi)\bar{\tilde{\tau}}(x^0,p_\phi)\nabla^2_{\boldsymbol{x}}\delta\Psi(x^\mu,p_\phi)\right\}. \tag{4.6}$$

From the dynamics derived in Sec. III C, we can straightforwardly obtain the dynamics of the background and the perturbed averaged volume. This will be done in the next two paragraphs.

1. Background volume

At the level of the background, the expectation value of the spatial volume operator in a classical limit (see assumption DC1) is given by

$$\bar{V}(x^0,p_\phi) = \tilde{\sigma}_0^2\mathrm{e}^{2\mu_+x^0}. \tag{4.7}$$

Performing derivatives with respect to the clock field value x^0, $\bar{V}$ satisfies

$$\frac{\bar{V}'}{3\bar{V}} = \frac{2}{3}\mu_+(p_\phi), \qquad \left(\frac{\bar{V}'}{3\bar{V}}\right)' = 0. \tag{4.8}$$

In terms of the Hubble parameter $\mathscr{H} = \bar{V}'/(3\bar{V})$ in harmonic gauge, these equations read as

$$\mathscr{H}^2 = \frac{4}{9}\mu_+^2(p_\phi), \qquad \mathscr{H}' = 0. \tag{4.9}$$

In GR, the Hubble parameter in harmonic gauge for a universe filled with MCMF scalar fields is also constant. Thus, the GFT background dynamics matches the classical GR ones if (see Sec. C 1)

$$\mu_+^2(p_\phi) = \frac{3}{8M_{\mathrm{Pl}}^2}\bar{\pi}_\phi^2, \tag{4.10}$$

where $\bar{\pi}_\phi$ is the constant momentum of the matter field ϕ at background level.

The factor of the Planck mass in Eq. (4.10) present at the level of the classical Einstein equations [see Eq. (C3)] and thus needs to be accounted for in the matching procedure. Working with fields of energy dimension 1, $[\phi] = 1$, and thus with conjugate momenta of energy dimension 2, $[\pi_\phi] = 2$; the matching is therefore consistent with the fact that $[\mu_+]$ is required to be 1.

2. Volume perturbations

To study the dynamics of δV [defined in Eq. (4.6)] it is convenient to perform a split of the complex-valued function $\delta\Psi$ into its modulus and phase, $\delta\Psi = R(x^\mu,p_\phi)\mathrm{e}^{i\Theta}$. We pose the condition that this phase is in fact constant; see also assumption DC4.[7] As a result, the overall phases of the first and second term inside the real parts of δV are respectively given by

$$\theta_1 = \Theta + \theta_f(x^0) - 2\tilde{\pi}_0^+x^0, \tag{4.11}$$

$$\theta_2 = \Theta. \tag{4.12}$$

Exploiting once more the dynamical freedom on $\delta\Phi$, and thus on the function $\mathrm{f}(\chi^0,\boldsymbol{\chi})$ entering Eq. (3.28), we set $\theta_f = \frac{\pi}{2} + 2\tilde{\pi}_0^+x^0$. In momentum space of the rod variable, which we consider for the remainder of Sec. IV, the resulting form of δV is given by

$$\frac{\delta V(x^0,k)}{2v\tilde{\sigma}_0\tilde{\tau}_0} = \left[\cos(\Theta)\mathrm{e}^{(\mu_++\mu_-)x^0}\left(J_0 - \frac{J_2}{2}k^2\right)\right.$$
$$\left.+\sin(\Theta)\mathrm{e}^{2\mu_+x^0}\frac{\tilde{\sigma}_0}{\tilde{\tau}_0}f\right]R. \tag{4.13}$$

Put in this form, the perturbed volume δV is directly related to the modulus R by a time- and momentum-dependent factor A,

[7] Assuming instead a time-dependent phase and splitting the equation into a real and imaginary part, one finds $\Theta' = c/R^2$ with some time-dependent factor c. Since R is however space dependent and we require Θ to be only time dependent, the function c must vanish, and we conclude that Θ is in fact constant.

$$\frac{\delta V(x^0,k)}{2v\tilde{\sigma}_0\tilde{\tau}_0} =: A(x^0,k)R. \tag{4.14}$$

Therefore, the dynamics of δV are essentially governed by the dynamics of R, which we discuss next.

Introducing the function

$$g_f := (\tilde{\sigma}_0 f \mathrm{e}^{\mu_+ x^0} + J_0\tilde{\tau}_0 \mathrm{e}^{\mu_- x^0}), \tag{4.15}$$

the dynamics of $\delta\Psi = R\mathrm{e}^{i\Theta}$ for a constant phase Θ are given by

$$0 = R'' + 2\frac{g_f'}{g_f}R' + \left(\frac{g_f''}{g_f} - \mu_+^2\right)R - \frac{\alpha\tilde{\tau}_0 \mathrm{e}^{\mu_- x^0}}{g_f}k^2 R, \tag{4.16}$$

which straightforwardly follows from Eq. (3.31) and the derivations of Sec. III C 2. Combining Eqs. (4.13) and (4.16), the dynamical equation for the perturbed volume δV is given by

$$\delta V'' + \left[2\frac{g_f'}{g_f} - 2\frac{A'}{A}\right]\delta V' + \left[\frac{g_f''}{g_f} - \mu_+^2 + 2\left(\frac{A'}{A}\right)^2 - \frac{A''}{A} - 2\frac{g_f'}{g_f}\frac{A'}{A}\right]\delta V - \frac{\alpha\tilde{\tau}_0 \mathrm{e}^{\mu_- x^0}}{g_f}k^2\delta V = 0. \tag{4.17}$$

The above equation, and thus any solution of it, clearly depends on the function g_f encoding the aforementioned mean-field dynamical freedom. Remarkably, however, this freedom can be fixed entirely by requiring the above equation to take the same functional form (at least in the late time, classical regime) of the corresponding GR one, given in Eq. (C18).

To see this explicitly, we start from the spatial derivative term, whose prefactor a^4, as mentioned in Sec. I, could not be recovered by considering a perturbed condensate of only spacelike tetrahedra [70]. Exactly because of the additional timelike degrees of freedom, and thus of the above dynamical freedom, here we can easily recover the appropriate prefactor, by simply requiring the function g_f to satisfy

$$-\frac{\alpha\tilde{\tau}_0 \mathrm{e}^{\mu_- x^0}}{g_f} = a^4 = \tilde{\sigma}_0^{8/3}\mathrm{e}^{8\mu_+ x^0/3}, \tag{4.18}$$

where a is the scale factor. The above condition corresponds to the following choice of f:

$$f = -\frac{\tilde{\tau}_0}{\tilde{\sigma}_0}\mathrm{e}^{(\mu_- - \mu_+)x^0}(J_0 + \alpha a^{-4}), \tag{4.19}$$

fixing the aforementioned dynamical freedom completely (see assumption DC3).[8]

As a result of this fixing, the function g_f satisfies the following derivative properties:

$$\frac{g_f'}{g_f} = \mu_- - \frac{8}{3}\mu_+, \qquad \frac{g_f''}{g_f} = \left(\mu_- - \frac{8}{3}\mu_+\right)^2. \tag{4.20}$$

Inserting the expression of f into the function $A(x^0,k)$, one obtains

$$\frac{A'}{A} = \mu_+ + \mu_- + \frac{8}{3}\mu_+ \frac{\alpha\frac{\tilde{\tau}_0}{\tilde{\sigma}_0}\sin(\Theta)a^{-4}}{\cos(\Theta)(J_0 - \frac{J_2}{2}k^2) - \frac{\tilde{\tau}_0}{\tilde{\sigma}_0}\sin(\Theta)(J_0 + \alpha a^{-4})}. \tag{4.21}$$

As we see from the above equation, in general A is a complicated function of the momenta k. As a consequence, the same holds for the factors in front of $\delta V'$ and δV in Eq. (4.17). This is in sharp contrast to what happens in GR; see again Eq. (C18). However, this undesired k dependence can be easily removed by choosing $\Theta = n\frac{\pi}{2}$ with odd integer n and assuming that J_0 is negligible with respect to αa^{-4} (assumptions KC3, DC4, and DC5). Notice that this is equivalent to $(\delta\pi_x/\epsilon\pi_0^+)^2 a^{-4} \gg 1$ which is ensured by the condition $\pi_x \gg \pi_0^+$. Under these assumptions, the derivatives of A take the form

$$\frac{A'}{A} = -\frac{5}{3}\mu_+ + \mu_-, \qquad \frac{A''}{A} = \left(-\frac{5}{3}\mu_+ + \mu_-\right)^2. \tag{4.22}$$

Combining Eqs. (4.20) and (4.22), the perturbed volume equation attains the form

$$\delta V'' - 3\mathscr{H}\delta V' + a^4k^2\delta V = 0, \tag{4.23}$$

where we identified $\mathscr{H} = \frac{2}{3}\mu_+$ from the background equations. Expressed instead in terms of the ratio $\delta V/\bar{V}$, the relative perturbed volume equation is given by

$$\left(\frac{\delta V}{\bar{V}}\right)'' + 3\mathscr{H}\left(\frac{\delta V}{\bar{V}}\right)' + a^4k^2\left(\frac{\delta V}{\bar{V}}\right) = 0. \tag{4.24}$$

Remarkably, the two coefficients in front of the zeroth and first derivative term in Eq. (4.23) are both completely fixed by the background parameter μ_+.[9] In fact, the parameter μ_-, characterizing the behavior of the timelike condensate, does not enter the perturbed volume equation at all. Even

[8]Note that the initial conditions for scale factor are chosen such that the present day value at time x_*^0 is normalized, i.e., $a(x_*^0) = 1$. Therefore, $a < 1$ for all times $x < x_*^0$ and therefore, the volume factor a^{-4} in the equation above is not negligible.

[9]The values of these two coefficients are a direct consequence of matching the spatial derivative term. If the exponent of a is chosen to be $\lambda \in \mathbb{R}$ instead of 4, the first derivative coefficient is given by $-2\mu_+(2\lambda + 1)$. Since the a^4 factor is crucial for obtaining the appropriate behavior of perturbations, we fix $\lambda = 4$.

though the inclusion of a timelike condensate allows one to nicely match the functional form of Eq. (4.24) (in particular solving all the issues reported in [70]) with that of the GR Eq. (4.67), the GFT volume perturbation equation does show some new intriguing features, which we investigate in detail in Sec. IV E.

3. Scale factor observables

Before closing this section, we would like to emphasize that the cosmological equations obtained at the level of background and perturbations do not depend on the fact that we choose the three volume to encode the (scalar, isotropic) geometric information. In fact, due to homogeneity and isotropy at the level of the background condensate, one could heuristically consider some geometric observable $\mathcal{O}$ whose expectation value is associated to $|\tilde{\sigma}|^2$ at the background level and which would classically be interpreted as an appropriate power of the scale factor: $\langle\mathcal{O}\rangle \sim a^d$. Examples other than the volume would be length for $d = 1$ or area for $d = 2$, although we remark that here we do not attempt to provide a rigorous definition for these operators, but rather to offer heuristic insights based on their collective nature (i.e., their properties as second quantized GFT operators). Reiterating the same derivation as above but now with general d, the relation of μ_+ and $\mathscr{H}$ is given by

$$\mathscr{H} = \frac{\mathcal{O}'}{\mathcal{O}d} = \frac{2\mu_+}{d}. \tag{4.25}$$

At the perturbed level, the GFT equations would show exactly the same behavior as for the volume, namely,

$$\delta\mathcal{O}''_{\mathrm{GFT}} - d\mathscr{H}\delta\mathcal{O}'_{\mathrm{GFT}} + a^4k^2\delta\mathcal{O}_{\mathrm{GFT}} = 0. \tag{4.26}$$

Perturbations of the classical quantity, $\delta\mathcal{O}_{\mathrm{GR}}$, would instead be described by

$$\delta\mathcal{O}''_{\mathrm{GR}} - 2d\mathscr{H}\delta\mathcal{O}'_{\mathrm{GR}} + d^2\mathscr{H}^2\delta\mathcal{O}_{\mathrm{GR}} + a^4k^2\delta\mathcal{O}_{\mathrm{GR}} = 0, \tag{4.27}$$

which shows that the differences in the equations of δV from GFT and GR do not arise from considering the "wrong" geometric observable.

B. Number of quanta

The number operator, introduced in Eq. (2.13), is clearly the simplest second-quantized operator available in the Fock space. However, as we mentioned above, it is extremely important to characterize the classical and continuum limit of the QG system. On the two-sector Fock space, one can define individual number operators $\hat{N}_\alpha$, counting the number of spacelike and timelike tetrahedra, respectively, or the total number operator $\hat{N} = \hat{N}_+ + \hat{N}_-$. Forming the expectation value of $\hat{N}_\alpha$ with respect to $|\Delta; x^0, \boldsymbol{x}\rangle$, the contributions of the background and perturbations are respectively given by

$$\bar{N}_+ = |\tilde{\sigma}(x^0, p_\phi)|^2, \tag{4.28}$$

$$\bar{N}_- = |\tilde{\tau}(x^0, p_\phi)|^2, \tag{4.29}$$

and

$$\delta N_+ = 2\mathfrak{Re}\left\{\int \mathrm{d}^4\chi \mathrm{d}\pi_\phi \delta\Psi(\chi^\mu, \pi_\phi)\left[\bar{\sigma}(\chi^0, \pi_\phi)\bar{\tau}(\chi^\mu, \pi_\phi) + \mathrm{f}(\chi^\mu)\bar{\sigma}^2(\chi^0, \pi_\phi)\right]\right\}, \tag{4.30}$$

$$\delta N_- = 2\mathfrak{Re}\left\{\int \mathrm{d}^4\chi \mathrm{d}\pi_\phi \left[\delta\Psi(\chi^\mu, \pi_\phi)\bar{\sigma}(\chi^0, \pi_\phi)\bar{\tau}(\chi^\mu, \pi_\phi) + \delta\Xi(\chi^\mu, \pi_\phi)\bar{\tau}^2(\chi^\mu, \pi_\phi)\right]\right\}, \tag{4.31}$$

wherein a regularization of an empty rod integration entering $\bar{N}_+$ is understood. By considering a single-spin condensate, the number operator on the spacelike sector is directly related to the volume operator by the factor of v, $\hat{V} = v\hat{N}_+$. Therefore, the expectation values of $\hat{N}_+$ and $\hat{V}$ are related by v at every order of perturbations.

The expectation value of the timelike number operator will be particularly important in the following for two main reasons. First, the matching conditions of the volume perturbations that we derived in Sec. IV A can be interpreted as a condition on δN_-, detailed in the last paragraph below. Second, the number operators enter the expressions for the matter observables which we analyze in Sec. IV C.

1. Background: Spacelike sector

The number of spacelike tetrahedra $\bar{N}_+$ at background level satisfies

$$\frac{\bar{N}'_+}{\bar{N}_+} = 2\mu_+, \qquad \left(\frac{\bar{N}'_+}{\bar{N}_+}\right)' = 0, \tag{4.32}$$

where μ_+ is related to the scalar field momentum $\bar{\pi}_\phi$ as determined by Eq. (4.10). Following from the single-spin assumption, spacelike particle number and volume are proportional to each other. Thus, the exponential form of $\bar{N}_+$ reflects the form of the scale factor (or the three volume) which is exponential in the relational clock χ^0.

2. Background: Timelike sector

At the background level, the number of timelike tetrahedra satisfies the equations

$$\frac{\bar{N}'_-}{\bar{N}_-} = 2\mu_-, \qquad \left(\frac{\bar{N}'_-}{\bar{N}_-}\right)' = 0. \tag{4.33}$$

Since the spatial background geometry is fully determined by the spacelike condensate, there are *a priori* no matching conditions for the parameter μ_- with respect to an observable of classical GR. This is also due to a lack of GFT observables that characterize the geometry of timelike slices. Such observables could also help in deciding whether the timelike condensate state is actually sufficient to characterize a timelike slice in a spatially, but not temporally homogeneous setting. Further research might reveal additional constraints on μ_-, as we discuss in more detail in Sec. V.

We also remark that the exponential behavior of $\bar{N}_+$ and $\bar{N}_-$ guarantees expectation values of background observables to be sharply peaked, necessary for a classical interpretation. For further references on such a *classicalization*, see [63,110,116].

