Weakly constrained double field theory as the double copy of Yang-Mills theory

Roberto Bonezzi,* Christoph Chiaffrino,† Felipe Díaz-Jaramillo,‡ and Olaf Hohm§
Institute for Physics, Humboldt University Berlin, Zum Großen Windkanal 6, D-12489 Berlin, Germany

(Received 18 September 2023; accepted 14 November 2023; published 21 March 2024)

The weakly constrained double field theory, in the sense of Hull and Zwiebach, captures the subsector of string theory on toroidal backgrounds that includes gravity, B-field, and dilaton together with all of their massive Kaluza-Klein and winding modes, which are encoded in doubled coordinates subject to the "weak constraint." Due to the complications of the weak constraint, this theory was only known to cubic order. Here we construct the quartic interactions for the case that all dimensions are toroidal and doubled. Starting from the kinematic C_∞ algebra $\mathcal{K}$ of pure Yang-Mills theory and its hidden Lie-type algebra, we construct the L_∞ algebra of weakly constrained double field theory on a subspace of the "double copied" tensor product space $\mathcal{K}\otimes\bar{\mathcal{K}}$, by doing homotopy transfer to the weakly constrained subspace and performing a nonlocal shift that is well-defined on the torus. We test the resulting three-brackets and establish their uniqueness up to cohomologically trivial terms, by verifying the Jacobi identities up to homotopy for the gauge sector.

DOI: 10.1103/PhysRevD.109.066020

I. INTRODUCTION

String theory is often thought of as something quite different from quantum field theory. There are, however, formulations of string theory as an "ordinary" field theory, known as string field theory, with the only somewhat unusual feature being that it carries an infinite number of component fields (see [1,2] for modern reviews). These component fields include familiar fields such as vector spin-1 gauge fields (in open string theory) or the tensor spin-2 fluctuations of gravity (in closed string field theory). Since string theory is UV-finite, we thus have with closed string field theory a quantum field theory of gravity that is at least perturbatively well-defined.

However, string theory and string field theory are technically infamously involved and also include numerous exotic ingredients such as extra dimensions and infinite towers of massive fields of ever-increasing spin. Undoubtedly, this state of affairs is part of the reason that so far no compelling scenario has emerged of how to connect string theory to real-world observations. At the same time, general qualitative aspects of the real-world physics encoded in the standard model of particle physics are naturally found in string theory. Yang-Mills theory, for instance, which governs all interactions in the realm of particle physics, can be obtained from open string field theory by eliminating (or rather integrating out) all massive string modes. Since Yang-Mills theory defines a perfectly good quantum field theory, without any need to pass to the full open string field theory, one may thus wonder, by analogy, whether there are consistent theories of quantum gravity that are "smaller" than the full closed string field theory, perhaps consistent duality-invariant subsectors of string theory that include only some of the massive string modes. (Since general relativity and the low-energy supergravity actions of string theory are certainly non-renormalizable and are unlikely to be UV-finite, it is clear that, in contrast to Yang-Mills theory, any putative quantum gravity theory has to include some, and most likely infinite towers of, extra states in order to improve the UV behavior.)

In this paper, we explicitly construct a gravity theory, known as the weakly constrained double field theory [3], that includes some infinite towers of massive string modes, which provide a promising subsector for two reasons: First, at the level of scattering amplitudes there are deep relations between open and closed string theory, the Kawai-Lewellen-Tye (KLT) relations [4], which have been shown by Bern, Carrasco, and Johansson (BCJ) to have field theory analogs [5,6]. Such so-called "double copy" constructions relate pure Yang-Mills theory to "$\mathcal{N}=0$ supergravity," i.e. Einstein-Hilbert gravity coupled to a two-form (B-field) and a scalar (dilaton). In view of the

*roberto.bonezzi@physik.hu-berlin.de
†chiaffrc@hu-berlin.de
‡felipe.diaz-jaramillo@hu-berlin.de
§ohohm@physik.hu-berlin.de

double copy this theory is most efficiently formulated as a (strongly constrained) double field theory (DFT) [7–10]. Second, DFT is believed to exist also in a weakly constrained version that features genuine massive string modes and is expected to exhibit an improved UV behavior. Concretely, weakly constrained DFT is defined on toroidal backgrounds and includes the massless fields of $\mathcal{N}=0$ supergravity together with all of their massive Kaluza-Klein and winding modes. Such a theory can in principle be derived from the full closed string field theory by integrating out all fields that do not belong to the DFT sector [11–13]. As argued by Sen, the weakly constrained DFT so obtained would inherit the UV finiteness of string theory [11]. Therefore, "bootstrapping" such a theory directly from Yang-Mills theory via double copy, plausibly upon also including α' corrections [14,15], appears to be a promising path toward quantum gravity.

The construction of weakly constrained DFT to be presented here, which was announced and outlined in [16], is based on homotopy algebras such as homotopy Lie or L_∞ algebras. In theoretical physics, such structures were first discovered in string field theory [17] and only later realized to govern also conventional field theories such as Yang-Mills theory; see in particular the early work of Zeitlin [18,19] (which remarkably already anticipated aspects of double copy [20,21]). Homotopy algebras also play a role in the formulation of quantum field theory due to Costello [22] and Gwilliam-Costello [23]. We refer to [24] for a self-contained introduction to L_∞ algebras and the general dictionary between field theories and L_∞ algebras.

Using this framework one may start from Yang-Mills theory, viewed as an L_∞ algebra, and give a perfectly precise meaning to the notion of "stripping off" color factors. While there is no such thing as a field theory of "color-stripped Yang-Mills fields," there is a homotopy algebra of such "fields," a C_∞ rather than an L_∞ algebra. Specifically, the vector space $X_{\rm YM}$, on which the L_∞ algebra of Yang-Mills theory is defined, can be decomposed as the tensor product $X_{\rm YM}=\mathcal{K}\otimes\mathfrak{g}$, where $\mathfrak{g}$ is the "color" Lie algebra of the gauge group, and the C_∞ algebra $\mathcal{K}$ encodes the "kinematics" of Yang-Mills theory [19]. C_∞ algebras are homotopy versions of *commutative associative* algebras, which means that the (graded commutative) product is only associative "up to homotopy," with the failure of associativity being governed by a "three-product." The kinematic algebra $\mathcal{K}$ of Yang-Mills theory is thus an "associative-type" algebra. However, the study of scattering amplitudes underlying the double copy indicates that there is also a hidden "Lie-type" algebra. At the level of any local off-shell Lagrangian formulation there is no such Lie algebra in the strict sense, not even up to homotopy, but a further relaxation of the Lie algebra axioms was proposed by Reiterer in [25] and proved to be realized in Yang-Mills theory in four (Euclidean) dimensions (with complexified fields and momenta). This algebra goes under the forbidding yet fitting name $BV_\infty^\square$ and is a generalization of a Batalin-Vilkovisky (BV) algebra. Here $\square$ refers to the flat space wave operator that, being of second order, is the origin of the obstructions that prevent $\mathcal{K}$ from carrying a homotopy Lie algebra. Nevertheless, the $BV_\infty^\square$ algebra seems to be at the core of the so-called color-kinematics duality of Yang-Mills scattering amplitudes. More recently, for Yang-Mills theory in arbitrary dimensions (and spacetime signature), we displayed this algebra up to trilinear maps [26].

In order to double copy Yang-Mills theory one considers the tensor product $\mathcal{K}\otimes\bar{\mathcal{K}}$ with a second copy $\bar{\mathcal{K}}$ of the kinematic algebra. This total space consists of functions of doubled (unconstrained) coordinates (coordinates x associated with $\mathcal{K}$ and coordinates $\bar{x}$ associated with $\bar{\mathcal{K}}$). This space inherits an algebra of precisely the same kind: a BV_∞^Δ algebra, where $\Delta=\frac{1}{2}(\square-\bar{\square})$ is the difference of the respective wave operators. This unconstrained space does not carry an unobstructed L_∞ algebra, and hence no consistent field theory, due to Δ being second order, but by restricting to a suitable subspace one can eliminate the obstructions. Identifying coordinates x with coordinates $\bar{x}$ implies $\Delta\equiv 0$, which yields the *strongly constrained* DFT that can be viewed as a duality invariant formulation of $\mathcal{N}=0$ supergravity. More precisely, so far this was established to quartic order in fields [26]. (See [27] for a quick and polemic introduction, [28–33] for DFT and double copy, and [34–39] for homotopy algebras and double copy.)

In this paper we will show how to construct *weakly constrained* DFT for toroidal and hence Euclidean backgrounds in which $\Delta=0$ is imposed as a constraint on fields, which then still genuinely depend on doubled coordinates and hence encode both physical winding and momentum modes. (Since all dimensions are toroidal and doubled, this theory does not yet include a noncompact time direction, which at least in conventional thinking should remain undoubled. We leave the construction of the full Lorentzian theory for future work.) To this end, one performs homotopy transfer to the subspace with $\Delta=0$ (see, e.g., [12] for a self-contained introduction to homotopy transfer). This still does not give an unobstructed L_∞ algebra, but by further imposing an algebraic constraint known from the level-matching constraints of string theory one can redefine the desired three-bracket by a nonlocal but perfectly well-defined shift so that one obtains a genuine L_∞ algebra to the order relevant for the quartic theory. This solves a problem that was outstanding since the modern inception of DFT by Hull and Zwiebach [3]. It should be emphasized that this solution of the problem is unique, up to cohomologically trivial redefinitions, given the "initial data" of the differential B_1 and two-bracket B_2 of weakly constrained DFT encoded in the cubic theory of [3].

What is perhaps the most striking aspect of this solution of the long-open problem of constructing weakly constrained DFT is that it relates to, and in some ways is almost identical with, deep hidden structures *that are present in Yang-Mills theory proper*, without any reference to gravity. While the conventional Lagrangian formulation of Yang-Mills theory relies only on the color Lie algebra $\mathfrak{g}$ and the kinematic C_∞ algebra, the computation of scattering amplitudes requires more structures, as for instance exhibited in gauge conditions. Given these extra structures, the kinematic vector space comes close to be a (homotopy) Lie algebra, but this is obstructed by the wave operator $\square$ being of second order. When computing scattering amplitudes one goes on-shell, so that $\square$ gives zero when acting on single fields (polarization vectors), but even then the algebraic structure is obstructed since the product of two on-shell fields is generally not on-shell. Thus, even on the subspace with $\square = 0$ the $\mathrm{BV}_\infty^\square$ algebra does not yield an unobstructed homotopy Lie algebra. In the amplitude literature it has been shown how to shift the kinematic numerators so that these obey Jacobi-type identities, a property known as color-kinematic duality. The problem of constructing weakly constrained DFT is therefore technically analogous to the problem of making color-kinematics manifest in Yang-Mills theory proper, just with $\mathrm{BV}_\infty^\Delta$ instead of $\mathrm{BV}_\infty^\square$. We hope to further explore this intriguing connection in the future.

The remainder of this paper is organized as follows. In Sec. II we introduce the C_∞ algebra of the kinematic space $\mathcal{K}$ of Yang-Mills theory, to the order of trilinear maps, and we introduce the $\mathrm{BV}_\infty^\square$ algebra. While to a large part this is a review of results presented in [26], we also introduce a more streamlined notation for objects of $\mathcal{K}$ and its multilinear maps, which is instrumental in order to efficiently compute the double copied maps in later sections. These results are useful additions to [26], even just for strongly constrained DFT. In Sec. III we prove, again to the order of trilinear maps relevant for the quartic theory, that the (unconstrained) doubled space $\mathcal{K} \otimes \bar{\mathcal{K}}$ carries a $\mathrm{BV}_\infty^\Delta$ algebra. Finally, in Sec. IV we construct weakly constrained DFT to quartic order, by first doing homotopy transfer to the subspace with $\Delta = 0$ and then performing a nonlocal but well-defined shift. We verify the inevitability of this nonlocal shift by computing the three-brackets of the gauge sector and by verifying the generalized Jacobi identities. We close with a summary and outlook in Sec. V. where we discuss possible applications and generalizations. In two Appendixes we collect all maps for Yang-Mills theory, and we give some of the technically challenging proofs.

II. THE KINEMATIC ALGEBRA OF YANG-MILLS

Here we start by reviewing the $\mathrm{BV}_\infty^\square$ algebra of Yang-Mills theory, up to its trilinear maps. In doing so we will introduce the necessary formalism and fix our conventions and notation. We follow closely the discussion in [26,40], although with some differences in the notation that, we believe, lead to a more streamlined treatment.

We employ a formulation of Yang-Mills theory with an auxiliary scalar field φ, whose action is given by [40]

$$S = \int d^d x \left[\frac{1}{2} A_a^\mu \square A_\mu^a - \frac{1}{2} \varphi_a \varphi^a + \varphi_a \partial^\mu A_\mu^a - f_{abc} \partial_\mu A_\nu^a A^{\mu b} A^{\nu c} - \frac{1}{4} f^e{}_{ab} f_{ecd} A_\mu^a A_\nu^b A^{\mu c} A^{\nu d} \right], \quad (2.1)$$

where f_{abc} are the structure constants of the color Lie algebra $\mathfrak{g}$. The cubic and quartic vertices are standard, and one recovers the usual action upon integrating out φ. In Yang-Mills theory all objects, including gauge parameters, fields, and field equations, take values in the Lie algebra $\mathfrak{g}$ of the gauge group. It is thus natural to view the space of Yang-Mills theory as the tensor product $\mathcal{K} \otimes \mathfrak{g}$, where elements of $\mathcal{K}$ are color-stripped local spacetime fields, which we still refer to as gauge parameters, fields, and so on.

A. The graded vector space $\mathcal{K}$

Let us describe in more detail the structure of the kinematic vector space $\mathcal{K}$. It is a graded vector space given by the direct sum of subspaces $\mathcal{K}_i$ of homogenous degree: $\mathcal{K} = \bigoplus_{i=0}^3 \mathcal{K}_i$. Elements in each $\mathcal{K}_i$ are identified as gauge parameters λ, fields $\mathcal{A}$, field equations $\mathcal{E}$, and Noether identities $\mathcal{N}$ according to the following diagram:

$$\begin{matrix} \mathcal{K}_0 & \mathcal{K}_1 & \mathcal{K}_2 & \mathcal{K}_3 \\ \lambda & \mathcal{A} & \mathcal{E} & \mathcal{N} \end{matrix}. \quad (2.2)$$

We take the field $\mathcal{A}$ to contain both the (color-stripped) gauge vector field A_μ and the scalar φ, and similarly the equations of motion $\mathcal{E}$ have a vector and a scalar component.

In order to display explicitly its degree structure, we find it useful to view $\mathcal{K}$ as the tensor product of a finite-dimensional graded vector space $\mathcal{Z} = \bigoplus_{i=0}^3 \mathcal{Z}_i$ with the space of smooth spacetime functions of degree zero: $\mathcal{K} = \mathcal{Z} \otimes C^\infty(\mathcal{M})$. Here $\mathcal{M}$ is flat d-dimensional Minkowski spacetime, but the signature is immaterial for the following discussion. The vector space $\mathcal{Z}$ is defined by giving a basis. To this end let us introduce a $(d+2)$-component graded vector $\theta_M = (\theta_+, \theta_\mu, \theta_-)$, where $\mu = 0, 1, \ldots, d-1$ is a Lorentz vector index. The degrees of the components are given by

$$|\theta_+| = 0, \qquad |\theta_\mu| = 1, \qquad |\theta_-| = 2, \quad (2.3)$$

which we sometimes summarize by writing $|\theta_M| = 1 - M$, where M means $+1$, 0, or -1, depending on the index. Next, we take a second copy of these vectors with degrees shifted by one, which we denote by $c\theta_M$, with $|c\theta_M| = 2 - M$, or

$$|c\theta_+| = 1, \qquad |c\theta_\mu| = 2, \qquad |c\theta_-| = 3. \tag{2.4}$$

A basis of $\mathcal{Z}$ is then given by

$$Z_A = (\theta_M, c\theta_M). \tag{2.5}$$

The above characterization of $\mathcal{Z}$ exhibits the manifest $\mathbb{Z}_2$ symmetry that exchanges θ_M and $c\theta_M$. This isomorphism between the subspaces generated by θ_M and $c\theta_M$, respectively, can be implemented by nilpotent operators b and c defined by their action on the basis Z_A:

$$\begin{aligned} c(\theta_M) &\coloneqq c\theta_M, \qquad c(c\theta_M) \coloneqq 0, \\ b(\theta_M) &\coloneqq 0, \qquad b(c\theta_M) \coloneqq \theta_M. \end{aligned} \tag{2.6}$$

The degrees of b and c are thus fixed to be $|c| = +1$ and $|b| = -1$, and from their definition one can see that they obey the algebra

$$c^2 = 0, \qquad b^2 = 0, \qquad bc + cb = 1. \tag{2.7}$$

The basis elements of $\mathcal{Z}$ can be displayed according to their degree in a way that emphasizes the $\mathbb{Z}_2$ symmetry:

$$\begin{array}{cccccccc} \mathcal{Z}_0 & & \mathcal{Z}_1 & & \mathcal{Z}_2 & & \mathcal{Z}_3 & \\ \theta_+ & & \theta_\mu & & \theta_- & & & \\ & \overset{b}{\nwarrow} & & \overset{b}{\nwarrow} & & \overset{b}{\nwarrow} & & \\ & & c\,\theta_+ & & c\,\theta_\mu & & c\,\theta_- & \end{array}, \tag{2.8}$$

where we have indicated the action of b (c acts by reversing the arrows).

In addition to the above $\mathbb{Z}_2$ symmetry, $\mathcal{Z}$ can be equipped with an odd symplectic bilinear form[1] ω of degree $|\omega| = -3$, satisfying

$$\omega(Z_1, Z_2) = (-1)^{Z_1 Z_2}\omega(Z_2, Z_1), \tag{2.9}$$

which is symmetric since it always pairs odd with even elements. We specify ω by giving its components $\omega(Z_A, Z_B)$ in the above basis:

$$\begin{aligned} \omega(\theta_+, c\theta_-) &= \omega(c\theta_-, \theta_+) = -1, \\ \omega(\theta_-, c\theta_+) &= \omega(c\theta_+, \theta_-) = +1, \\ \omega(\theta_\mu, c\theta_\nu) &= \omega(c\theta_\nu, \theta_\mu) = \eta_{\mu\nu}, \end{aligned} \tag{2.10}$$

where $\eta_{\mu\nu}$ is the d-dimensional Minkowski metric and all other pairings vanish.

Upon tensoring $\mathcal{Z}$ with smooth functions, we obtain the kinematic space $\mathcal{K}$ of Yang-Mills theory. The degree in $\mathcal{K}$ coincides with the one in $\mathcal{Z}$, meaning that for an homogeneous element $\psi = Zf(x)$ one has $|\psi| = |Z|$. An arbitrary element in $\mathcal{K}$ can thus be expanded as

$$\psi = Z_A \psi^A(x). \tag{2.11}$$

Comparing the degree structure (2.2) of $\mathcal{K}$ with (2.8), one infers that Yang-Mills fields, parameters, and so on are given by the following vectors in $\mathcal{K}$ with homogeneous degrees:

$$\begin{aligned} \lambda &= \theta_+ \lambda(x) \in \mathcal{K}_0, \\ \mathcal{A} &= \theta_\mu A^\mu(x) + c\theta_+ \varphi(x) \in \mathcal{K}_1, \\ \mathcal{E} &= \theta_- E(x) + c\theta_\mu E^\mu(x) \in \mathcal{K}_2, \\ \mathcal{N} &= c\theta_- \mathcal{N}(x) \in \mathcal{K}_3. \end{aligned} \tag{2.12}$$

The $\mathbb{Z}_2$ structure and the action of b and c are inherited from $\mathcal{Z}$. One can indeed draw the same diagram (2.8) in $\mathcal{K}$ to display the following:

$$\begin{array}{cccccccc} \mathcal{K}_0 & & \mathcal{K}_1 & & \mathcal{K}_2 & & \mathcal{K}_3 & \\ \lambda & & A^\mu & & E & & & \\ & \overset{b}{\nwarrow} & & \overset{b}{\nwarrow} & & \overset{b}{\nwarrow} & & \\ & & \varphi & & E^\mu & & \mathcal{N} & \end{array}, \tag{2.13}$$

where we omitted the Z_A and only wrote the component fields. The odd symplectic pairing ω induces a degree -3 inner product $\langle , \rangle$ in $\mathcal{K}$, defined by

$$\langle \psi_1, \psi_2 \rangle = \int d^d x\, \psi_1^A(x) \psi_2^B(x) \omega(Z_A, Z_B). \tag{2.14}$$

More specifically, using (2.10) one can see that the nonvanishing pairings are between fields $\mathcal{A}$ and field equations $\mathcal{E}$:

$$\langle \mathcal{A}, \mathcal{E} \rangle = \int d^d x [A_\mu(x) E^\mu(x) + \varphi(x) E(x)], \tag{2.15}$$

and between gauge parameters λ and Noether identities $\mathcal{N}$

$$\langle \lambda, \mathcal{N} \rangle = -\int d^d x\, \lambda(x) \mathcal{N}(x). \tag{2.16}$$

B. C_∞ algebra on $\mathcal{K}$

Having described $\mathcal{K}$ as a graded vector space, we now turn to reviewing the algebraic structures that can be defined on it. The consistency of Yang-Mills theory as a field theory (this includes, for instance, gauge covariance of the field equations and closure of the gauge algebra) is encoded, upon factoring out color, by a C_∞ algebra structure on $\mathcal{K}$ [19,35,40]. This is a homotopy

[1]This is closely related to the field theoretic odd symplectic structure of the Batalin-Vilkovisky formalism (see, e.g., [17,41–43]).

generalization of a commutative associative algebra where a graded vector space ($\mathcal{K}$ in the case at hand) is equipped with multilinear maps or products m_n obeying a set of quadratic relations. For the case of Yang-Mills theory, the only nonvanishing maps are an operator m_1 of degree $+1$, a bilinear product m_2 of degree zero, and a trilinear product m_3 of degree -1, summarized as $|m_n| = 2 - n$.

The nontrivial C_∞ relations to be satisfied then consist of

(i) Nilpotency of m_1:

$$m_1^2(\psi) = 0, \tag{2.17}$$

stating that m_1 is a differential, which makes $\mathcal{K}$ into a chain complex. Physically, $m_1^2 = 0$ encodes, in particular, gauge invariance of the free theory under linearized gauge transformations.

(ii) The differential m_1 acts as a derivation on m_2 (Leibniz rule):

$$m_1 m_2(\psi_1, \psi_2) = m_2(m_1\psi_1, \psi_2) + (-1)^{\psi_1} m_2(\psi_1, m_1\psi_2), \tag{2.18}$$

where ψ_i in exponents always denotes the degree $|\psi_i|$. This requirement ensures, upon tensoring with color, consistency of Yang-Mills theory up to cubic order.

