Universal IR holography, scalar fluctuations, and glueball spectra

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(Received 2 January 2024; accepted 9 February 2024; published 13 March 2024)

We show that the d'Alembertian operator with a possible mass term in the anti–de Sitter (AdS) soliton and more general confining gravity dual backgrounds admits infinitely many different spectra. These can be interpreted as different theories in the infrared and correspond to multitrace deformations of either the Dirichlet or the Neumann theory. We prove that all these fluctuations are normalizable and provide examples of their spectra. Therefore, the AdS soliton can be interpreted as giving a holographic Renormalization Group flow between a universal UV theory at the AdS boundary and these infinitely many possibilities in the IR, obtained by deformations. The massive spectrum of the double trace deformation in AdS_5 allows the matching of the large-N glueball masses of lattice QCD_3; the ratio of the ground states of the 2^{++} and 0^{++} channels are in full agreement with the lattice prediction. When considering AdS_7 and the four-dimensional pure glue theory, a remarkably general picture emerges, where we can write formulas for the fluctuations that are in agreement with ones from holographic high-energy scattering and from AdS/CFT with IR and UV cutoff. We point out that this log branch in the IR in D dimensions can be seen as the usual logarithmic branch of scalar fields saturating the Breitenlohner-Freedman bound in a conformally rescaled metric, with $\mathrm{AdS}_{D-1}\times S^1$ asymptotics.

DOI: 10.1103/PhysRevD.109.066011

I. INTRODUCTION AND DISCUSSION

Holography, in AdS/CFT and gauge/gravity duality, identifies the quantum states of supergravity in the bulk and those of the corresponding quantum field theory (QFT) on the boundary. These quantum states are required to be normalizable to belong to the relevant Hilbert spaces, hence normalizable modes of the supergravity fields are identified with the nonperturbative states in the QFT. However, a field can be logarithmically divergent and normalizable. The main objective of this letter is to analyze the rich physics that this logarithmic branch brings in.

Another, related, point we study is the generality of some simple formulas for the glueball spectra arising from gravity duals of QCD_3 (or rather, pure glue, YM_3) and QCD_4 (YM_4). The Witten model [1] for finite temperature AdS/CFT, arising from a certain $M\to\infty$ scaling of a black hole in AdS_D, can be written in Euclidean space as

$$ds^2=\frac{r^2}{\ell^2}\left[d\tau^2F(r)+d\vec{x}^2_{(D-2)}\right]+\ell^2\frac{dr^2}{r^2F(r)}$$
$$F(r)=1-\frac{r_0^{D-1}}{r^{D-1}},\tag{1.1}$$

and for $D=5$ (AdS_5), can be also obtained, once we add an extra S^5, as the near-horizon near-extremal D3-brane metric. If we compactify and reduce on the "Euclidean time" τ, we get a model dual to pure glue in three dimensions, which we call QCD_3. On the other hand, if we consider near-horizon near-extremal D4-branes in Euclidean time, with

$$\begin{aligned}ds^2&=\left(\frac{U}{\ell}\right)^{3/2}\left(F(U)d\tau^2+d\vec{x}^2_{(4)}\right)\\&\quad+\left(\frac{\ell}{U}\right)^{3/2}\left(\frac{dU^2}{F(U)}+U^2d\Omega_4^2\right)\\&=8\frac{\rho}{\ell}\left[\frac{\rho^2}{\ell^2}(F(\rho)d\tau^2+d\vec{x}^2_{(4)})+\ell^2\left(\frac{d\rho^2}{\rho^2F(\rho)}+d\Omega_4^2\right)\right],\end{aligned}\tag{1.2}$$

where $\rho=(\ell U)^{1/2}/2$ and $F(U)=1-U_0^3/U^3$, so is conformal to asymptotically $\mathrm{AdS}_6\times S^4$ (though, of course, the presence of the nontrivial, U-dependent dilaton $e^{\phi-\phi_0}=(U/\ell)^{3/4}$ means that the solution itself does not have AdS_6 symmetries), and if we compactify and reduce on

