Asymptotic safety in Lorentzian quantum gravity

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(Received 9 November 2023; accepted 9 February 2024; published 13 March 2024)

A recently introduced functional renormalization group (RG) provides a new tool to explore nonperturbative and covariant RG flows in Lorentzian spacetimes. We apply it for the first time to investigate the ultraviolet limit of quantum gravity. While the RG flow is state dependent, it is possible to evaluate state- and background-independent contributions to the flow. Taking into account only these universal terms, the RG flow exhibits a nontrivial fixed point in the Einstein-Hilbert truncation, providing a mechanism for asymptotic safety in Lorentzian quantum gravity.

DOI: 10.1103/PhysRevD.109.066012

I. INTRODUCTION

General relativity (GR) was discovered in 1915; quantum mechanics in 1926. The realization that the gravitational field should have been quantized along the same lines of the electromagnetic field came almost immediately: Already in 1916, Einstein pointed out that quantum effects would modify the theory of general relativity [1]. The search for a consistent quantum theory of gravity has fascinated generations of physicists ever since.

Among many conceptual puzzles, the main technical difficulty in the quantization of gravity is that the standard approach of quantum field theory (QFT) produces a quantum theory of gravity that is perturbatively nonrenormalizable [2–4].

Perturbative nonrenormalizability still leaves open the possibility of the asymptotic safety (AS) scenario [5,6], in which a QFT of the metric tensor is nonperturbatively renormalizable, thanks to the existence of a nontrivial fixed point in its RG flow. First realized in $2+\epsilon$ dimensions [7,8], the AS scenario in four dimensions has been explored through lattice simulations [9,10] and, in the continuum, through functional renormalization group (fRG) techniques [11–18]. While lattice computations are based on a background-independent regularization of the Lorentzian path integral, fRG approaches are mostly based on the Euclidean formulation of the Wetterich equation [19,20], with few exceptions.

In 2011, an fRG-based approach to Lorentzian quantum gravity (QG) has been put forward, providing the first evidence of a nontrivial fixed point in the RG flow in Lorentzian spacetimes [21]. The computation was carried out assuming an Arnowitt-Deser-Misner (ADM) foliation of the background geometry and a compact time direction, which allowed for a resummation of Matsubara frequencies in the propagator. The Lorentzian fRG based on the ADM formalism initiated a study of AS in foliated spacetimes [22,23]. More recently, fRG-based investigations have been carried out for the graviton spectral function in Minkowski [24,25]. However, all fRG-based approaches in Lorentzian spacetimes had to give up background independence in favor of Lorentzian signature.

In this paper, we provide the first evidence for a background-independent, nontrivial fixed point for quantum gravity in Lorentzian signature, in the Einstein-Hilbert truncation. The result is based on a novel Wetterich-type fRG equation (FRGE), directly developed in Lorentzian spacetimes with a covariant formalism and for any Hadamard state [26,27]. This new RG equation uses a local regulator in position, thus acting as an artificial mass, and a Hadamard regularization to subtract the UV divergences. Since it is written in terms of the interacting Feynman propagator, the Lorentzian FRGE exhibits state dependence [26]. The state is chosen for the free theory, and it acts as a background, fiducial quantity for the flow, similarly to the background geometry.

While a state for the graviton in general spacetimes is not known, here we show that the universal terms that must contribute in the FRGE for any state and, in all backgrounds, already determine the existence of a Reuter-type fixed point for Lorentzian quantum gravity.

II. QUANTUM GRAVITY AS A LOCALLY COVARIANT QFT

In order to apply the Lorentzian FRGE to gravity, we take as theoretical framework QG as a locally covariant

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QFT [28,29]. In this context, gravity is quantized on a fixed, globally hyperbolic spacetime $(\mathcal{M}, \bar{g})$ with background metric $\bar{g}$; our computation is background independent in the sense that $\mathcal{M}$ is fixed, but arbitrary, thus studying the RG equations in all spacetimes at once [30].

The space of off-shell configurations is $\mathscr{E}(\mathcal{M}) = \Gamma(T^*(\mathcal{M})^{\otimes 2}) \ni \hat{h}$, the space of symmetric bitensors. As usual, the configuration space must be extended to include the ghosts $\hat{c}$, the antighosts $\hat{\bar{c}}$, and the Nakanishi-Lautrup fields $\hat{b}$. We collect an element of the extended configuration space in the field multiplet $\varphi := \{\hat{h}, \hat{c}, \hat{b}, \hat{\bar{c}}\} \in \bar{\mathscr{E}}(\mathcal{M})$.

In the Batalin-Vilkovisky (BV) formalism [31–33], the configuration space is doubled to include the antifields, identified with the basis of the tangent space, $\varphi^\ddagger := \frac{\delta}{\delta\varphi}$. The classical BV algebra $\mathcal{BV}$ is, thus, the algebra of local functions on the odd cotangent bundle of the extended configuration space [34,35]. The *antibracket* is defined by $\{\varphi_A(x), \varphi^\ddagger_B(y)\} = \delta_{AB}\delta(x-y)$, with A, B indices on the field space, and extended to functions of the fields and antifields by the graded Leibniz rule.