3. Perturbations: Spacelike sector

At first order of perturbations, δN_+ is related to δV by a constant factor of v. This implies in particular that

$$\frac{\delta N_+}{\bar{N}_+} = \frac{\delta V}{\bar{V}}. \tag{4.34}$$

Given the dynamics of $\delta V/\bar{V}$ in Eq. (4.23) after matching with GR, the ratio $\delta N_+/\bar{N}_+$ satisfies

$$\left(\frac{\delta N_+}{\bar{N}_+}\right)'' + 3\mathscr{H}\left(\frac{\delta N_+}{\bar{N}_+}\right)' + a^4k^2\left(\frac{\delta N_+}{\bar{N}_+}\right) = 0. \tag{4.35}$$

4. Perturbations: Timelike sector and interpretation of matching conditions

In Secs. III C 3 and IV A we introduced some important conditions on the perturbed condensate wave function. This allowed us to simplify the intricate equations of motions of $\delta\Psi$ and to match the GR functional form of the perturbed volume equations. Remarkably, there is a direct physical interpretation of these conditions in terms of the perturbed timelike particle number, which we detail in the following.

Considering δN_- in Eq. (4.31), we perform again a split of all the complex-valued quantities into a modulus and a phase. For the first term, $\delta\Psi\bar{\sigma}\,\tau$, the choice of peaking parameters in Eq. (3.30) leads to a phase which only consists of Θ. Therefore, by setting $\Theta = n\frac{\pi}{2}$ with n an odd integer, this contribution to δN_- vanishes with the remaining term being

$$\delta N_- = 2\mathfrak{Re}\left\{\int \mathrm{d}\chi^0 \delta\Xi(\chi^0,\pi_\phi)\bar{\tilde{\tau}}^2(\chi^0,\pi_\phi)\eta^2_{\epsilon^+}(\chi^0 - x^0;\pi_0^+)\right\}. \tag{4.36}$$

Since $\delta\Xi$ is only time dependent, as we have shown in Sec. III C 3, it follows that $\delta N_-(x^\mu, p_\phi) \equiv \delta N_-(x^0, p_\phi)$ only depends on the relational time. Thus, from a relational perspective, the perturbation of the timelike tetrahedra number can be absorbed into the background and does not contribute to the space-dependent first-order perturbations.

Although this triviality of perturbations in the number of timelike tetrahedra seems to suggest some sort of effective "irrelevance" of purely timelike correlations, we note that the above result is due to matching conditions imposed only on a spacelike operator (i.e., the volume). It is conceivable that by considering classical matching conditions on both spacelike and timelike observables, purely timelike correlations would play a more important role.

C. Dynamics of matter observables

In this section, we will derive the dynamics of the "matter" (i.e., the only nonframe) scalar field ϕ. Its classical relational dynamics is captured by the expectation values of suitably defined matter and momentum operators

$$\hat{\Phi}_\alpha = \frac{1}{i}\int \mathrm{d}g_v \mathrm{d}X_\alpha \mathrm{d}\chi^\mu \mathrm{d}\pi_\phi \hat{\varphi}^\dagger(g_v, X_\alpha, \chi^\mu, \pi_\phi) \times \frac{\delta}{\delta\pi_\phi}\hat{\varphi}(g_v, X_\alpha, \chi^\mu, \pi_\phi), \tag{4.37}$$

$$\hat{\Pi}^\alpha_\phi = \int \mathrm{d}g_v \mathrm{d}X_\alpha \mathrm{d}\chi^\mu \mathrm{d}\pi_\phi \hat{\varphi}^\dagger(g_v, X_\alpha, \chi^\mu, \pi_\phi)\pi_\phi\hat{\varphi}(g_v, X_\alpha, \chi^\mu, \pi_\phi). \tag{4.38}$$

Notice that the scalar field operator is denoted by $\hat{\Phi}$ which is not to be confused with the spacelike-spacelike perturbation $\widehat{\delta\Phi}$ and its function Φ. We also recall that, in contrast to the reference fields χ^μ, we do not assume *a priori* that the scalar field propagates only along dual edges of a certain causal character.

In perfect analogy with Secs. IV A and IV B, we separate the expectation value of the above operators on the condensate states $|\Delta; x^0, \boldsymbol{x}\rangle$ in background and perturbations. Expectation values of $\hat{\Phi}_\alpha$ at the background and perturbed level evaluate to

$$\bar{\Phi}_+ = \frac{1}{i}\bar{\tilde{\sigma}}(x^0, \pi_\phi)\frac{\delta}{\delta\pi_\phi}\tilde{\sigma}(x^0, \pi_\phi)\Big|_{\pi_\phi = p_\phi}, \tag{4.39}$$

$$\bar{\Phi}_- = \frac{1}{i}\bar{\tilde{\tau}}(x^0, \pi_\phi)\frac{\delta}{\delta\pi_\phi}\tilde{\tau}(x^0, \pi_\phi)\Big|_{\pi_\phi = p_\phi}, \tag{4.40}$$

and

$$\delta\Phi_+ = \frac{1}{i}\int \mathrm{d}^4\chi \mathrm{d}\pi_\phi \left[\bar{\sigma}\partial_{\pi_\phi}(\delta\Phi\bar{\sigma}) + \overline{\delta\Phi}\sigma\partial_{\pi_\phi}\sigma + \bar{\sigma}\partial_{\pi_\phi}(\delta\Psi\bar{\tau}) + \overline{\delta\Psi}\tau\partial_{\pi_\phi}\sigma\right], \tag{4.41}$$

$$\delta\Phi_- = \frac{1}{i}\int \mathrm{d}^4\chi \mathrm{d}\pi_\phi \left[\bar{\tau}\partial_{\pi_\phi}(\delta\Psi\bar{\sigma}) + \overline{\delta\Psi}\sigma\partial_{\pi_\phi}\tau + \bar{\tau}\partial_{\pi_\phi}(\delta\Xi\bar{\tau}) + \overline{\delta\Xi}\tau\partial_{\pi_\phi}\tau\right], \tag{4.42}$$

respectively. Since we work in momentum space while peaking on momentum, the operators $\hat{\Pi}_\phi^\alpha$ and $\hat{N}_\alpha$ are closely defined, and thus, the corresponding expectation values are simply given by

$$\bar{\Pi}_\phi^\alpha = \bar{N}_\alpha(x^0, p_\phi)p_\phi, \tag{4.43}$$

and

$$\delta\Pi_\phi^\alpha = p_\phi \delta N_\alpha. \tag{4.44}$$

In the following two paragraphs, we analyze the dynamics of these expectation values and suggest a matching to the quantities ϕ and π_ϕ of general relativity.

1. Background part

To compute $\bar{\Phi}_\alpha$, we recall the decomposition of the condensate wave functions into a radial and angular part, $r_\alpha(x^0,\pi_\phi)$ and $\theta_\alpha(x^0,\pi_\phi)$, respectively. Keeping only dominant contributions in r_α, one obtains

$$\bar{\Phi}_\alpha = \bar{N}_\alpha \partial_{\pi_\phi}\theta_\alpha\big|_{\pi_\phi=p_\phi}. \tag{4.45}$$

Solutions of the background phases θ_α are given by

$$\theta_\alpha = \tilde{\pi}^\alpha x^0 - \frac{Q_\alpha}{\mu_\alpha r_\alpha^2} + C_\alpha, \tag{4.46}$$

where Q_α and C_α are integration constants. Then, the zeroth order expectation value of $\hat{\Phi}_\alpha$ is given by

$$\bar{\Phi}_\alpha = -\partial_{\pi_\phi}\left(\frac{Q_\alpha}{\mu_\alpha}\right) + 2\frac{Q_\alpha}{\mu_\alpha r_\alpha^2}(\partial_{\pi_\phi}\mu_\alpha)x^0 + \bar{N}_\alpha\partial_{\pi_\phi}C_\alpha\bigg|_{\pi_\phi=p_\phi}. \tag{4.47}$$

As a consequence of the peaking properties of σ and τ, the timelike condensate parameter μ_- is independent of π_ϕ, i.e., $\partial_{\pi_\phi}\mu_- = 0$. If we choose in addition C_α to be independent of π_ϕ, $\bar{\Phi}_\alpha$ is an intensive quantity for both α, as one would expect for a scalar field:

$$\bar{\Phi}_+ = -\partial_{\pi_\phi}\left(\frac{Q_+}{\mu_+}\right) + 2\frac{Q_+}{\mu_+}(\partial_{\pi_\phi}\mu_+)x^0\bigg|_{\pi_\phi=p_\phi},$$
$$\bar{\Phi}_- = -\frac{1}{\mu_-}\partial_{\pi_\phi}Q_-\bigg|_{\pi_\phi=p_\phi}. \tag{4.48}$$

In order to connect these expectation values to the scalar field variable ϕ of GR, one needs to define a way to combine the expectation values Φ_α. To that end, we notice that the scalar field is intensive and canonically conjugate to the extensive quantity $\hat{\Pi}_\phi$. In analogy to the chemical potential in statistical physics, one possible way to combine Φ_+ and Φ_- is to consider the weighted sum

$$\phi = \Phi_+\frac{N_+}{N} + \Phi_-\frac{N_-}{N}, \tag{4.49}$$

where all the quantities appearing are the full expectation values, containing zeroth- and first-order terms. N denotes the expectation value of the total number of GFT particles, i.e., $N = N_+ + N_-$. Expanding all the quantities to linear order, we identify the background scalar field as

$$\bar{\phi} = \bar{\Phi}_+\frac{\bar{N}_+}{\bar{N}} + \bar{\Phi}_-\frac{\bar{N}_-}{\bar{N}}. \tag{4.50}$$

Assuming that $\bar{N}_+ \gg \bar{N}_-$ at late times, corresponding to $\mu_+ > \mu_-$ (see also assumption DC6) and reflecting that the background is predominantly characterized by the spatial geometry, the matter field can be approximated as

$$\bar{\phi} \approx \bar{\Phi}_+. \tag{4.51}$$

Using Eq. (4.48), we see that the scalar field is linear in relational time, as expected classically. Thus, we can easily match the classical GR background equations for $\bar{\phi}$: imposing $Q_+ = \pi_\phi^2$, yields

$$\bar{\phi}' = p_\phi, \qquad \bar{\phi}'' = 0, \tag{4.52}$$

as required. Besides the relation $\mu_+ > \mu_-$, the background matching does not impose any further conditions on Q_- and the precise form of μ_-.

For Π_ϕ^α, we notice that this quantity grows with the system size, given by the respective number of tetrahedra

$\bar{N}_\alpha$. At lowest order, we therefore identify the classical quantity $\bar{\pi}_\phi$ as

$$\bar{\pi}_\phi = \frac{\bar{\Pi}_\phi^+ + \bar{\Pi}_\phi^-}{\bar{N}} = \frac{\bar{N}_+ + \bar{N}_-}{\bar{N}_+ + \bar{N}_-} p_\phi = p_\phi, \qquad (4.53)$$

which corresponds to the peaked matter momentum p_ϕ. With this identification, the GFT parameter μ_+ can be expressed by the peaked matter momentum as

$$M_{\mathrm{Pl}}^2 \mu_+^2(p_\phi) = \frac{8}{3}\bar{\pi}_\phi^2 = \frac{8}{3}p_\phi^2, \qquad (4.54)$$

where again a factor of Planck mass has been added to ensure the correct energy dimensions.

2. First-order perturbations

Given the expectation values $\delta\Phi_\alpha$ in Eqs. (4.41) and (4.42), we perform a partial integration in π_ϕ and only keep dominating terms, yielding

$$\delta\Phi_+ = 2\mathfrak{Re}\left\{\int \mathrm{d}^4\chi \mathrm{d}\pi_\phi \left[\delta\Phi\bar{\sigma}^2\partial_{\pi_\phi}\theta_+ + \delta\Psi\bar{\tau}^2\partial_{\pi_\phi}\theta_+\right]\right\}, \qquad (4.55)$$

$$\delta\Phi_- = 2\mathfrak{Re}\left\{\int \mathrm{d}^4\chi \mathrm{d}\pi_\phi \left[\delta\Psi\bar{\sigma}^2\partial_{\pi_\phi}\theta_- + \delta\Xi\bar{\tau}^2\partial_{\pi_\phi}\theta_-\right]\right\}. \qquad (4.56)$$

Using the relation of $\delta\Phi$ and $\delta\Psi$ in Eq. (3.28), as well as the assumptions on the peaking parameters of σ and τ in Eq. (3.30), the first-order expectation value $\delta\Phi_+$ evaluates to

$$\delta\Phi_+ = \delta N_+(x^\mu, \pi_\phi)\partial_{\pi_\phi}\theta_+\Big|_{\pi_\phi = p_\phi} = \frac{\delta N_+}{\bar{N}_+}\bar{\phi}. \qquad (4.57)$$

In contrast to $\delta\Phi_+$, the evaluation of $\delta\Phi_-$ is more intricate since the peaking properties of $\bar{\tau}^2$ yield a time derivative expansion when integrating over the reference field. However, as we show next, the perturbed scalar field $\delta\phi$ does not explicitly depend on $\delta\Phi_-$ under the assumption that $\mu_+ > \mu_-$. Following the definition of ϕ in Eq. (4.49), at linear order in perturbations, one obtains

$$\delta\phi \approx \bar{\phi}\left(\frac{\delta N_+ - \delta N_-}{\bar{N}_+}\right) + \bar{\Phi}_- \frac{\delta N_-}{\bar{N}_+}. \qquad (4.58)$$

Since the timelike number perturbation δN_- is only time dependent, and therefore part of the background, the factors of $\delta N_-/\bar{N}_+$ are negligible, and one is left with

$$\delta\phi = \left(\frac{\delta V}{\bar{V}}\right)\bar{\phi}. \qquad (4.59)$$

Applying Eqs. (4.24) and (4.52) for $\delta V/\bar{V}$ and $\bar{\phi}$, respectively, the dynamical equation for $\delta\phi$ from GFT is given by

$$\delta\phi'' + a^4k^2\delta\phi = (-3\mathscr{H}\bar{\phi} + 2\bar{\phi}')\left(\frac{\delta V}{\bar{V}}\right)'. \qquad (4.60)$$

Notice that the right-hand side of this partial differential equation constitutes a source term that is absent in the classical equation of ϕ_{GR}, given in Eq. (C28), formulated in harmonic gauge. We discuss the different features of GFT and GR solutions in Sec. IV E.

Let us consider now the first-order matter momentum variable $\delta\Pi_\phi^\alpha$ which, as for the background variable, scales with the system size. In order to connect this quantity to the intrinsic quantity $\delta\pi_\phi$ of GR, dividing $\delta\Pi_\phi^\alpha$ by the particle number is required. In principle, there are two different ways to do so, both of which we present in the following.

First, one can define $\delta\pi_\phi$ as the first-order term of

$$\delta\pi_\phi \overset{(1)}{=} \frac{\Pi_\phi^+ + \Pi_\phi^-}{N_+ + N_-} = 0, \qquad (4.61)$$

where all the quantities entering this expression contain both, zeroth- and first-order perturbations. However, in this case $\delta\pi_\phi = 0$. Operatively, this could be interpreted as a perturbation of the background momentum $\bar{\pi}_\phi$. Since this is a constant of motion, any such perturbation would vanish by construction.

Alternatively, one could perturb only the momenta and keep the particle numbers at zeroth order. In this case, $\delta\pi_\phi$ is given by

$$\delta\pi_\phi \overset{(2)}{=} \frac{\delta\Pi_\phi^+ + \delta\Pi_\phi^-}{\bar{N}_+ + \bar{N}_-} \approx p_\phi \frac{\delta N_+}{\bar{N}_+} = p_\phi \frac{\delta V}{\bar{V}}. \qquad (4.62)$$

None of the options above offer a matching to the classical perturbed momentum variable $\delta\pi_\phi^0$, defined in Eq. (C33) as the 0 component of the conjugate momentum of ϕ at linear order. The main difficulty in matching these two quantities is that the classical equation (C33) depends on the perturbation of the lapse function, A. To recover this quantity from the fundamental QG theory, one would need additional (relational) geometric operators other than the volume. We will return to this issue when in Sec. V.

D. A Mukhanov-Sasaki-like equation

In classical cosmology, physical information is encoded in perturbatively gauge-invariant quantities (see [119–124] for a review), such as the Bardeen [125,126] and the curvature perturbation variables ζ [126,127] and $\mathcal{R}$ [128], defined in Eq. (C38). The latter, usually called comoving curvature perturbation, is especially important in inflationary physics, being proportional to the so-called Mukhanov-Sasaki variable [129–131].