(iii) The product m_2 is associative up to homotopy:

$$\begin{aligned} &m_2(m_2(\psi_1, \psi_2), \psi_3) - m_2(\psi_1, m_2(\psi_2, \psi_3)) \\ &= m_1 m_3(\psi_1, \psi_2, \psi_3) + m_3(m_1\psi_1, \psi_2, \psi_3) \\ &\quad + (-1)^{\psi_1} m_3(\psi_1, m_1\psi_2, \psi_3) \\ &\quad + (-1)^{\psi_1+\psi_2} m_3(\psi_1, \psi_2, m_1\psi_3), \end{aligned} \tag{2.19}$$

which is responsible for consistency of the theory up to quartic order and thus fully, given that Yang-Mills theory has no higher vertices.

In a C_∞ algebra, the products m_n have to further obey symmetry constraints under permutations of arguments. Specifically, the m_n have to vanish under so-called shuffle permutations. For our purposes, the relevant symmetry properties are

$$\begin{aligned} &m_2(\psi_1, \psi_2) - (-1)^{\psi_1\psi_2} m_2(\psi_2, \psi_1) = 0, \\ &m_3(\psi_1, \psi_2, \psi_3) - (-1)^{\psi_1\psi_2} m_3(\psi_2, \psi_1, \psi_3) \\ &\quad + (-1)^{\psi_1(\psi_2+\psi_3)} m_3(\psi_2, \psi_3, \psi_1) = 0, \end{aligned} \tag{2.20}$$

which, for m_2, is the same as graded symmetry. For the following discussion we find it more convenient to work with a different representation of m_3, which we denoted $m_{3\mathrm{h}}$ in [26], defined as

$$\begin{aligned} m_{3\mathrm{h}}(\psi_1, \psi_2, \psi_3) &:= \frac{1}{3}\big(m_3(\psi_1, \psi_2, \psi_3) \\ &\qquad + (-1)^{\psi_1\psi_2} m_3(\psi_2, \psi_1, \psi_3)\big), \\ m_3(\psi_1, \psi_2, \psi_3) &= m_{3\mathrm{h}}(\psi_1, \psi_2, \psi_3) \\ &\qquad - (-1)^{\psi_1(\psi_2+\psi_3)} m_{3\mathrm{h}}(\psi_2, \psi_3, \psi_1). \end{aligned} \tag{2.21}$$

This is just a redefinition of m_3, not a projection, as it can be inverted explicitly by using the above formula. The redefined $m_{3\mathrm{h}}$ is graded symmetric in its first two arguments and vanishes upon total graded symmetrization. Otherwise stated, it is a graded hook representation in terms of Young diagrams.

Given the tensor product structure of $\mathcal{K} = \mathcal{Z} \otimes C^\infty(\mathcal{M})$ and the expansion (2.11) of arbitrary vectors, the m_n products act on elements of $\mathcal{K}$ as follows:

$$\begin{aligned} m_1(\psi) &= \hat{m}_1(Z_A)\psi^A(x), \\ m_2(\psi_1, \psi_2) &= \mu[\hat{m}_2(Z_A, Z_B)(\psi_1^A(x) \otimes \psi_2^B(x))], \\ m_{3\mathrm{h}}(\psi_1, \psi_2, \psi_3) &= \mu[\hat{m}_{3\mathrm{h}}(Z_A, Z_B, Z_C)(\psi_1^A(x) \\ &\qquad \otimes \psi_2^B(x) \otimes \psi_3^C(x))], \end{aligned} \tag{2.22}$$

where the operations on the right are defined as follows: First, the operators $\hat{m}_n(Z_{A_1}, \ldots, Z_{A_n})$ are $\mathcal{Z}$-valued multidifferential operators acting on the component fields as

$$\begin{aligned} \hat{m}_1(Z)&\colon C^\infty(\mathcal{M}) \to \mathcal{Z} \otimes C^\infty(\mathcal{M}), \\ \hat{m}_2(Z_1, Z_2)&\colon C^\infty(\mathcal{M}) \otimes C^\infty(\mathcal{M}) \\ &\qquad \to \mathcal{Z} \otimes (C^\infty(\mathcal{M}) \otimes C^\infty(\mathcal{M})), \\ \hat{m}_{3\mathrm{h}}(Z_1, Z_2, Z_3)&\colon C^\infty(\mathcal{M}) \otimes C^\infty(\mathcal{M}) \otimes C^\infty(\mathcal{M}) \\ &\qquad \to \mathcal{Z} \otimes (C^\infty(\mathcal{M}) \otimes C^\infty(\mathcal{M}) \otimes C^\infty(\mathcal{M})). \end{aligned} \tag{2.23}$$

Second, μ just denotes the local pointwise product:

$$\mu[f_1(x) \otimes \cdots \otimes f_n(x)] = f_1(x) \cdots f_n(x). \tag{2.24}$$

To clarify this notation, let us give some explicit examples (the explicit form of all m_n products can be found in [40]). Acting on the basis vectors of $\mathcal{Z}_1$ (corresponding to fields), we have

$$\begin{aligned} \hat{m}_1(\theta_\mu) &= c\theta_\mu \Box + \theta_- \partial_\mu, \\ \hat{m}_1(c\theta_+) &= -c\theta_\mu \partial^\mu - \theta_-. \end{aligned} \tag{2.25}$$

Using (2.22) one computes the action of the differential m_1 on a field $\mathcal{A} = \mathcal{Z}_A \psi^A = \theta_\mu A^\mu + c\theta_+ \varphi \in \mathcal{K}_1$ as

$$\begin{aligned} m_1(\mathcal{A}) &= \hat{m}_1(Z_A)\psi^A \\ &= \hat{m}_1(\theta_\mu)A^\mu + \hat{m}_1(c\theta_+)\varphi \\ &= c\theta_\mu(\Box A^\mu - \partial^\mu\varphi) + \theta_-(\partial\cdot A - \varphi), \end{aligned} \tag{2.26}$$

where we omitted the explicit spacetime dependence of the component fields and denoted the contraction of Lorentz indices with a dot. One can see that setting $m_1(\mathcal{A}) = 0$ corresponds to the free Maxwell equations upon solving for φ.

For the next example, the nonvanishing part of m_2 between fields $\mathcal{A}_1$ and $\mathcal{A}_2$ is encoded in the bidifferential operator

$$\begin{aligned} \hat{m}_2(\theta_\mu,\theta_\nu) = c\theta_\nu[(\partial_\mu\otimes\mathbb{1}) + 2(\mathbb{1}\otimes\partial_\mu)] \\ - c\theta_\mu[(\mathbb{1}\otimes\partial_\nu) + 2(\partial_\nu\otimes\mathbb{1})] \\ + c\theta_\rho\eta_{\mu\nu}[(\partial^\rho\otimes\mathbb{1}) - (\mathbb{1}\otimes\partial^\rho)]. \end{aligned} \tag{2.27}$$

This acts on $(A_1^\mu\otimes A_2^\nu)$ as

$$\begin{aligned} \hat{m}_2(\theta_\mu,\theta_\nu)(A_1^\mu\otimes A_2^\nu) = c\theta_\nu[(\partial\cdot A_1\otimes A_2^\nu) + 2(A_1^\mu\otimes\partial_\mu A_2^\nu)] \\ - c\theta_\mu[(A_1^\mu\otimes\partial\cdot A_2) + 2(\partial_\nu A_1^\mu\otimes A_2^\nu)] \\ + c\theta_\rho[(\partial^\rho A_1^\mu\otimes A_{2\mu}) - (A_1^\mu\otimes\partial^\rho A_{2\mu})]. \end{aligned} \tag{2.28}$$

The pointwise multiplication implemented by μ then yields (with a dot denoting contraction of Lorentz indices)

$$\begin{aligned} m_2(\mathcal{A}_1,\mathcal{A}_2) &= \mu[\hat{m}_2(\theta_\mu,\theta_\nu)(A_1^\mu\otimes A_2^\nu)] \\ &= c\theta_\mu(\partial\cdot A_1 A_2^\mu + 2A_1\cdot\partial A_2^\mu + \partial^\mu A_1\cdot A_2 \\ &\quad - (1\leftrightarrow 2)) \\ &\equiv c\theta_\mu(A_1\bullet A_2)^\mu, \end{aligned} \tag{2.29}$$

which gives the color-stripped cubic vertex of Yang-Mills as $\langle\mathcal{A}_3, m_2(\mathcal{A}_1,\mathcal{A}_2)\rangle$. Similarly, the only nonvanishing component of $m_{3\mathrm{h}}$ comes from the operator

$$\hat{m}_{3\mathrm{h}}(\theta_\mu,\theta_\nu,\theta_\rho) = (c\theta_\mu\eta_{\nu\rho} - c\theta_\nu\eta_{\mu\rho})(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}), \tag{2.30}$$

yielding

$$\begin{aligned} m_{3\mathrm{h}}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3) &= \mu[\hat{m}_{3\mathrm{h}}(\theta_\mu,\theta_\nu,\theta_\rho)(A_1^\mu\otimes A_2^\nu\otimes A_3^\rho)] \\ &= c\theta_\mu(A_1^\mu A_2\cdot A_3 - (1\leftrightarrow 2)), \end{aligned} \tag{2.31}$$

corresponding to the color-stripped quartic vertex. A complete list of the operators $\hat{m}_n$ can be found in Appendix A.

C. $\mathrm{BV}_\infty^\Box$ algebra on $\mathcal{K}$

While it is rather straightforward to determine that $\mathcal{K}$ carries a C_∞ structure (this is "just" rephrasing its usual consistency conditions), the next algebraic layer on $\mathcal{K}$ is highly nontrivial and plays a crucial role in the double copy construction. To see how this deeper structure arises on $\mathcal{K}$, one has to look at the interplay of the C_∞ algebra, given by the products m_n, with the b operator introduced before. From its definition in (2.6) and the expression of the differential m_1 one may verify that it obeys

$$b^2 = 0, \qquad bm_1 + m_1 b = \Box, \qquad |b| = -1, \tag{2.32}$$

where $\Box = \partial^\mu\partial_\mu$ is the wave operator. It turns out that (2.32) should be viewed as the general defining property of b, with our realization (2.6) and (2.13) being a particular case.[2] Although b does not play a role in the consistency relations encoded in the C_∞ algebra of the theory, it can be viewed as providing a gauge fixing condition as $b(\mathcal{A}) = 0$, as well as the related propagator as $\frac{b}{\Box}$ acting on the space of equations. The peculiar property of our realization (2.6) is that it acts on $\mathcal{K}$ as

$$b(\psi) = (bZ_A)\psi^A(x), \tag{2.33}$$

implying in particular that it is local and does not contain spacetime derivatives. This will be instrumental in order to construct a local theory from double copy.

With this second differential at our disposal, one can study its compatibility with the C_∞ products. Its graded commutator with m_1 is given in (2.32). Going one step further, b does not act as a derivation on m_2. Rather, the failure to do so defines a bracket b_2:

$$\begin{aligned} b_2(\psi_1,\psi_2) \coloneqq bm_2(\psi_1,\psi_2) - m_2(b\psi_1,\psi_2) \\ - (-1)^{\psi_1} m_2(\psi_1, b\psi_2), \end{aligned} \tag{2.34}$$

on which b acts as a derivation by construction. Given a product m_2 and a bracket b_2, one can ask if they are mutually compatible, i.e. if they obey the graded Poisson identity

$$\begin{aligned} b_2(\psi, m_2(\psi_1,\psi_2)) = m_2(b_2(\psi,\psi_1),\psi_2) \\ + (-1)^{\psi_1\psi_2} m_2(b_2(\psi,\psi_2),\psi_1). \end{aligned} \tag{2.35}$$

If this were the case (which also requires m_2 to be associative), b_2 would be a graded Lie bracket, and the triplet (b, m_2, b_2) would form a BV algebra. While this happens for Chern-Simons theory [36,45], it is not the case for Yang-Mills theory (at least in standard formulations). The compatibility (2.35) holds only up to an homotopy θ_3 and further $\Box$ deformations originating from (2.32). This prompts a cascade of higher relations, defined as a $\mathrm{BV}_\infty^\Box$ algebra in [25].

[2] For different realizations of the b operator in various gauge theories, see, e.g., [25,36,37,44,45].

In order to give all the relevant relations of the resulting $\mathrm{BV}_{\infty}^{\square}$ algebra (up to trilinear maps), we shall review a convenient input-free notation introduced in [26], which will allow us to establish the results of the forthcoming sections. In this part we will denote by $\mathcal{O}$ any linear operator in $\mathcal{K}$, of degree $|\mathcal{O}|$. Generic bilinear and trilinear maps will be denoted by $\mathcal{M}$ and $\mathcal{T}$, respectively, with arbitrary intrinsic degrees $|\mathcal{M}|$ and $|\mathcal{T}|$. Similar to (2.22), these generic maps act on elements of $\mathcal{K}$ as

$$\begin{aligned}\mathcal{O}(\psi) &= \hat{\mathcal{O}}(Z_A)\psi^A(x),\\ \mathcal{M}(\psi_1,\psi_2) &= \mu[\hat{\mathcal{M}}(Z_A,Z_B)(\psi_1^A(x)\otimes\psi_2^B(x))],\\ \mathcal{T}(\psi_1,\psi_2,\psi_3) &= \mu[\hat{\mathcal{T}}(Z_A,Z_B,Z_C)(\psi_1^A(x)\otimes\psi_2^B(x)\\ &\quad\otimes\psi_3^C(x))],\end{aligned} \tag{2.36}$$

where $\hat{\mathcal{O}}$, $\hat{\mathcal{M}}$, and $\hat{\mathcal{T}}$ are $\mathcal{Z}$-valued multidifferential operators acting on the component functions. We define the graded commutator of operators $\mathcal{O}_1$ and $\mathcal{O}_2$ by

$$[\mathcal{O}_1,\mathcal{O}_2](\psi) \coloneqq \mathcal{O}_1(\mathcal{O}_2\psi) - (-1)^{\mathcal{O}_1\mathcal{O}_2}\mathcal{O}_2(\mathcal{O}_1\psi), \tag{2.37}$$

where every symbol in exponents refers to the degree of a map or element. The commutators of an operator $\mathcal{O}$ with bilinear and trilinear maps $\mathcal{M}$ and $\mathcal{T}$ are the bilinear map $[\mathcal{O},\mathcal{M}]$ and trilinear map $[\mathcal{O},\mathcal{T}]$ given by

$$\begin{aligned}[\mathcal{O},\mathcal{M}](\psi_1,\psi_2) &\coloneqq \mathcal{O}\mathcal{M}(\psi_1,\psi_2) - (-1)^{\mathcal{O}\mathcal{M}}[\mathcal{M}(\mathcal{O}\psi_1,\psi_2)\\ &\quad + (-1)^{\psi_1\mathcal{O}}\mathcal{M}(\psi_1,\mathcal{O}\psi_2)],\\ [\mathcal{O},\mathcal{T}](\psi_1,\psi_2,\psi_3) &\coloneqq \mathcal{O}\mathcal{T}(\psi_1,\psi_2,\psi_3) - (-1)^{\mathcal{O}\mathcal{T}}\\ &\quad\times[\mathcal{T}(\mathcal{O}\psi_1,\psi_2,\psi_3)\\ &\quad + (-1)^{\mathcal{O}\psi_1}\mathcal{T}(\psi_1,\mathcal{O}\psi_2,\psi_3)\\ &\quad + (-1)^{\mathcal{O}(\psi_1+\psi_2)}\mathcal{T}(\psi_1,\psi_2,\mathcal{O}\psi_3)].\end{aligned} \tag{2.38}$$

The action of an operator $\mathcal{O}$ on a map (be it another operator, a bilinear or trilinear map) gives a map of the same kind, e.g.,

$$(\mathcal{O}\mathcal{M})(\psi_1,\psi_2) \coloneqq \mathcal{O}(\mathcal{M}(\psi_1,\psi_2)). \tag{2.39}$$

Finally, composition of bilinear maps is defined from the left and denoted by juxtaposition:

$$\mathcal{M}_1\mathcal{M}_2(\psi_1,\psi_2,\psi_3) \coloneqq \mathcal{M}_1(\mathcal{M}_2(\psi_1,\psi_2),\psi_3). \tag{2.40}$$

This is sufficient for our purposes, since all bilinear maps involved are graded symmetric. With this notation one can check that $[\mathcal{O},-]$ is a derivation on commutators and compositions, in the sense that it obeys

$$\begin{aligned}[\mathcal{O}_1,[\mathcal{O}_2,\mathcal{M}]] &= [[\mathcal{O}_1,\mathcal{O}_2],\mathcal{M}] + (-1)^{\mathcal{O}_1\mathcal{O}_2}[\mathcal{O}_2,[\mathcal{O}_1,\mathcal{M}]],\\ [\mathcal{O}_1,\mathcal{O}_2\mathcal{M}] &= [\mathcal{O}_1,\mathcal{O}_2]\mathcal{M} + (-1)^{\mathcal{O}_1\mathcal{O}_2}\mathcal{O}_2[\mathcal{O}_1,\mathcal{M}],\\ [\mathcal{O},\mathcal{M}_1\mathcal{M}_2] &= [\mathcal{O},\mathcal{M}_1]\mathcal{M}_2 + (-1)^{\mathcal{O}\mathcal{M}_1}\mathcal{M}_1[\mathcal{O},\mathcal{M}_2].\end{aligned} \tag{2.41}$$

We now turn to discuss the symmetry properties of trilinear maps $\mathcal{T}$. Since they are all graded symmetric in the first two arguments (this is the reason we chose to work with $m_{3\mathrm{h}}$ rather than m_3), they can be decomposed into a totally graded symmetric part, which we denote by $\mathcal{T}_{\mathrm{s}}$, and a graded hook part $\mathcal{T}_{\mathrm{h}} \coloneqq \mathcal{T} - \mathcal{T}_{\mathrm{s}}$. In terms of $\mathcal{T}$, the symmetrized map $\mathcal{T}_{\mathrm{s}}$ acts on three inputs as

$$\begin{aligned}\mathcal{T}_{\mathrm{s}}(\psi_1,\psi_2,\psi_3) &= \frac{1}{3}\{\mathcal{T}(\psi_1,\psi_2,\psi_3)\\ &\quad + (-1)^{\psi_1(\psi_2+\psi_3)}\mathcal{T}(\psi_2,\psi_3,\psi_1)\\ &\quad + (-1)^{\psi_3(\psi_1+\psi_2)}\mathcal{T}(\psi_3,\psi_1,\psi_2)\}.\end{aligned} \tag{2.42}$$

In line with (2.36) we want to associate a multidifferential operator $\hat{\mathcal{T}}_{\mathrm{s}}$ with $\mathcal{T}_{\mathrm{s}}$, such that

$$\mathcal{T}_{\mathrm{s}}(\psi_1,\psi_2,\psi_3) = \mu[\hat{\mathcal{T}}_{\mathrm{s}}(Z_A,Z_B,Z_C)(\psi_1^A\otimes\psi_2^B\otimes\psi_3^C)]. \tag{2.43}$$

To do so, we start by introducing a permutation operator Σ, which acts on trilinear operators as

$$(\mathcal{O}_1\otimes\mathcal{O}_2\otimes\mathcal{O}_3)\Sigma \coloneqq (\mathcal{O}_3\otimes\mathcal{O}_1\otimes\mathcal{O}_2), \tag{2.44}$$

and thus obeys

$$\begin{aligned}&\mu[\hat{\mathcal{T}}(Z_A,Z_B,Z_C)\Sigma(f_1\otimes f_2\otimes f_3)]\\ &\quad = \mu[\hat{\mathcal{T}}(Z_A,Z_B,Z_C)(f_2\otimes f_3\otimes f_1)],\\ &\mu[\hat{\mathcal{T}}(Z_A,Z_B,Z_C)\Sigma^2(f_1\otimes f_2\otimes f_3)]\\ &\quad = \mu[\hat{\mathcal{T}}(Z_A,Z_B,Z_C)(f_3\otimes f_1\otimes f_2)],\end{aligned} \tag{2.45}$$

and $\Sigma^3 = 1$. We then use this to define a projector π, obeying $\pi^2 = \pi$, so that the symmetrized and hook operators $\hat{\mathcal{T}}_{\mathrm{s}}$ and $\hat{\mathcal{T}}_{\mathrm{h}}$ are defined via

$$\hat{\mathcal{T}}_{\mathrm{s}} \coloneqq \hat{\mathcal{T}}\pi, \quad \hat{\mathcal{T}}_{\mathrm{h}} \coloneqq \hat{\mathcal{T}}(1-\pi), \quad \hat{\mathcal{T}} = \hat{\mathcal{T}}_{\mathrm{s}} + \hat{\mathcal{T}}_{\mathrm{h}}. \tag{2.46}$$

In terms of the permutation operator Σ, π is explicitly given by

$$\begin{aligned}\hat{\mathcal{T}}\pi(Z_A,Z_B,Z_C) &= \frac{1}{3}\{\hat{\mathcal{T}}(Z_A,Z_B,Z_C)\\ &\quad + (-1)^{Z_A(Z_B+Z_C)}\hat{\mathcal{T}}(Z_B,Z_C,Z_A)\Sigma\\ &\quad + (-1)^{Z_C(Z_A+Z_B)}\hat{\mathcal{T}}(Z_C,Z_A,Z_B)\Sigma^2\},\end{aligned} \tag{2.47}$$

which reproduces, upon using (2.43), the expression (2.42) for the map $\mathcal{T}_{\mathrm{s}}$. We will then use interchangeably the notation $\mathcal{T}_{\rm s} \equiv \mathcal{T}\pi$ for the symmetrized map as well. The operator $\hat{\mathcal{T}}_{\rm s}$ obeys the graded symmetry property

$$\hat{\mathcal{T}}_{\rm s}(Z_A, Z_B, Z_C) = (-1)^{Z_A(Z_B+Z_C)}\hat{\mathcal{T}}_{\rm s}(Z_B, Z_C, Z_A)\Sigma = (-1)^{Z_C(Z_A+Z_B)}\hat{\mathcal{T}}_{\rm s}(Z_C, Z_A, Z_B)\Sigma^2, \quad (2.48)$$

which implies the standard graded symmetry of the map $\mathcal{T}_{\rm s}(\psi_1, \psi_2, \psi_3)$ upon permutations of the inputs.