The dynamics is governed by the Euler-Lagrange equations of the action

$$I := I_{\mathrm{EH}} + I_{af} + \gamma\Psi = I_{\mathrm{EH}} + I_{af} + I_{gh} + I_{gf}, \quad (1)$$

where $I_{\mathrm{EH}} = 2\zeta^2 \int_{\mathcal{M}} \sqrt{-\det \hat{g}}(R(\hat{g}) - 2\Lambda)$ is the Einstein-Hilbert action in terms of the full metric $\hat{g} := \bar{g} + \hat{h}$ and $\zeta^2 = (32\pi G)^{-1}$, where G is Newton's constant. The antifield term is

$$I_{af} := \int_{\mathcal{M}} \sqrt{-\det \hat{g}}\mathcal{L}_{\hat{c}}\hat{g}^{ab}h^\ddagger_{ab} + c^b\partial_b c^a c^\ddagger_a + i\hat{b}^a\bar{c}^\ddagger_b,$$

where $\mathcal{L}_{\hat{c}}$ is the Lie derivative. The gauge-fixing fermion Ψ in the De-Donder gauge is

$$\Psi = i\int_{\mathcal{M}} \sqrt{-\det \bar{g}}\hat{\bar{c}}^b\left(\nabla^a\hat{h}_{ab} - \frac{1}{2}\nabla_b\hat{h}_{ac}\bar{g}^{ac}\right).$$

Finally, the *Becchi-Rouet-Stora-Tyutin (BRST) differential* is defined as $\gamma := \{\cdot, I_{af}\}$ [36–38], and the action satisfies the *classical master equation* $\{I, I\} = sI = 0$, where the *BV differential* is $s := \{\cdot, I\}$ [28,29,34,35].

Deformation quantisation proceeds splitting the action I into a term quadratic in the fields I_0 and a remaining, interacting term $V := I - I_0$. The free part I_0 is used to define the quantum products and the time-ordered products. The free algebra $\mathcal{A}$ is the $*-$ algebra of local functions $\mathcal{F}(\bar{\mathcal{E}})$ of the extended configuration space, together with the non-commutative product, and the involution defined by complex conjugation, $\overline{F(\varphi)} = F(\varphi)^*$:

$$\mathcal{A} := \{\mathcal{F}, \star, *\}. \quad (2)$$

As it is customary, we omit the noncommutative product in the following, denoting $F\star G = FG$.

The *Epstein-Glaser renormalisation procedure* constructs the time-ordered products of local functions at coincidence points [39–43]. Interacting observables are represented as formal power series in the $*-$algebra of free observables $\mathcal{A}$. Finally, a state is a linear, positive, normalised functional $\omega : \mathcal{A} \to \mathbb{C}$ mapping the observable to its expectation value [44,45].

In order to define the generating functionals, we introduce sources that couple linearly to the fields, $J := \int_{\mathcal{M}} j_A\varphi^A$, and classical BRST sources that couple to their BRST variations $\Sigma := \int_{\mathcal{M}} \sigma_A\gamma\varphi^A$. The contribution Σ can be understood as a shift of the antifield term I_{af}, so that, even if evaluating on a state ω sets the antifield to the zero configuration $\varphi^\ddagger = 0$, the generating functionals still depend nontrivially on σ.

Finally, we need to introduce the regulator terms Q_k. These are chosen as local terms quadratic in the fields, acting as artificial masses in the correlation functions [26,27]:

$$Q_k := -\frac{1}{2}\int_{\mathcal{M}} \sqrt{-\det \bar{g}(x)}[T(\hat{h}_{ab}(x)q_k{}^{abcd}(x)\hat{h}_{cd}(x)) + 2T\hat{\bar{c}}_a(x)\tilde{q}_k{}^{ab}(x)\hat{c}_b(x)], \quad (3)$$

where T is the time-ordering operator. Notice that the regulator are local in position, preserving causality and Lorentz invariance. The regulator kernels q_k and $\tilde{q}_k$ are chosen proportionally to the RG scale k and include a compactly supported smooth function $f(x) \in C_c^\infty$ which equals 1 on a given region $\mathcal{O} \subset \mathcal{M}$.

Together with the regulator term, we also introduce a source for its BRST variation:

$$H(\eta) := \frac{1}{2}\int_M \sqrt{-\det \bar{g}(x)} \times [\eta(x)\gamma(T\hat{h}_{ab}(x)\hat{h}^{ab}(x)) + \tilde{\eta}\gamma(T\hat{\bar{c}}_a(x)\hat{c}^a(x))]. \quad (4)$$

Introducing a scale-dependent BV differential $s_k := s + \int_{\mathcal{M}} q_{kA}\frac{\delta}{\delta\eta_A}$, the extended action $I_{\mathrm{ext}} := I + \Sigma + Q_k + H$ satisfies a symmetry identity, extending the BV invariance of the classical action I to [27]

$$s_k I_{\mathrm{ext}} = 0. \quad (5)$$

The regularised generating functional for time-ordered correlation functions is defined as

$$Z_k(\bar{g}; j, \sigma, \eta) := \omega(T(e^{-iT^{-1}V})T(e^{iT^{-1}(V+\Sigma+J+Q_k+H)})). \quad (6)$$

This definition of Z_k generalizes the usual path integral representation [46] to globally hyperbolic spacetimes and generic states ω. In flat space, both in Lorentzian or in Euclidean signature, there is a unique Poincaré (or Euclidean) invariant ground state, and it is usually chosen to evaluate correlation functions. However, in curved spacetimes there is no unique choice for a vacuum (as is known already for the scalar case, for example, in Schwarzschild spacetimes [47]), and the choice of a state ω needs to be taken explicitly into account.