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τ we get a model dual to pure glue in four dimensions, which we call QCD_4 (or YM_4), which was expanded by the introduction of D8-branes by Sakai and Sugimoto [2,3]. In both of these models, as well as in other holographic QCD constructions for $\mathcal{N} = 1$ SYM_4 [4–6] and $\mathcal{N} = 1$ SYM_3 [7], the metric in the IR is smooth, and locally equivalent to flat space, while in the UV it asymptotes to AdS space, perhaps log corrected [4,5] or with an equivalent nontrivial dilaton [6,7], as needed for gravity duals of QCD-like theories (see, for instance, chapter 21 of [8] for more details). A simpler model for QCD is the "hard-wall" model [9], where AdS space is cut off in the IR, while the Randall-Sundrum model [10,11] can also be understood as AdS/CFT with an extra UV cutoff, besides the IR cutoff. In all of these cases, the d'Alembertian operator $\Box$ in the gravity dual, corresponding to glueball spectra, reduces to a one-dimensional Schrödinger problem, and we will see that there are universal features, namely model independent, and how they reflect on the modes.

First, we will study a real scalar field in a general confining background. The discussion is quite generic, showing that always there is a solution with a logarithmic branch in the IR. Second, using the standard Klein-Gordon norm, we show that a scalar field with a logarithmic branch in the IR is indeed normalizable. Third, we show how to renormalize the action in the infrared and show how the logarithmic branch allows a holographic interpretation as a multi-trace deformation of the dual theory in the IR. Then we provide the spectra of several boundary conditions and we point out the relation of the log branch with that of a scalar field saturating the Breitenlohner-Freedman bound in $AdS_{D-1} \times S^1$. Finally, we consider the generic behavior of the UV and IR modes, from which we derive formulas for the mass spectrum that fit well the (lattice numerics) data, and show that they are compatible both with the Kaluza-Klein (KK) modes of the Randall-Sundrum model [10,11], reinterpreted through AdS/CFT, and with the spectrum derived from scattering of hadrons at high energy, via the Polchinski-Strassler scenario in the hard-wall model [9], found and argued to be asymptotically *exact* in [12–15].

The paper is organized as follows. In Sec. II we describe the IR logarithmic branch, its normalizability and its connection with the UV in $AdS_{D-1} \times S^1$, in Sec. III we study applications to QCD_3 and QCD_4 models, with several numerical results and explicit details. Below, in the tables and plots of this work we give the masses in units of $1/z_0$.

II. A LOGARITHMIC BRANCH IN THE IR

We consider the following gravity dual background, of which we see that the "Witten model," or "AdS soliton," is an example,[1]

$$ds^2 = \frac{\ell^2}{z^2}\left(F(z)d\theta^2 + \frac{dz^2}{F(z)} + \gamma_{ab}dx^a dx^b\right), \qquad (2.1)$$

where γ_{ab} is a Lorentzian metric independent of z and ℓ is the AdS radius, $z = \ell^2/r$ in (1.1), and $x^a = (t, \vec{x})$. We take $z \in [0, z_0]$. At the UV end point, $F(0) = 1$, and at the IR end point, $F(z_0) = 0$. Both are simple zeros of F and the function is otherwise positive. An important example in D dimensions is $F = 1 - (z/z_0)^{D-1}$ and $\gamma_{ab} = \eta_{ab}$ is the Minkowski metric, which is the well-known Witten model or AdS soliton [17]. It satisfies the Einstein equations with a negative cosmological constant $R_{\mu\nu} = -\frac{(D-1)}{\ell^2} g_{\mu\nu}$ and θ parametrizes a circle that has no conical singularities when $\theta \in [0, \theta_0]$. Other interesting and simple examples are the supersymmetric solitons of [18], which have $F = 1 - (z/z_0)^4$, for $D = 4$, $F = 1 - (z/z_0)^6$, for $D = 5$ and $\gamma_{ab} = \eta_{ab}$. Our discussion below on the existence and normalizability of the logarithmic mode in the IR depends only on the form of the metric in the neighborhood of z_0.