Let us illustrate the action of π with a concrete example. We consider the trilinear map $\mathcal{T}$ associated with the operator

$$\hat{\mathcal{T}}(\theta_\mu, \theta_\nu, \theta_\rho) = c\theta_- \eta_{\mu\nu}[(\mathbb{1} \otimes \partial_\rho \otimes \mathbb{1}) - (\partial_\rho \otimes \mathbb{1} \otimes \mathbb{1})], \quad (2.49)$$

which is part of the actual map $m_2 m_2(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$. Acting with (2.49) on $(A_1^\mu \otimes A_2^\nu \otimes A_3^\rho)$ and evaluating the pointwise product one obtains

$$\mathcal{T}(A_1, A_2, A_3) = c\theta_-(A_{1\mu}\partial_\rho A_2^\mu A_3^\rho - A_{2\mu}\partial_\rho A_1^\mu A_3^\rho), \quad (2.50)$$

where we abbreviated $A_i = \theta_\mu A_i^\mu$. According to the definition (2.47), the symmetrized operator $\hat{\mathcal{T}}\pi$ is given by

$$\begin{aligned}\hat{\mathcal{T}}\pi(\theta_\mu, \theta_\nu, \theta_\rho) = \frac{1}{3}c\theta_-(&\eta_{\mu\nu}(\mathbb{1} \otimes \partial_\rho \otimes \mathbb{1}) - \eta_{\mu\nu}(\partial_\rho \otimes \mathbb{1} \otimes \mathbb{1}) \\ &+ \eta_{\nu\rho}(\mathbb{1} \otimes \mathbb{1} \otimes \partial_\mu) - \eta_{\nu\rho}(\mathbb{1} \otimes \partial_\mu \otimes \mathbb{1}) \\ &+ \eta_{\mu\rho}(\partial_\nu \otimes \mathbb{1} \otimes \mathbb{1}) - \eta_{\mu\rho}(\mathbb{1} \otimes \mathbb{1} \otimes \partial_\nu)),\end{aligned} \quad (2.51)$$

yielding the symmetrized map

$$\begin{aligned}\mathcal{T}\pi(A_1, A_2, A_3) = \frac{1}{3}c\theta_-(&A_{1\mu}\partial_\rho A_2^\mu A_3^\rho - A_{2\mu}\partial_\rho A_1^\mu A_3^\rho \\ &+ A_{2\mu}\partial_\rho A_3^\mu A_1^\rho - A_{3\mu}\partial_\rho A_2^\mu A_1^\rho \\ &+ A_{3\mu}\partial_\rho A_1^\mu A_2^\rho - A_{1\mu}\partial_\rho A_3^\mu A_2^\rho).\end{aligned} \quad (2.52)$$

From the definition (2.38) of the commutator $[\mathcal{O}, \mathcal{T}]$, one can check that the action of $\mathcal{O}$ preserves the symmetry property of the map $\mathcal{T}$ in the sense that

$$[\mathcal{O}, \mathcal{T}]\pi = [\mathcal{O}, \mathcal{T}\pi]. \quad (2.53)$$

We conclude this review of the input-free formulation by focusing on the possible $\square$ obstructions. Since we are working on flat spacetime, the wave operator $\square$ commutes with all the multidifferential operators $\hat{\mathcal{O}}$, $\hat{\mathcal{M}}$, and $\hat{\mathcal{T}}$ in (2.36). Its commutators with the maps $\mathcal{O}$, $\mathcal{M}$, and $\mathcal{T}$ are thus entirely determined by the commutator of $\square$ on the pointwise product of functions. We thus define the following operators, acting on three local functions:

$$\begin{aligned} d_s(f_1 \otimes f_2 \otimes f_3) &\coloneqq 2(\partial^\mu f_1 \otimes \partial_\mu f_2 \otimes f_3), \\ d_\square(f_1 \otimes f_2 \otimes f_3) &\coloneqq 2(\partial^\mu f_1 \otimes \partial_\mu f_2 \otimes f_3) \\ &\quad + 2(f_1 \otimes \partial^\mu f_2 \otimes \partial_\mu f_3) \\ &\quad + 2(\partial^\mu f_1 \otimes f_2 \otimes \partial_\mu f_3). \end{aligned} \quad (2.54)$$

The subscript in d_s alludes to the Mandelstam variable s, and should not be confused with the symmetrization $\mathcal{T}_{\rm s}$. One can compose a $\mathcal{Z}$-valued tridifferential operator $\hat{\mathcal{T}}$ with d_s and $d_\square$, which we denote by juxtaposition:

$$\begin{aligned} \hat{\mathcal{T}}d_s &\coloneqq 2\hat{\mathcal{T}}\circ(\partial^\mu \otimes \partial_\mu \otimes \mathbb{1}), \\ \hat{\mathcal{T}}d_\square &\coloneqq 2\hat{\mathcal{T}}\circ\{(\partial^\mu \otimes \partial_\mu \otimes \mathbb{1}) + (\mathbb{1} \otimes \partial^\mu \otimes \partial_\mu) \\ &\quad + (\partial^\mu \otimes \mathbb{1} \otimes \partial_\mu)\}. \end{aligned} \quad (2.55)$$

These are also $\mathcal{Z}$-valued tridifferential operators which generate the corresponding maps $\mathcal{T}d_s$ and $\mathcal{T}d_\square$. For instance, one has $\mathcal{T}d_s(\psi_1, \psi_2, \psi_3) = 2\mathcal{T}(\partial^\mu\psi_1, \partial_\mu\psi_2, \psi_3)$ and so on. Under projection by π, d_s and $d_\square$ obey

$$\mathcal{T}d_\square\pi = \mathcal{T}\pi d_\square = 3\mathcal{T}\pi d_s\pi. \quad (2.56)$$

The $d_\square$ operator is always related to a total commutator with $\square$, in the sense that

$$[\square, \mathcal{T}] = \mathcal{T}d_\square, \quad (2.57)$$

while $\mathcal{T}d_s$ is not. Last, from the definition of $[\mathcal{O}, \mathcal{T}]$ it follows that d_s and $d_\square$ commute with linear operators $\mathcal{O}$:

$$[\mathcal{O}, \mathcal{T}]d_s = [\mathcal{O}, \mathcal{T}d_s], \qquad [\mathcal{O}, \mathcal{T}]d_\square = [\mathcal{O}, \mathcal{T}d_\square]. \quad (2.58)$$

With this notation at hand, we can summarize all the relevant $\mathrm{BV}_\infty^\square$ relations up to trilinear maps [26]:

$$\begin{aligned}
&m_1^2 = 0,\ b^2 = 0,\ [m_1, b] = \square, && \text{differentials and central obstruction,} \\
&[m_1, m_2] = 0,\ m_2 m_2(1-\pi) = [m_1, m_{3\rm h}], && C_\infty \text{ structure,} \\
&b_2 = [b, m_2],\ [m_1, b_2] = [\square, m_2], && \text{two-bracket and deformed Leibniz,} \\
&b_2 m_2 + m_2 b_2(1-3\pi) = [m_1, \theta_3] + m_{3\rm h}(d_\square - 3d_s\pi), && \text{deformed homotopy Poisson,} \\
&3b_2 b_2\pi + [m_1, b_3] + 3\theta_3 d_s\pi = 0, && \text{deformed homotopy Jacobi,} \\
&\theta_{3\rm h} + [b, m_{3\rm h}] = 0,\ b_3 + [b, \theta_{3\rm s}] = 0, && \text{compatibility of homotopies.}
\end{aligned} \quad (2.59)$$

The explicit maps for m_n and θ_3 can be found in the Appendix of [26], while b_2 and b_3 are easily derived from these by taking b-commutators. The corresponding differential operators $\hat{m}_n$ and $\hat{\theta}_3$ are listed in Appendix A.

From the above table one can see that the only consistent subsector is given by the original C_∞ algebra (m_1, m_2, m_{3h}). On the other hand, the brackets $(b_1 \equiv m_1, b_2, b_3)$ form an L_∞ algebra only up to $\Box$-deformations, governed at this level by m_2 and θ_3. Armed with this structure on $\mathcal{K}$, in the next section we will show that a natural $\mathrm{BV}_\infty^\Delta$ algebra exists on the tensor product $\mathcal{K} \otimes \bar{\mathcal{K}}$ of two copies of $\mathcal{K}$.

III. $\mathrm{BV}_\infty^\Delta$ ALGEBRA ON $\mathcal{K} \otimes \bar{\mathcal{K}}$

In this section we will consider two copies of the kinematic algebra $\mathcal{K}$ and show that the respective $\mathrm{BV}_\infty^\Box$ algebras give rise to a $\mathrm{BV}_\infty^\Delta$ algebra on the tensor product $\mathcal{X} \coloneqq \mathcal{K} \otimes \bar{\mathcal{K}}$, where $\Delta \coloneqq \frac{1}{2}(\Box - \bar{\Box})$. This will be used in the next sections to derive the three-brackets of DFT, both on arbitrary flat backgrounds in the strongly constrained sense and on a torus in the weakly constrained sense. At this point we should mention that the space $\mathcal{X}$ is *not* the L_∞ graded vector space of DFT, which we refer to as $\mathcal{V}_{\mathrm{DFT}}$ and which is a linear subspace of $\mathcal{X}$ to be described below.

A. Grading and maps on $\mathcal{K} \otimes \bar{\mathcal{K}}$

We start by spelling out the structure of the tensor product $\mathcal{X} = \mathcal{K} \otimes \bar{\mathcal{K}}$ as a graded vector space. From now on, we will denote all elements and maps of $\bar{\mathcal{K}}$ with a bar on the same symbols used for $\mathcal{K}$. Recalling that $\mathcal{K} = \mathcal{Z} \otimes C^\infty(\mathcal{M})$, one obtains that $\mathcal{X}$ similarly factorizes as a finite-dimensional graded vector space tensored with functions on a doubled spacetime:

$$\begin{aligned}\mathcal{X} = \mathcal{K} \otimes \bar{\mathcal{K}} &= (\mathcal{Z} \otimes C^\infty(\mathcal{M})) \otimes (\bar{\mathcal{Z}} \otimes C^\infty(\bar{\mathcal{M}})) \\ &\simeq (\mathcal{Z} \otimes \bar{\mathcal{Z}}) \otimes C^\infty(\mathcal{M} \times \bar{\mathcal{M}}),\end{aligned} \tag{3.1}$$

using that $C^\infty(\mathcal{M}) \otimes C^\infty(\bar{\mathcal{M}}) \simeq C^\infty(\mathcal{M} \times \bar{\mathcal{M}})$, which under a certain topological completion holds for the tensor product. Throughout this section, we will take $\mathcal{M}$ and $\bar{\mathcal{M}}$ to be d-dimensional flat spaces with an unspecified signature, but later on we will specialize to a Euclidean signature. Given the structure (3.1) of $\mathcal{X}$ and two copies of the finite-dimensional basis, i.e. $\{Z_A\}$ of $\mathcal{Z}$ and $\{\bar{Z}_{\bar{A}}\}$ of $\bar{\mathcal{Z}}$, we can expand an arbitrary element $\Psi \in \mathcal{X}$ as

$$\Psi(x, \bar{x}) = Z_A \bar{Z}_{\bar{B}} \Psi^{A\bar{B}}(x, \bar{x}), \tag{3.2}$$

where we denote coordinates of the doubled space by $(x^\mu, \bar{x}^{\bar{\mu}})$, and from now on we will use capital Ψ for elements in $\mathcal{X}$. The degree in $\mathcal{X}$ is defined as the sum of the degrees in $\mathcal{K}$ and $\bar{\mathcal{K}}$, with an additional shift by 2. Specifically, for a homogenous element $\Psi = Z\bar{Z}F(x, \bar{x})$ we set

$$|\Psi| = |Z| + |\bar{Z}| - 2, \tag{3.3}$$

where we recall that degrees in $\mathcal{Z}$ (the same for $\bar{\mathcal{Z}}$) are displayed in (2.8). The shift in degree is by an even amount, so it is immaterial for sign factors such as $(-1)^{|\Psi|}$ and thus strictly not necessary. However, the definition (3.3) complies with standard L_∞ degrees in the resulting double field theory.

Given the definition (2.36) and the expansion (3.2) we now proceed to lift the action of operators $\mathcal{O}: \mathcal{K} \to \mathcal{K}$ and $\bar{\mathcal{O}}: \bar{\mathcal{K}} \to \bar{\mathcal{K}}$ to $\mathcal{X}$ by defining

$$\begin{aligned}\mathcal{O}(\Psi) &\coloneqq \hat{\mathcal{O}}(Z_A)\bar{Z}_{\bar{B}}\Psi^{A\bar{B}}(x, \bar{x}), \\ \bar{\mathcal{O}}(\Psi) &\coloneqq (-1)^{Z_A \bar{\mathcal{O}}} Z_A \hat{\bar{\mathcal{O}}}(\bar{Z}_{\bar{B}})\Psi^{A\bar{B}}(x, \bar{x}),\end{aligned} \tag{3.4}$$

where the differential operators $\hat{\mathcal{O}}$ and $\hat{\bar{\mathcal{O}}}$ act on functions of x and $\bar{x}$ by taking $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\bar{\partial}_{\bar{\mu}} = \frac{\partial}{\partial \bar{x}^{\bar{\mu}}}$ derivatives, respectively. This allows us to sum operators from $\mathcal{K}$ and $\bar{\mathcal{K}}$ and yield well-defined operators on $\mathcal{X}$, such as $\mathcal{O} + \bar{\mathcal{O}}$. Similarly, tensor products of bilinear maps $\mathcal{M}$ and $\bar{\mathcal{M}}$ are defined to act on elements of $\mathcal{X}$ as follows:

$$\begin{aligned}(\mathcal{M} \otimes \bar{\mathcal{M}})(\Psi_1, \Psi_2) &= (\mathcal{M} \otimes \bar{\mathcal{M}})(Z_A \bar{Z}_{\bar{B}} \Psi_1^{A\bar{B}}, Z_C \bar{Z}_{\bar{D}} \Psi_2^{C\bar{D}}) \\ &\coloneqq (-1)^{Z_C \bar{Z}_{\bar{B}} + \bar{\mathcal{M}}(Z_A + Z_C)} \mu[\hat{\mathcal{M}}(Z_A, Z_C) \\ &\quad \times \hat{\bar{\mathcal{M}}}(\bar{Z}_{\bar{B}}, \bar{Z}_{\bar{D}})(\Psi_1^{A\bar{B}}(x, \bar{x}) \\ &\quad \otimes \Psi_2^{C\bar{D}}(x, \bar{x}))],\end{aligned} \tag{3.5}$$

with a completely analogous expression for $(\mathcal{T} \otimes \bar{\mathcal{T}}) \times (\Psi_1, \Psi_2, \Psi_3)$. With these definitions we can extend the input-free notation of the previous section to $\mathcal{X}$. It turns out that operators $\mathcal{O}$ and $\bar{\mathcal{O}}$ commute (in the graded sense). To show this we compute

$$\begin{aligned}\mathcal{O}(\bar{\mathcal{O}}\Psi) &= (-1)^{Z_A \bar{\mathcal{O}}} \mathcal{O}(Z_A \hat{\bar{\mathcal{O}}}(\bar{Z}_{\bar{B}})\Psi^{A\bar{B}}) \\ &= (-1)^{Z_A \bar{\mathcal{O}}} \hat{\mathcal{O}}(Z_A) \hat{\bar{\mathcal{O}}}(\bar{Z}_{\bar{B}}) \Psi^{A\bar{B}}(x, \bar{x}) \\ &= (-1)^{Z_A \bar{\mathcal{O}} + \bar{\mathcal{O}}(Z_A + \mathcal{O})} \bar{\mathcal{O}}(\hat{\mathcal{O}}(Z_A)\bar{Z}_{\bar{B}}\Psi^{A\bar{B}}) \\ &= (-1)^{\mathcal{O}\bar{\mathcal{O}}} \bar{\mathcal{O}}(\mathcal{O}\Psi),\end{aligned} \tag{3.6}$$

where we omitted the explicit dependence on $(x, \bar{x})$ in intermediate steps. This can be written as the input-free relation $[\mathcal{O}, \bar{\mathcal{O}}] = 0$, where the graded commutator is defined as in (2.37), albeit acting on elements of $\mathcal{K} \otimes \bar{\mathcal{K}}$. A similar computation using the definition (3.5) shows that operators of $\mathcal{K}$ commute with bilinear and trilinear maps of $\bar{\mathcal{K}}$ and vice versa, in the sense

$$\begin{aligned}[\mathcal{O}, \mathcal{M} \otimes \bar{\mathcal{M}}] &= [\mathcal{O}, \mathcal{M}] \otimes \bar{\mathcal{M}}, \\ [\bar{\mathcal{O}}, \mathcal{M} \otimes \bar{\mathcal{M}}] &= (-1)^{\mathcal{M}\bar{\mathcal{O}}} \mathcal{M} \otimes [\bar{\mathcal{O}}, \bar{\mathcal{M}}],\end{aligned} \tag{3.7}$$

with analogous formulas for commutators with $\mathcal{T}\otimes\bar{\mathcal{T}}$. Nesting of bilinear maps can also be extended naturally by defining

$$(\mathcal{M}_1\otimes\bar{\mathcal{M}}_1)(\mathcal{M}_2\otimes\bar{\mathcal{M}}_2) := (-1)^{\bar{\mathcal{M}}_1\mathcal{M}_2}\mathcal{M}_1\mathcal{M}_2\otimes\bar{\mathcal{M}}_1\bar{\mathcal{M}}_2, \tag{3.8}$$

where the composition $\mathcal{M}_1\mathcal{M}_2$ (same for the barred ones) is defined by (2.40).

Finally, one can introduce on $\mathcal{X}$ a symmetric projector Π, obeying $\Pi^2=\Pi$, via

$$\begin{aligned}(\mathcal{T}\otimes\bar{\mathcal{T}})\Pi(\Psi_1,\Psi_2,\Psi_3) = \frac{1}{3}(-1)^{\mathcal{E}}\mu[&(\hat{\mathcal{T}}(Z_A,Z_B,Z_C)\hat{\bar{\mathcal{T}}}(\bar{Z}_{\bar{A}},\bar{Z}_{\bar{B}},\bar{Z}_{\bar{C}})\\ &+(-1)^{Z_A(Z_B+Z_C)+\bar{Z}_{\bar{A}}(\bar{Z}_{\bar{B}}+\bar{Z}_{\bar{C}})}\hat{\mathcal{T}}(Z_B,Z_C,Z_A)\Sigma\hat{\bar{\mathcal{T}}}(\bar{Z}_{\bar{B}},\bar{Z}_{\bar{C}},\bar{Z}_{\bar{A}})\bar{\Sigma}\\ &+(-1)^{Z_C(Z_A+Z_B)+\bar{Z}_{\bar{C}}(\bar{Z}_{\bar{A}}+\bar{Z}_{\bar{B}})}\hat{\mathcal{T}}(Z_C,Z_A,Z_B)\Sigma^2\hat{\bar{\mathcal{T}}}(\bar{Z}_{\bar{C}},\bar{Z}_{\bar{A}},\bar{Z}_{\bar{B}})\bar{\Sigma}^2)(\Psi_1^{A\bar{A}}\otimes\Psi_2^{B\bar{B}}\otimes\Psi_3^{C\bar{C}})],\end{aligned} \tag{3.9}$$

where the global phase is $\mathcal{E}=Z_B\bar{Z}_{\bar{A}}+Z_C(\bar{Z}_{\bar{A}}+\bar{Z}_{\bar{B}})+(Z_A+Z_B+Z_C)\bar{\mathcal{T}}$. In terms of the map $\mathcal{T}\otimes\bar{\mathcal{T}}$, this results in

$$\begin{aligned}(\mathcal{T}\otimes\bar{\mathcal{T}})\Pi(\Psi_1,\Psi_2,\Psi_3) := \frac{1}{3}\{&(\mathcal{T}\otimes\bar{\mathcal{T}})(\Psi_1,\Psi_2,\Psi_3)+(-1)^{\Psi_1(\Psi_2+\Psi_3)}(\mathcal{T}\otimes\bar{\mathcal{T}})(\Psi_2,\Psi_3,\Psi_1)\\ &+(-1)^{\Psi_3(\Psi_1+\Psi_2)}(\mathcal{T}\otimes\bar{\mathcal{T}})(\Psi_3,\Psi_1,\Psi_2)\},\end{aligned} \tag{3.10}$$

which makes the graded symmetry manifest. One can lift the definition of the single copy π or $\bar{\pi}$ to a trilinear map $\mathcal{T}\otimes\bar{\mathcal{T}}$ on $\mathcal{X}$ by

$$(\mathcal{T}\otimes\bar{\mathcal{T}})\pi := (\mathcal{T}\pi)\otimes\bar{\mathcal{T}}, \qquad (\mathcal{T}\otimes\bar{\mathcal{T}})\bar{\pi} := \mathcal{T}\otimes(\bar{\mathcal{T}}\bar{\pi}), \tag{3.11}$$

and using (2.43) and (2.47) for the single copy symmetrized maps. From this it follows that $\pi\Pi=\bar{\pi}\Pi=\pi\bar{\pi}$, which further implies the decomposition

$$\begin{aligned}\Pi &= [\pi\bar{\pi}+(1-\pi)(1-\bar{\pi})]\Pi,\\ 1-\Pi &= [\pi(1-\bar{\pi})+(1-\pi)\bar{\pi}+(1-\pi)(1-\bar{\pi})](1-\Pi).\end{aligned} \tag{3.12}$$

B. BV_∞^Δ algebra on $\mathcal{X}$

We are now ready to show that, given the $BV_\infty^\square$ algebras on $\mathcal{K}$ and $\bar{\mathcal{K}}$, a natural BV_∞^Δ algebra arises on $\mathcal{X}$. As for the single copies, for the moment we will work this out up to trilinear maps. The starting point is the C_∞ sector of (2.59). With the differentials m_1 and $\bar{m}_1$ and the two-products m_2 and $\bar{m}_2$ we define a differential M_1 and two-product M_2 on $\mathcal{X}$ by

$$\begin{aligned}M_1 &:= m_1+\bar{m}_1,\\ M_2 &:= m_2\otimes\bar{m}_2.\end{aligned} \tag{3.13}$$

From the graded symmetry of m_2 and $\bar{m}_2$ one can easily show that M_2 is graded symmetric in $\mathcal{X}$. Due to the degree shift (3.3), one has $|M_1|=+1$ and $|M_2|=+2$. Upon using (3.6) and (3.7) it is immediate to see that M_1 is nilpotent and acts as a derivation on M_2:

$$\begin{aligned}M_1^2 &= (m_1+\bar{m}_1)^2 = m_1\bar{m}_1+\bar{m}_1m_1 = 0,\\ &= [m_1+\bar{m}_1, m_2\otimes\bar{m}_2]\\ &= [m_1,m_2]\otimes\bar{m}_2+m_2\otimes[\bar{m}_1,\bar{m}_2] = 0.\end{aligned} \tag{3.14}$$

We next study the associativity of M_2 by computing its associator. Due to the graded symmetry of M_2, the latter is equivalent to the hook projection $M_2M_2(1-\Pi)$. Using the definition (3.13), the property (3.12) of projectors, and the homotopy associativity of m_2 and $\bar{m}_2$, we can compute

$$\begin{aligned}M_2M_2(1-\Pi) &= m_2m_2\otimes\bar{m}_2\bar{m}_2(\pi(1-\bar{\pi})+(1-\pi)\bar{\pi}+(1-\pi)(1-\bar{\pi}))(1-\Pi)\\ &= \{m_2m_2\pi\otimes[\bar{m}_1,\bar{m}_{3h}]+[m_1,m_{3h}]\otimes\bar{m}_2\bar{m}_2\bar{\pi}+[m_1,m_{3h}]\otimes[\bar{m}_1,\bar{m}_{3h}]\}(1-\Pi)\\ &= [M_1, m_2m_2\pi\otimes\bar{m}_{3h}+m_{3h}\otimes\bar{m}_2\bar{m}_2\bar{\pi}](1-\Pi)+[m_1,m_{3h}]\otimes[\bar{m}_1,\bar{m}_{3h}](1-\Pi),\end{aligned} \tag{3.15}$$

where in the last step we used $[m_1,m_2m_2]=0$ (and the barred relation) to extract a total differential M_1. At this stage an ambiguity arises on how to treat the last term, since

$$[m_1,m_{3h}]\otimes[\bar{m}_1,\bar{m}_{3h}] = \left[M_1, \left(\frac{1}{2}-\xi\right)m_{3h}\otimes[\bar{m}_1,\bar{m}_{3h}]+\left(\frac{1}{2}+\xi\right)[m_1,m_{3h}]\otimes\bar{m}_{3h}\right] \tag{3.16}$$

for arbitrary ξ. Keeping ξ arbitrary leads to a one-parameter family of three-products, differing by an M_1-exact term (which for maps means a total M_1 commutator). This is expected, since in a homotopy associative algebra the three-product is defined only up to an M_1-closed quantity. For simplicity we choose $\xi = 0$ and obtain

$$M_{3h} = \frac{1}{2}\{m_{3h} \otimes \bar{m}_2\bar{m}_2(1+\bar{\pi}) + m_2m_2(1+\pi) \otimes \bar{m}_{3h}\}(1-\Pi), \tag{3.17}$$

obeying homotopy associativity in the form $M_2M_2(1-\Pi) = [M_1, M_{3h}]$. Even though the original C_∞ algebras on $\mathcal{K}$ and $\bar{\mathcal{K}}$ have no higher products than m_{3h}, the tensor algebra $\mathcal{X}$ are expected to have infinitely many higher M_n, which we will not explore further.