The effective average action (EAA) $\Gamma_k(\bar{g};\phi,\sigma,\eta)$ is defined in the standard way. The regularized generating functional for connected Green's functions $W_k(j) = -i\log Z_k$ defines the classical fields $\phi = W_k^{(1)} = \langle\varphi\rangle$ as functions of the sources j. The mean value operator is defined by

$$\langle F\rangle = Z_k^{-1}\omega(T(e^{iT^{-1}V})^{-1}T(e^{iT^{-1}(V+\Sigma+J+Q_k+H)}F)). \quad (7)$$

The relation between the sources and the fields can be inverted in $W^{(1)}(j_\phi) = \phi$ [26], so that the Legendre transform $\tilde{\Gamma}_k = W_k(j_\phi) - J_\phi(\phi)$ is well defined. The EAA is the modified Legendre transform of the regularized generating functional of connected Green's functions $\Gamma_k \coloneqq \tilde{\Gamma}_k - Q_k(\phi)$. The EAA is, thus, a function of the classical fields $\phi \coloneqq \{h, c, b, \bar{c}\}$, of σ, and the scale k.

Thanks to the extended symmetry of Eq. (5), the Legendre EAA satisfies the extended Slavnov-Taylor identity [27]

$$\int_{\mathcal{M}} \frac{1}{\sqrt{-\det\bar{g}(x)}}\left[\frac{\delta\tilde{\Gamma}_k}{\delta\sigma_A(x)}\frac{\delta\tilde{\Gamma}_k}{\delta\phi^A(x)} + q_k^A(x)\frac{\delta\tilde{\Gamma}_k}{\delta\eta^A(x)}\right] = 0. \quad (8)$$

The EAA is then constrained by the cohomology of the BRST operator γ in ghost number zero; from the solution of the Wess-Zumino consistency condition [48,49], it follows that the EAA must be a BRST-invariant functional of the full classical metric $g \coloneqq \bar{g} + h$ [27].

III. RENORMALIZATION GROUP FLOW EQUATIONS

The RG flow equations for gravity are derived in complete analogy with the gauge theory case [27,50]. They read

$$\partial_k\Gamma_k(\bar{g};\phi) = \frac{i}{2}\int_{\mathcal{M}} \mathrm{Tr}\{\partial_k q_k(x) : G_k : (x,x)\}. \quad (9)$$

The trace is over Lorentz and field indices. The equations are written in terms of $\Gamma_k(\bar{g},\phi) = \Gamma_k(\bar{g},\phi,\sigma=0,\eta=0)$, with the field b integrated out, and the *interacting propagator*, satisfying

$$\frac{\delta^2}{\delta\phi(x)\delta\phi(z)}(\Gamma_k + Q_k)G_k(z,y) = -\delta(x,y)\mathbb{I}, \quad (10)$$

where $\mathbb{I}$ denotes an appropriate tensor identity.

Notice that, contrary to standard practice in the asymptotic safety literature in Euclidean space [14], the regulator terms are local in position. In Euclidean signature, the regulator is usually chosen to be a nonlocal function in position. This guarantees that the RG equation remains finite, without additional regularizations. However, in Lorentzian spacetimes, there is no known example of a regulator that satisfies at the same time the requirements of Lorentz invariance, causality, and finiteness of the FRGE [51]. In the case of cosmological backgrounds, an alternative is the use of a regulator depending on the spatial momenta only, since the background already breaks Lorentz invariance [52].

Here, as the background is kept arbitrary, we choose a simple regulator local in position, $q_k(x) = k^2 f(x)$. Recall that $f \in C_c^\infty$ and equals 1 on a region $\mathcal{O} \subset \mathcal{M}$. The advantage of such a regulator is that it preserves Lorentz invariance and causality. Moreover, the cutoff function f acts as an infrared cutoff, since the rhs of the FRGE (9) is proportional to f itself.

Ultraviolet finiteness is instead guaranteed by the normal-ordering prescription, arising from the definition of the EAA in terms of time-ordered quantities. It follows that the FRGE is both ultraviolet and infrared finite by definition.

In fact, by direct computation, one can see that the normal-ordered interacting propagator is given by

$$\begin{aligned}\lim_{y\to x} -i : G_k : &\coloneqq \langle\lim_{y\to x}(\varphi(x)\varphi(y) - h_F(x,y))\rangle \\ &= \lim_{y\to x}\langle\varphi(x)\varphi(y)\rangle - H_k.\end{aligned}$$

The subtraction of the Hadamard parametrix $h_F(x,y)$, encoding the divergences of the Feynman propagator in the coincidence limit, follows from the insertion of the time-ordering operator in the regulator, Eq. (3) [26]. The commutation of the limit with the mean value operator, thus, defines the counterterm H_k, and it guarantees that FRGE are ultraviolet finite.