However, as discussed in Appendix C, $\mathcal{R}$ cannot be constructed out of volume and matter observables only. That is because, as emphasized already at the beginning of Sec. IV A, the volume $\delta V/\bar{V}$ is composed of both perturbation functions ψ and E. In order to single out the function E and identifying it with expectation values of GFT operators, one would have to relax isotropy (see assumption KS2) and introduce anisotropic observables, such as the areas of orthogonal two surfaces. Until such operators are defined, the importance of which we highlight in Sec. V B, we introduce a "curvature-like" variable $\tilde{\mathcal{R}}$,

$$\tilde{\mathcal{R}} \coloneqq -\frac{\delta V}{3\bar{V}} + \mathscr{H}\frac{\delta\phi}{\bar{\phi}'}, \tag{4.63}$$

in analogy to $\mathcal{R}$. As remarked above, classically, $\mathcal{R}$ is gauge invariant under infinitesimal transformations $x^\mu \mapsto x^\mu + \xi^\mu$. In contrast, $\tilde{\mathcal{R}}$, as defined above [see also Eq. (C40)], is classically gauge invariant only in the superhorizon limit. In the context of GFT however, the quantity $\tilde{\mathcal{R}}_{\mathrm{GFT}}$ defined above is obtained by combining effectively relational observables (obtained via averages on CPSs), and thus it is (effectively) gauge invariant by construction. Still, as for the volume and matter observables, its dynamics can be directly compared with that of GR in harmonic gauge.

Applying the GFT dynamics for $\delta V/\bar{V}$ and $\delta\phi$ to $\tilde{\mathcal{R}}$, given in Eqs. (4.24) and (4.60), respectively, yields

$$\tilde{\mathcal{R}}'' + a^4k^2\tilde{\mathcal{R}} = \left[3\mathscr{H} - \frac{1}{4M_{\mathrm{Pl}}^2}(\bar{\phi}^2)'\right]\left(\frac{\delta V}{\bar{V}}\right)'. \tag{4.64}$$

Similar to the perturbed matter equation from GFT, the differential equation for $\tilde{\mathcal{R}}$ contains a source term on the right-hand side that is not present in the classical case, Eq. (C41). We explicitly compare the solutions of $\tilde{\mathcal{R}}_{\mathrm{GR}}$ and $\tilde{\mathcal{R}}_{\mathrm{GFT}}$ in Sec. IV E, showing that the discrepancies are negligible under certain assumptions on the initial conditions.

We close this section by presenting a different expression for the Mukhanov-Sasaki-like equation which is closer to standard cosmology formulations. As mentioned above, since Eq. (4.64) is expressed in a fully relational fashion using the physical scalar reference frame, it is most naturally compared to the GR-equation formulated in harmonic coordinates, given in Eq. (C41). However, in standard cosmology, it is common practice to use conformal-longitudinal coordinates instead. Since we work in a manifestly coordinate-independent setting, we need to introduce a parametrization of the reference fields to make the connection to this representation in GFT. Adapted to the scalar reference frame, we introduce harmonic coordinates and then change to the conformal-longitudinal system (see Appendix C 3 for more details). As a relational quantity, $\tilde{\mathcal{R}}_{\mathrm{GFT}}$ is manifestly gauge invariant and thus behaves as a scalar under the parametrization change. Moreover, as it is a first-order quantity, only the background change from harmonic time to conformal time τ matters, yielding

$$\frac{\mathrm{d}^2\tilde{\mathcal{R}}}{\mathrm{d}\tau^2} + 2\mathcal{H}\frac{\mathrm{d}\tilde{\mathcal{R}}}{\mathrm{d}\tau} + k^2\tilde{\mathcal{R}} = \left[3\mathcal{H} - \frac{1}{4M_{\mathrm{Pl}}^2}\frac{\mathrm{d}}{\mathrm{d}\tau}(\bar{\phi}^2)\right]\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\delta V}{\bar{V}}\right), \tag{4.65}$$

where $\mathcal{H}$ is the Hubble parameter with respect to conformal time.

E. Solutions of GFT and GR perturbations

In this section, we provide a direct comparison of emergent perturbations from GFT and classical perturbations from GR. In particular, we study solutions of the GFT and GR equations to see how the differences in the respective differential equations are reflected in their solutions. This allows one to determine conditions on initial values under which the GFT curvaturelike variable $\tilde{\mathcal{R}}_{\mathrm{GFT}}$ shows good agreement with $\tilde{\mathcal{R}}_{\mathrm{GR}}$ even for intermediate and subhorizon modes $a^2k \gtrsim \mathscr{H}$.

1. Volume perturbations

In momentum space for the rod variable, the dynamics of the relative volume perturbation $\delta V/\bar{V}$ are captured by

$$\left(\frac{\delta V}{\bar{V}}\right)''_{\mathrm{GFT}} + a^4k^2\left(\frac{\delta V}{\bar{V}}\right)_{\mathrm{GFT}} = -3\mathscr{H}\left(\frac{\delta V}{\bar{V}}\right)'_{\mathrm{GFT}}, \tag{4.66}$$

$$\left(\frac{\delta V}{\bar{V}}\right)''_{\mathrm{GR}} + a^4k^2\left(\frac{\delta V}{\bar{V}}\right)_{\mathrm{GR}} = 0, \tag{4.67}$$

for GFT and GR, respectively. A derivation of the classical equation is given explicitly in Appendix C 1. These equations can be analytically solved, yielding

$$\left(\frac{\delta V}{\bar{V}}\right)_{\mathrm{GFT}} = e^{-3\mathscr{H}x^0/2}\left[c_1^{\mathrm{GFT}}J_{-3/4}\left(\frac{a^2k}{2\mathscr{H}}\right) + c_2^{\mathrm{GFT}}J_{3/4}\left(\frac{a^2k}{2\mathscr{H}}\right)\right], \tag{4.68}$$

$$\left(\frac{\delta V}{\bar{V}}\right)_{\mathrm{GR}} = c_1^{\mathrm{GR}}J_0\left(\frac{a^2k}{2\mathscr{H}}\right) + c_2^{\mathrm{GR}}Y_0\left(\frac{a^2k}{2\mathscr{H}}\right), \tag{4.69}$$

where J_n and Y_n are Bessel functions of the first and second kind, respectively. Requiring that classical GR perturbations are constant in the superhorizon regime $a^2k/\mathscr{H} \ll 1$ imposes the condition $c_2^{\mathrm{GR}} = 0$. Matching the GFT perturbations in the superhorizon limit, one obtains the two conditions

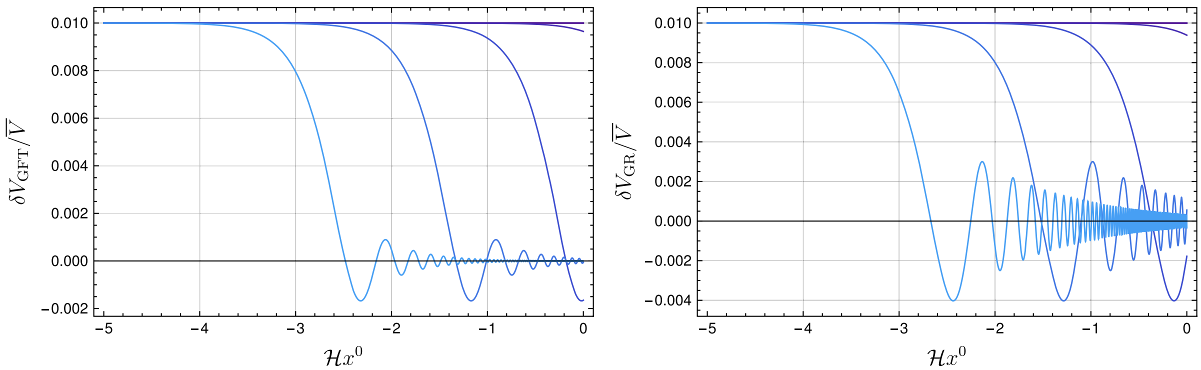

FIG. 2. Evolution of relative volume perturbations $\delta V/\bar{V}$ as dictated by GFT, respectively GR. Here, we set the integration constant $c_1^{(\rm GR)} = 10^{-2}$. Notice, that the quantities $(x^0, \mathscr{H})$ and $(k, \mathscr{H})$ only appear as a product, respectively a quotient, setting the two scales for the system $\mathscr{H}x^0$ and $k/\mathscr{H}$. The drawn momenta lie in the range $k/\mathscr{H} \in \{10^{-3}, \dots, 10^2\}$, where lighter blue corresponds to larger modes. Qualitatively, this range captures superhorizon, intermediate, and subhorizon modes, respectively.

$$c_1^{\rm GFT} = 0, \qquad c_2^{\rm GFT} = \left(4\frac{\mathscr{H}}{k}\right)^{3/4}\Gamma(7/4)c_1^{\rm GR}. \tag{4.70}$$

With these choices of initial conditions, we thus have

$$\left(\frac{\delta V}{\bar{V}}\right)_{\rm GFT} = c_1^{\rm GR}\Gamma\left(\frac{7}{4}\right)\left(\frac{4\mathscr{H}}{k}\right)^{3/4}\mathrm{e}^{-3\mathscr{H}x^0/2}J_{3/4}\left(\frac{a^2k}{2\mathscr{H}}\right), \tag{4.71}$$

$$\left(\frac{\delta V}{\bar{V}}\right)_{\rm GR} = c_1^{\rm GR}J_0\left(\frac{a^2k}{2\mathscr{H}}\right), \tag{4.72}$$

depicted in Fig. 2 for $c_1^{\rm GR} = 0.01$. By assumption, the perturbations are smaller than the background volume, and thus, $c_1^{\rm GR}$ is required to be smaller than 1. As we are going to show in the following paragraphs, this is consistent with finding good matching of matter and curvaturelike perturbations.

The deviations between GFT and GR volume perturbations are twofold, and become relevant for modes $a^2k/\mathscr{H} \gtrsim 1$. First, as the perturbations cross the Hubble horizon, the ratio $(\delta V/\bar{V})_{\rm GFT}$ is more strongly suppressed by an additional exponential factor of $\mathrm{e}^{-3\mathscr{H}x^0/2}$. Second, the phase of the two functions is shifted. This becomes apparent by considering an asymptotic expansion of $\delta V/\bar{V}$ in terms of large modes $a^2k/\mathscr{H} \gg 1$:

$$\left(\frac{\delta V}{\bar{V}}\right)_{\rm GFT} \xrightarrow[a^2k/\mathscr{H}\gg 1]{} \sqrt{\pi}c_1^{\rm GR}\Gamma\left(\frac{7}{4}\right)\left(\frac{4\mathscr{H}}{k}\right)^{5/4}\mathrm{e}^{-5\mathscr{H}x^0/2}\sin\left(\frac{\pi}{8} - \frac{a^2k}{2\mathscr{H}}\right), \tag{4.73}$$

$$\left(\frac{\delta V}{\bar{V}}\right)_{\rm GR} \xrightarrow[a^2k/\mathscr{H}\gg 1]{} \sqrt{\pi}c_1^{\rm GR}\left(\frac{4\mathscr{H}}{k}\right)^{1/2}\mathrm{e}^{-\mathscr{H}x^0}\cos\left(\frac{\pi}{4} - \frac{a^2k}{2\mathscr{H}}\right). \tag{4.74}$$

We will give a physical interpretation for this subhorizon deviations at the end of this section.

The initial conditions have been chosen so that the matching of $\delta V_{\rm GFT}/\bar{V}$ and $\delta V_{\rm GR}/\bar{V}$ holds for superhorizon modes. Attempting a matching with different initial conditions naturally leads to matching conditions which are inconsistent under relational time evolution. More precisely, matching the two perturbations at a certain time x_*^0 and at a certain scale $k/\mathscr{H}$ which is not superhorizon, the volume perturbations will only match at this instance of time and show strong deviations for all $x^0 \neq x_*^0$. Therefore, we conclude that one obtains the closest matching with initial conditions assigned in the superhorizon regime, which is in fact common practice in standard cosmology [119–121,123,124]. In the following paragraphs, we will therefore assume that $\delta V/\bar{V}$ is given by Eqs. (4.71) and (4.72) for GFT and GR, respectively.

2. Matter perturbations

Matter perturbations in GFT and GR are respectively governed by

$$\delta\phi''_{\rm GFT} + a^4k^2\delta\phi_{\rm GFT} = (-3\mathscr{H}\bar{\phi} + 2\bar{\phi}')\left(\frac{\delta V}{\bar{V}}\right)'_{\rm GFT}, \tag{4.75}$$

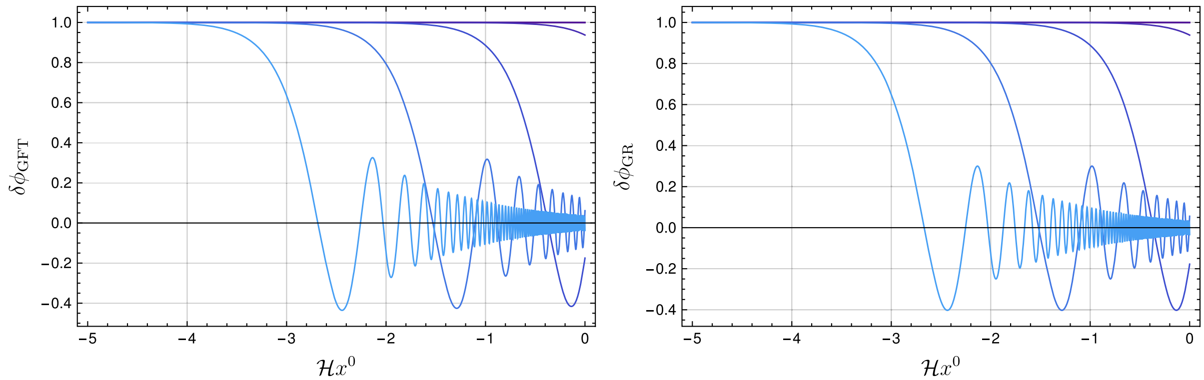

FIG. 3. Solutions of matter perturbations derived from GFT and GR, respectively, where we set $d_1^{\rm GR} = 1$. In this plot, the modes $k/\mathscr{H}$ lie again in the set $\{10^{-3}, \ldots, 10^2\}$ with the same color coding as for δV. Furthermore, the initial condition of the relative volume perturbations is fixed to $c_1^{\rm GR} = 10^{-2}$.

$$\delta\phi''_{\rm GR} + a^4k^2\delta\phi_{\rm GR} = 0, \tag{4.76}$$

where we refer to Appendix C 2 for a derivation of the classical equation. While the GR matter equation forms a closed system, the matter perturbation equation of GFT does not close, since the relative volume perturbation enters as a source term on the right-hand side. However, as we show in this paragraph, these inhomogeneities are controlled by the parameter $c_1^{\rm GR}$. To make this statement explicit, notice that the solution of $\delta\phi_{\rm GR}$ is given by

$$\delta\phi_{\rm GR}(x^0, k) = d_1^{\rm GR} J_0\left(\frac{a^2k}{2\mathscr{H}}\right). \tag{4.77}$$

For simplicity, we set $d_1^{\rm GR} = 1$ in the remainder, such that the difference of initial conditions for $\delta\phi$ and $\delta V/\bar{V}$ is captured by $c_1^{\rm GR}$. The differential equation for $\delta\phi_{\rm GFT}$ is solved numerically. Given initial conditions $\delta\phi_{\rm GFT}(-5) = d_1^{\rm GR}$ and $\delta\phi'_{\rm GFT}(-5) = 0$, the dynamics of GFT and GR matter perturbations show good agreement, as Fig. 3 visualizes. A more direct comparison of the two perturbations for a fixed mode, say $k/\mathscr{H} = 10^3$, and for varying constant $c_1^{\rm GR}$ is given in Fig. 4.

3. Curvaturelike perturbations

Similar to the matter perturbations, the equations for the curvaturelike perturbation $\tilde{\mathcal{R}}$

$$\tilde{\mathcal{R}}''_{\rm GFT} + a^4k^2\tilde{\mathcal{R}}_{\rm GFT} = \left[3\mathscr{H} - \frac{1}{4M_{\rm Pl}^2}(\bar{\phi}^2)'\right]\left(\frac{\delta V}{\bar{V}}\right)'_{\rm GFT}, \tag{4.78}$$

$$\tilde{\mathcal{R}}''_{\rm GR} + a^4k^2\tilde{\mathcal{R}}_{\rm GR} = 0, \tag{4.79}$$

derived from GFT and GR, respectively, differ in a source term on the right-hand side of the GFT equation. Since $\tilde{\mathcal{R}}$ is by definition a linear combination of the volume perturbation ratio $\delta V/\bar{V}$ and the matter perturbation $\delta\phi$, the initial conditions must be chosen accordingly.