In order to go beyond the C_∞ structure, one has to identify the analog of the b differential on the tensor space. Two natural candidates are the linear combinations

$$b^{\pm} := \frac{1}{2}(b \pm \bar{b}). \tag{3.18}$$

Both $b^{\pm}$ are nilpotent and (anti)commute with each other. Their commutators with M_1, which determine the obstruction Δ, are given by

$$[M_1, b^{\pm}] = \frac{1}{2}(\Box \pm \bar{\Box}). \tag{3.19}$$

For establishing a $\mathrm{BV}_\infty^\Delta$ algebra on $\mathcal{X}$, both choices $b^{\pm}$ for the second differential are equivalent. Our choice is dictated by the goal of constructing double field theory on a suitable subspace of $\mathcal{X}$ which, in particular, requires an unobstructed L_∞ algebra. In view of the fact that the subspace $\mathcal{V}_{\mathrm{DFT}}$ is partly determined by constraining $(\Box - \bar{\Box})\Psi = 0$, the natural choice for the b operator is b^-, yielding

$$[M_1, b^-] = \Delta, \qquad \Delta := \frac{1}{2}(\Box - \bar{\Box}). \tag{3.20}$$

Since Δ arises as the above commutator, it is guaranteed to commute with both M_1 and b^-: $[M_1, \Delta] = [b^-, \Delta] = 0$. With this choice one can construct a degree $+1$ two-bracket B_2, in perfect analogy with the single copy version (2.34):

$$\begin{aligned} B_2 := [b^-, M_2] &= \frac{1}{2}[b - \bar{b}, m_2 \otimes \bar{m}_2] \\ &= \frac{1}{2}(b_2 \otimes \bar{m}_2 - m_2 \otimes \bar{b}_2), \end{aligned} \tag{3.21}$$

where in the second line we emphasized its tensor product structure that is of the schematic form "Lie $\otimes$ Commutative." While b^- is trivially a derivation for B_2, M_1 is not. The obstruction is easily computed and takes the same form as in (2.59):

$$\begin{aligned} [M_1, B_2] &= [M_1, [b^-, M_2]] \\ &= [[M_1, b^-], M_2] - [b^-, [M_1, M_2]] \\ &= [\Delta, M_2], \end{aligned} \tag{3.22}$$

where we used $[M_1, M_2] = 0$. For the Δ-obstructions on trilinear maps we define $D_\Delta := \frac{1}{2}(d_\Box - \bar{d}_{\bar{\Box}})$ and $D_s := \frac{1}{2}(d_s - \bar{d}_s)$ which act on three functions $F_i(x, \bar{x})$ as

$$\begin{aligned} D_s(F_1 \otimes F_2 \otimes F_3) &= (\partial^\mu F_1 \otimes \partial_\mu F_2 \otimes F_3) \\ &\quad - (\bar{\partial}^{\bar{\mu}} F_1 \otimes \bar{\partial}_{\bar{\mu}} F_2 \otimes F_3), \\ D_\Delta(F_1 \otimes F_2 \otimes F_3) &= (\partial^\mu F_1 \otimes \partial_\mu F_2 \otimes F_3) \\ &\quad - (\bar{\partial}^{\bar{\mu}} F_1 \otimes \bar{\partial}_{\bar{\mu}} F_2 \otimes F_3) \\ &\quad + (F_1 \otimes \partial^\mu F_2 \otimes \partial_\mu F_3) \\ &\quad - (F_1 \otimes \bar{\partial}^{\bar{\mu}} F_2 \otimes \bar{\partial}_{\bar{\mu}} F_3) \\ &\quad + (\partial^\mu F_1 \otimes F_2 \otimes \partial_\mu F_3) \\ &\quad - (\bar{\partial}^{\bar{\mu}} F_1 \otimes F_2 \otimes \bar{\partial}_{\bar{\mu}} F_3). \end{aligned} \tag{3.23}$$

Given the definition of D_s and D_Δ and the projector (3.9), they obey

$$(\mathcal{T} \otimes \bar{\mathcal{T}})D_\Delta\Pi = (\mathcal{T} \otimes \bar{\mathcal{T}})\Pi D_\Delta = 3(\mathcal{T} \otimes \bar{\mathcal{T}})\Pi D_s\Pi, \tag{3.24}$$

similar to (2.56) for d_s and $d_\Box$.

Given the product M_2 and the bracket $B_2 = [b^-, M_2]$, the Poisson compatibility condition [defined as in (2.35) upon replacing $m_2 \to M_2$ and $b_2 \to B_2$] can be formulated as

$$B_2M_2 + M_2B_2(1-3\Pi) \equiv [b^-, M_2M_2] - 3M_2[b^-, M_2]\Pi \stackrel{?}{=} 0. \tag{3.25}$$

The first form of (3.25) emphasizes the Poisson relation between B_2 and M_2, while the second form shows that this is equivalent to b^- being second order with respect to M_2. Since the single copy b_2 and m_2 obey

$$[b, m_2m_2] - 3m_2b_2\pi = [m_1, \theta_3] + m_{3h}(d_\Box - 3d_s\pi), \tag{3.26}$$

one does not expect (3.25) to vanish, but rather to obey a similar relation in terms of M_{3h} and a Θ_3 yet to be determined. To show that this is indeed the case, it is convenient to split the computation of (3.25) into its totally symmetric and hook parts. We start from the hook, which is simple to determine:

$$\begin{aligned} &\{[b^-, M_2M_2] - 3M_2B_2\Pi\}(1-\Pi) \\ &= [b^-, M_2M_2](1-\Pi) \\ &= [b^-, M_2M_2(1-\Pi)] = [b^-, [M_1, M_{3h}]] \\ &= -[M_1, [b^-, M_{3h}]] + [[M_1, b^-], M_{3h}] \\ &= -[M_1, [b^-, M_{3h}]] + [\Delta, M_{3h}] \\ &= [M_1, \Theta_{3h}] + M_{3h}D_\Delta, \end{aligned} \tag{3.27}$$

where we identified $\Theta_{3h} = -[b^-, M_{3h}]$. One can see that (3.27) has the same form as the hook projection of (3.26), and the relation between Θ_{3h} and M_{3h} is the same as in (2.59) for compatibility of the homotopies. Computing the symmetric projection of (3.25) is considerably more involved. We spell out the computation in detail in Appendix B, which results in

$$\{[b^-,M_2M_2]-3M_2B_2\Pi\}\Pi=[M_1,\Theta_{3s}]-3M_{3h}D_s\Pi, \tag{3.28}$$

where Θ_{3s} is determined in terms of Yang-Mills maps as

$$\Theta_{3s}:=\frac{1}{2}\Big(\theta_{3s}\otimes\bar m_2\bar m_2-m_2m_2\otimes\bar\theta_{3s}+3m_{3h}\otimes\bar m_2\bar b_2 +3m_2b_2\otimes\bar m_{3h}-\frac{3}{2}m_{3h}\otimes\bar m_{3h}(d_s+\bar d_s) -[b^-,m_2m_2\otimes\bar m_{3h}+m_{3h}\otimes\bar m_2\bar m_2]\Big)\Pi. \tag{3.29}$$

Given the Poisson relation, one can determine the Jacobiator $3B_2B_2\Pi$ of the two-bracket B_2 by taking a b^- commutator. This is also presented in Appendix B and yields the deformed L_∞ relation

$$3B_2B_2\Pi+[M_1,B_3]+3\Theta_3D_s\Pi=0, \tag{3.30}$$

where the three-bracket is given by $B_3=-[b^-,\Theta_{3s}]$.

This concludes the $\mathrm{BV}_\infty^\Delta$ relations up to trilinear maps, which we summarize in the following table, analogous to the single copy one (2.59):

$$\begin{aligned}
&M_1^2=0,\ (b^-)^2=0,\ [M_1,b^-]=\Delta, && \text{differentials and central obstruction,}\\
&[M_1M_2]=0,\ M_2M_2(1-\Pi)=[M_1,M_{3h}], && C_\infty\ \text{structure,}\\
&B_2=[b^-,M_2],\ [M_1,B_2]=[\Delta,M_2], && \text{two-bracket and deformed Leibniz,}\\
&B_2M_2+M_2B_2(1-3\Pi)=[M_1,\Theta_3]+M_{3h}(D_\Delta-3D_s\Pi), && \text{deformed homotopy Poisson,}\\
&3B_2B_2\Pi+[M_1,B_3]+3\Theta_3D_s\Pi=0, && \text{deformed homotopy Jacobi,}\\
&\Theta_{3h}+[b^-,M_{3h}]=0,\ B_3+[b^-,\Theta_{3s}]=0, && \text{compatibility of homotopies.}
\end{aligned} \tag{3.31}$$

In particular, notice that the brackets of the L_∞ sector (albeit obstructed) are all determined in terms of other structures as $B_1\equiv M_1$, $B_2=[b^-,M_2]$, $B_3=-[b^-,\Theta_{3s}]$. This is in close analogy with closed string field theory, where all the string brackets, apart from B_1, are b-exact [17].

We conclude this section by collecting the expressions of the relevant maps in terms of Yang-Mills building blocks:

$$\begin{aligned}
M_1&:=m_1+\bar m_1,\qquad b^\pm:=\frac{1}{2}(b\pm\bar b),\\
M_2&:=m_2\otimes\bar m_2,\\
M_{3h}&:=\frac{1}{2}(m_{3h}\otimes\bar m_2\bar m_2(1+\bar\pi)+m_2m_2(1+\bar\pi)\otimes\bar m_{3h})(1-\Pi),\\
\Theta_{3s}&:=\frac{1}{2}\Big(\theta_{3s}\otimes\bar m_2\bar m_2-m_2m_2\otimes\bar\theta_{3s}+3m_{3h}\otimes\bar m_2\bar b_2+3m_2b_2\otimes\bar m_{3h}\\
&\quad-\frac{3}{2}m_{3h}\otimes\bar m_{3h}(d_s+\bar d_s)-[b^-,m_2m_2\otimes\bar m_{3h}+m_{3h}\otimes\bar m_2\bar m_2]\Big)\Pi.
\end{aligned} \tag{3.32}$$

IV. WEAKLY CONSTRAINED DOUBLE FIELD THEORY

In this section we will start from the $\mathrm{BV}_\infty^\Delta$ algebra on $\mathcal{X}$ and construct the L_∞ algebra of weakly constrained DFT on a spatial torus, up to and including the three-bracket, which encodes all the quartic structures of the theory. The obvious issue is that the L_∞ sector of (3.31) is obstructed on $\mathcal{X}$, due to Δ. The idea is that these obstructions should be milder when considering the relevant subspace of weakly constrained fields, obeying $\Delta\Psi=0$.

Let us start by giving the precise definition of the graded vector space $\mathcal{V}_{\mathrm{DFT}}$ carrying the L_∞ algebra of double field theory. This is given by the following linear subspace of $\mathcal{X}=\mathcal{K}\otimes\bar{\mathcal{K}}$:

$$\begin{aligned}
\mathcal{V}_{\mathrm{DFT}}&:=\{\Psi\in\mathcal{X},\ \text{s.t.}\,\Delta\Psi=0,b^-\Psi=0\}\\
&=(\ker\Delta\cap\ker b^-)\subset\mathcal{X}.
\end{aligned} \tag{4.1}$$

Given that an arbitrary element of $\mathcal{X}$ can be expanded as in (3.2), with the graded vectors $Z_A=(\theta_M,c\theta_M)$ and $\bar Z_{\bar A}=(\bar\theta_{\bar M},\bar c\bar\theta_{\bar M})$, one can explicitly characterize the elements of $\mathcal{V}_{\mathrm{DFT}}$ as

$$\begin{aligned}
&\Psi=\theta_M\bar\theta_{\bar N}\psi^{M\bar N}(x,\bar x)+c^+\theta_M\bar\theta_{\bar N}\chi^{M\bar N}(x,\bar x),\\
&\Delta\psi^{M\bar N}=\Delta\chi^{M\bar N}=0,
\end{aligned} \tag{4.2}$$

where $c^+:=c+\bar c$, so that $c^+\theta_M\bar\theta_{\bar N}=c\theta_M\bar\theta_{\bar N}+(-1)^{|\theta_M|}\theta_M\bar c\bar\theta_{\bar N}$. From the degree assignment (3.3), one can see that the degrees in $\mathcal{V}_{\mathrm{DFT}}$ are given by

$$|\theta_M\bar{\theta}_{\bar{N}}| = -(M+\bar{N}), \quad |c^+\theta_M\bar{\theta}_{\bar{N}}| = 1-(M+\bar{N}), \quad (4.3)$$

where we recall that we associate degrees $(+1, 0, -1)$ with $M = (+, \mu, -)$. One thus finds, for instance, the DFT gauge parameters in degree -1 to be given by

$$\Lambda = \theta_+\bar{\theta}_{\bar{\mu}}\bar{\lambda}^{\bar{\mu}} - \theta_\mu\bar{\theta}_+\lambda^\mu - 2c^+\theta_+\bar{\theta}_+\eta, \quad (4.4)$$

which thus consist of two vector parameters, related to generalized diffeomorphisms, and a Stückelberg scalar parameter. Similarly, one has the fields in degree zero,

$$\psi = \theta_\mu\bar{\theta}_{\bar{\nu}}e^{\mu\bar{\nu}} + 2\theta_+\bar{\theta}_-\bar{e} + 2\theta_-\bar{\theta}_+e + 2c^+\theta_\mu\bar{\theta}_+f^\mu + 2c^+\theta_+\bar{\theta}_{\bar{\mu}}\bar{f}^{\bar{\mu}}, \quad (4.5)$$

comprising the tensor fluctuation $e_{\mu\bar{\nu}}$, two scalars (one combination is related to the dilaton, and the other is pure gauge), and two auxiliary vectors. This is precisely the field content of DFT as first introduced in [3].

In order to construct an L_∞ algebra on $\mathcal{V}_{\rm DFT}$, we will proceed in two steps: we will first transport the $\mathrm{BV}_\infty^\Delta$ structure of $\mathcal{X}$ to the subspace $\bar{\mathcal{X}} \coloneqq \ker\Delta$ via homotopy transfer, to be discussed momentarily, and in the second step we will restrict the maps to act on $\ker b^-$. In this last step the $\mathrm{BV}_\infty^\Delta$ structure will be lost, leaving an unobstructed L_∞ algebra on $\mathcal{V}_{\rm DFT}$.

A. Homotopy transfer to $\ker\Delta$

In order to perform homotopy transfer, we shall find a suitable projector $\mathcal{P}_\Delta$ to $\ker\Delta$, together with an homotopy operator h of degree $|h| = -1$, obeying

$$[M_1, h] = 1 - \mathcal{P}_\Delta. \quad (4.6)$$

Since $\bar{\mathcal{X}} = \ker\Delta \subset \mathcal{X}$ is a subspace of $\mathcal{X}$, we consider the projector $\mathcal{P}_\Delta\colon \mathcal{X} \to \bar{\mathcal{X}}$ as an operator in $\mathcal{X}$, implicitly assuming a trivial inclusion map $\iota\colon \bar{\mathcal{X}} \to \mathcal{X}$ (see, e.g., [12] for an introduction to homotopy transfer). The projector has to be properly normalized and is required to be a chain map, meaning that it should obey

$$\mathcal{P}_\Delta^2 = \mathcal{P}_\Delta, \qquad \mathcal{P}_\Delta M_1 = M_1\mathcal{P}_\Delta. \quad (4.7)$$

We further require that the homotopy h obeys the so-called side conditions:

$$h\mathcal{P}_\Delta = \mathcal{P}_\Delta h = 0, \qquad h^2 = 0. \quad (4.8)$$

In order to define $\mathcal{P}_\Delta$, we specialize the signature and topology of our underlying doubled space to be a doubled Euclidean square torus. In particular, we have two copies of the Euclidean metric $\delta_{\mu\nu}$ and $\delta_{\bar{\mu}\bar{\nu}}$ and we identify coordinates with periodicity 2π. Any function on the doubled torus can be expanded in discrete Fourier modes as

$$f(x,\bar{x}) = \sum_{k,\bar{k}}\tilde{f}(k,\bar{k})e^{ik\cdot x + i\bar{k}\cdot\bar{x}}, \quad (4.9)$$

with an unconstrained sum over discrete momenta $(k^\mu, \bar{k}^{\bar{\mu}}) \in \mathbb{Z}^{2d}$. The obstruction Δ acts on (4.9) as

$$\Delta f(x,\bar{x}) = -\frac{1}{2}\sum_{k,\bar{k}}(k^2 - \bar{k}^2)\tilde{f}(k,\bar{k})e^{ik\cdot x + i\bar{k}\cdot\bar{x}}. \quad (4.10)$$

This allows us to write the projectors $\mathcal{P}_\Delta$ and $(1-\mathcal{P}_\Delta)$ explicitly in terms of the Fourier expansion by inserting suitable Kronecker deltas:

$$\begin{aligned}(\mathcal{P}_\Delta f)(x,\bar{x}) &= \sum_{k,\bar{k}}\delta_{k^2,\bar{k}^2}\tilde{f}(k,\bar{k})e^{ik\cdot x + i\bar{k}\cdot\bar{x}} \\ &= \sum_{k^2=\bar{k}^2}\tilde{f}(k,\bar{k})e^{ik\cdot x + i\bar{k}\cdot\bar{x}}, \\ ((1-\mathcal{P}_\Delta)f)(x,\bar{x}) &= \sum_{k,\bar{k}}(1-\delta_{k^2,\bar{k}^2})\tilde{f}(k,\bar{k})e^{ik\cdot x + i\bar{k}\cdot\bar{x}} \\ &= \sum_{k^2\neq\bar{k}^2}\tilde{f}(k,\bar{k})e^{ik\cdot x + i\bar{k}\cdot\bar{x}}. \quad (4.11)\end{aligned}$$

This operator clearly projects to $\ker\Delta$, since

$$\Delta\mathcal{P}_\Delta = \mathcal{P}_\Delta\Delta = 0, \quad (4.12)$$

and squares to itself: $\mathcal{P}_\Delta^2 = \mathcal{P}_\Delta$. Moreover, being a function of Δ, it commutes with M_1, thus complying with the properties (4.7). To construct the homotopy operator, we shall first introduce the "propagator" G. Given a function orthogonal to $\ker\Delta$, meaning it obeys $\mathcal{P}_\Delta f = 0$, one can invert Δ by means of the propagator G, defined as

$$\begin{aligned}(Gf)(x,\bar{x}) &= -\sum_{k,\bar{k}}\frac{2}{k^2-\bar{k}^2}\tilde{f}(k,\bar{k})e^{ik\cdot x + i\bar{k}\cdot\bar{x}}, \\ \tilde{f}(k,\bar{k}) &\equiv 0 \quad \forall\ k^2 = \bar{k}^2. \quad (4.13)\end{aligned}$$

Such an operator is clearly not defined on $\ker\Delta$. Nevertheless, the following operator relations hold on the full space $\mathcal{X}$:

$$G\Delta = 1 - \mathcal{P}_\Delta, \qquad \Delta G(1-\mathcal{P}_\Delta) = 1 - \mathcal{P}_\Delta, \quad (4.14)$$

as one may quickly verify. Since $\Delta = [M_1, b^-]$, it is now straightforward to find the homotopy:

$$h \coloneqq b^- G(1-\mathcal{P}_\Delta), \quad (4.15)$$

which indeed obeys the fundamental relation (4.6) and the side conditions (4.8). To verify this one uses that b^- is nilpotent and commutes with $\mathcal{P}_\Delta$ and G.

Equipped with the projector and homotopy maps we are now ready to transport the $\mathrm{BV}_\infty^\Delta$ structure to $\bar{\mathcal{X}}$. Homotopy transfer is well-established for L_∞ and A_∞ algebras but, to the best of our knowledge, it has not been discussed for $\mathrm{BV}_\infty^\square$ or $\mathrm{BV}_\infty^\Delta$ algebras. We will thus proceed step by step in a constructive way. From now on, we will denote all transferred maps in $\bar{\mathcal{X}}$ with an overline, which should not be confused with the second copy $\bar{\mathcal{K}}$, since the underlying Yang-Mills maps will not play a role anymore. Here all inputs are intended as $\bar{\Psi}_i$, living in $\ker\Delta$. Since we do not display any input, we shall denote by $\mathcal{M}|_{\bar{\mathcal{X}}}$ the restriction of any multilinear map $\mathcal{M}$ (including operators) to act on elements of the subspace $\bar{\mathcal{X}}\subset\mathcal{X}$.