Operationally, the normal ordering of $:G_k:$ can be computed by a point-splitting procedure. Formally divergent quantities, such as $\langle\hat{h}^{ab}(x)\hat{h}_{cd}(x)\rangle = -iG_k^{hh}{}_{ab}{}^{cd}(x,x)$, are replaced by point-split expressions $G_k^{hh}{}_{ab}{}^{a'b'}(x,y)$, for y in the vicinity of x and spacelike separated. The singular terms in the coincidence limit H_k are subtracted, obtaining the regularized corresponding quantity $:G_k^{hh}{}_{ab}{}^{ab}: \coloneqq G_k - H_k$.

Despite the use of a local regulator, it is still possible to prove that [26]

$$\lim_{k\to\infty}\Gamma_k = I(\phi) + C,$$

where C is a (finite) arbitrary constant. It follows that the EAA interpolates between the quantum action Γ_0 in the IR and the bare classical action I in the UV. The FRGE, thus, describes an RG flow, even if strictly speaking it is derived as a flow of the EAA under rescalings of the mass parameter.

Mass-type regulators appeared already in the literature with the name of *Callan-Symanzik cutoffs* [24,53]. The FRGE (9), first derived in Refs. [26,27], shares some similarities with the recently introduced *renormalized spectral flows* [51]. The difference is in the definition of the counterterms H_k: In the case of renormalized spectral flows, they arise from the dependence of the regulator function Q_k on an additional, UV cutoff scale Λ.

IV. STATE DEPENDENCE

In Lorentzian spacetimes, Eq. (10) admits an infinite family of solutions G_k. This is the main difference from the Euclidean case, where the EAA admits a unique inverse. It follows that the FRGE depends on the choice of the interacting propagator G_k. Different propagators G_k and G'_k differ by a smooth contribution $w - w'$, and they give raise to different RG flows.

The ambiguity in the choice of the interacting propagator can be fixed choosing a state for the free theory ω. Here, we recall the main argument and results; a detailed discussion can be found in Refs. [26,27,54]. Consider a region of the spacetime where the interaction is turned off, $V = 0$. Then, the EAA reduces to $\Gamma_k(\phi) = I_0(\phi) + C$, for some finite constant C, where I_0 is the quadratic part of the bare action I. By direct computation, the interacting propagator is then proportional to $\Delta_{F,k}$, the Feynman propagator for the free theory with a mass modified by the regulator term q_k. The Feynman propagator is fixed by the choice of a state ω, as it is given by the time-ordered, connected two-point function. Moreover, if the state satisfies the Hadamard condition, the Feynman propagator has a universal UV singular structure h_k [44,55–57]. It follows that the choice of a Hadamard state fixes the smooth contributions to the Feynman propagator $w_k \coloneqq \Delta_{F,k} - h_k$.

When the interaction V is turned on, the EAA can be decomposed into $\Gamma_k = I_0 + U_k(\phi)$, where U_k incorporates all the quantum corrections, including nonlocal and higher derivative terms. The construction of the full interacting propagator G_k follows from the free case by a perturbative-type construction, and, in particular, it is possible to prove that [26]

$$-i{:}G_k{:} = \sum_{n=0}^{\infty} (i\Delta_{F,k} U_k^{(2)})^n (\Delta_{F,k} - h_k). \qquad (11)$$

The series is uniquely fixed by the starting element $w_k = \Delta_{F,k} - h_k$, and the requirement that G_k is a fundamental solution for $\Gamma_k^{(2)} - q_k$. The interacting propagator and, by extension, the FRGE, thus, depend on the choice of the smooth contribution w_k, which uniquely fixes a quasifree Hadamard state for the free theory, as quasifree states are determined by their two-point function. In this way, the FRGE inherits a dependence on the state for the free theory.

V. HADAMARD SUBTRACTION AND LOCAL POTENTIAL APPROXIMATION

We now assume that the operator $\Gamma_k^{(2)} - q_k$ is Green hyperbolic, with the kinetic term, apart from a possible wave function renormalization Z_k, given by the free part of the action: $\Gamma_k^{(2)} - q_k = Z_k D - q_k + U_k^{(2)}$, where $D = I_0^{(2)}$. In this approximation, the *effective potential* $U_k^{(2)}$ does not contain derivatives of the Dirac delta.

In this case, it is known that the interacting propagator coincides with the propagator of the free theory, with a mass modified by $q_k - U_k$ [26,58], by an exact resummation of the series in Eq. (11). Therefore, in the LPA the FRGE reduces to

$$\partial_k \Gamma_k(\bar{g}; \phi) = -\frac{1}{16\pi^2 \zeta_k^2} \int_M \mathrm{Tr}\partial_k q_k(x) \omega_k({:}\varphi^2{:}(x)), \qquad (12)$$

and the computation of ${:}G_k{:}$ reduces to the problem of computing the renormalised Wick square, in the state ω_k of the free theory, with a mass modified by $q_k - U_k^{(2)}$ [26].