With the choices for $\delta V_{\rm GR}/\bar{V}$ and $\delta\phi_{\rm GR}$ of the previous paragraphs, being $d_1^{\rm GR} = 1$ and $c_1^{\rm GR}$ not fixed, the curvaturelike perturbation equation of GR is solved by

$$\tilde{\mathcal{R}}_{\rm GR} = \left(\frac{1}{\sqrt{6}} - \frac{c_1^{\rm GR}}{3}\right) J_0\left(\frac{a^2k}{2\mathscr{H}}\right). \tag{4.80}$$

It is important to notice that $c_1^{\rm GR}$ enters the expression of the classical quantity $\tilde{\mathcal{R}}_{\rm GR}$ explicitly, in contrast to the classical matter perturbations $\delta\phi_{\rm GR}$. To find the behavior of $\tilde{\mathcal{R}}_{\rm GFT}$, we solve its governing differential equation numerically, with initial conditions $\tilde{\mathcal{R}}(-5) = \frac{1}{\sqrt{6}} - \frac{c_1^{\rm GR}}{3}$ and $\tilde{\mathcal{R}}'(-5) = 0$. The result is depicted on the left-hand side of Fig. 5, next to the analytical solution of GR.

Matching of GFT and GR solutions is controlled by the parameter $c_1^{\rm GR}$, which now enters both $\tilde{\mathcal{R}}_{\rm GR}$ and $\tilde{\mathcal{R}}_{\rm GFT}$. A comparison of the classical and the GFT solutions for a fixed mode $k/\mathscr{H} = 10^3$ and different values of $c_1^{\rm GR}$ is depicted in Fig. 6. As explained above, $\tilde{\mathcal{R}}_{\rm GFT}$ is gauge

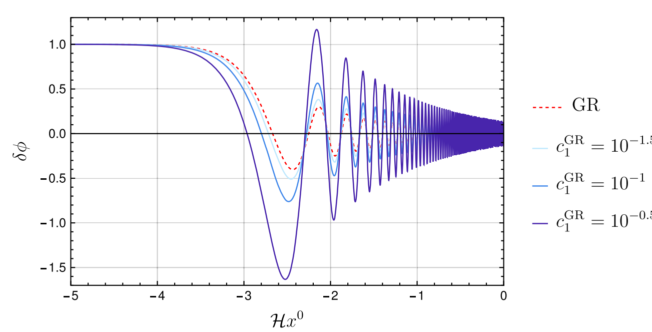

FIG. 4. For fixed mode $k/\mathscr{H} = 10^3$, this plot shows a comparison of the GFT (blue) and GR (dashed red) solutions depending on the value of $c_1^{\rm GR}$. Clearly, the two solutions closely agree for small $c_1^{\rm GR}$.

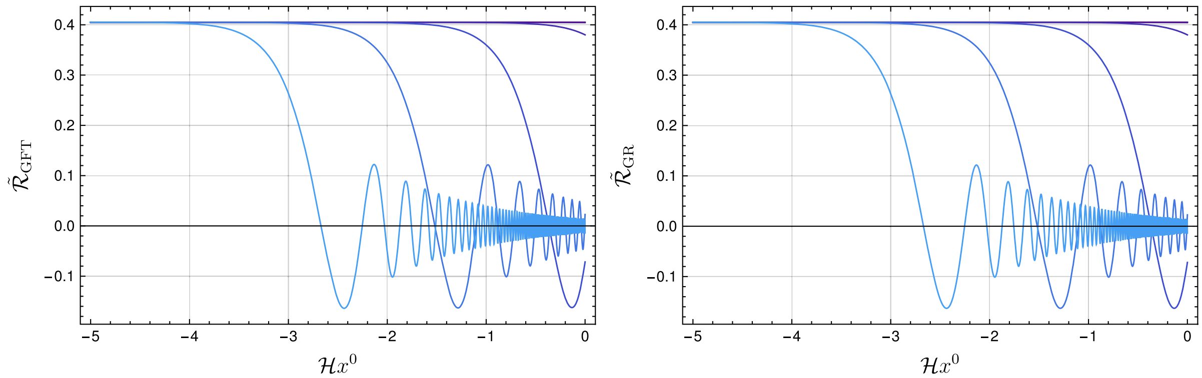

FIG. 5. Solutions of curvaturelike perturbations $\tilde{\mathcal{R}}$ in GFT and GR, respectively, where the integration constant is fixed to $c_1^{\mathrm{GR}} = 1$. The range of modes and the associated coloring in the plots are as for δV and $\delta\phi$ above.

invariant as a relational quantity, while $\tilde{\mathcal{R}}_{\mathrm{GR}}$ changes under gauge transformations as $\tilde{\mathcal{R}}_{\mathrm{GR}} \mapsto \tilde{\mathcal{R}}_{\mathrm{GR}} + k^2\xi$. One can exploit this gauge freedom on the GR side to improve the matching for a fixed mode. However, a consistent improvement of matching between $\tilde{\mathcal{R}}_{\mathrm{GFT}}$ and $\tilde{\mathcal{R}}_{\mathrm{GR}}$ by choosing a certain ξ is not possible for all modes. Therefore, there does not exist a gauge where the matching of GFT and GR perturbations is perfect at all modes.

Summarizing, the GFT perturbation equations for $\delta V/\bar{V}$, $\delta\phi$ and $\tilde{\mathcal{R}}$ do not close in that the relative volume perturbation enters the equations for $\delta\phi$ and $\tilde{\mathcal{R}}$ as a source term. While solutions for $\delta V/\bar{V}$ can be fully matched only in the superhorizon limit, both $\delta\phi$ and $\tilde{\mathcal{R}}$ can be matched for a wide range of modes by requiring the ratio $c_1^{\mathrm{GR}}/d_1^{\mathrm{GR}}$ to be a small value (as required for the self-consistency of the perturbative setting).

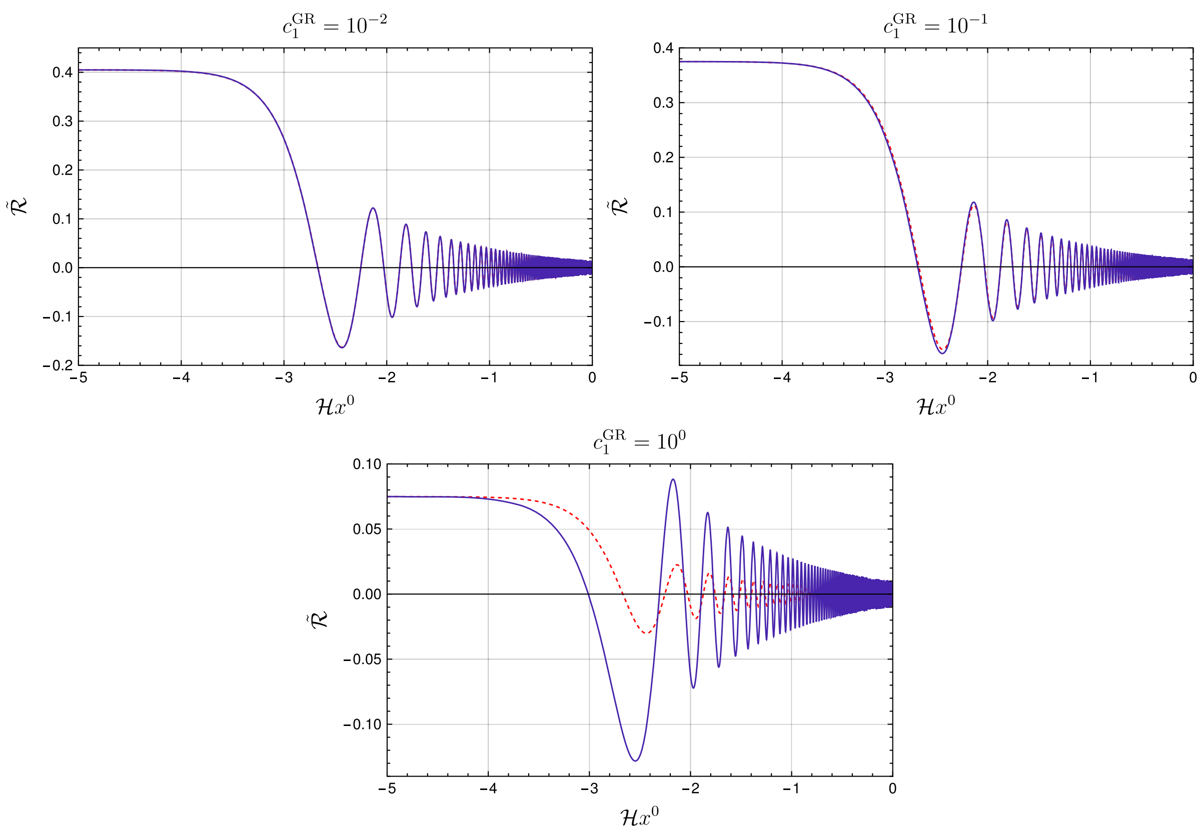

FIG. 6. Comparison of classical (dashed red) and GFT solution (blue) for different values of c_1^{GR} at a fixed mode $k/\mathscr{H} = 10^3$. For $c_1^{\mathrm{GR}} = 10^{-2}$, the curves show almost perfect matching and are therefore visually indistinguishable.

It should be emphasized that under the above conditions, the mismatch between GFT and GR solutions is in general only important for trans-Planckian modes, as explicitly shown in [132]. For smaller modes (and in particular for any mode of cosmological interest), the GFT perturbation dynamics are perfectly consistent with those of GR [132]. However, as we emphasized above, the GFT and GR dynamics differ only in the subhorizon regime. This is because, as we argue below, there is a natural correspondence between the subhorizon $a^2k/\mathscr{H} \gg 1$ and trans-Planckian modes when the cosmological system is relationally described in terms of a physical reference frame consisting of four massless scalar fields.

4. Subhorizon and trans-Planckian modes

Particularly important in this paragraph, we work in units of $8\pi G = 1$ and explicitly keep track of factors of Planck masses. Starting this classical analysis, we introduce harmonic coordinates $\chi^\mu = \kappa^\mu x^\mu$ (no summation over μ) [70], with κ^μ dimensionful constants. As explained in Appendix C, harmonic coordinates are, at the background, defined by the harmonic gauge condition $a^3/N = c_H =$ const. In this paragraph, we keep the constant c_H arbitrary and set it to unity otherwise. Then, the energy momentum tensor, defined in Eq. (C24), is given by

$$M_{\rm Pl}^2 T_{\mu\nu} = \sum_{\lambda=0}^{3} (\kappa^\lambda)^2 \left(\delta_\mu^\lambda \delta_\nu^\lambda - \frac{g_{\mu\nu}}{2} g^{\rho\sigma} \delta_\rho^\lambda \delta_\sigma^\lambda \right) + \partial_\mu \phi \partial_\nu \phi - \frac{g_{\mu\nu}}{2} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi, \tag{4.81}$$

with background (00) component

$$M_{\rm Pl}^2 \bar{T}_{00} = \frac{1}{2}\left[(\kappa^0)^2 + (\bar{\phi}')^2 + 3a^4 \frac{\kappa^2}{c_H^2} \right]. \tag{4.82}$$

which can be expressed equivalently via the canonical conjugate momenta as

$$M_{\rm Pl}^2 \bar{T}_{00} = \frac{1}{2}\left[\left(\frac{\pi_\phi}{c_H}\right)^2 + \left(\frac{\pi_{\chi^0}}{c_H}\right)^2 + 3\frac{(\pi_{\chi^i})^2}{a^4} \right]. \tag{4.83}$$

Notice that we set $\kappa \equiv \kappa^i$, motivated by relational isotropy (see assumption KS2).[10] Because of the particular alignment of coordinates and reference frame, the constants $\partial_\mu \kappa^i x^i$ already enter at the background level. Notice, that this strongly differs from cosmology in coordinates, where the spatial coordinates only enter the perturbed quantities. At zeroth order in perturbations, the Einstein equations are given by[11]

$$3\mathscr{H}^2 = \frac{1}{2M_{\rm Pl}^2}\left[(\kappa^0)^2 + (\bar{\phi}')^2 + 3a^4 \frac{\kappa^2}{c_H^2} \right]. \tag{4.84}$$

As one can check explicitly from, e.g., Eq. (C17), following the definition of $\mathscr{H}^2$ above, the subhorizon condition is re-expressed as

$$a^4 k^2 \gg \mathscr{H}^2 \geq \frac{a^4 \kappa^2}{2M_{\rm Pl}^2 c_H^2}. \tag{4.85}$$

Now, in order for the coordinates x^μ to have mass dimension of -1 (given that $[\chi^\mu] = 1$), the mass dimension of the κ^μ needs to be 2. This is consistent with the fact that κ^μ is proportional to the canonical conjugate momentum which has a mass dimension of 2. Thus, we can rewrite κ^μ in terms of the Planck-mass and some dimensionless constant $\tilde{\kappa}^\mu$, i.e., $\kappa^\mu = \tilde{\kappa}^\mu M_{\rm pl}^2$. The subhorizon condition can therefore be recast into

$$k^2 \gg \frac{\tilde{\kappa}^2}{c_H^2} M_{\rm pl}^2. \tag{4.86}$$

Choosing a time coordinate that is perfectly adapted to the physical clock suggests $\tilde{\kappa}^0 = 1$. Furthermore, we choose $\tilde{\kappa}/c_H = \tilde{\kappa}^0$, such that the rod contribution to the energy momentum is equal to that of the clock at present time when $a = 1$. In this case, the subhorizon condition implies that

$$k^2 \gg M_{\rm pl}^2, \tag{4.87}$$

which suggests that the subhorizon regime is in fact trans-Planckian in the physical reference frame adopted. Importantly, this conclusion is based on the rod contribution to $\bar{T}_{00}$ which, as argued in Appendix C 3, is generic and not only a particular feature of the harmonic coordinate system.

Notice that this issue becomes only apparent when one considers inhomogeneities, since in the homogeneous setting there is no notion of superhorizon and subhorizon regimes. This is why choosing $\kappa^\mu = 1$ or $\kappa^\mu = M_{\rm pl}^2$ did not make any practical difference in previous studies (e.g., in [62] $\kappa^\mu = 1$ was imposed when comparing the effective GFT equations with GR). Clearly, the result that subhorizon modes are trans-Planckian in the standard effective field

[10] Importantly, we define perturbations at the level of the on shell energy-momentum tensor rather than perturbing the fields and then splitting $T_{\mu\nu}$. This procedure is more natural from the perspective of perturbing the Einstein equations.

[11] Notice, that this is the form of the Einstein equations when considering all of the matter momenta. For the argument that we make here, it is important to keep the contributions of the $(\kappa^\mu)^2$. When matching the homogeneous GFT equations to background cosmology, we assume that π_ϕ dominates the matter content; see also DS5.

theory treatment of linear cosmological perturbations in the relational frame used here calls for a better understanding and further investigations. Specifically, it should be checked if this also holds for the more realistic case of a dust frame. We comment on the relation to the so-called *Trans-Planckian Censorship Conjecture* (TCC) in Sec. V A.

5. Summary

Summarizing this section, we compared geometric and matter perturbations in GFT and GR, which we combined to construct a curvaturelike variable $\tilde{\mathcal{R}}$. The GFT effective dynamics of these perturbations (when the ratio of initial conditions $c_1^{\rm GR}/d_1^{\rm GR}$ is small) are in remarkable agreement with GR for non-Planckian modes, as explicitly shown in [132]. Here, we have shown that these trans-Planckian corrections (with a clear quantum gravitational origin[12]) manifest themselves prominently at subhorizon scales, in virtue of a correspondence between subhorizon and trans-Planckian modes due to the relational description that we are adopting.

Notice, that all perturbation quantities remain small under relational time evolution, implying that the perturbative framework defined here is in fact self-consistent. Finally, classical corrections to the standard cosmological perturbation equations coming from the presence of reference fields only enter proportionally to the rod variable κ^2, which in turn can be consistently neglected (see Appendix C 2 for further details).

V. SUMMARY AND CONCLUSION

In this section, we provide a summary and discussion of the main results obtained in this article. First, in Sec. V A, we review the main ideas of this work and the procedures applied to obtain our results. Thereafter, in Sec. V B, we discuss in more detail the two main results and how these can serve as an important step toward a better understanding of the microscopic nature of cosmic perturbations and toward connecting GFT perturbations with cosmological observations. Moreover, we provide an overview of pursuing research directions that we consider fruitful.

A. Summary

Throughout this paper, our explorations have been guided by two overarching principles, both of which are facilitated by the extended causal structure of the complete BC model [17]. The first principle is that the causal properties of frame fields should be faithfully transferred to the quantum theory. The second principle is that inhomogeneities of cosmological observables emerge from quantum entanglement between GFT quanta.

Following the first principle, in Sec. II A we introduced the complete BC GFT model for Lorentzian quantum gravity, including both spacelike and timelike tetrahedra. This is a minimal framework in which the first principle above can be consistently implemented. The inclusion of timelike tetrahedra necessitates an extension of the Fock space structure, which we realized by means of a tensor product between the spacelike and timelike sector. We extended the algebra of creation and annihilation operators to the tensor product Fock space. Still at the level of kinematics, we introduced in Sec. II B five minimally coupled massless free scalar fields to the GFT model, serving as a relational frame and matter content, respectively. Here, we made the first guiding principle manifest by restricting the kinetic kernels of the GFT in such a way that the clock and the rods only propagate along timelike, respectively spacelike dual edges.