We begin with the differential, which is unchanged thanks to $[M_1,\mathcal{P}_\Delta]=0$:

$$\bar{M}_1 \coloneqq M_1|_{\bar{\mathcal{X}}}. \tag{4.16}$$

Next, the transferred two-product M_2 is simply obtained by restricting the map to the subspace and projecting the output back to the subspace,

$$\bar{M}_2 \coloneqq \mathcal{P}_\Delta M_2|_{\bar{\mathcal{X}}}, \tag{4.17}$$

which obeys $\Delta\bar{M}_2=0$ by definition. Similarly, $\bar{B}_2$ is given by projection, and it retains its BV relation with $\bar{M}_2$:

$$\bar{B}_2 \coloneqq \mathcal{P}_\Delta B_2|_{\bar{\mathcal{X}}} = \mathcal{P}_\Delta[b^-,M_2]|_{\bar{\mathcal{X}}} = [b^-,\mathcal{P}_\Delta M_2|_{\bar{\mathcal{X}}}] = [b^-,\bar{M}_2]. \tag{4.18}$$

The original failure of M_1 to be a derivation of B_2 is now cured for $\bar{B}_2$:

$$[\bar{M}_1,\bar{B}_2] = [M_1,[b^-,\bar{M}_2]] = [\Delta,\bar{M}_2] = 0, \tag{4.19}$$

where we used the fact that acting on elements of $\ker\Delta$ one has $[\Delta,\bar{M}_2]=\Delta\bar{M}_2=0$.

We move on to the first trilinear map, which requires the homotopy operator. In order to find $\bar{M}_{3\mathrm{h}}$, we compute the associator of $\bar{M}_2$ [we leave the restriction $(\cdots)|_{\bar{\mathcal{X}}}$ implicit]:

$$\begin{aligned}\bar{M}_2\bar{M}_2(1-\Pi) &= \mathcal{P}_\Delta M_2\mathcal{P}_\Delta M_2(1-\Pi)\\ &= \mathcal{P}_\Delta(M_2M_2 - M_2[M_1,h]M_2)(1-\Pi)\\ &= \mathcal{P}_\Delta([M_1,M_{3\mathrm{h}}]-[M_1,M_2hM_2])(1-\Pi)\\ &= [\bar{M}_1,\bar{M}_{3\mathrm{h}}],\end{aligned} \tag{4.20}$$

with the transferred three-product

$$\bar{M}_{3\mathrm{h}} = \mathcal{P}_\Delta(M_{3\mathrm{h}} - M_2hM_2(1-\Pi))|_{\bar{\mathcal{X}}}. \tag{4.21}$$

Notice that we used $[M_1,M_2]=0$ to compute

$$[M_1,M_2hM_2] = M_2[M_1,hM_2] = M_2[M_1,h]M_2. \tag{4.22}$$

We now define the Poisson compatibility of the transferred $\bar{M}_2$ and $\bar{B}_2$ as the one in $\mathcal{X}$ [see (3.25)] by

$$\bar{B}_2\bar{M}_2 + \bar{M}_2\bar{B}_2(1-3\Pi) \equiv [b^-,\bar{M}_2\bar{M}_2] - 3\bar{M}_2\bar{B}_2\Pi. \tag{4.23}$$

Since now both $\bar{M}_2$ and $\bar{B}_2$ commute with M_1, one immediately has that the expression (4.23) is M_1-closed. Its hook part is also exact, since

$$\begin{aligned}\{[b^-,\bar{M}_2\bar{M}_2]-3\bar{M}_2\bar{B}_2\Pi\}(1-\Pi) &= [b^-,\bar{M}_2\bar{M}_2(1-\Pi)] = [b^-,[M_1,\bar{M}_{3\mathrm{h}}]]\\ &= -[M_1,[b^-,\bar{M}_{3\mathrm{h}}]] + [\Delta,\bar{M}_{3\mathrm{h}}]\\ &= [M_1,\bar{\Theta}_{3\mathrm{h}}]\end{aligned} \tag{4.24}$$

for $\bar{\Theta}_{3\mathrm{h}} = -[b^-,\bar{M}_{3\mathrm{h}}]$. The Δ-obstruction above vanishes, since $[\Delta,\bar{M}_{3\mathrm{h}}]=\Delta\bar{M}_{3\mathrm{h}}=0$ when acting on inputs in $\bar{\mathcal{X}}$. Computing the symmetric part of (4.23) is, as usual, more complicated. We obtain

$$\begin{aligned}\{[b^-,\bar{M}_2\bar{M}_2]-3\bar{M}_2\bar{B}_2\Pi\}\Pi &= \mathcal{P}_\Delta([b^-,M_2\mathcal{P}_\Delta M_2]-3M_2\mathcal{P}_\Delta B_2)\Pi\\ &= \mathcal{P}_\Delta([b^-,M_2M_2]-3M_2B_2-[b^-,M_2[M_1,h]M_2]+3M_2[M_1,h]B_2)\Pi.\end{aligned} \tag{4.25}$$

Now we can use

$$\begin{aligned}M_2[M_1,h]M_2 &= [M_1,M_2hM_2],\\ [M_1,hB_2] &= [M_1,h]B_2 - h[M_1,B_2] = [M_1,h]B_2 - h[\Delta,M_2],\end{aligned} \tag{4.26}$$

in order to pull out some M_1-commutators. Then, equation (4.25) becomes

$$\begin{aligned}&\mathcal{P}_\Delta([b^-,M_2M_2]-3M_2B_2-[b^-,[M_1,M_2hM_2]]+3[M_1,M_2hB_2]+3M_2h[\Delta,M_2])\Pi\\ &= \mathcal{P}_\Delta([b^-,M_2M_2]-3M_2B_2+[M_1,[b^-,M_2hM_2]+3M_2hB_2]+3M_2h[\Delta,M_2])\Pi,\end{aligned} \tag{4.27}$$

where we used $\mathcal{P}_\Delta[\Delta, M_2hM_2]|_{\bar{\mathcal{X}}} = \mathcal{P}_\Delta\Delta M_2hM_2|_{\bar{\mathcal{X}}} = 0$. The last term above can be further manipulated as follows:

$$\begin{aligned} 3\mathcal{P}_\Delta M_2h[\Delta, M_2]\Pi &= 3\mathcal{P}_\Delta M_2hM_2D_s\Pi = 3pM_2hM_2(\Pi + 1 - \Pi)D_s\Pi \\ &= \mathcal{P}_\Delta M_2hM_2D_\Delta\Pi + 3\mathcal{P}_\Delta M_2hM_2(1-\Pi)D_s\Pi \\ &= \mathcal{P}_\Delta[\Delta, M_2hM_2]\Pi + 3\mathcal{P}_\Delta M_2hM_2(1-\Pi)D_s\Pi \\ &= 3\mathcal{P}_\Delta M_2hM_2(1-\Pi)D_s\Pi, \end{aligned} \tag{4.28}$$

where we used (3.24) and the vanishing of total Δ-commutators under $\mathcal{P}_\Delta$. We can now use the homotopy Poisson relation (B13) on $\mathcal{X}$ to rewrite (4.27) as

$$\begin{aligned} &\mathcal{P}_\Delta([M_1, \Theta_{3s}] - 3M_{3h}D_s + [M_1, [b^-, M_2hM_2] + 3M_2hB_2] + 3M_2hM_2(1-\Pi)D_s)\Pi \\ &= [M_1, \mathcal{P}_\Delta(\Theta_{3s} + [b^-, M_2hM_2] + 3M_2hB_2)\Pi] - 3\mathcal{P}_\Delta(M_{3h} - M_2hM_2(1-\Pi))D_s\Pi \\ &= [\bar{M}_1, \bar{\Theta}_{3s}] - 3\bar{M}_{3h}D_s\Pi, \end{aligned} \tag{4.29}$$

with the transferred $\bar{\Theta}_3$ given by

$$\bar{\Theta}_3 = \mathcal{P}_\Delta(\Theta_3 + [b^-, M_2hM_2] + 3M_2hB_2\Pi)|_{\bar{\mathcal{X}}}. \tag{4.30}$$

The full homotopy Poisson relation thus reads

$$[b^-, \bar{M}_2\bar{M}_2] - 3\bar{M}_2\bar{B}_2\Pi = [\bar{M}_1, \bar{\Theta}_3] - 3\bar{M}_{3h}D_s\Pi. \tag{4.31}$$

This is telling us that the $\mathrm{BV}_\infty^\Delta$ algebra is well-behaved under homotopy transfer, but this is not enough to completely remove the obstructions. The reason is that Δ is not zero as an operator on $\bar{\mathcal{X}}$. Rather, only total Δ-commutators are transferred to zero.

The main difference of (4.31) compared to the identity (B13) is that both sides of (4.31) are M_1-closed. This is obvious for the left-hand side, since $\bar{M}_1$ commutes with both $\bar{M}_2$ and $\bar{B}_2$, while checking it for the right-hand side requires some computation:

$$\begin{aligned} [\bar{M}_1, \bar{M}_{3h}D_s\Pi] &= [\bar{M}_1, \bar{M}_{3h}]D_s\Pi \\ &= \bar{M}_2\bar{M}_2(1-\Pi)D_s\Pi \\ &= -\frac{1}{3}\bar{M}_2\bar{M}_2D_\Delta\Pi + \bar{M}_2\bar{M}_2D_s\Pi \\ &= -\frac{1}{3}[\Delta, \bar{M}_2\bar{M}_2]\Pi + \bar{M}_2[\Delta, \bar{M}_2]\Pi = 0. \end{aligned} \tag{4.32}$$

We thus see that the Δ-obstruction in (4.31) is M_1-closed. If it were exact, one could shift $\bar{\Theta}_3$ to obtain a genuine homotopy Poisson identity. If this is not the case, on the other hand, $\bar{M}_{3h}D_s\Pi$ would be a genuine cohomological obstruction.

For the last step, let us determine the homotopy Jacobi relation. Due to $\bar{B}_2 = [b^-, \bar{M}_2]$, the Jacobiator is given by taking a b^--commutator of the left-hand side of (4.31). The computation is entirely analogous to the one leading to (B15), and we obtain

$$3\bar{B}_2\bar{B}_2\Pi + [\bar{M}_1, \bar{B}_3] + 3\bar{\Theta}_3D_s\Pi = 0, \tag{4.33}$$

including the deformation. It is interesting to note that only the hook part of $\bar{\Theta}_3$ (and thus $\bar{M}_{3h}$) contributes to (4.33), since

$$\bar{\Theta}_{3s}D_s\Pi = \bar{\Theta}_3\Pi D_s\Pi = \frac{1}{3}\bar{\Theta}_3D_\Delta\Pi = 0. \tag{4.34}$$

The transferred three-bracket is given by $\bar{B}_3 = -[b^-, \bar{\Theta}_{3s}]$, as dictated by $\mathrm{BV}_\infty^\Delta$. Upon using the expression (4.30), one can see that $\bar{B}_3$ has also the standard form in terms of homotopy transfer of L_∞ algebras:

$$\bar{B}_3 = \mathcal{P}_\Delta(B_3 - 3B_2hB_2\Pi)|_{\bar{\mathcal{X}}}. \tag{4.35}$$

In the following subsection we will study the obstruction to the homotopy Jacobi identity encoded in the last term of (4.33) upon restricting the inputs to $\ker b^-$. Only the L_∞ brackets $\bar{B}_n$ will restrict to $\mathcal{V}_{\mathrm{DFT}}$, and we will show that a well-defined albeit nonlocal deformation of $\bar{B}_3$ can be used to remove the obstruction.

B. Restriction to $\ker b^-$ and three-bracket

In the last section we have shown that the space $\bar{\mathcal{X}}$ of weakly constrained fields still carries a $\mathrm{BV}_\infty^\Delta$ structure with milder, but nonvanishing, Δ deformations. The DFT vector space (4.1) can be viewed, in terms of $\bar{\mathcal{X}}$, as the subspace $\mathcal{V}_{\mathrm{DFT}} = \ker b^- \subset \bar{\mathcal{X}}$. In the following we will denote the restriction of maps $\bar{\mathcal{M}}$ (which already act on inputs in $\bar{\mathcal{X}}$) to act on elements in $\ker b^-$ by $\bar{\mathcal{M}}|_{\ker b^-}$. Let us stress that here we are merely restricting the inputs to lie in $\ker b^-$, but no homotopy transfer is involved. While generic operators and maps of the $\mathrm{BV}_\infty^\Delta$ algebra on $\bar{\mathcal{X}}$ are not well-defined on $\mathcal{V}_{\mathrm{DFT}}$ upon restricting the inputs, the brackets of the L_∞ sector are, as we will show in a moment.

From now on we will focus on the (obstructed) L_∞ sector on $\bar{\mathcal{X}}$, with brackets $\bar{B}_n$ given by

$$\bar{B}_1 = \bar{M}_1,\quad \bar{B}_2 = [b^-, \bar{M}_2],\quad \bar{B}_3 = -[b^-, \bar{\Theta}_{3s}], \tag{4.36}$$

obeying the following quadratic relations:

$$\bar{B}_1^2 = 0,\quad [\bar{B}_1, \bar{B}_2] = 0,\quad 3\bar{B}_2\bar{B}_2\Pi + [\bar{B}_1, \bar{B}_3] = -3\bar{\Theta}_{3h}D_s\Pi. \tag{4.37}$$

Before studying the obstruction of the homotopy Jacobi identity, we recall that restricting the inputs to $\ker b^-$ gives us a well-defined differential and two-bracket on $\mathcal{V}_{\rm DFT}$ [40], which we denote by $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively,

$$\mathcal{B}_1 \coloneqq \bar{B}_1|_{\ker b^-},\quad \mathcal{B}_2 \coloneqq \bar{B}_2|_{\ker b^-} = [b^-, \bar{M}_2]|_{\ker b^-} = b^-\bar{M}_2|_{\ker b^-}. \tag{4.38}$$

As we have mentioned, $\mathcal{B}_1$ and $\mathcal{B}_2$ are well-defined on $\ker b^-$, since they obey $b^-\mathcal{B}_i = 0$. While it is trivial to see this for $\mathcal{B}_2$ from (4.38), for the differential it can be shown by computing

$$b^-\mathcal{B}_1 = b^-\bar{B}_1|_{\ker b^-} = [b^-, \bar{B}_1]|_{\ker b^-} = [b^-, M_1]|_{\ker \Delta}|_{\ker b^-} = \Delta|_{\ker \Delta}|_{\ker b^-} = 0. \tag{4.39}$$

This confirms that $\mathcal{B}_1\colon \mathcal{V}_{\rm DFT} \to \mathcal{V}_{\rm DFT}$ and $\mathcal{B}_2\colon \mathcal{V}_{\rm DFT}^{\otimes 2} \to \mathcal{V}_{\rm DFT}$ restrict correctly and obey nilpotency and the Leibniz relation, thus defining a consistent field theory (DFT) to cubic order.

We now move on to the Jacobi identity of $\mathcal{B}_2$, which is given by restriction of the corresponding relation (4.37):

$$3\mathcal{B}_2\mathcal{B}_2\Pi + [\mathcal{B}_1, \bar{B}_3|_{\ker b^-}] = \mathcal{O},\quad \text{where } \mathcal{O} \coloneqq -3\bar{\Theta}_{3h}D_s\Pi|_{\ker b^-}, \tag{4.40}$$

with the obstruction denoted by $\mathcal{O}$. Taking into account the restriction to $\ker b^-$, one can express the obstruction as

$$\mathcal{O} = -3\bar{\Theta}_{3h}D_s\Pi|_{\ker b^-} = 3[b^-, \bar{M}_{3h}]D_s\Pi|_{\ker b^-} = 3b^-\bar{M}_{3h}D_s\Pi|_{\ker b^-}, \tag{4.41}$$

where we used that $[b^-, \mathcal{T}]|_{\ker b^-} = b^-\mathcal{T}|_{\ker b^-}$. This expression can be further manipulated by using the definition (4.21) of the transferred $\bar{M}_{3h}$:

$$\begin{aligned}\mathcal{O} &= 3b^-\bar{M}_{3h}D_s\Pi|_{\ker b^-}\\ &= 3\mathcal{P}_\Delta b^-(M_{3h} - M_2hM_2(1-\Pi))D_s\Pi|_{\ker b^-\cap\ker\Delta}\\ &= 3\mathcal{P}_\Delta b^-(M_{3h}D_s\Pi - M_2\Delta hM_2\Pi)|_{\ker b^-\cap\ker\Delta},\end{aligned} \tag{4.42}$$

where we used $M_2hM_2D_s|_{\ker\Delta} = M_2[\Delta, hM_2]|_{\ker\Delta} = M_2\Delta hM_2|_{\ker\Delta}$. The homotopy (4.15) obeys $\Delta h = b^-(1-\mathcal{P}_\Delta)$ and, for inputs in $\ker b^-$, we can also write

$$b^-M_2b^-M_2|_{\ker b^-} = [b^-, M_2]b^-M_2|_{\ker b^-} = B_2b^-M_2|_{\ker b^-} = B_2B_2|_{\ker b^-}, \tag{4.43}$$

yielding a simpler expression for the obstruction:

$$\mathcal{O} = 3\mathcal{P}_\Delta(b^-M_{3h}D_s - B_2(1-\mathcal{P}_\Delta)B_2)\Pi|_{\ker b^-\cap\ker\Delta}, \tag{4.44}$$

which we will use to determine whether it can be removed.

First of all, the obstruction is closed: $[\mathcal{B}_1, \mathcal{O}] = 0$, as can be seen by taking a $\mathcal{B}_1$-commutator of (4.40). However, $\mathcal{O}$ is given by projection $\mathcal{P}_\Delta$ of an otherwise not closed quantity:

$$\mathcal{O} = \mathcal{P}_\Delta\tilde{\mathcal{O}},\qquad [\mathcal{B}_1, \tilde{\mathcal{O}}] = \Delta\mathcal{W} \Rightarrow [\mathcal{B}_1, \mathcal{O}] = 0, \tag{4.45}$$

with explicit $\tilde{\mathcal{O}}$ and $\mathcal{W}$ given by

$$\begin{aligned}\tilde{\mathcal{O}} &= 3(b^-M_{3h}D_s - B_2(1-\mathcal{P}_\Delta)B_2)\Pi|_{\ker b^-\cap\ker\Delta},\\ \mathcal{W} &= (3M_{3h}D_s + b^-M_2M_2 - 3M_2(1-\mathcal{P}_\Delta)B_2)\Pi|_{\ker b^-\cap\ker\Delta}.\end{aligned} \tag{4.46}$$

Since $\tilde{\mathcal{O}}$ is not closed, it certainly cannot be exact. It is thus hard to expect that one can extract a $\mathcal{B}_1$-commutator from $\mathcal{O}$ in a simple way.

In order to prove that $\mathcal{O}$ is, in fact, exact, we shall consider the Laplacian corresponding to the Euclidean signature:

$$\Delta_+ \coloneqq \frac{1}{2}(\partial^\mu\partial_\mu + \bar{\partial}^{\bar{\mu}}\bar{\partial}_{\bar{\mu}}),\qquad [\mathcal{B}_1, b^+] = \Delta_+, \tag{4.47}$$

which acts on the Fourier expansion (4.9) as

$$\Delta_+ f(x, \bar{x}) = -\frac{1}{2}\sum_{k,\bar{k}}(k^2 + \bar{k}^2)\tilde{f}(k, \bar{k})e^{ik\cdot x + i\bar{k}\cdot\bar{x}}. \tag{4.48}$$

Since the metric in both k^2 and $\bar{k}^2$ is Euclidean, Δ_+ is almost invertible. The only solution to $\Delta_+ f(x, \bar{x}) = 0$ is the doubled zero mode $\tilde{f}(0, 0)$, which is allowed on the doubled torus due to its nontrivial topology. We can thus associate a zero mode projector $\mathcal{P}_0$ with $\ker\Delta_+$, with a corresponding homotopy

$$h_0 \coloneqq b^+\frac{1}{\Delta_+}(1-\mathcal{P}_0),\qquad [\mathcal{B}_1, h_0] = 1 - \mathcal{P}_0. \tag{4.49}$$

At this stage, it is important to notice that nonlinear combinations of fields of the form

$$((1-\mathcal{P}_\Delta)f\mathcal{P}_\Delta g)(x,\bar{x})\sum_{k^2\neq\bar{k}^2,l^2=\bar{l}^2}\tilde{f}(k,\bar{k})\tilde{g}(l,\bar{l})e^{i(k+l)\cdot x+i(\bar{k}+\bar{l})\cdot\bar{x}}, \tag{4.50}$$

do not contain zero modes. This is easily seen from the fact that in the sum above $(k^\mu,\bar{k}^{\bar{\mu}})\neq-(l^\mu,\bar{l}^{\bar{\mu}})$, given that $k^2\neq\bar{k}^2$, while $l^2=\bar{l}^2$. The total momentum above is thus $(k^\mu+l^\mu,\bar{k}^{\bar{\mu}}+\bar{l}^{\bar{\mu}})\neq(0,0)$. On such combinations one has $(1-\mathcal{P}_0)=1$, yielding

$$(1-\mathcal{P}_\Delta)f\mathcal{P}_\Delta g=[\mathcal{B}_1,h_0]((1-\mathcal{P}_\Delta)f\mathcal{P}_\Delta g). \tag{4.51}$$

In order to see that we can apply this argument to our obstruction, let us act with $\mathcal{O}$ in (4.44) on three arbitrary inputs (Ψ_1,Ψ_2,Ψ_3) in $\mathcal{V}_{\rm DFT}$:

$$\begin{aligned}\mathcal{O}(\Psi_1,\Psi_2,\Psi_3)\overset{(123)}{=}&3\mathcal{P}_\Delta b^-(M_{3\rm h}(\partial^\mu\Psi_1,\partial_\mu\Psi_2,\Psi_3)\\&-M_{3\rm h}(\bar{\partial}^{\bar{\mu}}\Psi_1,\bar{\partial}_{\bar{\mu}}\Psi_2,\Psi_3))\\&-3\mathcal{P}_\Delta B_2((1-\mathcal{P}_\Delta)B_2(\Psi_1,\Psi_2),\Psi_3),\end{aligned} \tag{4.52}$$

where by (123) we denote graded symmetrization in the labels. The B_2B_2 term above has momenta of the form (4.50), given the explicit projector $(1-\mathcal{P}_\Delta)$ and recalling $\mathcal{P}_\Delta\Psi_i=\Psi_i$. The $M_{3\rm h}$ term falls in the same category, since $\mathcal{P}_\Delta$ only acts on input functions, and one has

$$\begin{aligned}\mu[D_s(F_1\otimes F_2\otimes F_3)]&=\mu[(\partial^\mu F_1\otimes\partial_\mu F_2\otimes F_3)\\&\quad-(\bar{\partial}^{\bar{\mu}}F_1\otimes\bar{\partial}_{\bar{\mu}}F_2\otimes F_3)]\\&=\partial^\mu F_1\partial_\mu F_2F_3-\bar{\partial}^{\bar{\mu}}F_1\bar{\partial}_{\bar{\mu}}F_2F_3\\&=\Delta(F_1F_2)F_3\\&=(1-\mathcal{P}_\Delta)[\Delta(F_1F_2)]\mathcal{P}_\Delta F_3,\end{aligned} \tag{4.53}$$

for weakly constrained functions $F_i(x,\bar{x})$ obeying $\Delta F_i=0$.