This in particular means that, if $\Delta_{F,k}$ satisfies the Hadamard condition, G_k is Hadamard as well. Thus, for y in a normal convex neighbourhood of a given x, the interacting propagator must have the same Hadamard singularity structure of the free propagator:

$$G_k = \frac{i}{8\pi^2 \zeta_k^2} (H_k + W), \qquad (13)$$

written in terms of a smooth contribution W and the *Hadamard parametrix*, capturing its universal UV singularity structure:

$$H_k(x, y) = \frac{i}{8\pi^2 \zeta_k^2} \lim_{\epsilon \to 0^+} \left[\frac{\Delta^{1/2}}{\sigma_\epsilon(x, y)} \mathbb{I} + V \log \frac{\sigma_\epsilon(x, y)}{\mu^2} \right].$$

In the last equations, $\zeta_k^2 \coloneqq Z_k \zeta^2$, $\sigma(x, y)$ is the squared geodesic distance taken with sign between x and y and $\sigma_\epsilon(x, y) = \sigma(x, y) + i\epsilon$, Δ is the van Vleck–Morette determinant. $\mathbb{I}$ is an appropriate tensor structure, depending on whether G_k describes the gravitational or the ghost propagator.

The distributions V and W can be expanded in an covariant Taylor expansion as $V = \sum_{n=0} V_n \sigma^n$ and $W = \sum_{n=0} W_n \sigma^n$; the *Hadamard recursion relations* determine higher orders in the expansion from the zeroth order [59]. The zeroth term V_0 is completely determined by the quantum wave operator and the background geometry by the formula [60]

$$V_0 = -\frac{1}{2} \frac{\delta^2}{\delta\phi\delta\phi} (\Gamma_k + Q_k) \Delta^{1/2} \mathbb{I}, \qquad (14)$$

and the coincidence values $\Delta^{1/2}(x, x) = 1$ and $\nabla_a \nabla_b \Delta^{1/2}(x, x) = 1/6 \bar{R}_{ab}$ [61]. On the other hand, the smooth contribution W_0 remains arbitrary; once W_0 is fixed, it locally identifies the state.

The subtraction of the Hadamard parametrix defines the normal-ordered quantity $:G_k: \coloneqq G_k - H_k$, smooth in the coincidence limit; the FRGE for Γ_k thus becomes

$$\partial_k \Gamma_k(\bar{g};\phi) = -\frac{1}{16\pi^2 \zeta_k^2} \int_{\mathcal{M}} \mathrm{Tr}\left\{ \partial_k q_k(x) \left[S_0 + V_0 \log \frac{M^2}{\mu^2} \right] \right\}. \tag{15}$$

The logarithmic term $\log M^2$ is a smooth contribution coming from the arbitrary function W, and it is necessary to make the logarithm in Eq. (21) dimensionless; S_0 is the remaining smooth contribution in the coincidence limit.

Equation (15) holds for a local regulator and a Hadamard interacting propagator G_k. In Euclidean space, it is possible to derive a completely analogous equation, with the fundamental difference that the smooth contributions in the rhs of Eq. (15) are uniquely fixed by Eq. (10). Moreover, in Euclidean space the coefficients V_n can be equivalently computed by heat kernel techniques. However, the Hadamard expansion is preferable for two reasons in Lorentzian spacetimes. First, the heat kernel is ill-defined on Lorentzian spacetimes, as it relies on the positivity of the wave operator; secondly, heat kernel computations typically discard the smooth contributions to the 2-point function [62]. In Lorentzian spacetimes, these smooth contributions correspond to the freedom in the choice of a state, and they contribute to the FRGE (9).

VI. EINSTEIN-HILBERT TRUNCATION

The *Einstein-Hilbert truncation* assumes an ansatz for the effective average action in the form

$$\begin{aligned}\Gamma_k(\bar{g};\phi,\sigma,\eta) = {} & \Gamma_k^{\mathrm{EH}}(\bar{g},g) + \Gamma_k^{gh}(\bar{g},h,c,\bar{c}) \\ & + \Gamma_k^{gf}(\bar{g},h,b,c,\bar{c}) + \Sigma(\bar{g};\phi,\sigma) \\ & + H(\bar{g};\phi,\eta).\end{aligned} \tag{16}$$

The Einstein-Hilbert contribution is

$$\Gamma_k^{\mathrm{EH}} = 2\zeta_k^2 \int_{\mathcal{M}} \sqrt{-\det g}(R(g) - 2\Lambda_k).$$

In terms of the fluctuation field $h \coloneqq \langle \hat{h} \rangle = g - \bar{g}$, the ghost and gauge-fixing terms are, respectively,

$$\Gamma_k^{gh} = \zeta_k^2 \int_{\mathcal{M}} \sqrt{-\det \bar{g}} \bar{c}_a (\bar{g}^{ab}\Box + \bar{R}^{ab}(\bar{g})) c_b,$$

$$\Gamma_k^{gf} = -\zeta_k^2 \int_{\mathcal{M}} \sqrt{-\det \bar{g}} b^a \left(\nabla^b h_{ab} - \frac{1}{2} \nabla_a \bar{g}^{bc} h_{bc} \right),$$

and Σ and H correspond to the classical contributions.

The equations for the interacting propagators are derived expanding the effective average action up to second-order in a Taylor expansion, $\Gamma_k(\bar{g}+h) = \Gamma_k(\bar{g}) + \mathcal{O}(h) + \Gamma_k^{\mathrm{quad}}(h,\bar{g})$.