Pick a real scalar field in this background with the standard, Lorentzian action

$$S_\phi = -\frac{1}{2}\int_0^{z_0} dz \int_0^{\theta_0} d\theta \int d^{D-2}x\sqrt{-g}\left((\partial\phi)^2 + m^2\phi^2\right) + S_\partial, \qquad (2.2)$$

where we include possible boundary terms, S_∂, which will render the action principle well defined. The novelty is that S_∂ arises in the IR; in the UV, we assume that the standard considerations apply, see the review [19] and references therein. Let us remark that in the IR, the procedure is not completely new: in the context of AdS/CFT in Minkowski space at finite temperature, Son and Starinets [20] proposed to write boundary terms both in the UV and IR, and further Iqbal and Liu [21] used a similar procedure to calculate transport coefficients at the horizon via the membrane paradigm; this became a standard tool in AdS/CMT, see the review [22] and the book [23] for more details. The wave equation is $\Box_g\phi - m^2\phi = 0$, when evaluated in (2.1) with the Frobenius ansatz $\phi = (z_0 - z)^\Delta \Phi_M(t, \vec{x})\cos(p\theta)$ and $\Box_\gamma \Phi_M - M^2\Phi_M = 0$ yields $\Delta = \pm p c_0^{-1}$ under the assumption that the metric and the scalar field potential admits a Taylor series around z_0 of the form $F(z) = c_0(z_0 - z) + O(z_0 - z)^2$. It follows that when $p = 0$ there is a logarithmic branch. This is the case of our interest. Hence, in what follows we restrict our study to $p = 0$ and the following expansion of the scalar field in the IR:

$$\phi_i = \alpha_i(t, \vec{x})\sum_{n=0}^{\infty} a_n\left(1 - \frac{z}{z_0}\right)^n + \beta_i(t, \vec{x})\ln\left(1 - \frac{z}{z_0}\right) \times \sum_{n=0}^{\infty} b_n\left(1 - \frac{z}{z_0}\right)^n, \qquad (2.3)$$

[1]The condition for such a background to be dual to a confining theory can be found from [16].

where we fix the redundancy in the parametrization setting $a_0 = b_0 = 1$. Moreover, this logarithmic branch is generic for all solitons, in all theories, in all dimensions provided that the background admits this expansion; for several examples see Refs. [18,24–29].

The Klein-Gordon current in D dimensions is $J_\mu = -i(\phi_1^* \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1^*)$, where ϕ_i are two different solutions to the Klein-Gordon equation. For a detailed discussion of normalizability in AdS/CFT, see, for instance, [30]. Let us prove that the logarithmic mode is normalizable with this norm.[2] To do this we first show that it is possible to define a conserved and finite norm from J^μ independently of z including the logarithmic branch. To pass from the current to the norm it is useful to note that, because the fluctuation of interest is independent of θ, $J_\theta = 0$, and

$$\int_0^{z_0} dz \sqrt{-g} \nabla_z (J^z) = J_z g^{zz} \sqrt{-g} \Big|_0^{z_0} = i c_0 (\beta_2 \alpha_1^* - \beta_1^* \alpha_2) \frac{\ell^{D-2}}{z_0^{D-2}} - \mathrm{UV}. \quad (2.4)$$

We suppose that the UV term vanishes as the fluctuations are normalizable there. The IR term vanishes with Dirichlet boundary conditions $\alpha = 0$, Neumann boundary conditions $\beta = 0$, or double trace boundary conditions $\beta = \lambda\alpha$. We will justify the denomination "double-trace" below. With these IR conditions the constant time integral of $\nabla_\mu J^\mu = 0$ yields a conserved norm

$$0 = \int dz d^{D-3} x \sqrt{-g} \nabla_\mu J^\mu = \frac{d}{dt} \int dz d^{D-3} x g^{tt} J_t \sqrt{-g}. \quad (2.5)$$

We note that for black holes (in Lorentzian signature) g^{tt} ($\propto F(z)^{-1}$) is singular around $F(z) = 0$, whereas for solitons ($= z^2/l^2$) it is regular in the IR, which allows for a finite result. Therefore, the z integrals are manifestly finite in the IR, as follows from the expansion (2.3). Also note that, while scalars of interest might be real, the standard prescription for the KK soliton (here AdS soliton), as in [17,18,31], allows for discarding a part of the scalar field (with negative energy), and thus keeping a complex part.