Having shown that Δ_+ is invertible on $\mathcal{O}$, we can prove that $\mathcal{O}$ is exact:

$$\begin{aligned}\mathcal{O}&=\left[\mathcal{B}_1,\frac{b^+}{\Delta_+}\right]\mathcal{O}=\left[\mathcal{B}_1,\frac{b^+}{\Delta_+}\mathcal{O}\right]+\frac{b^+}{\Delta_+}[\mathcal{B}_1,\mathcal{O}]\\&=\left[\mathcal{B}_1,\frac{b^+}{\Delta_+}\mathcal{O}\right],\end{aligned} \tag{4.54}$$

where we used $[\mathcal{B}_1,\mathcal{O}]=0$.

Since we have shown that the obstruction is exact, we can shift the original $\bar{B}_3|_{\ker b^-}$ appearing in (4.40) by $\frac{b^+}{\Delta_+}\mathcal{O}$ and obtain a genuine L_∞ relation on $\mathcal{V}_{\rm DFT}$:

$$3\mathcal{B}_2\mathcal{B}_2\Pi+[\mathcal{B}_1,\mathcal{B}_3]=0, \tag{4.55}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are given by (4.38), and the final three-bracket reads

$$\begin{aligned}\mathcal{B}_3=\mathcal{P}_\Delta\Big(B_3-3\frac{b^+}{\Delta_+}(b^-M_{3\rm h}D_s\\-B_2(1-\mathcal{P}_\Delta)B_2)\Pi\Big)\Big|_{\ker b^-\cap\ker\Delta}.\end{aligned} \tag{4.56}$$

Notice that the standard homotopy part B_2hB_2 in the definition (4.35) of the transported $\bar{B}_3$ drops on $\ker b^-$, due to $h\propto b^-$ and $B_2\propto b^-$. We have thus succeeded in constructing the three-bracket of weakly constrained DFT on a purely spatial torus. Since the whole construction is fairly abstract and intricate, in the next section we will provide an explicit check of the above results by computing the gauge algebra.

C. Gauge algebra

We now compute explicitly a consistent subsector of the gauge algebra of weakly constrained DFT, which is encoded in the homotopy Jacobi relation

$${\rm Jac}(\Lambda_1,\Lambda_2,\Lambda_2)+[\mathcal{B}_1,\mathcal{B}_3](\Lambda_1,\Lambda_2,\Lambda_3)=0, \tag{4.57}$$

with the Jacobiator ${\rm Jac}(\Lambda_1,\Lambda_2,\Lambda_3)$ defined as

$$\begin{aligned}{\rm Jac}(\Lambda_1,\Lambda_2,\Lambda_3)&\coloneqq3\mathcal{B}_2(\mathcal{B}_2(\Lambda_{[1},\Lambda_2),\Lambda_{3]})\\&\overset{[123]}{=}3\mathcal{B}_2(\mathcal{B}_2(\Lambda_1,\Lambda_2),\Lambda_3).\end{aligned} \tag{4.58}$$

The input labels inside of the square brackets [123] on top of the last equals sign denote antisymmetrization of the labels. In the above equation, as in the remainder of the paper, we employ the convention

$$\begin{aligned}3\mathcal{B}_2(\mathcal{B}_2(\Lambda_{[1},\Lambda_2),\Lambda_{3]})&=\mathcal{B}_2(\mathcal{B}_2(\Lambda_1,\Lambda_2),\Lambda_3)\\&\quad+\mathcal{B}_2(\mathcal{B}_2(\Lambda_2,\Lambda_3),\Lambda_1)\\&\quad+\mathcal{B}_2(\mathcal{B}_2(\Lambda_3,\Lambda_1),\Lambda_2),\end{aligned} \tag{4.59}$$

where we used the antisymmetry of $\mathcal{B}_2$ when acting on gauge parameters: $\mathcal{B}_2(\Lambda_1,\Lambda_2)=-\mathcal{B}_2(\Lambda_2,\Lambda_1)$.

In order to check the identity (4.57), it will be convenient to rewrite the individual terms of the Jacobiator in a different but equivalent way. One can rewrite the inner projector of the nested brackets in the Jacobiator as $\mathcal{P}_\Delta=1-(1-\mathcal{P}_\Delta)$ while keeping the external projector untouched. Doing so yields

$$\begin{aligned}{\rm Jac}(\Lambda_1,\Lambda_2,\Lambda_3)&\overset{[123]}{=}3\mathcal{P}_\Delta B_2(B_2(\Lambda_1,\Lambda_2),\Lambda_3)\\&\quad-3\mathcal{P}_\Delta B_2((1-\mathcal{P}_\Delta)B_2(\Lambda_1,\Lambda_2),\Lambda_3)\\&\overset{[123]}{=}3\mathcal{P}_\Delta B_2(B_2(\Lambda_1,\Lambda_2),\Lambda_3)\\&\quad-3\mathcal{P}_\Delta B_2(\{B_2(\Lambda_1,\Lambda_2)\}_\perp,\Lambda_3),\end{aligned} \tag{4.60}$$

where here and in what follows it is understood that all the maps are acting on $\ker b^-\cap\ker\Delta$ and in order to simplify our notation we introduced the perpendicular projector $\{B_2\}_\perp$ in the second term in the last line which denotes $(1-\mathcal{P}_\Delta)B_2$. The above split will be useful once we compute the part of the homotopy Jacobi relation that contains the three-bracket $\mathcal{B}_3$ because it contains a term with $\mathcal{P}_\Delta B_2(1-\mathcal{P}_\Delta)B_2$.

As presented in Eq. (4.4), a generic gauge parameter in double field theory has three components: two vector components and one scalar component. However, in order to simplify the computation we will restrict our attention to vanishing $\bar\lambda^{\bar\nu}$ and η while only keeping λ^μ. For this reason from now on we consider the consistent subsector of the gauge algebra with parameters of the form

$$\Lambda = -\theta_\mu\bar\theta_+\lambda^\mu. \tag{4.61}$$

The homotopy Jacobi relation (4.57) takes values in the space of gauge parameters, and hence consists of three components. For this reason we will check the gauge algebra explicitly displaying the basis elements of the DFT space $Z_A\bar Z_{\bar B}$. This will allow us to keep track of the different components of the relation.

We now turn to finding the two-bracket between gauge parameters using the technology developed in Sec. III. For gauge parameters defined as in (4.61), we have

$$\begin{aligned}
\mathcal{B}_2(\Lambda_1,\Lambda_2) &= \mathcal{P}_\Delta b^- m_2\otimes\bar m_2(\theta_\mu\bar\theta_+\lambda_1^\mu,\theta_\nu\bar\theta_+\lambda_2^\nu)\\
&= \mathcal{P}_\Delta b^-\mu[\hat m_2(\theta_\mu,\theta_\nu)\hat{\bar m}_2(\theta_+,\theta_+)(\lambda_1^\mu\otimes\lambda_2^\nu)]\\
&= \mathcal{P}_\Delta b^-\mu\{[c\theta_\nu[(\partial_\mu\otimes\mathbb{1})+2(\mathbb{1}\otimes\partial_\mu)]-c\theta_\mu[(\mathbb{1}\otimes\partial_\nu)+2(\partial_\nu\otimes\mathbb{1})]\\
&\quad + c\theta_\rho\eta_{\mu\nu}[(\partial^\rho\otimes\mathbb{1})-(\mathbb{1}\otimes\partial^\rho)]\bar\theta_+(\mathbb{1}\otimes\mathbb{1})](\lambda_1^\mu\otimes\lambda_2^\nu)\}\\
&= \mathcal{P}_\Delta b^- c\theta_\rho\bar\theta_+(\partial\cdot\lambda_1\lambda_2^\rho+2\lambda_1\cdot\partial\lambda_2^\rho+\partial^\rho\lambda_1\cdot\lambda_2-(1\leftrightarrow 2))\\
&= \frac{1}{2}\mathcal{P}_\Delta\theta_\rho\bar\theta_+(\partial\cdot\lambda_1\lambda_2^\rho+2\lambda_1\cdot\partial\lambda_2^\rho+\partial^\rho\lambda_1\cdot\lambda_2-(1\leftrightarrow 2))\\
&\equiv \frac{1}{2}\mathcal{P}_\Delta\theta_\rho\bar\theta_+(\lambda_1\bullet\lambda_2)^\rho\in\mathcal{V}_{-1},
\end{aligned}\tag{4.62}$$

and we used the component form of $\hat m_2(\theta_\mu,\theta_\nu)$ and $\hat{\bar m}_2(\bar\theta_+,\bar\theta_+)$, which can be found in Appendix A in Eq. (A4). Using the above expression for the two-bracket B_2 we obtain the following Jacobiator:

$$\begin{aligned}
\mathrm{Jac}(\Lambda_1,\Lambda_2,\Lambda_3) \overset{[123]}{=} &-\frac{3}{2}\mathcal{P}_\Delta\theta_\mu\bar\theta_+[\partial^\mu(\lambda_{1\rho}\partial^\rho\lambda_{2\nu}\lambda_3^\nu)+2\partial_\rho\lambda_{1\nu}\lambda_2^\nu\partial^\rho\lambda_3^\mu+\Delta_+\lambda_{1\rho}\lambda_2^\rho\lambda_3^\mu\\
&+2\partial_\rho\lambda_1^\rho\lambda_2^\nu\partial_\nu\lambda_3^\mu+\partial_\rho\lambda_1^\rho\partial^\mu\lambda_{2\nu}\lambda_3^\nu+\lambda_2^\nu\partial_\nu\partial_\rho\lambda_1^\rho\lambda_3^\mu]+\frac{3}{4}\mathcal{P}_\Delta\theta_\mu\bar\theta_+[\{\lambda_1\bullet\lambda_2\}_\perp\bullet\lambda_3]^\mu,
\end{aligned}\tag{4.63}$$

where we use $\mathcal{P}_\Delta\Box=\mathcal{P}_\Delta\Delta_+$.

Having the Jacobiator (4.63) at our disposal, in order to verify the homotopy Jacobi relation (4.57) we need the following components of $\mathcal{B}_3$: first, $\mathcal{B}_3$ on three gauge parameters, whose only nontrivial part can be found with the following computation:

$$\begin{aligned}
\mathcal{B}_3(\Lambda_1,\Lambda_2,\Lambda_3) &= \mathcal{P}_\Delta B_3(\Lambda_1,\Lambda_2,\Lambda_2)\\
&= -\frac{1}{2}\mathcal{P}_\Delta b^-\theta_{3s}\otimes\bar m_2\bar m_2\Pi(\Lambda_1,\Lambda_2,\Lambda_3)\\
&\overset{[123]}{=} \frac{1}{2}\mathcal{P}_\Delta b^-\theta_{3s}\otimes\bar m_2\bar m_2(\theta_\mu\bar\theta_+\lambda_1^\mu,\theta_\nu\bar\theta_+\lambda_2^\nu,\theta_\rho\bar\theta_+\lambda_3^\rho\\
&\overset{[123]}{=} \frac{1}{2}\mathcal{P}_\Delta b^-\mu[\hat\theta_{3s}(\theta_\mu,\theta_\nu,\theta_\rho)\bar\theta_+(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1})(\lambda_1^\mu\otimes\lambda_2^\nu\otimes\lambda_3^\rho)]\\
&\overset{[123]}{=} \frac{1}{2}\mathcal{P}_\Delta b^-\mu\{c\theta_+[\eta_{\mu\nu}(\partial_\rho\otimes\mathbb{1}\otimes\mathbb{1})-\eta_{\mu\nu}(\mathbb{1}\otimes\partial_\rho\otimes\mathbb{1})+\eta_{\nu\rho}(\mathbb{1}\otimes\partial_\mu\otimes\mathbb{1})\\
&\quad -\eta_{\nu\rho}(\mathbb{1}\otimes\mathbb{1}\otimes\partial_\mu)+\eta_{\mu\rho}(\mathbb{1}\otimes\mathbb{1}\otimes\partial_\nu)-\eta_{\mu\rho}(\partial_\nu\otimes\mathbb{1}\otimes\mathbb{1})]\bar\theta_+(\lambda_1^\mu\otimes\lambda_2^\nu\otimes\lambda_3^\rho)\}\\
&\overset{[123]}{=} 3\mathcal{P}_\Delta b^- c\theta_+\bar\theta_+\{\lambda_{1\rho}\partial^\rho\lambda_{2\nu}\lambda_3^\nu\}\\
&\overset{[123]}{=} \frac{3}{2}\mathcal{P}_\Delta\theta_+\bar\theta_+\{\lambda_{1\rho}\partial^\rho\lambda_{2\nu}\lambda_3^\nu\}\in\mathcal{V}_{-2},
\end{aligned}\tag{4.64}$$

where we used the component form $\hat{\theta}_{3s}(\theta_\mu, \theta_\nu, \theta_\rho)$ shown in Eq. (A12). Second, we need to find $\mathcal{B}_3(\Lambda_1, \Lambda_2, \Psi)$. From a computation analogous to the above, we find

$$\begin{aligned}\mathcal{B}_3(\Lambda_1, \Lambda_2, \Psi) \overset{[12]}{=} & -\frac{1}{2}\mathcal{P}_\Delta \theta_\mu \bar{\theta}_+ [2f_\rho \lambda_1^\rho \lambda_2^\mu + 4e\lambda_1^\nu \partial_\nu \lambda_2^\mu + 2\lambda_1^\nu \partial_\nu e \lambda_2^\mu + 2e\partial^\mu \lambda_{1\nu}\lambda_2^\nu \\ & - e^{\mu\bar{\nu}}\bar{\partial}_{\bar{\nu}}\lambda_{1\rho}\lambda_2^\rho + \bar{\partial}_{\bar{\nu}}\lambda_1^\mu e^{\rho\bar{\nu}}\lambda_{2\rho} - (2f^\rho + \bar{\partial}_{\bar{\nu}} e^{\rho\bar{\nu}})\lambda_{1\rho}\lambda_2^\mu] \\ & + \mathcal{P}_\Delta \frac{1}{\Delta_+}\theta_\mu \bar{\theta}_+ \bar{\partial}^{\bar{\nu}} [\partial_\rho \lambda_1^\mu \partial^\rho \lambda_2^\nu e_{\nu\bar{\nu}} + \lambda_1^\nu \partial_\rho \lambda_2^\mu \partial^\rho e_{\nu\bar{\nu}} + \partial_\rho e^\mu{}_{\bar{\nu}} \partial^\rho \lambda_{1\nu}\lambda_2^\nu \\ & - \bar{\partial}_{\bar{\rho}}\lambda_1^\mu \bar{\partial}^{\bar{\rho}}\lambda_2^\nu e_{\nu\bar{\nu}} - \lambda_1^\nu \bar{\partial}_{\bar{\rho}}\lambda_2^\mu \bar{\partial}^{\bar{\rho}} e_{\nu\bar{\nu}} - \bar{\partial}_{\bar{\rho}} e^\mu{}_{\bar{\nu}} \bar{\partial}^{\bar{\rho}}\lambda_{1\nu}\lambda_2^\nu] \\ & - \frac{1}{4}\mathcal{P}_\Delta \frac{1}{\Delta_+}\theta_\mu \bar{\theta}_+ \bar{\partial}^{\bar{\nu}} [\{\lambda_1 \bullet \lambda_2\}_\perp \bullet e_{\bar{\nu}} + 2\lambda_2 \bullet \{\lambda_1 \bullet e_{\bar{\nu}}\}_\perp]^\mu \\ & - \frac{1}{2}\mathcal{P}_\Delta \theta_+ \bar{\theta}_{\bar{\nu}} [\lambda_1^\rho \partial_\rho \lambda_{2\nu} e^{\nu\bar{\nu}} + \lambda_2^\rho \partial_\rho e^{\nu\bar{\nu}}\lambda_{1\nu} + e^{\nu\bar{\nu}}\partial_\nu \lambda_{1\rho}\lambda_2^\rho] \\ & - \frac{1}{2}\mathcal{P}_\Delta c^+ \theta_+ \bar{\theta}_+ \bar{\partial}_{\bar{\nu}} [\lambda_1^\rho \partial_\rho \lambda_{2\nu} e^{\nu\bar{\nu}} + \lambda_2^\rho \partial_\rho e^{\nu\bar{\nu}}\lambda_{1\nu} + e^{\nu\bar{\nu}}\partial_\nu \lambda_{1\rho}\lambda_2^\rho] \in \mathcal{V}_{-1}. \end{aligned} \tag{4.65}$$

From this expression one infers by inspection of the third to fifth line that the nonlocality inherent in $\frac{1}{\Delta_+}$ is unavoidable: there is no overall Δ_+ that can be factored out to cancel it, as $\bar{\partial}^{\bar{\nu}}$ is contracted with $e_{\mu\bar{\nu}}$ and not with a derivative. This changes after replacing the field in (4.65) by $\mathcal{B}_1(\Lambda)$, which is the next step in order to verify the homotopy Jacobi relation. For instance, in the last line in Eq. (4.65) one obtains

$$\mathcal{P}_\Delta \frac{1}{\Delta_+}\bar{\partial}^{\bar{\nu}}\{\{\lambda_{[1}\bullet\lambda_2\}_\perp \bullet \bar{\partial}_{\bar{\nu}}\lambda_{3]} + 2\lambda_{[2}\bullet\{\lambda_1 \bullet \bar{\partial}_{\bar{\nu}}\lambda_{3]}\}_\perp\}^\mu = \mathcal{P}_\Delta \frac{1}{\Delta_+}\bar{\partial}^{\bar{\nu}}\bar{\partial}_{\bar{\nu}}\{\{\lambda_{[1}\bullet\lambda_2\}_\perp \bullet \lambda_{3]}\}^\mu, \tag{4.66}$$

where the equality follows using the Leibniz rule and the antisymmetry of the labels. Under the projector $\mathcal{P}_\Delta$ we can then use the weak constraint $\bar{\partial}_{\bar{\nu}}\bar{\partial}^{\bar{\nu}} \equiv \bar{\square} = \square$ together with $\mathcal{P}_\Delta \square = \mathcal{P}_\Delta \Delta_+$ to cancel $\frac{1}{\Delta_+}$. Doing so for the other terms and appropriate antisymmetrizations of the inputs leads to

$$\begin{aligned} 3\mathcal{B}_3(\Lambda_{[1}, \Lambda_2, \mathcal{B}_1(\Lambda_{3]})) \overset{[123]}{=} & \frac{3}{2}\mathcal{P}_\Delta \theta_\mu \bar{\theta}_+ \{\Delta_+ \lambda_{3\rho}\lambda_1^\rho \lambda_2^\mu + 2\partial_\rho \lambda_3^\rho \lambda_1^\nu \partial_\nu \lambda_2^\mu + \lambda_1^\nu \partial_\nu \partial_\rho \lambda_3^\rho \lambda_2^\mu + \partial_\rho \lambda_3^\rho \partial^\mu \lambda_{1\nu}\lambda_2^\nu + 2\partial_\rho \lambda_1^\nu \lambda_{2\nu}\partial^\rho \lambda_3^\mu\} \\ & - \frac{3}{4}\mathcal{P}_\Delta \theta_\mu \bar{\theta}_+ \{\{\lambda_1 \bullet \lambda_2\}_\perp \bullet \lambda_3\}^\mu - \frac{3}{2}\mathcal{P}_\Delta \theta_+ \bar{\theta}_{\bar{\nu}} \bar{\partial}^{\bar{\nu}} \{\lambda_{1\rho}\partial^\rho \lambda_{2\nu}\lambda_3^\nu\} - \frac{3}{2}\mathcal{P}_\Delta c^+ \theta_+ \bar{\theta}_+ \Delta_+ \{\lambda_{1\rho}\partial^\rho \lambda_{2\nu}\lambda_3^\nu\}, \end{aligned} \tag{4.67}$$

which has no nonlocalities.

Next, we act with the differential on $\mathcal{B}_3(\Lambda_1, \Lambda_2, \Lambda_3)$, which yields

$$\begin{aligned} \mathcal{B}_1\mathcal{B}_3(\Lambda_1, \Lambda_2, \Lambda_3) \overset{[123]}{=} & \frac{3}{2}\mathcal{P}_\Delta \theta_\mu \bar{\theta}_+ \partial^\mu (\lambda_{1\rho}\partial^\rho \lambda_{2\nu}\lambda_3^\nu) \\ & + \frac{3}{2}\mathcal{P}_\Delta \theta_+ \bar{\theta}_{\bar{\nu}} \bar{\partial}^{\bar{\nu}} (\lambda_{1\rho}\partial^\rho \lambda_{2\nu}\lambda_3^\nu) \\ & + \frac{3}{2}\mathcal{P}_\Delta c^+ \theta_+ \bar{\theta}_+ \Delta_+ (\lambda_{1\rho}\partial^\rho \lambda_{2\nu}\lambda_3^\nu). \end{aligned} \tag{4.68}$$

Finally, adding up (4.63), (4.67), and (4.68) one verifies the homotopy Jacobi relation (4.57).

V. CONCLUSIONS AND OUTLOOK

In this paper we have explicitly constructed weakly constrained double field theory to quartic order in fields, encoded in the three-brackets of the corresponding L_∞ algebra. Due to the weak constraint originating from the level-matching constraints of string theory, the construction of such a theory is a highly nontrivial problem and requires an essential nonlocality, which is also present in the full string theory. Specifically, the weak constraint requires that all fields are annihilated by Δ, the second-order Laplacian with respect to the flat metric of signature (d, d). It is precisely the second-order character of Δ that complicates the construction of an algebra of fields, since the product of two fields satisfying the weak constraint in general does not satisfy the weak constraint. Rather, the naive product has to be modified by projecting the output to the $\Delta = 0$ subspace, an operation that singles out certain Fourier modes and is hence nonlocal. Consequently, the resulting product is nonassociative.

It is relatively straightforward to solve the resulting consistency problems to cubic order [3,46], which is essentially due to a "kinematical accident," but to quartic and higher order it is highly nontrivial to construct a consistent field theory (an L_∞ algebra). In this paper we give the corresponding L_∞ algebra up to and including three-brackets, constructed via a double copy procedure from Yang-Mills theory. Apart from the nonlocality inherent in the weak constraint we found the need for additional nonlocalities in the form of inverses of Δ_+, the positive definite flat-space Laplacian, but they only show up in terms where they are fully well-defined on the torus. We verified that these new nonlocalities are inevitable given the problem we set out to solve: finding the three-brackets B_3 so that the Jacobiator relation involving $B_2 B_2$ is obeyed. Since B_1 and B_2 are fixed from the cubic theory of Hull and Zwiebach [3],[3] which agrees with our double copy [40], the only freedom is the definition of B_3 (which is only well-defined up to cohomologically trivial contributions that drop out from $[B_1, B_3]$). We have verified for the gauge sector that the inverses of Δ_+ are essential.