We can now specify the regulator terms q_k and $\widetilde{q}_k$. They are chosen to act as artificial masses for the fields, dressing the d'Alembertians as $\Box \to \Box - k^2$:

$$q_k{}^{ab}{}_{cd} = \zeta_k^2 k^2 K^{ab}{}_{cd}, \qquad \tilde{q}_{k_{ab}} = \zeta_k^2 k^2 \bar{g}_{ab}, \tag{17}$$

where $K_{abcd} = 1/2(\bar{g}_{ac}\bar{g}_{bd} + \bar{g}_{bc}\bar{g}_{ad} - \bar{g}^{ab}\bar{g}_{cd})$.

The graviton propagator may be decomposed in the sum of a tensor G_k^T and a scalar $G_k^S = \bar{g}^{ab}\bar{g}_{c'd'}G_k^{hh}{}_{ab}{}^{cd}$ contribution [63]. The equations of motion then read

$$\begin{aligned}&\zeta_k^2 \left[\bar{g}_{ac}\bar{g}_{bd} \left(\Box - k^2 + 2\Lambda_k - \frac{1}{2}\bar{R} \right) - P_{abcd} \right] G_k^{T\,abc'd'} \\ &\quad = -\frac{1}{2}(\bar{g}_c{}^{c'}\bar{g}_d{}^{d'} + \bar{g}_d{}^{c'}\bar{g}_c{}^{d'} - \bar{g}_{cd}\bar{g}^{c'd'})\delta(x,y),\end{aligned} \tag{18}$$

$$-\frac{\zeta_k^2}{2}(\Box - k^2 + 2\Lambda_k) G_k^S = -\delta(x,y), \tag{19}$$

$$\zeta_k^2 [\bar{g}_{ab}(\Box - k^2) + \bar{R}_{ab}] \tilde{G}_k^{ab'} = -\bar{g}_b{}^{b'}\delta(x,y). \tag{20}$$

The tensor $P_{ab}{}^{cd} \coloneqq -2\bar{R}_{(a}{}^{c}{}_{b)}{}^{d} - 2\bar{g}^{(c}{}_{(a}\bar{R}^{d)}{}_{b)} + \bar{g}^{cd}\bar{R}_{ab} + \bar{g}_{ab}\bar{R}^{cd}$ is a potential term. In the last relation, all curvature quantities are constructed from the background metric $\bar{g}$; the d'Alembertian is $\Box = \bar{g}(\nabla,\nabla)$.

Each propagator has a corresponding Hadamard expansion:

$$G_k^S = -\frac{i}{4\pi^2 \zeta_k^2}\{H_k^S + V_0^S \log M_S^2 + S_0^S\}, \tag{21}$$

$$G_k^{T\,abc'd'} = \frac{i}{8\pi^2\zeta_k^2}\{H_k^{T\,abc'd'} + V_0^{T\,abc'd'} \log M_T^2 + S_0^{T\,abc'd'}\}, \tag{22}$$

$$\tilde{G}_k^{ab'} = \frac{i}{8\pi^2\zeta_k^2}\{\tilde{H}_k^{ab'} + \tilde{V}_0{}^{ab'} \log \tilde{M}^2 + \tilde{S}_0{}^{ab'}\}. \tag{23}$$

The terms V_0^T, V_0^S, and $\tilde{V}_0$ arising from Eqs. (18)–(20) can be computed from Eq. (14) and are given by [63,64]

$$V_0^S = \frac{1}{2}(k^2 - 2\Lambda_k) - \frac{1}{12}\bar{R}, \tag{24}$$

$$\begin{aligned}V^T{}_{0ab}{}^{cd} = {} & \frac{1}{2}\left(k^2 - 2\Lambda_k + \frac{1}{3}\bar{R} \right) K_{ab}{}^{cd} \\ & + \frac{1}{2}\left(P_{ab}{}^{cd} - \frac{1}{2}\bar{g}^{cd} P_{abe}{}^{e} \right),\end{aligned} \tag{25}$$

$$\tilde{V}_0{}^{ab} = -\frac{1}{12}\bar{g}^{ab}\bar{R} + \frac{1}{2}(k^2\bar{g}^{ab} - \bar{R}^{ab}). \tag{26}$$

VII. UNIVERSAL TERMS AND STATE DEPENDENCE

The RG equations (15) depend on the choice of a state. This is the main difficulty in applying the Lorentzian RG equations, in comparison with their Euclidean counterpart.

In particular, Hadamard states for the graviton are not known in general spacetimes but only in specific geometries [65–71]. The construction of a Hadamard vacuum state for the graviton is well beyond the scope of this short note. Thus, here we take into account only universal contributions to the evolution equations, that are present in any Hadamard state and in all backgrounds. The evaluation of state-dependent contributions is possible only selecting a class of backgrounds, and it will be addressed in future works.

To solve the FRGE (15), we need to evaluate $S_0 = \{S_0^S, S_0^T, \tilde{S}_0\}$ and M_S^2, M_T^2, and $\tilde{M}^2$. First of all, the smooth functions S_0 vanish in the flat space limit [66,72]. Moreover, any k-independent term can be removed by a redefinition of the effective average action, while terms proportional to the scale k can be removed by an appropriate choice of the renormalization ambiguities [41–43]. Since the remaining contributions are completely state dependent, here we neglect S_0.