In order to define a standard QFT, the states with positive energy should have positive norm. It should be clear from the previous discussion that a solution with a well-defined mass $\phi = \Phi_M(t, \vec{x}) X_M(z)$ has a positive norm whenever they have a positive energy, $\partial_t \Phi_M = -i\omega \Phi_M$. Indeed, in this case the conserved norm (2.5) reduces to the standard Klein-Gordon norm in flat space for Φ_M times a manifestly positive integral,

$$N_M \equiv -\int_0^{z_0} dz \sqrt{-g} g^{tt} X_M(z)^2 > 0. \quad (2.6)$$

The variation of the action (2.2) with these new boundary conditions gets a contribution in the IR. Hence, counterterms need to be included there, a divergent and a finite one,

$$S_\partial = -4\pi \frac{\ell^{D-2}}{z_0^{D-2}} \int d^{D-2} x \sqrt{-\gamma} \left(\frac{\beta^2}{2} \ln \epsilon + S_F \right), \quad (2.7)$$

where ϵ is an IR regulator defined as $z = z_0 - \epsilon z_0$, such that the IR is located at $\epsilon = 0$, and S_F is a finite counterterm that one can add and that changes the IR theory, where we can have, for instance,

$$S_F = 0 \Rightarrow \delta S_\phi = 4\pi \frac{\ell^{D-2}}{z_0^{D-2}} \int d^{D-2} x \sqrt{-\gamma} \boldsymbol{\beta \delta \alpha},$$
$$S_F = \alpha\beta \Rightarrow \delta S_\phi = -4\pi \frac{\ell^{D-2}}{z_0^{D-2}} \int d^{D-2} x \sqrt{-\gamma} \boldsymbol{\alpha \delta \beta},$$
$$S_F = \frac{\beta^2}{2\lambda} \Rightarrow \delta S_\phi = 4\pi \frac{\ell^{D-2}}{z_0^{D-2}} \int d^{D-2} x \sqrt{-\gamma} \boldsymbol{\beta (\delta\alpha - \delta\beta/\lambda)} \equiv 4\pi \frac{\ell^{D-2}}{z_0^{D-2}} \int d^{D-2} x \sqrt{-\gamma} \beta \delta J_\beta,$$
$$S_F = \alpha\beta - \frac{\lambda\alpha^2}{2} \Rightarrow \delta S_\phi = -4\pi \frac{\ell^{D-2}}{z_0^{D-2}} \int d^{D-2} x \sqrt{-\gamma} \boldsymbol{\alpha (\delta\beta - \lambda \delta\alpha)} \equiv -4\pi \frac{\ell^{D-2}}{z_0^{D-2}} \int d^{D-2} x \sqrt{-\gamma} \alpha \delta J_\alpha, \quad (2.8)$$

where $J_\beta = \alpha - \frac{\beta}{\lambda}$ and $J_\alpha = \beta - \lambda\alpha$ are generalized sources and λ is a constant. As the renormalized action is finite, one should consider λ as a *renormalized* coupling constant. One can see that the possibilities in the last two cases, with $S_F = S_F(\lambda)$, are double trace deformations of the dual theory in the IR, in the sense of [32].