The second principle was implemented instead in Sec. III, where we introduced the concept of perturbed coherent peaked states. These extend the concept of spacelike CPSs [57,63] in two ways. At the level of the background, we complemented the spacelike CPS by introducing a condensate also on the timelike sector. The specific form of the condensate wave functions and their peaking properties have been guided by geometric and relational isotropy (see assumption KS2) as well as to simplify the ensuing computations of perturbed observables. The second important difference between these states and the spacelike CPSs is that in the former, perturbations are associated with the action of three types of two-body operators, which encode correlations between and within spacelike and timelike sectors. This is how the second principle above is implemented in our framework. Thereupon, in Sec. III C, we derived dynamical equations for the perturbed condensate in a relational manner as the expectation value of the full quantum dynamics with respect to the perturbed CPS.

In Sec. IV, we derived the background and perturbed dynamics of observables by computing expectation values of GFT operators with respect to the perturbed CPS. To compare such emergent quantities to observables of GR, we introduced matching conditions that relate macroscopic GFT observables with cosmological quantities. At the background level, we reproduced the results of [57,65], showing that for late relational times, the dynamics of the spatial volume $\bar{V}$ and the scalar field $\bar{\phi}$ in GFT and GR agree. We computed the dynamics of the perturbed spatial volume δV and the perturbed matter field $\delta\phi$ in Secs. IV A and IV C, respectively. Their effective equations show a

[12]Note that because of the single-spin assumption DC2, the source term appearing in Eqs. (4.66), (4.75), and (4.78), which produces the above corrections, can be equivalently written as proportional to $(\delta V/\bar{V})'$ and $(\delta N_+/\bar{N}_+)'$. Although it is not possible to distinguish between the two quantities within this approximation, we note that there is a crucial conceptual difference between them, since the latter has no classical counterpart. Should further studies identify the source term as proportional to $(\delta N_+/\bar{N}_+)'$, this would provide further evidence that the deviations discussed in this section are quantum gravitational in nature.

significant improvement compared to [57] as the harmonic spatial derivative term of GR, entering with a factor of a^4, is now faithfully reproduced. Also, we combined geometric and matter perturbations into a curvaturelike quantity $\tilde{\mathcal{R}}$, the dynamics of which again show an improvement with respect to previous work. Despite the similarities between the effective dynamics of GFT and GR perturbations, deviations in the form of source terms are present in the differential equations for $\delta V, \delta\phi$, and $\tilde{\mathcal{R}}$. To study these differences further we provided a comparison between solutions of perturbations in Sec. IV E. Remarkably, the curvaturelike observable $\tilde{\mathcal{R}}_{\mathrm{GFT}}$ shows very good agreement with $\tilde{\mathcal{R}}_{\mathrm{GR}}$ for sub-Planckian modes if initial conditions are chosen appropriately. Quantum gravity corrections become important instead for trans-Planckian modes [132], which we have shown to be naturally associated with subhorizon scales in the relational framework we employ.

B. Discussion and outlook

In this section we discuss the two main results of this work. First, that from the perspective of the underlying quantum gravity theory, cosmological perturbations can be seen as arising from quantum correlations. Second, that the macroscopic effective dynamics of these perturbations show some deviations from GR at scales that we interpret as trans-Planckian in our physical scalar frame.

1. Cosmological perturbations and quantum entanglement

The first of the two results mentioned above provides crucial insights into the intrinsically quantum nature of cosmological perturbations. Moreover, it concretely substantiates the intuition, shared by many approaches to QG, that nontrivial geometries are associated with entanglement between the fundamental geometric degrees of freedom; see for instance [133–143].

In this work, entanglement is encoded in relational two-body nearest neighbor correlations between GFT atoms, and it is lifted to the macroscopic level (and thus associated with cosmological quantities) by the properties of two-body coherent states. These states deviate substantially from one-body condensate states, and their differences can be seen perturbatively as entangled "out-of-condensate" components. We leave it to future research to investigate the associated entanglement entropy.

We emphasize however that this is the only quantum effect that has been incorporated since the microscopic dynamics are still obtained within a mean-field analysis. As shown in Appendix B, this is a very robust approximation as long as GFT interactions are negligible. Interactions will eventually become important at very late times [57,62], and thus their study will be crucial for determining the self-consistency of the framework. When interactions are present, the inclusion of out-of-condensate perturbations will require a systematic analysis of the quantum properties of the GFT field. This scenario (in the simpler deparametrized setting of [144]) is studied in [145] by perturbatively splitting the GFT field into a classical homogeneous part and an inhomogeneous quantum field.

Finally, we leave it as an interesting avenue of future research to extend our analysis to a causally complete GFT formulation of the EPRL model [23,86]. Since recent results suggest that the BC and EPRL models could lie in the same universality class [36,52,65,87,89,146,147], we do not expect substantial differences between the perturbation theory of these two models. However, once our framework matured sufficiently to model tensor perturbation modes, it will be interesting to investigate if these are anomalously polarized in the EPRL model, as can be expected from the parity oddness of the Holst term [148–154].

2. Emergent effective dynamics of cosmological perturbations

Including timelike tetrahedra turned out to be crucial to obtain effective dynamics of cosmological perturbations similar to GR. However, several parameters associated with the timelike sector of our states were not involved at all in the GR matching and are thus completely unconstrained. This "passive" behavior of timelike quanta, exemplified by the fact that the GR matching forces their perturbed number to be trivial (see Sec. IV B), admits two possible interpretations. First, it could reflect mathematically that our Universe can be fully described at the background level by evolving spacelike quantities only. On the other hand, it could be due to the fact that only spacelike quantities have been considered in this work. In this case, one would expect that extending the analysis to other types of geometric observables (especially the extrinsic curvature) would constrain the free parameters of the timelike sector of our states.

However, the presence of such "irrelevant" free parameters in the timelike sector is a blessing in disguise. Indeed, it implies that the emergent perturbation equations are determined by only a set of parameters which turn out to be fixed by the requirement of consistency with the background dynamics. In other words, the predictions made by these models under a mean-field approximation are in principle easily falsifiable by comparison with cosmological observations. There are three different levels (in increasing order of conceptual and computational complexity) at which one could try to make contact with observations.

3. Phenomenology

First, one could try to phenomenologically incorporate the modified dynamics of trans-Planckian modes arising from GFT into the Standard Cosmological Model. Since, for an inflationary phase slightly longer than the minimum period, all modes observed today were originally trans-Planckian [155], this could already produce nontrivial observable effects. Note that this would not technically violate the TCC [156–158], since in principle the quantum gravity effects captured by these phenomenological models need not be described in terms of an effective field theory.

4. Extraction of full cosmological dynamics

A more systematic approach would be to *derive* the emergent cosmological perturbation dynamics from the full QG theory. However, this requires (i) the construction of additional operators to extract the full effective (anisotropic) geometry, (ii) the inclusion of the appropriate matter content, and (iii) the generalization of the analysis performed here to early times. Let us discuss the above three points in some more detail.

(i) The construction of additional geometric operators is extremely important for several reasons. First, it would allow us to extend our analysis to other cosmological observables such as the comoving curvature $\mathcal{R}$, requiring the introduction of anisotropic observables (see Sec. IV D). Such observables would also be important to reconstruct the full effective metric, including not only scalar perturbations, but also vector and, most importantly tensor perturbations. The construction of anisotropic operators would require one to represent only relational observables in the GFT Fock space, which is, however, a highly nontrivial task [57]. Finally, it is conceivable that modeling tensor perturbations requires the inclusion of lightlike tetrahedra as these are expected to propagate along lightlike dual edges. Studies in GFT cosmology mostly focused on minimally coupled massless free scalar fields [62,106] although recently, the analysis has been extended to include a nontrivial potential [159]. To move toward a more realistic matter content, one would have to include cosmic fluids. Considerable effort has been devoted to the study of dust in classical and quantum gravity (see, e.g., [99,100] and [105,160–162] for cosmological applications), since it not only constitutes a key component of the Universe, but also serves as a natural physical reference frame. In fact, it is the reference frame in which the cosmological principle is formulated, and thus in which background and perturbations are defined. This is particularly important, since the mixing between subhorizon and trans-Planckian modes emphasized in Sec. IV E hinges on the definition of perturbations with respect to our physical scalar frame. Coupling dust to GFT models would therefore allow us to set up more realistic cosmological models, in which one could explore delicate issues such as the subhorizon/trans-Planckian mixing. Importantly, it would also allow us to test the physical covariance (or violation thereof) of the emergent cosmological dynamics. As a final comment on this point, we report that matter components may also emerge due to the underlying GFT dynamics; see [113] for an example.

(ii) Generalizing the current analysis to earlier times would be important to understand the imprint of the quantum gravity bounce on the perturbations. Moreover, for the perturbation theory developed here to be self-consistent also in this regime it will be important to check if the energy density of the perturbations remains bounded and small compared to the background quantum geometry so that back reaction effects can be ignored. This concerns in particular perturbations of trans-Planckian wavelength and was dubbed the "real trans-Planckian issue" in another setting [163,164]. In general, however, the mean-field equations become considerably more complicated as the density of the background condensate decreases [70]. In particular, recent results [165] seem to suggest that, in the superhorizon limit, the early times emergent dynamics cannot be reconciled with that of any modified gravity theory.

5. Initial conditions from the fundamental theory

Finally, in both the methods discussed so far, the primordial power spectrum would be obtained by a Fock quantization of the macroscopic degrees of freedom. This is a strategy that is followed, for example, in both condensed matter physics [166] and loop quantum cosmology [163,167–175] (see [164,176] for a review). From the perspective of QG, however, it would be natural to look for a mechanism for generating the cosmological initial conditions that can be derived entirely from the fundamental theory. If the statistical properties of the cosmological inhomogeneities and anisotropies are ultimately due to quantum gravity fluctuations, their description may require going beyond the effective relational framework (which is based on observable averages) used here, and considering relational observables rigorously defined on the GFT Fock space. Similarly, one could explore the possibility that the inflationary mechanism is purely quantum geometric in nature, although this possibility has already been ruled out in some phenomenological GFT models [111,112].

In closing, we hope that our results on the coupling of a physical Lorentzian reference frame and on the exploration of the link between quantum geometric entanglement and cosmological perturbations will impact on a wider set of quantum gravity approaches. In particular, we believe that our work strengthens the argument for models that incorporate a causally complete set of discrete Lorentzian geometries such as causal dynamical triangulations [29,30,32], causal sets [71], Lorentzian Regge calculus [89–92,177], or Lorentzian spin foams and LQG [78,82–85].

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APPENDIX A: LIST OF ASSUMPTIONS

In this Appendix, we present an exhaustive list of the assumptions made in order to arrive at the results we obtained in the main body of this work. Naturally, the assumptions can be split into kinematic and dynamic ones. Furthermore, we split the assumptions into two categories, namely structural ones and those stemming from matching with GR perturbations. The former category contains assumptions that are motivated either conceptually or that simplify technically challenging computations.

1. Kinematic assumptions

Kinematic approximations are related to the properties of the specific states we are considering.

a. Structural

KS1 (Perturbed condensate states). In this paper, we model a spatially homogeneous and isotropic spacetime with scalar perturbations by perturbed condensate states, which extend the notion of usual condensate states [55,56,58–60,62,110,178] in two ways. First, given the causally extended structure including timelike tetrahedra, the background contains an additional timelike condensate, introduced in Sec. III A 4. Second, perturbations are encoded in two-body operators that create an entanglement within and between the spacelike and timelike sector. As discussed in Sec. V and Appendix B, this is a first step in the direction of out-of-condensate perturbations.

KS2 (Geometrical and relational isotropy). The spacelike and timelike condensate wave functions as well as the correlation functions $\delta\Phi, \delta\Psi$, and $\delta\Xi$ are required to satisfy a quantum analog of isotropy, realized by setting all the area eigenvalues of the faces f of a tetrahedron to the same value, $\rho_f = \rho$. Moreover, the spacelike peaking encoded in the condensate wave function τ is isotropic by choosing the same peaking parameters for all of the three spatial directions. This also ensures that in a derivative expansion for the effective relational equations, only the Laplace operator with respect to the rod variables enters.

KS3 (Peaking and frame-dependence). Following the strategy of [57,63,70], the condensate wave function factorizes into a peaking term and a reduced wave function. On the spacelike sector, the peaking function only contains the clock variable, while the timelike condensate is peaked on clock and rods. This ensures that the spatial derivative, acting on the volume perturbations, is only associated to spacelike dual edges. The peaking functions are Gaussians with a nontrivial phase and a small width. As argued in [57,63], a nonvanishing ϵ as well as a nonvanishing phase are required in order to guarantee that all quantum fluctuations of observables associated to the reference fields are small in the classical regime. Both of the reduced condensate wave functions, $\tilde{\sigma}(\chi^0)$ and $\tilde{\tau}(\chi^0)$, only depend on relational time as they are assumed to be part of the background. For the two-body correlations $\delta\Phi, \delta\Psi$, and $\delta\Xi$, no peaking is encoded, and an explicit rod dependence is assumed in order to render these functions part of the perturbations. Notice that in spite of assuming a rod dependence of $\delta\Xi$, the dynamical equations, together with Eq. (3.28), render this function only time dependent as we argue in Sec. III C 2.

KS4 (Signature of tetrahedra and faces). We extend the spacelike Barrett-Crane model by timelike tetrahedra and leave the inclusion of lightlike tetrahedra for future research. For the timelike condensate, entering the perturbed CPS in Eq. (3.2), we assume that the corresponding timelike tetrahedra contain spacelike faces only. This assumption is supported by two reasons: (*i*) Recent studies in [115] suggest that only spacelike faces contribute to a condensate phase and (*ii*) timelike and spacelike tetrahedra can interact only via spacelike faces, as shown in [17].

KS5 (Regularization). At several points of the analysis, infinities appear that need to be treated appropriately. Extended closure in Eq. (2.2) yields empty $\mathrm{SL}(2,\mathbb{C})$ integrations in the definition of the GFT action, two-body operators and the equations of motion. On the other hand, due to spatial homogeneity and the fact that σ is peaked only on the clock, the spacelike background equations in Sec. III C 1 as well as the background expectation value of the three-volume operator in Sec. IV A contain empty rod integrations. Both types of divergencies are considered to be unphysical, and we consider only the finite factors of these expressions.

b. Motivated by classical matching

KC1 (Peaking on matter momenta). We assume the condensate wave functions σ and τ to be peaked on the matter momentum π_ϕ, realized by a Gaussian function without phase. This is necessary in order to recover the Friedmann equations at the background level as well as to render feasible perturbation equations for the condensate and the observables in Secs. III C and IV, respectively. As pointed out in [70], a deeper physical intuition for this peaking may be obtained when considering a scalar field with nontrivial potential.

KC2 (Local perturbation functions). *A priori*, the two-body correlations $\delta\Phi$, $\delta\Psi$, and $\delta\Xi$ are bilocal functions of the relational reference frame and the matter momentum π_ϕ. To make contact to localized perturbations, we assume that the two arguments are identified via a δ distribution. Following the simplicial gravity picture, this assumption would correspond to correlations within the same four simplex with momentum conservation across tetrahedra.

KC3 (Peaking parameters). To render the computations of Sec. IV feasible and to allow for a more straightforward matching of GFT and GR perturbations, we assume that the peaking parameters $\epsilon^\pm$ and $\pi_0^\pm$ of the spacelike and timelike sector are related as $\epsilon^+ = \epsilon^-$ and $\pi_0^+ = -\pi_0^-$. Furthermore, we assume that the phase parameters π_x and π_0^+ satisfy the strong inequality $\pi_x \gg \pi_0^+$, which yields a drastic simplification for the dynamical equation of the perturbed volume in Eq. (4.17).

2. Dynamic assumptions

Dynamic approximations are related to the details of the GFT action and on how background and perturbed equations for the condensate and the observables are obtained.

a. Structural

DS1 (GFT action and causal building blocks). We choose to work with an extension of the Lorentzian Barrett-Crane model [17,65] and here study its cosmological implications while including spacelike and timelike tetrahedra and excluding lightlike ones. Thus, the configurations are the two group fields φ_+ and φ_- which are distinguished in their domain and in the form of the simplicity constraint, given in Eq. (2.3).