The research presented here should be generalized in many directions, which include the following:

(i) So far we have given the three-brackets only in the case that all dimensions are toroidal and hence Euclidean, with all coordinates being doubled. It remains to include an undoubled time coordinate or, more generally, an arbitrary number of dimensions for the external or noncompact space. Thus, the theory presented here should be thought of as the internal sector of a split (or Hamiltonian-type) formulation as in [50,51] for double field theory (or in [52–54] for the closely related U-duality invariant exceptional field theory). It would be interesting to see whether such split formulations can be interpreted as a tensor product between internal and external algebras along the lines of [55,56].

(ii) One of the potentially most important applications, and one of strongest original motivations, of a weakly constrained double field theory is in the realm of cosmology. One may imagine massive string modes being excited in the very early universe that leave an imprint on the cosmic microwave background (CMB). For instance, the string gas cosmology proposal of Brandenberger-Vafa invokes the winding modes that must be present if some of the spatial dimensions in cosmology are toroidal [57] (see [58] for a recent review). Generalizing the previous item, it remains to find a weakly constrained double field theory on time-dependent Friedmann-Robertson-Walker backgrounds, generalizing the cubic perturbation theory of [46] to quartic and higher orders.

(iii) Independent of the inclusion of noncompact dimensions, the arguably most important outstanding problem is to generalize the construction to higher order in fields, even just for the internal or compact dimensions. Since the quartic theory exists, it is virtually certain that the theory exists to all orders, but since the detailed construction is already quite involved for the three-brackets, we need a more efficient formulation for the kinematic $\mathrm{BV}_\infty^\square$ structures that are present in Yang-Mills theory proper in order to display the algebra and its double copy to all orders.[4] It is intriguing that this is a problem already in pure Yang-Mills theory, which thus displays a complexity comparable to that of gravity.

(iv) Our double copy procedure developed in [26,40], which is based on the additional structures involving the b-ghost, may appear rather special and only applicable to peculiar formulations, but this is not so. We hope to be able to illustrate this with further examples in the future and to develop the general theory further. Notably, in this paper we have been cavalier about the cyclic structure of the L_∞ algebra, which is needed in order to write an action. Thus, the results presented here are strictly applicable to the equations of motion only. We leave the detailed construction of the cyclic L_∞ brackets, which might differ from the ones presented here by cohomologically trivial shifts (that, however, in the language of BV, are not symplectomorphisms) to future work.

(v) The weakly constrained double field theory constructed here to quartic order is quite complicated and nonlocal. While above we have emphasized that the nonlocalities are inevitable given the fixed starting point encoded in B_1 and B_2 of the cubic theory of Hull and Zwiebach, it is conceivable that there are simpler versions that carry more propagating fields, which would manifest themselves already to quadratic order, and that are effectively integrated out in the theory encountered here. One may wonder if there are versions with weaker constraints or perhaps even no level-matching constraints, as recently explored in string field theory [59,60].

[3] A small caveat is that we do not include so-called cocycle factors, which are claimed to be necessary in the full string theory [47–49] (see also Sec. 3.3 in [46]), since in the present approach they appear unnecessary.

[4] In order to describe these structures to all orders it would be helpful to understand $\mathrm{BV}_\infty^\square$ algebras in terms of so-called operads. We thank Bruno Vallette for explaining to us how a strict $\mathrm{BV}^\square$-algebra can indeed be described in the language of differential graded operads.

(vi) It would be very interesting to generalize weakly constrained double field theory to theories including massive M-theory states of the kind required by U-duality invariance. The strongly constrained versions are known as exceptional field theory (see, e.g., [52–54,61,62]). One of the challenges here is that there is no immediate analog of the double copy construction from Yang-Mills theory, but one may speculate that there are exotic field theories waiting to be constructed that could serve as similar building blocks [63].

ACKNOWLEDGMENTS

We thank Alex Arvanitakis, Chris Hull, Victor Lekeu, Bruno Vallette, Ashoke Sen, and Barton Zwiebach for discussions and collaborations on related topics. O. H. and R. B. thank Barton Zwiebach and MIT for hospitality during the final stages of this project. O. H. thanks Dennis Sullivan for discussions and the CUNY Graduate Center New York for hospitality. This work is supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (Grant Agreement No. 771862). The work of R. B. is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)–Projektnummer 524744955. F. D. is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)—Projektnummer 417533893/GRK2575 "Rethinking Quantum Field Theory."

APPENDIX A: YANG-MILLS MAPS AND OPERATORS

In this appendix we collect all the relevant operators associated with the maps of the $\mathrm{BV}_\infty^\square$ algebra of $\mathcal{K}$. We start from the operators $\hat{m}_1$, $\hat{m}_2$, and $\hat{m}_{3\mathrm{h}}$ in (2.22) corresponding to the C_∞ subalgebra. The differential m_1 is related to $\hat{m}_1$ via

$$m_1(\psi) = \hat{m}_1(Z_A)\psi^A(x), \tag{A1}$$

where the complete list of operators $\hat{m}_1(Z_A)$ is given by

$$\begin{aligned} &\hat{m}_1(\theta_+) = \theta_\mu\partial^\mu + c\theta_+\square, \qquad \hat{m}_1(c\theta_+) = -c\theta_\mu\partial^\mu - \theta_-,\\ &\hat{m}_1(\theta_\mu) = c\theta_\mu\square + \theta_-\partial_\mu, \qquad \hat{m}_1(c\theta_\mu) = -c\theta_-\partial_\mu,\\ &\hat{m}_1(\theta_-) = c\theta_-\square, \qquad \hat{m}_1(c\theta_-) = 0. \end{aligned} \tag{A2}$$

We continue with the two-product m_2, which acts as

$$m_2(\psi_1,\psi_2) = \mu[\hat{m}_2(Z_A,Z_B)(\psi_1^A(x)\otimes\psi_2^B(x))], \tag{A3}$$

and the nonvanishing bidifferential operators $\hat{m}_2(Z_A,Z_B)$ read

$$\begin{aligned} \hat{m}_2(\theta_+,\theta_+) &= \theta_+(\mathbb{1}\otimes\mathbb{1}),\\ \hat{m}_2(\theta_\mu,\theta_\nu) &= c\theta_\nu[(\partial_\mu\otimes\mathbb{1}) + 2(\mathbb{1}\otimes\partial_\mu)]\\ &\quad - c\theta_\mu[(\mathbb{1}\otimes\partial_\nu) + 2(\partial_\nu\otimes\mathbb{1})]\\ &\quad + c\theta_\rho\eta_{\mu\nu}[(\partial^\rho\otimes\mathbb{1}) - (\mathbb{1}\otimes\partial^\rho)], \end{aligned} \tag{A4}$$

for "diagonal" (Z_A,Z_B), while for "off-diagonal" ones we give both orderings explicitly:

$$\begin{aligned} \hat{m}_2(\theta_\mu,\theta_+) &= \theta_\mu(\mathbb{1}\otimes\mathbb{1}) + c\theta_+(\partial_\mu\otimes\mathbb{1} + \mathbb{1}\otimes\partial_\mu), & \hat{m}_2(\theta_+,\theta_\mu) &= \hat{m}_2(\theta_\mu,\theta_+),\\ \hat{m}_2(\theta_+,c\theta_\mu) &= c\theta_\mu(\mathbb{1}\otimes\mathbb{1}), & \hat{m}_2(c\theta_\mu,\theta_+) &= \hat{m}_2(\theta_+,c\theta_\mu),\\ \hat{m}_2(\theta_+,\theta_-) &= -c\theta_\mu(\mathbb{1}\otimes\partial^\mu), & \hat{m}_2(\theta_-,\theta_+) &= -c\theta_\mu(\partial^\mu\otimes\mathbb{1}),\\ \hat{m}_2(\theta_\mu,c\theta_\nu) &= -c\theta_-\eta_{\mu\nu}(\mathbb{1}\otimes\mathbb{1}), & \hat{m}_2(c\theta_\nu,\theta_\mu) &= \hat{m}_2(\theta_\mu,c\theta_\nu),\\ \hat{m}_2(\theta_\mu,\theta_-) &= c\theta_-(\mathbb{1}\otimes\partial_\mu), & \hat{m}_2(\theta_-,\theta_\mu) &= c\theta_-(\partial_\mu\otimes\mathbb{1}),\\ \hat{m}_2(\theta_+,c\theta_-) &= c\theta_-(\mathbb{1}\otimes\mathbb{1}), & \hat{m}_2(c\theta_-,\theta_+) &= \hat{m}_2(\theta_+,c\theta_-), \end{aligned} \tag{A5}$$

which enforce graded symmetry of the map m_2. The C_∞ maps are exhausted with the three-product

$$\begin{aligned} m_{3\mathrm{h}}(\psi_1,\psi_2,\psi_3) = \mu[\hat{m}_{3\mathrm{h}}(Z_A,Z_B,Z_C)(\psi_1^A(x)\\ \otimes\psi_2^B(x)\otimes\psi_3^C(x))], \end{aligned} \tag{A6}$$

whose only a nonvanishing component is associated with the operator

$$\hat{m}_{3\mathrm{h}}(\theta_\mu,\theta_\nu,\theta_\rho) = (c\theta_\mu\eta_{\nu\rho} - c\theta_\nu\eta_{\mu\rho})(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}). \tag{A7}$$

Coming to the $\mathrm{BV}_\infty^\square$ structure, the two-bracket b_2 is associated with a bidifferential operator $\hat{b}_2$ exactly as in (A4):

$$b_2(\psi_1,\psi_2) = \mu[\hat{b}_2(Z_A,Z_B)(\psi_1^A(x)\otimes\psi_2^B(x))], \tag{A8}$$

but we do not give the explicit form of the operators $\hat{b}_2(Z_A,Z_B)$, since they can be straightforwardly derived from $b_2 = [b, m_2]$. The homotopy Poisson map θ_3 is related to tridifferential operators $\hat{\theta}_3(Z_A,Z_B,Z_C)$ by

$$\begin{aligned} \theta_3(\psi_1,\psi_2,\psi_3) = \mu[\hat{\theta}_3(Z_A,Z_B,Z_C)(\psi_1^A(x)\\ \otimes\psi_2^B(x)\otimes\psi_3^C(x))]. \end{aligned} \tag{A9}$$

The following operators correspond to totally graded symmetric maps:

$$
\begin{aligned}
\hat\theta_3(\theta_+,\theta_+,\theta_-) &= \theta_+(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}),\\
\hat\theta_3(\theta_+,\theta_+,c\theta_-) &= -c\theta_+(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}),\\
\hat\theta_3(\theta_+,\theta_\mu,c\theta_\nu) &= c\theta_+\eta_{\mu\nu}(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}),\\
\hat\theta_3(\theta_+,\theta_\mu,\theta_-) &= \theta_\mu(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}) + c\theta_+(\partial_\mu\otimes\mathbb{1}\otimes\mathbb{1}),\\
\hat\theta_3(\theta_+,c\theta_+,\theta_-) &= c\theta_+(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}),\\
\hat\theta_3(\theta_+,\theta_-,\theta_-) &= \theta_-(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}),\\
\hat\theta_3(\theta_+,\theta_-,c\theta_\mu) &= c\theta_\mu(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}),\\
\hat\theta_3(\theta_+,\theta_-,c\theta_-) &= c\theta_-(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}),\\
\hat\theta_3(\theta_\mu,\theta_-,\theta_-) &= 2c\theta_-(\mathbb{1}\otimes\partial_\mu\otimes\mathbb{1} + \mathbb{1}\otimes\mathbb{1}\otimes\partial_\mu),\\
\hat\theta_3(\theta_\mu,c\theta_\nu,\theta_-) &= -c\theta_-\eta_{\mu\nu}(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}),\\
\hat\theta_3(c\theta_+,\theta_-,\theta_-) &= c\theta_-(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}). \qquad \text{(A10)}
\end{aligned}
$$

Given the ordering of (Z_A, Z_B, Z_C) above, the operator corresponding to the exchange of the first two Z's, i.e. $\hat\theta_3(Z_B, Z_A, Z_C)$, is obtained by just exchanging the first two factors in $(\mathcal{O}_1\otimes\mathcal{O}_2\otimes\mathcal{O}_3)$, since the sign $(-1)^{Z_A Z_B}$ in all these cases is $+1$. For instance, given the above expression for $\hat\theta_3(\theta_+,\theta_\mu,\theta_-)$, one has

$$\hat\theta_3(\theta_\mu,\theta_+,\theta_-) = \theta_\mu(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}) + c\theta_+(\mathbb{1}\otimes\partial_\mu\otimes\mathbb{1}). \qquad \text{(A11)}$$

This ensures the graded symmetry of the map $\theta_3(\psi_1,\psi_2,\psi_3)$ in the first two arguments. Since all the maps associated with (A10) are totally graded symmetric, the corresponding operators obey $\hat\theta_3 = \hat\theta_{3\mathrm{s}} = \widehat{\theta_3\pi}$. The remaining permutations of the arguments (Z_A, Z_B, Z_C) can then be recovered from (2.48). The next $\hat\theta_3$ operators have both a totally graded symmetric part $\hat\theta_{3\mathrm{s}} = \widehat{\theta_3\pi}$ and a hook part $\hat\theta_{3\mathrm{h}} = \hat\theta_3 - \widehat{\theta_3\pi}$, which we give separately:

$$
\begin{aligned}
\hat\theta_{3\mathrm{s}}(\theta_\mu,\theta_\nu,\theta_\rho) = c\theta_+[&\eta_{\mu\nu}(\partial_\rho\otimes\mathbb{1}\otimes\mathbb{1}) - \eta_{\mu\nu}(\mathbb{1}\otimes\partial_\rho\otimes\mathbb{1}) + \eta_{\nu\rho}(\mathbb{1}\otimes\partial_\mu\otimes\mathbb{1})\\
&- \eta_{\nu\rho}(\mathbb{1}\otimes\mathbb{1}\otimes\partial_\mu) + \eta_{\mu\rho}(\mathbb{1}\otimes\mathbb{1}\otimes\partial_\nu) - \eta_{\mu\rho}(\partial_\nu\otimes\mathbb{1}\otimes\mathbb{1})],\\
\hat\theta_{3\mathrm{h}}(\theta_\mu,\theta_\nu,\theta_\rho) = (&\theta_\nu\eta_{\mu\rho} - \theta_\mu\eta_{\nu\rho})(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}), \qquad \text{(A12)}
\end{aligned}
$$

and one can see that they are antisymmetric in the simultaneous exchange of $\mu\leftrightarrow\nu$ and $\mathcal{O}_1\leftrightarrow\mathcal{O}_2$ in the factors $\mathcal{O}_1\otimes\mathcal{O}_2\otimes\mathcal{O}_3$. The last group of nonvanishing $\hat\theta_3$ also has totally graded symmetric and hook components, given by

$$
\begin{aligned}
\hat\theta_{3\mathrm{s}}(\theta_\mu,\theta_\nu,c\theta_\rho) &= (c\theta_\nu\eta_{\mu\rho} - c\theta_\mu\eta_{\nu\rho})(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}),\\
\hat\theta_{3\mathrm{s}}(\theta_\mu,\theta_\nu,\theta_-) &= c\theta_\nu(\mathbb{1}\otimes\mathbb{1}\otimes\partial_\mu) - c\theta_\mu(\mathbb{1}\otimes\mathbb{1}\otimes\partial_\nu) + 2c\theta_\nu(\mathbb{1}\otimes\partial_\mu\otimes\mathbb{1}) - 2c\theta_\mu(\partial_\nu\otimes\mathbb{1}\otimes\mathbb{1})\\
&\quad + c\theta_\rho\eta_{\mu\nu}[(\partial^\rho\otimes\mathbb{1}\otimes\mathbb{1}) - (\mathbb{1}\otimes\partial^\rho\otimes\mathbb{1})],\\
\hat\theta_{3\mathrm{s}}(\theta_\mu,c\theta_+,\theta_-) &= -\hat\theta_{3\mathrm{s}}(c\theta_+,\theta_\mu,\theta_-) = -c\theta_\mu(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}),\\
\hat\theta_{3\mathrm{h}}(\theta_\mu,\theta_\nu,c\theta_\rho) &= (c\theta_\nu\eta_{\mu\rho} - c\theta_\mu\eta_{\nu\rho})(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}),\\
\hat\theta_{3\mathrm{h}}(c\theta_\rho,\theta_\mu,\theta_\nu) &= \hat\theta_{3\mathrm{h}}(\theta_\mu,c\theta_\rho,\theta_\nu) = (c\theta_\mu\eta_{\nu\rho} - c\theta_\rho\eta_{\mu\nu})(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}). \qquad \text{(A13)}
\end{aligned}
$$

As for the two-bracket b_2, we do not give the explicit form of the operators $\hat b_3$ corresponding to the three-bracket, since they can be derived from $b_3 = -[b,\theta_{3\mathrm{s}}]$.

APPENDIX B: DERIVATION OF Θ_3 AND B_3

In this appendix we compute the symmetric projection of the Poisson relation (3.25), in order to determine the symmetric part of the homotopy Θ_3. We then use this to compute the Jacobiator of the bracket B_2, yielding the deformed homotopy Jacobi identity.

We begin by writing the maps in terms of their Yang-Mills building blocks:

$$
\begin{aligned}
\{[b^-, M_2M_2] - 3M_2B_2\Pi\}\Pi &= \frac{1}{2}\{[b-\bar b,(m_2\otimes\bar m_2)(m_2\otimes\bar m_2)] - 3(m_2\otimes\bar m_2)(b_2\otimes\bar m_2 - m_2\otimes\bar b_2)\}\Pi\\
&= \frac{1}{2}\{[b-\bar b, m_2m_2\otimes\bar m_2\bar m_2] - 3(m_2b_2\otimes\bar m_2\bar m_2 - m_2m_2\otimes\bar m_2\bar b_2)\}\Pi\\
&= \frac{1}{2}\{([b,m_2m_2] - 3m_2b_2)\otimes\bar m_2\bar m_2 - m_2m_2\otimes([\bar b,\bar m_2\bar m_2] - 3\bar m_2\bar b_2)\}\Pi. \qquad \text{(B1)}
\end{aligned}
$$

We continue by substituting the Poisson relation (3.26) for $[b, m_2m_2]$ and its barred counterpart:

$$\begin{aligned}\{[b^-, M_2M_2] - 3M_2B_2\Pi\}\Pi &= \frac{1}{2}\{([m_1,\theta_3] + m_{3h}(d_\square - 3d_s\pi) - 3m_2b_2(1-\pi)) \otimes \bar{m}_2\bar{m}_2 \\ &\quad - m_2m_2 \otimes ([\bar{m}_1,\bar{\theta}_3] + \bar{m}_{3h}(\bar{d}_{\bar{\square}} - 3\bar{d}_s\bar{\pi}) - 3\bar{m}_2\bar{b}_2(1-\bar{\pi}))\}\Pi \\ &= \left[M_1, \frac{1}{2}(\theta_3 \otimes \bar{m}_2\bar{m}_2 - m_2m_2 \otimes \bar{\theta}_3)\Pi\right] + \frac{1}{2}\{(m_{3h}(d_\square - 3d_s\pi) - 3m_2b_2(1-\pi)) \otimes \bar{m}_2\bar{m}_2 \\ &\quad - m_2m_2 \otimes (\bar{m}_{3h}(\bar{d}_{\bar{\square}} - 3\bar{d}_s\bar{\pi}) - 3\bar{m}_2\bar{b}_2(1-\bar{\pi}))\}\Pi, \end{aligned} \tag{B2}$$

where we have used $[m_1, m_2m_2] = 0$ to extract a total differential M_1, which gives the first contribution to Θ_{3s}. For the next steps we will repeatedly use the projector relations $\pi\Pi = \bar{\pi}\Pi$ and $(1-\pi)\Pi = (1-\bar{\pi})\Pi$. The terms involving m_2b_2 and $\bar{m}_2\bar{b}_2$ in (B2) can be further manipulated as

$$\begin{aligned}&-\frac{3}{2}\{m_2b_2(1-\pi) \otimes \bar{m}_2\bar{m}_2 - m_2m_2 \otimes \bar{m}_2\bar{b}_2(1-\bar{\pi})\}\Pi \\ &\quad = -\frac{3}{2}\{m_2b_2 \otimes \bar{m}_2\bar{m}_2(1-\bar{\pi}) - m_2m_2(1-\pi) \otimes \bar{m}_2\bar{b}_2\}\Pi \\ &\quad = -\frac{3}{2}\{m_2b_2 \otimes [\bar{m}_1, \bar{m}_{3h}] - [m_1, m_{3h}] \otimes \bar{m}_2\bar{b}_2\}\Pi \\ &\quad = \frac{3}{2}[M_1, (m_2b_2 \otimes \bar{m}_{3h} + m_{3h} \otimes \bar{m}_2\bar{b}_2)\Pi] - \frac{3}{2}\{[m_1, m_2b_2] \otimes \bar{m}_{3h} - m_{3h} \otimes [\bar{m}_1, \bar{m}_2\bar{b}_2]\}\Pi \\ &\quad = \frac{3}{2}[M_1, (m_2b_2 \otimes \bar{m}_{3h} + m_{3h} \otimes \bar{m}_2\bar{b}_2)\Pi] - \frac{3}{2}\{m_2[\square, m_2] \otimes \bar{m}_{3h} - m_{3h} \otimes \bar{m}_2[\bar{\square}, \bar{m}_2]\}\Pi, \end{aligned} \tag{B3}$$

with the total M_1-commutator contributing to Θ_{3s}. Let us now consider the above terms containing $\square$-commutators, together with the $m_{3h}d_s\pi$ terms from (B2). Using the projector relations and rewriting $d_s = \frac{1}{2}(d_s + \bar{d}_s) + \frac{1}{2}(d_s - \bar{d}_s)$, $\bar{d}_s = \frac{1}{2}(d_s + \bar{d}_s) - \frac{1}{2}(d_s - \bar{d}_s)$, we obtain