On the other hand, while the specific expressions for the functions M_T^2, M_S^2, and $\tilde{M}^2$ are state dependent, they must be present in any Hadamard state. They are functions of mass dimension 2, analytic in the physical parameters. Moreover, the expression

$$V_0 \log M^2 \tag{27}$$

must have well defined limits for the vanishing of the running cosmological constant, of the Ricci scalar, and of the scale $k \to 0$ independently, so that the interacting propagator (and thus the RG flow) is well-defined also in flat or Ricci flat spacetimes, or in the absence of the regulator. The only dimension-2 term in the Hadamard expansion for the interacting propagator is V_0; we, thus, choose

$$M_S^2 = V_0^S, \quad M_T^2 = {V^T}_{0ab}{}^{cd} {I^{ab}}_{cd}, \quad \tilde{M}^2 = {\tilde{V}_0}{}^{ab} \bar{g}_{ab}. \tag{28}$$

These choices completely fix W_0^S, W_0^T, and $\tilde{W}_0$, and, thus, they fix a vacuumlike state through the Hadamard recursion relations. In the case of the scalar field, this choice coincides with the Minkowski vacuum state [26].

The last term to be fixed is the arbitrary mass μ. Contrary to the mass terms M_T^2, M_S^2, and $\tilde{M}^2$, depending on the choice of the state, this term is actually an arbitrary mass contribution coming from the choice of the Hadamard parametrix. Thus, we are free to choose a running Hadamard mass $\mu^2 = k^2$, adjusting the UV regularization to the renormalization scale k.

With these choices, the FRGE (15) is written in terms of state-independent, universal quantities. Of course, state-dependent terms in specific backgrounds can significantly alter the FRGE.

VIII. PHASE DIAGRAM

We can now compute the β functions for the dimensionless constants g_k and λ_k, related to the dimensionful running Newton's and cosmological constants by canonical rescalings:

$$(32\pi\zeta_k^2)^{-1} = G_k = k^{-2} g_k, \qquad \Lambda_k = k^2 \lambda_k.$$

Substituting the values for the V_0 coefficients, Eqs. (24)–(26), the mass functions Eq. (28), and setting the smooth contributions S_0 to 0 in the RG flow (15), we get a flow equation for the EAA written in terms of the Ricci scalar $\bar{R}$ and the coupling constants ζ_k^2 and Λ_k. Notice that, thanks to the truncation, the rhs of Eq. (15) depends on spacetime points only through the cutoff function f, and the trace is a simple trace over Lorentz and field indices. Thus, the functional derivatives on both sides of Eq. (15), with respect to $\sqrt{-\det g}$ and with respect to $\bar{R}$ at vanishing background fields, give the zeroth and first order in the Ricci scalar expansion, resulting in the evolution equations for $\zeta_k^2 \Lambda_k$ and ζ_k^2, respectively. The evolution equations are proportional to f, and we can take the adiabatic limit $f = 1$ over the whole spacetime $\mathcal{M}$. Substituting the dimensionless coupling constants then give the β functions for the dimensionless couplings g_k and λ_k:

$$k\partial_k g_k = (\eta_N + 2) g_k, \tag{29}$$

$$k\partial_k \lambda_k = -(2-\eta_N)\lambda_k + \frac{g_k}{4\pi}(2-\eta_N)\Big\{4\log 4 + (1-2\lambda_k) \times \Big[8\log[4(1-2\lambda_k)] + \log\Big[\frac{1}{2}(1-2\lambda_k)\Big]\Big]\Big\}, \tag{30}$$

in terms of the anomalous dimension $\eta_N \coloneqq G_k^{-1} k\partial_k G_k$:

$$\eta_N(g_k, \lambda_k) = \frac{g_k}{6\pi} \frac{27\log(1-2\lambda_k) + 7 + 37\log 2}{1 + \frac{g_k}{12\pi}(37\log 2 + 27\log(1-2\lambda_k))}. \tag{31}$$

The β functions Eqs. (29) and (30) can now be numerically integrated, producing the phase diagram in Fig. 1.

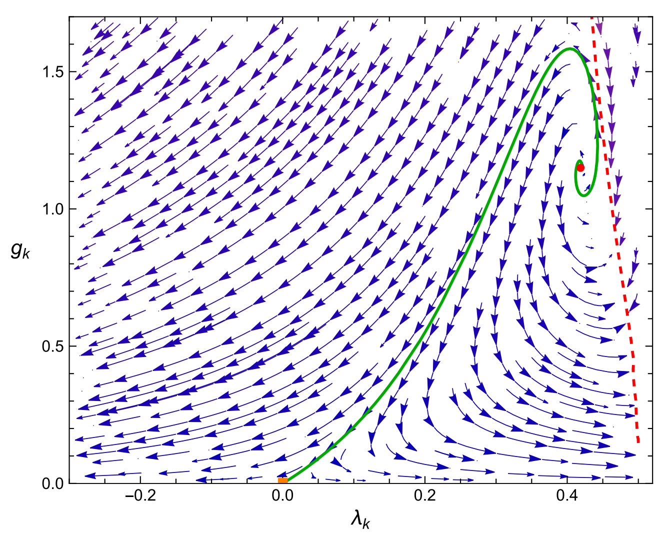

FIG. 1. Phase diagram obtained by numerical integration of the β functions (29) and (30). The solid line is the separatrix, connecting the non-Gaussian fixed point (circle) to the Gaussian one (square); the dashed line denotes the locus where η_N diverges.