DS2 (Scalar field coupling). Clock and rod fields χ^μ as well as the matter field ϕ are coupled to the GFT such that the Feynman amplitudes correspond to a simplicial gravity path integral with the fields propagating along dual edges. As discussed in [106], this coupling is obtained in a semiclassical limit, which is an assumption that should be kept in mind. In order to turn the scalar reference frame into a physical Lorentzian reference frame, we align its causal character with that of the geometry by imposing restrictions on the kernels in Eqs. (2.26). As elaborated in Sec. II B, the physical picture of these conditions is that the clock propagates along timelike dual edges and the rods propagate along spacelike dual edges.

DS3 (Mean-field dynamics). We assume the effective dynamics to be well approximated by the mean-field equations. In fact, as we show in Appendix B and discuss in the conclusion, the mean-field equations solve the higher-order Schwinger-Dyson equations if negligible interactions and linear perturbation theory are assumed.

DS4 (Negligible interactions). When taking expectation values with respect to the perturbed CPS $|\Delta; x^0, \boldsymbol{x}\rangle$, interaction terms are expected to be negligible in the effective dynamics. It has been shown in [62], that the error induced by this assumption grows with the number of GFT particles. This assumption can therefore be consistently implemented at late but not very late times. See also [58] for a complementary discussion on this matter.

DS5 (Classical matter content). We assume that the matter content of the classical theory, given by the five fields χ^μ and ϕ, is dominated by ϕ. In this way, perturbations of χ^μ are considered to be negligible, and inhomogeneities of matter and geometry can be defined unambiguously with respect to the clock and rod fields. In Appendix C 2, we compute the corrections to the classical Einstein equations if one would not neglect the frame contributions to the energy-momentum tensor.

b. Motivated by classical matching

DC1 (Mesoscopic regime). Classical dynamics are obtained in a mesoscopic regime, where the averaged number of particles of the system is taken to be large enough to allow for both a continuum interpretation of the expectation values of relevant operators and classical behavior, but not too large that interactions are dominating. Furthermore, the perturbation equations of Secs. III C and IV simplify significantly in this regime, as the condensate wave functions are solved by simple exponential functions; see Eqs. (3.24) and (3.25).

DC2 (Single-spin dominance). We assume that coefficients of σ, τ and the two-body correlations $\delta\Phi, \delta\Psi$, and $\delta\Xi$ in spin representation are dominated by a single representation label ρ, mostly suppressed in the notation. For the background condensates, this assumption is supported by the studies of Refs. [65,110,114,179], which show that under rather generic assumptions on the kinetic kernels of the GFT, a dominant representation emerges dynamically.

DC3 (Dynamical freedom). Exploiting the dynamical freedom of having two first-order equations for three functions, we relate the spacelike-spacelike correlation $\delta\Phi$ and the spacelike-timelike correlation $\delta\Psi$ via a function f. This function is chosen such that the dynamical equation for $\delta\Psi$ as well as the definition of δV simplify drastically. Finally, when matching the a^4 term in front of the spatial derivative of δV in Eq. (4.17), the function f is fixed completely. As we detail in Sec. V B, it is left open for future studies to include interactions (see for instance Ref. [145] for further details) such that the number of independent first-order equations matches the number of perturbation functions and no such dynamical freedom is present.

DC4 (Constant phase). Splitting the perturbation function $\delta\Psi$ into modulus R and phase Θ, we assume the phase to be constant, rendering the dynamical equations of $\delta\Psi$ and δV feasible. Moreover, Θ is set to $n\frac{\pi}{2}$ in Sec. IV A, simplifying the dynamical equation for δV in Eq. (4.17). One can show that if one relaxes the constant-Θ condition to a pure time dependence, then consistency requires that Θ is in fact constant. Clearly, the most general case is given for Θ carrying a space dependence. However, the resulting equations of motion for the perturbation function as well as for observables take a highly intricate form, which does not allow for further analytical studies.

DC5 (Normalized scale factor). As commonly done in standard cosmology, the scale factor a is normalized to 1 at present relational time, such that it takes smaller values for all earlier relational times. Notice, that this assumption is not in contradiction with a large GFT-particle number or, equivalently, a large volume. That is because the volume is given by a^3 times a fiducial volume factor, which can be much larger than 1.

DC6 (Timelike particle number). When computing the dynamics of matter observables, we assumed that the effective mass parameters of the spacelike and timelike sector satisfy $\mu_+ > \mu_-$. This assumption is guided by the intuition that the background is predominantly described by the spacelike condensate. Indeed, as a consequence, one finds that the background number of spacelike GFT particles dominates over the timelike particle number in the classical limit. In Sec. IV C, this leads to a simplification of the emergent equations for the perturbed matter field $\delta\phi$ and the perturbed matter momentum $\delta\pi_\phi$.

APPENDIX B: GOING BEYOND MEAN FIELD IN THE ABSENCE OF INTERACTIONS

In this Appendix, we show that in the absence of interactions, higher-order Schwinger-Dyson equations reduce to powers of the lowest mean-field equation. This result is of interest for the perturbed coherent states introduced in Sec. III B, as these states constitute out-of-condensate perturbations only if interactions are taken into account. Furthermore, this result is exactly the reason for the dynamical freedom of one of the two-body correlation functions which we exploited in Sec. IV to match the perturbations of GR.

For a polynomially bounded functional $\mathcal{O}[\varphi_\alpha, \bar{\varphi}_\alpha]$ on field space, where α refers to either the spacelike or the timelike group field, the Schwinger-Dyson equations are given by

$$\left\langle \frac{\delta\mathcal{O}}{\delta\bar{\varphi}_\alpha} - \mathcal{O}\frac{\delta S}{\delta\bar{\varphi}_\alpha} \right\rangle = 0. \tag{B1}$$

The polynomial expansion of the operator $\mathcal{O}$ is given by

$$\mathcal{O}[\varphi_\alpha, \bar{\varphi}_\alpha] = \sum_{k,l,m,n} \mathcal{O}_{klmn} \bar{\varphi}_+^k \varphi_+^l \otimes \bar{\varphi}_-^m \varphi_-^n, \tag{B2}$$

with $\mathcal{O}_{klmn}$ being the kernel. Given the perturbed state $|\Delta\rangle$, defined in Sec. III B, the expectation value of $\mathcal{O}$ up to first order in perturbations is schematically given by

$$\begin{aligned} \langle\Delta|\mathcal{O}|\Delta\rangle = \mathcal{O}|_{\sigma,\tau} &+ \left(2\bar{\varphi}_+ \frac{\delta\mathcal{O}}{\delta\varphi_+}\bigg|_{\sigma,\tau} + \frac{\delta^2\mathcal{O}}{\delta\varphi_+^2}\bigg|_{\sigma,\tau}\right)\delta\Phi \\ &+ \left(\bar{\varphi}_- \frac{\delta\mathcal{O}}{\delta\varphi_+}\bigg|_{\sigma,\tau} + \bar{\varphi}_+ \frac{\delta\mathcal{O}}{\delta\varphi_-}\bigg|_{\sigma,\tau}\right)\delta\Psi \\ &+ \left(2\bar{\varphi}_- \frac{\delta\mathcal{O}}{\delta\varphi_+}\bigg|_{\sigma,\tau} + \frac{\delta^2\mathcal{O}}{\delta\varphi_-^2}\bigg|_{\sigma,\tau}\right)\delta\Xi + \text{c.c.}, \end{aligned} \tag{B3}$$

where a vertical line indicates that the operator $\mathcal{O}$ is functionally evaluated on the condensate wave functions σ and τ.

Building up on this result, we consider now the Schwinger-Dyson equation under the assumption that interactions are absent, i.e., $S = \sum_\alpha \bar{\varphi}_\alpha \mathcal{K}_\alpha \varphi_\alpha$. At zeroth order, the equations then take the form

$$\frac{\delta\mathcal{O}}{\delta\bar{\varphi}_\alpha}\bigg|_{\sigma,\tau} = \mathcal{O}\mathcal{K}_\alpha\varphi_\alpha|_{\sigma,\tau}, \tag{B4}$$

which, for $\mathcal{O} = \mathbb{1}$, reproduces the mean-field equation studied in Sec. III C 1.

At first order, we look at the coefficients in front of the same perturbation function. For instance, the $\delta\Psi$ term is given by

$$\bar{\varphi}_{-}\frac{\delta^2\mathcal{O}}{\delta\varphi_{+}\delta\bar{\varphi}_{\alpha}}\bigg|_{\sigma,\tau}+\bar{\varphi}_{+}\frac{\delta^2\mathcal{O}}{\delta\varphi_{-}\delta\bar{\varphi}_{\alpha}}\bigg|_{\sigma,\tau} = \bar{\varphi}_{-}\frac{\delta(\mathcal{O}\mathcal{K}_\alpha\varphi_\alpha)}{\delta\varphi_{+}}\bigg|_{\sigma,\tau}+\bar{\varphi}_{+}\frac{\delta(\mathcal{O}\mathcal{K}_\alpha\varphi_\alpha)}{\delta\varphi_{-}}\bigg|_{\sigma,\tau}, \tag{B5}$$

with the other coefficients given in a similar form. Clearly, upon solutions of the zeroth-order equation, also the first-order equations are satisfied. Hence, higher-order Schwinger-Dyson equations do not yield additional dynamical equations if one works (*i*) in a perturbative setting and (*ii*) in the absence of interactions. This can be also shown by computing n-point Green functions which, in perturbation theory and in the absence of interactions, factorize into one-point functions corresponding to the mean field. This is demonstrated for a single-sector GFT in [58]. As a consequence, we find in the analysis of the two-body correlation functions in Sec. III C a dynamical freedom for one of the variables. The results of this Appendix suggest that in order to obtain higher-order out-of-condensate equations, one would need to take interactions into account. We comment on this matter further in Sec. V.

APPENDIX C: CLASSICAL PERTURBATION THEORY

In this Appendix, we provide an overview of the perturbation equations for geometry and matter in classical general relativity. In order to allow for a simpler comparison with relational GFT results, we mostly use harmonic coordinates $\{x^\mu\}$ which are adapted to the reference field $\{\chi^\mu\}$ via the relation $\chi^\mu = \kappa^\mu x^\mu$ (no summation over μ), where κ^μ are some dimensionful proportionality factors [107]. In harmonic coordinates, the reference fields are assumed to satisfy the Klein-Gordon equation at all orders of perturbations. This can equivalently be rewritten as

$$\Gamma^\lambda_{\mu\nu}g^{\mu\nu} = 0, \tag{C1}$$

which poses a condition on the metric.

1. Geometry

At zeroth order, the line element of a spatially flat FLRW spacetime with signature of $(-,+,+,+)$ is given by

$$\mathrm{d}s^2 = \bar{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + a^2\mathrm{d}\boldsymbol{x}^2, \tag{C2}$$

where N is the lapse function, a is the scale factor, and $\mathrm{d}\boldsymbol{x}^2$ the line element of three-dimensional Euclidean flat space. Imposing harmonic gauge on the background yields $a^3/N = c_H$, where c_H is an integration constant. For the remainder, we set $c_H = 1$, and we assume that the matter content is dominated by the matter field ϕ with conjugate momentum π_ϕ.

Within these assumptions, the dynamics of the geometry at background level are captured by

$$3\mathscr{H}^2 = \frac{1}{2M_{\mathrm{Pl}}^2}\bar{\pi}_\phi^2, \qquad \mathscr{H}' = 0, \tag{C3}$$

where $\mathscr{H} = a'/a$ is the Hubble parameter in harmonic coordinates and $\bar{\pi}_\phi$ is the background contribution of the canonical conjugate of the scalar field, defined in Eq. (C29). Introducing the background volume $\bar{V} = a^3$, the geometric equations can be recast to

$$3\left(\frac{\bar{V}'}{3\bar{V}}\right)^2 = \frac{1}{2M_{\mathrm{Pl}}^2}\bar{\pi}_\phi^2, \qquad \left(\frac{\bar{V}'}{3\bar{V}}\right)' = 0. \tag{C4}$$

To derive perturbed volume equations from GR, we consider in the following first-order scalar perturbations of the FLRW metric. Using the background harmonic gauge condition, the line element is given by

$$\mathrm{d}s^2 = -a^6(1+2A)\mathrm{d}t^2 + a^4\partial_i B\mathrm{d}t\mathrm{d}x^i + a^2((1-2\psi)\delta_{ij}+2\partial_i\partial_j E)\mathrm{d}x^i\mathrm{d}x^j, \tag{C5}$$

with scalar perturbation functions A, B, ψ, and E. Einstein's equations at linear order yield [70,180]

$$\frac{1}{2M_{\mathrm{Pl}}^2}\bar{\phi}'\delta\phi' + 3\mathscr{H}\psi' - a^4\nabla^2\psi - \mathscr{H}\nabla^2(E'-a^2B) = 0, \tag{C6}$$

$$\mathscr{H}A + \psi' - \frac{1}{2M_{\mathrm{Pl}}^2}\bar{\phi}'\delta\phi = 0, \tag{C7}$$

$$E'' - a^4\nabla^2 E = 0, \tag{C8}$$

where $\delta\phi$ is the scalar field perturbation. Combining Eq. (C6) and the time derivative of Eq. (C7), we obtain

$$\psi'' = -\mathscr{H}A' - 3\mathscr{H}\psi' + a^4\nabla^2\psi + \mathscr{H}\nabla^2(E'-a^2B). \tag{C9}$$

To obtain an equation for the perturbed volume, which is an observable accessible also from the GFT side, consider on a slice of constant time the local volume element

$$\sqrt{-g_{(3)}} = \bar{V} + \delta V = a^3(1 - 3\psi + \nabla^2 E). \tag{C10}$$

Thus, we identify the perturbed spatial volume as

$$\frac{\delta V}{\bar{V}} = -3\psi + \nabla^2 E. \tag{C11}$$

Taking the second derivative of $\delta V/\bar{V}$ and using Eqs. (C8) and (C9), one obtains

$$\left(\frac{\delta V}{\bar{V}}\right)'' + 3\mathscr{H}\left(\frac{\delta V}{\bar{V}}\right)' - a^4\nabla^2\left(\frac{\delta V}{\bar{V}}\right) = 3\mathscr{H}(A' + a^2\nabla^2 B). \tag{C12}$$

a. Harmonic gauge

At first order in perturbations, the harmonic gauge conditions given by [180]

$$0 = A' + 3\psi' - \nabla^2(E' - a^2 B), \tag{C13a}$$

$$0 = (a^2 B)' + a^4(A - \psi - \nabla^2 E), \tag{C13b}$$

which, imposed on Einstein's equations (C6)–(C8), yields [70,180]

$$\psi'' - a^4\nabla^2\psi = 0, \quad A'' - a^4\nabla^2 A + 4a^4\nabla^2\psi = 0 \tag{C14}$$

$$E'' - a^4\nabla^2 E = 0, \quad (a^2B)'' - a^4\nabla^2(a^2B) - 8a^2(a^2\psi)' = 0. \tag{C15}$$

Expressed in terms of the volume, the first harmonic gauge condition is expressed as

$$A' + a^2\nabla^2 B = \left(\frac{\delta V}{\bar{V}}\right)', \tag{C16}$$

such that the volume equation becomes

$$\left(\frac{\delta V}{\bar{V}}\right)'' - a^4\nabla^2\left(\frac{\delta V}{\bar{V}}\right) = 0 \tag{C17}$$

or equivalently

$$\delta V'' - 6\mathscr{H}\delta V' + 9\mathscr{H}^2\delta V - a^4\nabla^2\delta V = 0. \tag{C18}$$

To change to Fourier space in the rod variable, heavily employed in Sec. IV, one can simply perform the substitution $\nabla^2 \to -k^2$ here and in the following.

Following [180], there is a residual gauge freedom in performing a coordinate transformation

$$\xi^\mu \mapsto x^\mu + \xi^\mu, \tag{C19}$$

with $\xi^\mu = (\xi^0, \partial^i\xi)$ satisfying

$$(\xi^0)'' - a^4\nabla^2\xi^0 = \xi'' - a^4\nabla^2\xi = 0, \tag{C20}$$

such that harmonicity is conserved. Under this transformation, the perturbation functions transform as [57,180]

$$\psi \mapsto \psi + \mathscr{H}\xi^0, \qquad A \mapsto A - (\xi^0)' - 3\mathscr{H}\xi^0, \tag{C21}$$

$$E \mapsto E - \xi, \qquad B \mapsto B + a^2\xi^0 - a^{-2}\xi'. \tag{C22}$$

After introducing the matter equations in the following, we combine the geometric and matter quantities in a single fully gauge-invariant quantity, the so-called curvature perturbation $\mathcal{R}$.