$$\begin{aligned}&-\frac{3}{2}\{m_{3h}d_s\pi \otimes \bar{m}_2\bar{m}_2 - m_2m_2 \otimes \bar{m}_{3h}\bar{d}_s\bar{\pi} + m_2[\square, m_2] \otimes \bar{m}_{3h} - m_{3h} \otimes \bar{m}_2[\bar{\square}, \bar{m}_2]\}\Pi \\ &\quad = -\frac{3}{2}\{m_{3h}d_s\pi \otimes \bar{m}_2\bar{m}_2 - m_2m_2 \otimes \bar{m}_{3h}\bar{d}_s\bar{\pi} + m_2m_2d_s \otimes \bar{m}_{3h} - m_{3h} \otimes \bar{m}_2\bar{m}_2\bar{d}_s\}\Pi \\ &\quad = -\frac{3}{2}\{m_{3h}d_s \otimes \bar{m}_2\bar{m}_2\bar{\pi} - m_2m_2\pi \otimes \bar{m}_{3h}\bar{d}_s + m_2m_2d_s \otimes \bar{m}_{3h} - m_{3h} \otimes \bar{m}_2\bar{m}_2\bar{d}_s\}\Pi \\ &\quad = -\frac{3}{4}\{m_2m_2(1-\pi) \otimes \bar{m}_{3h}\bar{d}_s + m_2m_2(1-\pi)d_s \otimes \bar{m}_{3h} \\ &\qquad - m_{3h} \otimes \bar{m}_2\bar{m}_2(1-\bar{\pi})\bar{d}_s - m_{3h}d_s \otimes \bar{m}_2\bar{m}_2(1-\bar{\pi})\}\Pi \\ &\qquad - \frac{3}{4}\{m_{3h}d_s \otimes \bar{m}_2\bar{m}_2\bar{\pi} - m_{3h} \otimes \bar{m}_2\bar{m}_2\bar{\pi}\bar{d}_s + m_2m_2\pi d_s \otimes \bar{m}_{3h} - m_2m_2\pi \otimes \bar{m}_{3h}\bar{d}_s \\ &\qquad + m_2m_2d_s \otimes \bar{m}_{3h} - m_2m_2 \otimes \bar{m}_{3h}\bar{d}_s + m_{3h}d_s \otimes \bar{m}_2\bar{m}_2 - m_{3h} \otimes \bar{m}_2\bar{m}_2\bar{d}_s\}\Pi \\ &\quad = -\frac{3}{4}\{[m_1, m_{3h}] \otimes \bar{m}_{3h}(d_s + \bar{d}_s) - m_{3h} \otimes [\bar{m}_1, \bar{m}_{3h}](d_s + \bar{d}_s)\}\Pi \\ &\qquad - \frac{3}{2}\{m_{3h} \otimes \bar{m}_2\bar{m}_2\bar{\pi}D_s + m_2m_2\pi \otimes \bar{m}_{3h}D_s + m_2m_2 \otimes \bar{m}_{3h}D_s + m_{3h} \otimes \bar{m}_2\bar{m}_2D_s\}\Pi \\ &\quad = -\frac{3}{4}[M_1, (m_{3h} \otimes \bar{m}_{3h})(d_s + \bar{d}_s)\Pi] - \frac{3}{2}\{m_{3h} \otimes \bar{m}_2\bar{m}_2(1+\bar{\pi}) + m_2m_2(1+\pi) \otimes \bar{m}_{3h}\}D_s\Pi, \end{aligned} \tag{B4}$$

with a new contribution to Θ_{3s}. We now combine the obstructions above, proportional to D_s, with the remaining $m_{3h}d_\square$ terms from (B2), and rewrite $d_\square = \frac{1}{2}(d_\square + \bar{d}_{\bar{\square}}) + \frac{1}{2}(d_\square - \bar{d}_{\bar{\square}})$ and $\bar{d}_{\bar{\square}} = \frac{1}{2}(d_\square + \bar{d}_{\bar{\square}}) - \frac{1}{2}(d_\square - \bar{d}_{\bar{\square}})$. Using the definition $D_\Delta = \frac{1}{2}(d_\square - \bar{d}_{\bar{\square}})$, this yields

$$
\begin{aligned}
&\frac{1}{2}\{m_{3h}d_{\square}\otimes\bar{m}_2\bar{m}_2-m_2m_2\otimes\bar{m}_{3h}\bar{d}_{\bar{\square}}\}\Pi\\
&\quad-\frac{3}{2}\{m_{3h}\otimes\bar{m}_2\bar{m}_2(1+\bar{\pi})+m_2m_2(1+\pi)\otimes\bar{m}_{3h}\}D_s\Pi\\
&=\frac{1}{4}\{m_{3h}d_{\square}\otimes\bar{m}_2\bar{m}_2+m_{3h}\otimes\bar{m}_2\bar{m}_2\bar{d}_{\bar{\square}}-m_2m_2\otimes\bar{m}_{3h}\bar{d}_{\bar{\square}}-m_2m_2d_{\square}\otimes\bar{m}_{3h}\}\Pi\\
&\quad+\frac{1}{2}\{m_{3h}\otimes\bar{m}_2\bar{m}_2D_{\Delta}+m_2m_2\otimes\bar{m}_{3h}D_{\Delta}\}\Pi\\
&\quad-\frac{3}{2}\{m_{3h}\otimes\bar{m}_2\bar{m}_2(1+\bar{\pi})+m_2m_2(1+\pi)\otimes\bar{m}_{3h}\}(1-\Pi+\Pi)D_s\Pi\\
&=\frac{1}{4}\{m_{3h}d_{\square}\otimes\bar{m}_2\bar{m}_2+m_{3h}\otimes\bar{m}_2\bar{m}_2\bar{d}_{\bar{\square}}-m_2m_2\otimes\bar{m}_{3h}\bar{d}_{\bar{\square}}-m_2m_2d_{\square}\otimes\bar{m}_{3h}\}\Pi\\
&\quad+\frac{1}{2}\{m_{3h}\otimes\bar{m}_2\bar{m}_2D_{\Delta}+m_2m_2\otimes\bar{m}_{3h}D_{\Delta}\}\Pi\\
&\quad-\frac{1}{2}\{m_{3h}\otimes\bar{m}_2\bar{m}_2(1+\bar{\pi})+m_2m_2(1+\pi)\otimes\bar{m}_{3h}\}D_{\Delta}\Pi-3M_{3h}D_s\Pi,
\end{aligned}
\tag{B5}
$$

where in the last line we used (3.24) and recognized M_{3h} from (3.17). One can now use the implicit projection $m_{3h}=m_{3h}(1-\pi)$ to see that $m_{3h}\otimes\bar{m}_2\bar{m}_2\bar{\pi}\Pi=0$ and $m_2m_2\pi\otimes\bar{m}_{3h}\Pi=0$ in the last line above. From (B5) we are thus left with

$$
\begin{aligned}
&\frac{1}{4}\{m_{3h}d_{\square}\otimes\bar{m}_2\bar{m}_2+m_{3h}\otimes\bar{m}_2\bar{m}_2\bar{d}_{\bar{\square}}-m_2m_2\otimes\bar{m}_{3h}\bar{d}_{\bar{\square}}-m_2m_2d_{\square}\otimes\bar{m}_{3h}\}\Pi-3M_{3h}D_s\Pi\\
&=\frac{1}{4}[\square+\bar{\square},m_{3h}\otimes\bar{m}_2\bar{m}_2-m_2m_2\otimes\bar{m}_{3h}]\Pi-3M_{3h}D_s\Pi\\
&=\frac{1}{2}[[M_1,b^+],m_{3h}\otimes\bar{m}_2\bar{m}_2-m_2m_2\otimes\bar{m}_{3h}]\Pi-3M_{3h}D_s\Pi\\
&=\frac{1}{2}[M_1,[b^+,m_{3h}\otimes\bar{m}_2\bar{m}_2-m_2m_2\otimes\bar{m}_{3h}]]\Pi+\frac{1}{2}[b^+,[M_1,m_{3h}\otimes\bar{m}_2\bar{m}_2-m_2m_2\otimes\bar{m}_{3h}]]\Pi-3M_{3h}D_s\Pi.
\end{aligned}
\tag{B6}
$$

We now use $m_{3h}\otimes\bar{m}_2\bar{m}_2\Pi=m_{3h}\otimes\bar{m}_2\bar{m}_2(1-\bar{\pi})\Pi$ to show that the b^+-commutator above vanishes:

$$
\begin{aligned}
(m_{3h}\otimes\bar{m}_2\bar{m}_2-m_2m_2\otimes\bar{m}_{3h})\Pi&=(m_{3h}\otimes\bar{m}_2\bar{m}_2(1-\bar{\pi})-m_2m_2(1-\pi)\otimes\bar{m}_{3h})\Pi\\
&=(m_{3h}\otimes[\bar{m}_1,\bar{m}_{3h}]-[m_1,m_{3h}]\otimes\bar{m}_{3h})\Pi\\
&=-[M_1,m_{3h}\otimes\bar{m}_{3h}]\Pi,
\end{aligned}
\tag{B7}
$$

thus reducing (B6) to

$$
\frac{1}{2}[M_1,[b^+,m_{3h}\otimes\bar{m}_2\bar{m}_2-m_2m_2\otimes\bar{m}_{3h}]]\Pi-3M_{3h}D_s\Pi.
\tag{B8}
$$

Collecting all the M_1-commutators from (B2), (B3), (B4), and (B8), we finally find that the symmetric projection of (3.25) obeys

$$
\{[b^-,M_2M_2]-3M_2B_2\Pi\}\Pi=[M_1,\Theta_{3s}]-3M_{3h}D_s\Pi,
\tag{B9}
$$

which has the same structure as the symmetric projection of (3.26), where the symmetric part of Θ_3 is given by

$$
\begin{aligned}
\Theta_{3s}=\frac{1}{2}\bigg\{&\theta_3\otimes\bar{m}_2\bar{m}_2-m_2m_2\otimes\bar{\theta}_3+3m_2b_2\otimes\bar{m}_{3h}+3m_{3h}\otimes\bar{m}_2\bar{b}_2-\frac{3}{2}m_{3h}\otimes\bar{m}_{3h}(d_s+\bar{d}_s)\\
&+[b^+,m_{3h}\otimes\bar{m}_2\bar{m}_2-m_2m_2\otimes\bar{m}_{3h}]\bigg\}\Pi.
\end{aligned}
\tag{B10}
$$

The contributions to Θ_{3s} involving θ_3 and $\bar{\theta}_3$ are not separately projected in their single copy constituents. We do so by splitting $\theta_3 = \theta_{3s} + \theta_{3h} = \theta_{3s} - [b, m_{3h}]$. This produces

$$\theta_3 \otimes \bar{m}_2\bar{m}_2 - m_2m_2 \otimes \bar{\theta}_3 = \theta_{3s} \otimes \bar{m}_2\bar{m}_2 - m_2m_2 \otimes \bar{\theta}_{3s} - [b, m_{3h} \otimes \bar{m}_2\bar{m}_2] + [\bar{b}, m_2m_2 \otimes \bar{m}_{3h}], \tag{B11}$$

which, inserted in (B10), gives the final expression for Θ_{3s}:

$$\Theta_{3s} = \frac{1}{2}\left\{\theta_{3s} \otimes \bar{m}_2\bar{m}_2 - m_2m_2 \otimes \bar{\theta}_{3s} + 3m_2b_2 \otimes \bar{m}_{3h} + 3m_{3h} \otimes \bar{m}_2\bar{b}_2 - \frac{3}{2}m_{3h} \otimes \bar{m}_{3h}(d_s + \bar{d}_s)\right.$$
$$\left. - [b^-, m_{3h} \otimes \bar{m}_2\bar{m}_2 + m_2m_2 \otimes \bar{m}_{3h}]\right\}\Pi. \tag{B12}$$

Collecting the two projections (3.27) and (B9), we find the complete homotopy Poisson relation

$$[b^-, M_2M_2] - 3M_2B_2\Pi = [M_1, \Theta_3] + M_{3h}(D_\Delta - 3D_s\Pi), \tag{B13}$$

analogous to the single copy one (3.26).

Given the Poisson relation, the homotopy Jacobi identity of B_2 is fixed by taking a b^--commutator of (B13):

$$\begin{aligned} 0 &= [b^-, 3M_2B_2\Pi - [b^-, M_2M_2] + [M_1, \Theta_3] + M_{3h}(D_\Delta - 3D_s\Pi)] \\ &= 3B_2B_2\Pi + [b^-, [M_1, \Theta_3]] + [b^-, M_{3h}](D_\Delta - 3D_s\Pi) \\ &= 3B_2B_2\Pi - [M_1, [b^-, \Theta_3]] + \Theta_3D_\Delta - \Theta_{3h}(D_\Delta - 3D_s\Pi). \end{aligned} \tag{B14}$$

Upon decomposing $\Theta_3 = \Theta_3(\Pi + 1 - \Pi)$ and using $3\Pi D_s\Pi = \Pi D_\Delta$, we can further manipulate the above expression as follows:

$$\begin{aligned} 0 &= 3B_2B_2\Pi - [M_1, [b^-, \Theta_3]] + \Theta_3D_\Delta - \Theta_{3h}(D_\Delta - 3D_s\Pi) \\ &= 3B_2B_2\Pi - [M_1, [b^-, \Theta_{3s}]] + \Theta_3\Pi D_\Delta + 3\Theta_3(1 - \Pi)D_s\Pi \\ &= 3B_2B_2\Pi + [M_1, B_3] + 3\Theta_3D_s\Pi, \end{aligned} \tag{B15}$$

which is the deformed homotopy Jacobi relation. Above we have defined the three-bracket B_3 as $B_3 \coloneqq -[b^-, \Theta_3] = -[b^-, \Theta_{3s}]$, where the last equality follows from $\Theta_{3h} = -[b^-, M_{3h}]$. This exhausts the relations of the $\mathrm{BV}_\infty^\Delta$ algebra up to trilinear maps.

---

[1] H. Erbin, String field theory: A modern introduction, Lect. Notes Phys. **980**, 1 (2021).

[2] M. Doubek, B. Jurčo, M. Markl, and I. Sachs, Algebraic structure of string field theory, Lect. Notes Phys. **973**, 1 (2020).

[3] C. Hull and B. Zwiebach, Double field theory, J. High Energy Phys. 09 (2009) 099.

[4] H. Kawai, D. C. Lewellen, and S. H. H. Tye, A relation between tree amplitudes of closed and open strings, Nucl. Phys. **B269**, 1 (1986).

[5] Z. Bern, J. J. M. Carrasco, and H. Johansson, New relations for gauge-theory amplitudes, Phys. Rev. D **78**, 085011 (2008).

[6] Z. Bern, J. J. Carrasco, M. Chiodaroli, H. Johansson, and R. Roiban, The duality between color and kinematics and its applications, arXiv:1909.01358.

[7] W. Siegel, Superspace duality in low-energy superstrings, Phys. Rev. D **48**, 2826 (1993).

[8] O. Hohm, C. Hull, and B. Zwiebach, Background independent action for double field theory, J. High Energy Phys. 07 (2010) 016.

[9] O. Hohm, C. Hull, and B. Zwiebach, Generalized metric formulation of double field theory, J. High Energy Phys. 08 (2010) 008.

[10] O. Hohm, On factorizations in perturbative quantum gravity, J. High Energy Phys. 04 (2011) 103.

[11] A. Sen, Wilsonian effective action of superstring theory, J. High Energy Phys. 01 (2017) 108.

[12] A. S. Arvanitakis, O. Hohm, C. Hull, and V. Lekeu, Homotopy transfer and effective field theory I: Tree-level, Fortschr. Phys. **70**, 2200003 (2022).

[13] A. S. Arvanitakis, O. Hohm, C. Hull, and V. Lekeu, Homotopy transfer and effective field theory II: Strings and double field theory, Fortschr. Phys. **70**, 2200004 (2022).
[14] O. Hohm, W. Siegel, and B. Zwiebach, Doubled α'-geometry, J. High Energy Phys. 02 (2014) 065.
[15] O. Hohm and B. Zwiebach, Double field theory at order α', J. High Energy Phys. 11 (2014) 075.
[16] R. Bonezzi, C. Chiaffrino, F. Diaz-Jaramillo, and O. Hohm, Weakly constrained double field theory: The quartic theory, arXiv:2306.00609.
[17] B. Zwiebach, Closed string field theory: Quantum action and the B-V master equation, Nucl. Phys. **B390**, 33 (1993).
[18] A. M. Zeitlin, Homotopy Lie superalgebra in Yang-Mills theory, J. High Energy Phys. 09 (2007) 068.
[19] A. M. Zeitlin, Conformal field theory and algebraic structure of gauge theory, J. High Energy Phys. 03 (2010) 056.
[20] A. M. Zeitlin, Quasiclassical Lian-Zuckerman homotopy algebras, courant algebroids and gauge theory, Commun. Math. Phys. **303**, 331 (2011).
[21] A. M. Zeitlin, Beltrami-Courant differentials and G_∞-algebras, Adv. Theor. Math. Phys. **19**, 1249 (2015).
[22] K. Costello, *Renormalization and Effective Field Theory*, Mathematical Surveys and Monographs (American Mathematical Society), https://books.google.de/books?id=9iM3NSy_xCUC.
[23] K. Costello and O. Gwilliam, *Factorization Algebras in Quantum Field Theory* (Cambridge University Press, Cambridge, England, 2016).
[24] O. Hohm and B. Zwiebach, L_∞ algebras and field theory, Fortschr. Phys. **65**, 1700014 (2017).
[25] M. Reiterer, A homotopy BV algebra for Yang-Mills and color-kinematics, arXiv:1912.03110.
[26] R. Bonezzi, C. Chiaffrino, F. Diaz-Jaramillo, and O. Hohm, Gauge invariant double copy of Yang-Mills theory: The quartic theory, Phys. Rev. D **107**, 126015 (2023).
[27] R. Bonezzi, C. Chiaffrino, F. Diaz-Jaramillo, and O. Hohm, Gravity = Yang-Mills, arXiv:2306.14788.
[28] F. Diaz-Jaramillo, O. Hohm, and J. Plefka, Double field theory as the double copy of Yang-Mills theory, Phys. Rev. D **105**, 045012 (2022).
[29] K. Lee, Kerr-Schild double field theory and classical double copy, J. High Energy Phys. 10 (2018) 027.
[30] W. Cho and K. Lee, Heterotic Kerr-Schild double field theory and classical double copy, J. High Energy Phys. 07 (2019) 030.
[31] D. S. Berman, K. Kim, and K. Lee, The classical double copy for M-theory from a Kerr-Schild ansatz for exceptional field theory, J. High Energy Phys. 04 (2021) 071.
[32] E. Lescano and S. Roychowdhury, Heterotic Kerr-Schild double field theory and its double Yang-Mills formulation, J. High Energy Phys. 04 (2022) 090.
[33] E. Lescano, G. Menezes, and J. A. Rodríguez, Aspects of conformal gravity and double field theory from a double copy map, arXiv:2307.14538.
[34] R. Bonezzi, F. Diaz-Jaramillo, and S. Nagy, Gauge independent kinematic algebra of self-dual Yang-Mills, Phys. Rev. D **108**, 065007 (2023).
[35] L. Borsten, H. Kim, B. Jurčo, T. Macrelli, C. Saemann, and M. Wolf, Double copy from homotopy algebras, Fortschr. Phys. **69**, 2100075 (2021).
[36] L. Borsten, B. Jurco, H. Kim, T. Macrelli, C. Saemann, and M. Wolf, Kinematic lie algebras from twistor spaces, Phys. Rev. Lett. **131**, 041603 (2023).
[37] L. Borsten, B. Jurco, H. Kim, T. Macrelli, C. Saemann, and M. Wolf, Tree-level color-kinematics duality from pure spinor actions, arXiv:2303.13596.
[38] L. Borsten, B. Jurco, H. Kim, T. Macrelli, C. Saemann, and M. Wolf, Double copy from tensor products of metric $BV^{\blacksquare}$-algebras, arXiv:2307.02563.
[39] L. Borsten, B. Jurco, H. Kim, T. Macrelli, C. Saemann, and M. Wolf, Double-copying self-dual Yang-Mills theory to self-dual gravity on twistor space, arXiv:2307.10383.
[40] R. Bonezzi, F. Diaz-Jaramillo, and O. Hohm, The gauge structure of double field theory follows from Yang-Mills theory, Phys. Rev. D **106**, 026004 (2022).
[41] G. Barnich and M. Grigoriev, Hamiltonian BRST and Batalin-Vilkovisky formalisms for second quantization of gauge theories, Commun. Math. Phys. **254**, 581 (2005).
[42] G. Barnich, M. Grigoriev, A. Semikhatov, and I. Tipunin, Parent field theory and unfolding in BRST first-quantized terms, Commun. Math. Phys. **260**, 147 (2005).
[43] M. Grigoriev and D. Rudinsky, Notes on the L_∞-approach to local gauge field theories, J. Geom. Phys. **190**, 104863 (2023).
[44] M. Ben-Shahar and M. Guillen, 10D super-Yang-Mills scattering amplitudes from its pure spinor action, J. High Energy Phys. 12 (2021) 014.
[45] M. Ben-Shahar and H. Johansson, Off-shell color-kinematics duality for Chern-Simons, J. High Energy Phys. 08 (2022) 035.
[46] O. Hohm and A. F. Pinto, Cosmological perturbations in double field theory, J. High Energy Phys. 04 (2023) 073.
[47] H. Hata, K. Itoh, T. Kugo, H. Kunitomo, and K. Ogawa, Gauge string field theory for torus compactified closed string, Prog. Theor. Phys. **77**, 443 (1987).
[48] M. Maeno and H. Takano, Derivation of the cocycle factor of vertex in closed bosonic string field theory on torus, Prog. Theor. Phys. **82**, 829 (1989).
[49] T. Kugo and B. Zwiebach, Target space duality as a symmetry of string field theory, Prog. Theor. Phys. **87**, 801 (1992).
[50] O. Hohm and H. Samtleben, Gauge theory of Kaluza-Klein and winding modes, Phys. Rev. D **88**, 085005 (2013).
[51] U. Naseer, Canonical formulation and conserved charges of double field theory, J. High Energy Phys. 10 (2015) 158.
[52] O. Hohm and H. Samtleben, U-duality covariant gravity, J. High Energy Phys. 09 (2013) 080.
[53] O. Hohm and H. Samtleben, Exceptional form of $D = 11$ supergravity, Phys. Rev. Lett. **111**, 231601 (2013).
[54] O. Hohm and H. Samtleben, Exceptional field theory I: $E_{6(6)}$ covariant form of M-theory and type IIB, Phys. Rev. D **89**, 066016 (2014).
[55] J. Greitz, P. Howe, and J. Palmkvist, The tensor hierarchy simplified, Classical Quantum Gravity **31**, 087001 (2014).
[56] R. Bonezzi and O. Hohm, Duality hierarchies and differential graded Lie algebras, Commun. Math. Phys. **382**, 277 (2021).
[57] R. H. Brandenberger and C. Vafa, Superstrings in the early universe, Nucl. Phys. **B316**, 391 (1989).

[58] R. Brandenberger, Superstring cosmology—A complementary review, J. Cosmol. Astropart. Phys. 11 (2023) 019.

[59] H. Erbin and M. Médevielle, Closed string theory without level-matching at the free level, J. High Energy Phys. 03 (2023) 091.

[60] Y. Okawa and R. Sakaguchi, Closed string field theory without the level-matching condition, arXiv:2209.06173.

[61] C. Hillmann, Generalized $E_{7(7)}$ coset dynamics and $D = 11$ supergravity, J. High Energy Phys. 03 (2009) 135.

[62] D. S. Berman and M. J. Perry, Generalized geometry and M theory, J. High Energy Phys. 06 (2011) 074.

[63] A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, and S. Nagy, A magic pyramid of supergravities, J. High Energy Phys. 04 (2014) 178.