It is interesting to note that, for the third case in (2.8), with source $J_\beta = \alpha - \lambda^{-1}\beta$ and deformation $S_F = \lambda^{-1}\beta^2/2$, the coupling constants are related by $\frac{1}{\lambda} + \ln\epsilon = \frac{1}{\lambda_0}$, where λ_0 is the bare coupling constant. Let us set $\epsilon = \frac{\mu}{\Lambda}$, with $\Lambda \gg \mu$,[3] then one can recover the typical relation between the renormalized coupling and the bare coupling in a theory with a beta function running only at one loop,

$$\lambda = \frac{\lambda_0}{1 + \lambda_0 \ln \frac{\Lambda}{\mu}}. \quad (2.9)$$

This discussion is the same as the one given in [32], giving the same (2.9), but there in the UV. However, in [32],

[2] This is the norm that is finite and conserved in time for solutions with arbitrary time dependence of the wave equation; otherwise other norms could be imagined. We thank Carlos Núñez for a discussion on this point.

[3] In this language it is natural to think of the IR scale of the theory as $\Lambda = z_0$ and the renormalization scale as $\mu = z_0 - z$ with fixed z.

λ is in the numerator of S_F and Λ is a UV cutoff. Here λ is in the denominator of the deformation S_F, and we have an *IR cutoff*, which is $\Lambda_{\rm QCD}$ in this holographic context, so it is natural to replace $1/\epsilon = \Lambda/\mu$ and $\lambda = \lambda(\Lambda)$ with $1/\epsilon = \mu/\Lambda_{\rm QCD}$ and $\lambda = \lambda(\mu)$, as noted. The fact that the coupling constant appears in the denominator of S_F makes it natural to identify $\lambda = \pm g^2_{\rm QFT}$, where $g^2_{\rm QFT}$ is the standard coupling constant of a non-Abelian gauge theory, with the beta function

$$\mu \frac{\partial g_{\rm QFT}}{\partial \mu} = -\frac{\pm 1}{2} g^3_{\rm QFT}. \tag{2.10}$$

It is not completely clear what is the exact interpretation of the formula (2.9) in field theory, though one possible interpretation would be that there is an effective quantum IR description, according to the rules above, which is one-loop exact (since presumably, higher loop corrections would modify the formula).

In conclusion, we replace an UV renormalization with an IR renormalization, which at one loop still looks formally the same. In these double trace deformations, (both $J_\beta = 0$ and $J_\alpha = 0$ imply that) we have $\alpha = \lambda^{-1}\beta$, as an *on-shell* relation (at zero source). We notice that the behavior is that of asymptotic freedom in the IR for $\lambda > 0$ and the theory is strongly coupled in the IR for $\lambda < 0$. We have chosen our conventions such that the sign of λ indicates the sign of the contribution to the energy. Hence, the behavior of the fluctuations with positive or negative λ should be very different. This will be shown to be the case in what follows.

A. Conformal relation to $\mathrm{AdS}_{D-1} \times S^1$

The IR structure we propose here is analogous to what happens at the usual AdS boundary. Therefore, one might wonder whether the IR is related to a geometry with an AdS factor. This is indeed the case, as we will show below. Furthermore, we will see how to map the dynamics of the fluctuations in one metric to the other in the neighborhood of $z = z_0$.

First, note that the conformally rescaled metric (2.1),

$$ds^2_{\mathrm{AdS}_{D-1}\times S^1} = \frac{ds_g^2}{F(z)} = \frac{\ell^2}{z^2}\left(d\theta^2 + \frac{dz^2}{F(z)^2} + \frac{1}{F(z)}\gamma_{ab}dx^a dx^b\right), \tag{2.11}$$

is asymptotically $\mathrm{AdS}_{D-1} \times S^1$. This can be easily seen by going to the neighborhood of $z = z_0$, setting $F(z) = c_0(z_0 - z)$, and changing the coordinates as

$$z = -\frac{1}{4}c_0 Z^2 + z_0, \tag{2.12}$$

which shows that (2.11) is asymptotically $\mathrm{AdS}_{D-1} \times S^1$ around $Z = 0$. The radius of this new AdS_{D-1} is $L = \frac{2\ell}{z_0 c_0}$, which is also the radius of the S^1. Hence, we find that when $F(z) = c_0(z_0 - z)$ we have