2. Matter

The matter content of the classical theory consists of four reference scalar fields χ^μ as well as one additional free minimally coupled real scalar field ϕ, defined by the continuum action

$$S[\chi^\mu, \phi] = -\frac{1}{2M_{\mathrm{Pl}}^2}\int \mathrm{d}^4x\sqrt{-g}g^{ab}\left(\partial_a\phi\partial_b\phi + \sum_{\mu=0}^{3}\partial_a\chi^\mu\partial_b\chi^\mu\right). \tag{C23}$$

In this form, the action poses a well defined variational principle, yielding the Klein-Gordon equations for appropriate boundary conditions. One such admissible condition is von Neumann boundary conditions which assume vanishing variation of the gradients at the boundary. For reference fields in harmonic coordinates, as used in the remainder of this subsection, $\chi^\mu = \kappa^\mu x^\mu$, this clearly applies since $\partial_\mu\chi^\nu = \delta^\nu_\mu\kappa^\nu$ is constant and thus has vanishing variation.

The energy momentum tensor in arbitrary coordinates is given by

$$M_{\mathrm{Pl}}^2 T_{ab} = \sum_{\lambda=0}^{3}\left(\partial_a\chi^\lambda\partial_b\chi^\lambda - \frac{g_{ab}}{2}g^{mn}\partial_m\chi^\lambda\partial_n\chi^\lambda\right) + \partial_a\phi\partial_b\phi - \frac{g_{ab}}{2}g^{mn}\partial_m\phi\partial_n\phi, \tag{C24}$$

which we assume to be dominated by the matter field ϕ. The full equations of motion for ϕ are given by the massless Klein-Gordon equation

$$\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) = 0. \tag{C25}$$

Linearizing in both, the scalar field and the metric, we obtain the zeroth-order equation

$$\bar{\phi}'' = 0 \tag{C26}$$

and the first-order perturbation equation

$$\delta\phi'' - a^4\nabla^2\delta\phi = \left[A' + 3\psi' - \nabla^2 E' + a^2\nabla^2 B\right]\bar{\phi}', \tag{C27}$$

respectively. Supplementing the latter with the harmonic gauge condition in Eq. (C13a), $\delta\phi$ satisfies

$$\delta\phi'' - a^4\nabla^2\delta\phi = 0. \tag{C28}$$

To define the GR counterpart of the GFT observables $\hat{\Pi}^\alpha_\phi$, defined in Eq. (4.38), we introduce the momentum conjugate to the scalar field, commonly defined as

$$\pi^\mu_\phi := \frac{\partial\tilde{\mathcal{L}}}{\partial(\partial_\mu\phi)} = -\sqrt{-g}g^{\mu\nu}(\partial_\nu\phi), \tag{C29}$$

where $\tilde{\mathcal{L}}$ is the Lagrangian density, defined by the matter field action above. Expanding up to linear order, the scalar field momentum π^μ_ϕ is given by

$$\pi^\mu_\phi = -\overline{\sqrt{-g}}\bar{g}^{\mu 0}\partial_0\bar{\phi} - \delta\sqrt{-g}\bar{g}^{\mu 0}\partial_0\bar{\phi} - \overline{\sqrt{-g}}(\delta g^{\mu 0}\partial_0\bar{\phi} + \bar{g}^{\mu\nu}\partial_\nu\delta\phi), \tag{C30}$$

which can be split into background and perturbed part,

$$\pi^\mu_\phi = \bar{\pi}^\mu_\phi + \delta\pi^\mu_\phi. \tag{C31}$$

At the background level and in harmonic gauge, $\bar{\pi}^\mu_\phi$ is given by

$$\bar{\pi}^0_\phi = \partial_0\bar{\phi}, \qquad \bar{\pi}^i_\phi = 0. \tag{C32}$$

The perturbed part of π^μ_ϕ is given by

$$\delta\pi^0_\phi = (-A - 3\psi + \nabla^2 E)\bar{\phi}' + \delta\phi', \tag{C33}$$

$$\delta\pi^i_\phi = a^2\bar{\phi}'\partial^i B - a^4\partial^i\delta\phi. \tag{C34}$$

Applying the zeroth- and first-order equations for ϕ, the perturbed momentum satisfies the relativistic energy-momentum conservation equation

$$\partial_\mu\delta\pi^\mu_\phi = (\delta\pi^0_\phi)' + \partial_i\delta\pi^i_\phi = 0. \tag{C35}$$

Re-expressing this equation in terms of observables that are available in GFT, being $\bar{V}, \delta V, \bar{\phi}$, and $\delta\phi$, we find

$$(\delta\pi^0_\phi)' - \delta\phi'' - \left(\frac{\delta V}{\bar{V}}\right)'\bar{\phi}' = -A'\bar{\phi}'. \tag{C36}$$

While the left-hand side is given in terms of variables available in GFT, the right-hand side contains the variable A, which is not accessible by the GFT observable that is available at the present state. In Sec. V B, we comment on the importance of defining additional geometrical observables in GFT.

a. Classsical Mukhanov-Sasaki-like equation

As the transformations of Eqs. (C21) and (C22) show, the harmonic gauge conditions leaves a residual gauge freedom. Under these transformations, the perturbed scalar field $\delta\phi$ changes as

$$\delta\phi \mapsto \delta\phi - \bar{\phi}'\xi^0. \tag{C37}$$

Given this transformation behavior, one can combine ψ and $\delta\phi$ to a fully gauge-invariant quantity, the so called gauge-invariant curvature perturbation

$$\mathcal{R} := \psi + \mathscr{H}\frac{\delta\phi}{\bar{\phi}'}. \tag{C38}$$

Since in harmonic gauge, ψ and $\delta\phi$ satisfy the same equation, $\mathcal{R}$ satisfies [180]

$$\mathcal{R}'' - a^4\nabla^2\mathcal{R} = 0. \tag{C39}$$

In the context of GFT, one does not have direct access to the quantity ψ but rather to the perturbed volume δV. For comparison of classical and GFT mechanics, we define the *curvaturelike perturbation* $\tilde{\mathcal{R}}$ as

$$\tilde{\mathcal{R}} := -\frac{\delta V}{3\bar{V}} + \mathscr{H}\frac{\delta\phi}{\bar{\phi}'}. \tag{C40}$$

Again, since $\delta V/\bar{V}$ and $\delta\phi$ satisfy the same equation, $\tilde{\mathcal{R}}$ obeys

$$\tilde{\mathcal{R}}'' - a^4\nabla^2\tilde{\mathcal{R}} = 0. \tag{C41}$$

Notice however, that $\tilde{\mathcal{R}}$ is not gauge invariant but changes as

$$\tilde{\mathcal{R}} \mapsto \tilde{\mathcal{R}} - \nabla^2\xi. \tag{C42}$$

Still, since ξ is assumed to satisfy the equation above, the equation for $\tilde{\mathcal{R}}$ does not change under gauge transformations.

b. Reference field corrections to the perturbation equations

In this Appendix, we have so far assumed that the matter content is dominated by the scalar field ϕ, neglecting contributions of the clock and rod fields χ^μ. However, in order to rule out the possibility that the source term in the GFT perturbation equations (4.24), (4.60), and (4.64) is associated to the potential contribution of the physical reference frame, it is instructive to consider the background and perturbed field equations with all matter terms, similarly to what has been done, e.g., in [162] (albeit with a different physical frame).

At the background level, the Einstein equations take the form

$$3\mathscr{H}^2 = \frac{1}{2M_{\rm Pl}^2}(\bar{\phi}'^2 + (\kappa^0)^2 + 3\kappa^2 a^4), \qquad \mathscr{H}' = \frac{1}{M_{\rm Pl}^2}\kappa^2 a^4, \tag{C43}$$

which are readily dependent. Clearly, if one assumes π_ϕ^2 to dominate, consistency requires $\mathscr{H}' = 0$.

To compute the perturbed Einstein equations, we add the frame contributions to the perturbed energy-momentum tensor, the components of which are explicitly given by

$$M_{\rm Pl}^2(\delta T^\chi)^0{}_0 = \frac{(\kappa^0)^2}{a^6}A - \frac{\kappa^2}{a^2}(3\psi - \nabla^2 E), \tag{C44}$$

$$M_{\rm Pl}^2(\delta T^\chi)^0{}_i = \frac{\kappa^2}{a^4}\partial_i B, \tag{C45}$$

$$M_{\rm Pl}^2(\delta T^\chi)^i{}_j = \delta^i_j\left(-\frac{(\kappa^0)^2}{a^6}A + \frac{\kappa^2}{a^2}(-\psi + \nabla^2 E)\right) - 2\frac{\kappa^2}{a^2}\partial^i\partial_j E. \tag{C46}$$

Using the Einstein tensor components of [180], the perturbed equations of motion are given by

$$\frac{1}{2M_{\rm Pl}^2}\bar{\phi}'\delta\phi' + 3\mathscr{H}\psi' - a^4\nabla^2\psi - \mathscr{H}\nabla^2(E' - a^2 B) + \frac{1}{2M_{\rm Pl}^2}a^4\kappa^2(3(A+\psi) - \nabla^2 E) = 0, \tag{C47}$$

$$\mathscr{H}A + \psi' + \frac{1}{2M_{\rm Pl}^2}(\kappa^2 a^2 B - \bar{\phi}'\delta\phi) = 0, \tag{C48}$$

$$E'' - a^4\nabla^2 E + \frac{2}{M_{\rm Pl}^2}\kappa^2 a^4 E = 0, \tag{C49}$$

complemented by the harmonic gauge conditions in Eqs. (C13). Already at this point, one observes that the clock contributions, entering with $(\kappa^0)^2$, will cancel out upon the background equations as these only enter with the perturbation function A. We leave a clarification of this intriguing cancellation open for future investigations.

Following similar steps as above, we obtain a differential equation for ψ:

$$\psi'' - a^4\nabla^2\psi + \frac{2}{M_{\rm Pl}^2}\kappa^2 a^4(A+\psi) = 0, \tag{C50}$$

that is modified by the presence of the rod contribution, entering with κ^2. Using the definition of the relative perturbed volume as well as Eqs. (C49) and (C50), we obtain

$$\left(\frac{\delta V}{\bar{V}}\right)'' - a^4\nabla^2\left(\frac{\delta V}{\bar{V}}\right) = -\frac{2}{M_{\rm Pl}^2}\kappa^2 a^4\left(\frac{\delta V}{\bar{V}} - 3A\right), \tag{C51}$$

which is modified compared to Eq. (C17) by a term that is controlled via the rod variable κ.

The equation for the perturbed matter field $\delta\phi$ is not altered by the additional clock and rod contributions. Thus, using the volume and matter equation, we obtain an equation for $\tilde{\mathcal{R}}$ in the presence of frame contributions:

$$\tilde{\mathcal{R}}'' - a^4\nabla^2\tilde{\mathcal{R}} = \frac{2}{M_{\rm Pl}^2}\kappa^2 a^4\left[\tilde{\mathcal{R}} - A + 3\frac{\mathscr{H}}{\bar{\phi}'}\delta\phi + 2\frac{\delta\phi'}{\bar{\phi}'}\right]. \tag{C52}$$

Clearly, all of the corrections compared to Eq. (C40) enter with the rod variable κ and contain a mixture of geometric and matter perturbations. Structurally, these corrections are therefore quite different from those obtained in [162], possibly as a result of the different gauge fixing. Importantly, this result substantiates the interpretation of the source terms in the perturbation equations obtained from GFT that we give in Sec. IV E. That is, the source terms are not artifacts of reference frame contributions but are rather interpreted as genuine quantum gravity corrections.

3. Change of gauge

In ordinary cosmology, formulated without the use of reference fields as relational coordinates, the most commonly used coordinates are so-called conformal longitudinal (CL). In this section, we discuss the change from harmonic to CL coordinates and the consequences for the interpretation of clock and rod fields. To start with, the line element is given by [122,123]

$$\mathrm{d}s^2 = a^2(\tau)\left[-(1+2\Psi(\tau,\boldsymbol{x}))\mathrm{d}\tau^2 + (1-2\Phi(\tau,\boldsymbol{x}))\mathrm{d}\boldsymbol{x}^2\right], \tag{C53}$$

in CL coordinates, where Ψ and Φ are the gauge-invariant Bardeen variables. In order to transform from harmonic coordinates $\{x^\mu_{\rm H}\}$ to CL coordinates $\{x^\mu_{\rm CL}\}$, we have to perform two transformations, first at the background to conformal time and then a transformation at the level of perturbations. For the background transformation we consider

$$x^0_{\rm H} \to \tau(x^0_{\rm H}) = \int_0^{x^0_{\rm H}} \mathrm{d}\tilde{x}^0 a^2(\tilde{x}^0), \tag{C54}$$

such that $a^2\mathrm{d}x^0_{\rm H} = \mathrm{d}\tau$.

Following Eqs. (C21) and (C22), the perturbation functions change upon the conformal transformation as

$$\psi \mapsto \psi + \mathcal{H}\zeta^0, \qquad A \mapsto A - \frac{\mathrm{d}}{\mathrm{d}\tau}\zeta^0 - 3\mathcal{H}\zeta^0, \tag{C55}$$

$$E \mapsto E - \xi, \qquad B \mapsto B + \zeta^0 - \frac{\mathrm{d}}{\mathrm{d}\tau}\xi, \tag{C56}$$

where we introduced the rescaled variable $\zeta^0 = a^2\xi^0$ and the Hubble parameter in conformal time, $\mathcal{H} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\tau}$. The line element of Eq. (C53) is then obtained via the transformation

$$\zeta^0 = E' - B, \tag{C57}$$

$$\xi = E. \tag{C58}$$

where A and ψ transform to

$$A \mapsto A - 3\mathcal{H}(E' - B) - (E' - B)' \equiv \Psi, \tag{C59}$$

$$\psi \mapsto \psi + \mathcal{H}(E' - B) \equiv -\Phi. \tag{C60}$$

Indeed as one can easily verify, the perturbation functions Ψ and Φ are fully gauge invariant.

In the new coordinates, the gauge-invariant curvature satisfies

$$\frac{\mathrm{d}^2\mathcal{R}}{\mathrm{d}\tau^2} + 2\mathcal{H}\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\tau} - \nabla^2\mathcal{R} = 0. \tag{C61}$$

Since $\tilde{\mathcal{R}}$ is, in contrast to $\mathcal{R}$, not gauge invariant, it changes under the transformation above to

$$\tilde{\mathcal{R}} \mapsto \tilde{\mathcal{R}} - \nabla^2 E. \tag{C62}$$

However, since the equation for E in Eq. (C8) is valid manifestly (not only in harmonic gauge) and is the same as that of $\mathcal{R}$, the equation for $\tilde{\mathcal{R}}$ in conformal longitudinal gauge is given by

$$\frac{\mathrm{d}^2\tilde{\mathcal{R}}}{\mathrm{d}\tau^2} + 2\mathcal{H}\frac{\mathrm{d}\tilde{\mathcal{R}}}{\mathrm{d}\tau} - \nabla^2\tilde{\mathcal{R}} = 0. \tag{C63}$$

As a result, $\tilde{\mathcal{R}}$ satisfies the same equation in harmonic and CL coordinates, but solutions of $\tilde{\mathcal{R}}$ change accordingly.

We consider next the four reference fields $\{\chi^\mu\}$, which satisfy the Klein-Gordon (KG) equation

$$\partial_a\left(\sqrt{-g}g^{ab}\partial_b\right)\chi^\mu = 0. \tag{C64}$$

In harmonic coordinates, we chose adapted solutions of the form $\chi^\mu = \kappa^\mu x^\mu$. Plugging this ansatz into the KG equation, one reobtains the harmonic gauge conditions

$$c^\mu = \partial_\alpha\left(\sqrt{-g}g^{\alpha\beta}\partial_\beta\right)\kappa^\mu x^\mu \stackrel{!}{=} 0. \tag{C65}$$

Crucially, since coordinates do in general not transform as scalars nor do the four constraints c^μ. That can also be seen by the relation of c^μ and the Christoffel symbols, $c^\mu = \Gamma^\mu_{\alpha\beta}g^{\alpha\beta}$, which are well known to not transform as tensors. As a consequence, $c^\mu = 0$ is only satisfied in harmonic gauge and is violated in general in other coordinates $\{x^a\}$, i.e., $c^\mu(x^a) \neq 0$.

The behavior of clocks and rods differs significantly in different coordinates. To see that explicitly, note that the ansatz $\chi^0 = \kappa^0 x^0$ is only a solution of the KG equation in harmonic coordinates and is invalid in particular in conformal-longitudinal coordinates. In contrast, plugging the ansatz $\chi^i = \kappa^i x^i_{\mathrm{CL}}$ into the KG equation in conformal-longitudinal coordinates yields

$$\begin{aligned}\partial_i(\sqrt{-g}g^{ij}\partial_j\chi^l) &= \kappa^l\partial_i\left[a^2(1+\Psi-\Phi)\delta^{ij}\partial_j x^l_{\mathrm{CL}}\right]\\ &= a^4\kappa^l\partial_l(\Psi-\Phi) = 0,\end{aligned} \tag{C66}$$

where we expanded the geometric quantities up to first order. The last equation above holds true on shell and by assuming vanishing shear which is the case in our system.