The resulting phase diagram shares many similarities with its Euclidean counterpat, first obtained in Ref. [73]. The flow exhibits one nontrivial fixed point for $g_* = 1.15, \lambda_* = 0.42$, realizing the analog of the Reuter fixed point in Lorentzian spacetimes. The critical coefficients for the Lorentzian fixed point are a pair of complex conjugate values, $\theta_{1,2} = 5.11 \pm 11.59i$; therefore, λ_k and g_k are two relevant directions, agreeing again with Euclidean results. These values can be compared to those obtained in the ADM formalism in Ref. [21] that are $(g_*^{\mathrm{ADM}}, \lambda_*^{\mathrm{ADM}}) = (0.21, 0.3)$ and $\theta_{1,2}^{\mathrm{ADM}} = 0.94 \pm 3.1i$. The Euclidean values are [74] $(g_*^E, \lambda_*^E) = (0.34, 0.3)$ and $\theta_{1,2}^E = 1.55 \pm 3.83i$. While the numerical values are roughly of the same order of magnitude, their difference is expected from the different spacetime signatures (Lorentzian versus Euclidean), choice of regulators (local versus nonlocal), and computational technique (Hadamard expansion scheme versus heat kernel techniques).

In the ADM formalism in Ref. [21], the interacting propagator is computed from a resummation of Matsubara frequencies in the compact time direction. The difference between the ADM-based formalism and the covariant formalism presented here should lie in different reference states ω. In fact, the smooth contributions W selecting a Hadamard state are related to the choice of positive frequencies along a selected time direction. The resummation of Matsubara frequencies suggests that the computation in Ref. [21] is performed with respect to a thermal Kubo-Martin-Schwinger (KMS) state at finite inverse temperature $\beta = k$. The computation presented here instead captures universal contributions to the RG flow, that are present in any state and in any background. The similarity between the two phase diagrams suggests that the choice of a thermal state can alter the precise values of the coupling constants at the fixed point and the critical exponents, but it leaves unaltered the existence and qualitative features of the fixed point.

The detailed connection between the two formalisms will be performed in future works, to highlight state and background dependence of the RG flow in Lorentzian spacetimes. However, the qualitative picture of a non-Gaussian fixed point in the positive quadrant with critical exponents arises in all cases. The fixed point (g_*, λ_*), thus, provides a realization of the AS scenario in Lorentzian spacetimes.

IX. CONCLUSIONS

The novel RG framework allows for the investigation of Lorentzian flows in a nonperturbative regime for gravity. In this note, we have seen that the contribution of universal, background-independent terms in the flow of the Einstein-Hilbert truncation supports the evidence that gravity is nonperturbatively renormalizable also in the Lorentzian case.

To preserve background independence, we have restricted our attention to contributions to the flow coming only from universal terms. The important question now is if the nontrivial fixed point persists when state-dependent terms are taken into account. The investigation of state-dependent terms, however, requires one to select a background. The Lorentzian FRGE (15) then allows for a systematic investigation of these state-dependent contributions in specific background geometries.

The RG flow state dependence can also be put in contact with the different runnings of the coupling constants in the effective field theory approach to quantum gravity [75]. In fact, Newton's constant and the cosmological constant have different scalings in different scattering processes. Since the Lorentzian RG flow is state dependent, it is possible to study the flow of couplings in different states nonperturbatively.

The new formalism is tailored to Lorentzian spacetimes. The Hadamard expansion allows for quick generalizations to more advanced truncations. Universal terms, in particular, can be easily computed from the V_0 and $\tilde{V}_0$ terms in the Hadamard expansion in terms of the EAA thanks to Eq. (14). The use of a local regulator and the Hadamard expansion of the interacting propagator allow for a relatively simple computation scheme for the contributions to the FRGE, preserving general covariance.

The main novel result is that, in all backgrounds and for all Hadamard states, universal contributions are sufficient to identify a nontrivial fixed point in the RG flow, thus providing a universal mechanism for asymptotic safety in quantum gravity, at least in the Einstein-Hilbert truncation. The result is of particular relevance in a Lorentzian context, where there is an infinite family of interacting propagators for any given effective average action, indexed by a smooth function. Whether the choice of specific backgrounds and states can significantly alter this mechanism will be addressed in future works.

Finally, while the EAA is a gauge-dependent quantity, gauge-invariant relational observables have been already studied in the context of locally covariant QG [76–79] and in Euclidean fRG flows [80]. In future works, we plan to investigate the RG flow of gauge-invariant observables in Lorentzian quantum gravity.

ACKNOWLEDGMENTS

I thank Renata Ferrero, Paolo Meda, and Nicola Pinamonti for useful discussions. I am supported by a Ph.D. scholarship of the University of Genoa and by the GNFM-INdAM Progetto Giovani *Non-linear sigma models and the Lorentzian Wetterich equation*, CUP_E53C22001930001.

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