Consequently, when plugging the on shell solution $\chi^i = \kappa^i x^i_{\mathrm{CL}}$ into the energy-momentum tensor, the background components $\bar{T}_{\mu\nu}(x_{\mathrm{CL}})$ contain a contribution of the rods. Explicitly, we find in CL coordinates the inequalities

$$k^2 \gg \mathcal{H}^2 \geq \frac{\kappa^2}{2M^2_{\mathrm{Pl}}}, \tag{C67}$$

where $\mathcal{H}$ is again the Hubble parameter in conformal time, such that for $\kappa = M^2_{\mathrm{Pl}}$, one obtains

$$k^2 \gg M^2_{\mathrm{Pl}}. \tag{C68}$$

Following the arguments of the last paragraph of Sec. IV E, these inequalities show the mixing of subhorizon and trans-Planckian modes. Concluding, the computations of this section demonstrate that the observations of Sec. IV E are not a particular feature of harmonic coordinates but are also present using the commonly used conformal-longitudinal coordinates.

APPENDIX D: DERIVATION OF CONDENSATE DYNAMICS

In this Appendix we provide the detailed derivations of the dynamical equations for the perturbed condensate introduced in Sec. III B. To that end, we consider an expansion of the kinetic kernels

$$\mathcal{K}_+((\chi^0)^2, \pi^2_\phi) = \sum_{n=0}^\infty \frac{\mathcal{K}^{(2n)}_+(\pi^2_\phi)}{(2n)!}(\chi^0)^{2n}, \tag{D1}$$

$$\mathcal{K}_-(|\boldsymbol{\chi}|^2,\pi_\phi^2)=\sum_{n=0}^{\infty}\frac{\mathcal{K}_-^{(2n)}(\pi_\phi^2)}{(2n)!}|\boldsymbol{\chi}|^{2n}, \tag{D2}$$

the existence of which is supported by the studies of [106]. Notice that the reference fields are coupled to the GFT model via the Eqs. (2.26a) and (2.26b), such that their expansion differs slightly from that discussed in [70]. The reduced condensate wave functions $\tilde{\sigma}$ and $\tilde{\tau}$ are expanded in derivatives

$$\tilde{\sigma}(\chi^0+x^0,\pi_\phi)=\sum_{n=0}^{\infty}\frac{\tilde{\sigma}^{(n)}(x^0,\pi_\phi)}{n!}(\chi^0)^n, \tag{D3}$$

$$\tilde{\tau}(\chi^0+x^0,\pi_\phi)=\sum_{n=0}^{\infty}\frac{\tilde{\tau}^{(n)}(x^0,\pi_\phi)}{n!}(\chi^0)^n, \tag{D4}$$

where $\tilde{\sigma}^{(n)}$ denotes the n-th derivative with respect to the clock argument, applying similarly to $\tilde{\tau}$. These expansions will be employed both, for the background as well as the perturbed part of the equations of motion.

1. Background equations

a. Spacelike part

Using the expansions of Eqs. (D1) and (D3), the (regularized) spacelike background equation (3.18) evaluates to

$$\begin{aligned}0&=\int \mathrm{d}\chi^0\mathcal{K}_+((\chi^0)^2,\pi_\phi^2)\tilde{\sigma}(\chi^0+x^0,\pi_\phi)\eta_{\epsilon^+}(\chi^0,\pi_0^+)\\&=\sum_{m,n}\frac{\mathcal{K}_+^{(2m)}(\pi_\phi^2)\tilde{\sigma}^{(n)}(x^0,\pi_\phi)}{(2m)!n!}\int \mathrm{d}^4\chi\eta_{\epsilon^+}(\chi^0,\pi_0^+)(\chi^0)^{2m+n}\\&\approx\mathcal{K}_+^{(0)}\left[\left(I_0+I_2\frac{\mathcal{K}_+^{(2)}}{2\mathcal{K}_+^{(0)}}\right)\tilde{\sigma}(x^0,\pi_\phi)+I_1\partial_0\tilde{\sigma}(x^0,\pi_\phi)+\frac{1}{2}I_2\partial_0^2\tilde{\sigma}(x^0,\pi_\phi)\right]\int \mathrm{d}^3\chi.\end{aligned} \tag{D5}$$

Following [70], we introduced the function $I_{2m+n}(\epsilon^+,\pi_0^+)$, defined as the χ^0 integration, which can be explicitly evaluated to

$$I_n(\epsilon^+,\pi_0^+)=\mathcal{N}_{\epsilon^+}\sqrt{2\pi\epsilon^+}\left(i\sqrt{\frac{\epsilon^+}{2}}\right)^n\mathrm{e}^{-z_+^2}H_n\left(\sqrt{\frac{\epsilon^+}{2}}\pi_0^+\right), \tag{D6}$$

where H_n are the Hermite polynomials and $z_+^2=\epsilon^+(\pi_0^+)^2/2$. We truncated the expansion at order ϵ^+, leading to the condition that only terms with $2m+n\leq 2$ contribute. Introducing the quantities

$$E_+^2(\pi_\phi)\coloneqq\frac{2}{\epsilon^+(2z_+^2-1)}-\frac{\mathcal{K}_+^{(2)}}{\mathcal{K}_+^{(0)}}, \tag{D7}$$

$$\tilde{\pi}_0^+\coloneqq\frac{\pi_0^+}{2z_+^2-1}, \tag{D8}$$

we finally obtain

$$\partial_0^2\tilde{\sigma}(x^0,\pi_\phi)-2i\tilde{\pi}_0^+\partial_0\tilde{\sigma}(x^0,\pi_\phi)-E_+^2(\pi_\phi)\tilde{\sigma}(x^0,\pi_\phi)=0. \tag{D9}$$

b. Timelike part

On the timelike sector, the procedure to obtain the equations of motion differs slightly because of the different peaking properties of τ and the mere rod dependence of the timelike kernel $\mathcal{K}_-$. Starting with Eq. (3.19) and inserting the expansions of Eqs. (D2) and (D4), we obtain

$$\begin{aligned}0&=\int \mathrm{d}^4\chi\mathcal{K}_-(|\boldsymbol{\chi}|^2,\pi_\phi^2)\tilde{\tau}(\chi^0+x^0,\pi_\phi)\eta_{\epsilon^-}(\chi^0,\pi_0^-)\eta_\delta(|\boldsymbol{\chi}|,\pi_x)\\&=\sum_n\frac{\tilde{\tau}^{(n)}(x^0,\pi_\phi)}{n!}\int \mathrm{d}\chi^0\eta_{\epsilon^-}(\chi^0,\pi_0^-)(\chi^0)^n\int \mathrm{d}^3\chi\mathcal{K}_-(|\boldsymbol{\chi}|^2,\pi_\phi)\eta_\delta(|\boldsymbol{\chi}|,\pi_x).\end{aligned} \tag{D10}$$

Assuming that the spatial integral is nonzero, the equations factorize. Truncating at linear order in ϵ^- finally yields

$$I_0^-\tilde{\tau}(x^0,\pi_\phi)+I_1^-\partial_0\tilde{\tau}(x^0,\pi_\phi)+\frac{1}{2}I_2^-\partial_0^2\tilde{\tau}(x^0,\pi_\phi)\approx 0, \tag{D11}$$

where I_n^- is defined equivalently to Eq. (D6) but evaluated on the timelike peaking parameters ϵ^- and π_0^-. Introducing

$$E_-^2 := \frac{2}{\epsilon^-(2z_-^2-1)}, \tag{D12}$$

$$\tilde{\pi}_0^- := \frac{\pi_0^-}{2z_-^2-1}, \tag{D13}$$

the background equation for the timelike reduced condensate wave function reads as

$$\partial_0^2\tilde{\tau}(x^0,\pi_\phi) - 2i\tilde{\pi}_0^-\partial_0\tilde{\tau}(x^0,\pi_\phi) - E_-^2\tilde{\tau}(x^0,\pi_\phi) = 0. \tag{D14}$$

Notice that due to the interplay of peaking and kernel dependencies, the quantity E_- does not carry a matter momentum dependence, in contrast to $E_+(\pi_\phi)$.

2. Perturbation equations

Continuing the analysis of the equations of motion, we derive in this section the perturbed equations of motion for the spacelike and then the timelike sector.

a. Spacelike part

The starting point is Eq. (3.26), which we complement by the peaking properties of the condensate wave functions σ and τ. As for the background, we expand $\mathcal{K}_+, \tilde{\sigma}$, and $\tilde{\tau}$ according to Eqs. (D1), (D3), and (D4). Also we use the relation of $\delta\Psi$ and $\delta\Phi$ in Eq. (3.28) and the relation of peaking parameters in Eq. (3.30). Truncating then at linear order in ϵ^+ and δ, one obtains

$$\begin{aligned}0 = \mathcal{K}_+^{(0)}(p_\phi^2)\Bigg[&\left(I_0 + I_2\frac{\mathcal{K}_+^{(2)}}{2\mathcal{K}_+^{(0)}}\right)\delta\Psi(J_{0,\mathbf{0}}\bar{\tilde{\tau}} + f\mathrm{e}^{i\theta_f}\bar{\tilde{\sigma}}) + I_1\partial_0(\delta\Psi(J_{0,\mathbf{0}}\bar{\tilde{\tau}} + f\mathrm{e}^{i\theta_f}\bar{\tilde{\sigma}}))\\ &+\frac{I_2}{2}\partial_0^2(\delta\Psi(J_{0,\mathbf{0}}\bar{\tilde{\tau}} + f\mathrm{e}^{i\theta_f}\bar{\tilde{\sigma}})) + \bar{\tilde{\tau}}I_0\frac{J_{0,(0,0,2)}}{2}\nabla_{\boldsymbol{x}}^2\delta\Psi\Bigg].\end{aligned} \tag{D15}$$

All fields, $\tilde{\sigma}$, $\tilde{\tau}$, and $\delta\Psi$ are evaluated at x^0, respectively x^i, and the peaked matter momentum p_ϕ. Notice that the first-order time derivative enters with a coefficient I_1 and not its complex conjugated because of the relation between π_0^+ and π_0^- as well as the phase factor of the function f in Eq. (3.29). The functions I_0 and I_2 are the functions of temporal peaking parameters defined in the section above, evaluated on the + parameters. Owing to the spatial peaking of the timelike condensate τ, coefficients $J_{m,(n_1,n_2,n_3)}$ appear in the expression above, defined as

$$J_{m,(n_1,n_2,n_3)} = \int \mathrm{d}^3\chi\eta_\delta(|\chi|,\pi_x)|\chi|^{2m}\prod_{i=1}^{3}(\chi^i)^{n_i}. \tag{D16}$$

The relevant coefficients for the derivation of the equations of motion are $J_{0,\mathbf{0}}$, $J_{2,\mathbf{0}}$, and $J_{0,(0,0,2)}$, explicitly defined as [70]

$$J_{0,\mathbf{0}} = -2\mathcal{N}_\delta\sqrt{2\pi}\pi^2\delta^{3/2}z^2\mathrm{e}^{-z^2}, \tag{D17}$$

$$J_{2,\mathbf{0}} = 4\mathcal{N}_\delta\sqrt{2\pi}\pi^2\delta^{5/2}z^4\mathrm{e}^{-z^2}, \tag{D18}$$

$$J_{0,(0,0,2)} = \frac{16}{3}\mathcal{N}_\delta\sqrt{2\pi}\pi\delta^{5/2}z^4\mathrm{e}^{-z^2}, \tag{D19}$$

keeping only first-order contributions in the peaking parameter δ, where $z^2 = \delta\pi_x^2/2$.

Factorizing $I_2/2$ from the spacelike perturbed equations of motion above, we finally obtain

$$\begin{aligned}0 = {}&\partial_0^2(\delta\Psi(J_{0,\mathbf{0}}\bar{\tilde{\tau}} + f\mathrm{e}^{i\theta_f}\bar{\tilde{\sigma}})) - 2i\tilde{\pi}_0^+\partial_0(\delta\Psi(J_{0,\mathbf{0}}\bar{\tilde{\tau}} + f\mathrm{e}^{i\theta_f}\bar{\tilde{\sigma}}))\\ &+ -E_+^2\delta\Psi(J_{0,\mathbf{0}}\bar{\tilde{\tau}} + f\mathrm{e}^{i\theta_f}\bar{\tilde{\sigma}}) + \alpha\tilde{\tau}\nabla_{\boldsymbol{x}}^2\delta\Psi,\end{aligned} \tag{D20}$$

where the parameter α is defined as

$$\alpha := \frac{I_0J_{0,(0,0,2)}}{I_2}. \tag{D21}$$

b. Timelike part

To derive the perturbed condensate equation on the timelike sector, given in Eq. (3.34), our starting point is Eq. (3.33). We use the expansions of Eqs. (D2), (D3), and (D4) as well as the relations of Eqs. (3.28) and (3.30) to arrive at

$$0 = \int \mathrm{d}^3\chi \mathcal{K}_-(|\chi|^2, p_\phi^2)\left[I_0\delta\Psi\bar{\bar{\sigma}} + \bar{I}_1\partial(\delta\Psi\bar{\bar{\sigma}}) + \frac{I_2}{2}\partial_0^2(\delta\Psi\bar{\bar{\sigma}})\right] + \mathcal{K}_-^{(0)}(p_\phi^2)\left[\left(I_0J_{0,\mathbf{0}} + I_0J_{2,\mathbf{0}}\frac{\mathcal{K}_-^{(2)}}{\mathcal{K}_-^{(0)}}\right)\delta\Xi\bar{\bar{\tau}} + J_{0,\mathbf{0}}I_1\partial(\delta\Xi\bar{\bar{\tau}}) + J_{0,\mathbf{0}}\frac{I_2}{2}\partial_0^2(\delta\Xi\bar{\bar{\tau}}) + I_0\frac{J_{0,(0,0,2)}}{2}\bar{\bar{\tau}}\nabla_{\boldsymbol{x}}^2\delta\Xi\right], \tag{D22}$$

where the coefficients $I_0, I_2, J_{0,\mathbf{0}}, J_{2,\mathbf{0}}$, and $J_{0,(0,0,2)}$ are defined as above. Using the background equations of motion in the classical limit, with solutions given by Eqs. (3.24) and (3.25), and factorizing $J_{0,\mathbf{0}}I_2/2$, one obtains

$$0 = \bar{\bar{\sigma}}\int \mathrm{d}^3\chi \mathcal{K}_-(|\boldsymbol{\chi}|, p_\phi^2)\left[\left(\frac{2I_0}{I_2} + (\pi_0^+)^2 + \mu_+^2\right)\delta\Psi + 2\mu_+\partial_0\delta\Psi + \partial_0^2\delta\Psi\right] + \mathcal{K}_-^{(0)}J_{0,\mathbf{0}}\bar{\bar{\tau}}\left[\left(\frac{2I_0}{I_2} + \frac{I_0J_{2,\mathbf{0}}}{I_2J_{0,\mathbf{0}}}\frac{\mathcal{K}_-^{(2)}}{\mathcal{K}_-^{(0)}} + (\pi_0^+)^2 + \mu_-^2\right) + 2\mu_-\partial_0\delta\Xi + \partial_0^2\delta\Xi + \frac{\alpha}{J_{0,\mathbf{0}}}\nabla_{\boldsymbol{x}}^2\delta\Xi\right]. \tag{D23}$$

Using the definition of μ_-^2 and introducing

$$\beta \coloneqq -\frac{I_0J_{2,\mathbf{0}}}{I_2J_{0,\mathbf{0}}}\frac{\mathcal{K}_-^{(2)}}{\mathcal{K}_-^{(0)}}, \qquad \gamma \coloneqq \frac{\alpha}{J_{0,\mathbf{0}}}, \tag{D24}$$

the perturbed equation of motion on the timelike sector is finally given by

$$0 = \bar{\bar{\sigma}}\int \mathrm{d}^3\chi \mathcal{K}_-(|\boldsymbol{\chi}|, p_\phi^2)\left[\partial_0^2\delta\Psi + 2\mu_+\partial_0\delta\Psi - \frac{\mathcal{K}_+^{(2)}}{\mathcal{K}_+^{(0)}}\delta\Psi\right] + \bar{\bar{\tau}}\mathcal{K}_-^{(0)}J_{0,\mathbf{0}}\left[\partial_0^2\delta\Xi + 2\mu_-\partial_0\delta\Xi - \beta\delta\Xi + \gamma\nabla_{\boldsymbol{x}}^2\delta\Xi\right]. \tag{D25}$$

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