$$ds^2_{\mathrm{AdS}_{D-1}\times S^1} \simeq L^2 d\tilde{\theta}^2 + \frac{L^2}{Z^2}\left(dZ^2 + \gamma_{ab}dx^a dx^b\right), \tag{2.13}$$

with $\tilde{\theta} \in [0, 2\pi)$. For fluctuations independent of the angle θ or equivalently of $\tilde{\theta}$, we find that

$$F(z)^{\frac{2-D}{4}}\left(\Box_{\mathrm{AdS}_{D-1}\times S^1}\psi - \mu^2\psi\right) - F(z)\left(\Box_g\phi - m^2\phi\right) = O(z_0 - z), \tag{2.14}$$

provided that $\psi = \phi F(z)^{\frac{D-2}{4}}$, and that the mass of the fluctuation ψ in the conformally rescaled metric saturates the Breitenlohner-Freedman bound in that space, $\mu^2 = -\frac{(D-2)^2}{4L^2}$.[4] This observation seems interesting, since it might help to export holographic techniques and concepts to other space-times. In particular, in this case we see that the log branch we have discussed so far is, asymptotically, the usual logarithmic branch of a scalar field saturating the Breithenloner-Freedman bound in $\mathrm{AdS}_{D-1} \times S^1$.

This discussion is instructive to see that from the point of view of the $\mathrm{AdS}_{D-1} \times S^1$, the α and β of (2.3) have the same conformal dimension, Δ, corresponding to a scalar field saturating the Breithenloner-Freedman bound, ψ, namely $\Delta = \frac{D-2}{2}$. From the point of view of the original metric, and scalar field $\phi = \psi Z^{-\frac{D-2}{2}}$, it follows that α and β are invariant under rescaling in Z. Therefore the deformations discussed above, (2.8), can be thought of as relevant, with Z as the energy scale. Then a particular IR theory corresponds to a particular value for these α and β.

III. APPLICATIONS

A. Massless modes and validity of the probe limit

Our previous discussion has been rather general; now we pass to discuss concrete theories. The first important issue would be whether these modes can be treated as probes on a geometry as they can potentially deform it due to its logarithmic IR behavior. This turns out to be intrinsically related to the existence of a possible massless mode for certain deformations. It is relatively simple to construct such mode in the case of the AdS soliton for a scalar field of the form $\Box\Phi = 0$, the massless mode is $\Phi = \Phi_0(t, \vec{x}) \ln\left(1 - \left(\frac{z}{z_0}\right)^{D-1}\right)$, with $\eta^{ab}\partial_a\partial_b\Phi_0 = 0$. This mode is indeed normalizable, since the norm $N_M = N_0$ satisfies

$$N_0 \le \zeta(3), \tag{3.1}$$

[4] Note that in (2.14) the m^2 term becomes subleading in this IR region of the original metric, and instead of it, we now have μ^2, leading in (the UV of) $\mathrm{AdS}_{D-1} \times S^1$.

with ζ the Riemann zeta function and equality saturated for $D = 3$. This massless mode has IR boundary conditions $\alpha = \beta \ln(D-1)$. However, whenever this massless mode exists there is also a zero mode that changes the geometry. The most general static solutions of the Einstein-massless scalar system with a cosmological constant are the naked singularities found in [33]. When the scalar field is normalizable in the UV, all these solutions have the IR boundary conditions $\alpha = \beta \ln(D-1)$.[5] These pathological cases are therefore excluded if the boundary condition of the gravity theory is such that $\lambda^{-1} \neq \ln(D-1)$. Hence, when the probe limit is valid these new IR boundary conditions exclude massless modes on the brane.[6] This can be understood as a no-hair theorem for massless scalar fields in AdS. Namely, in the Einstein-massless Klein Gordon-AdS system the only solution that is normalizable in the UV satisfies $\alpha = \beta \ln(D-1)$ in the IR. Therefore if the case $\alpha = \beta \ln(D-1)$ is excluded, the only static solution is that of pure general relativity with cosmological constant. Hence, the probe limit is valid provided on-shell the scalar field satisfies $\lambda^{-1} \neq \ln(D-1)$ and is normalizable in the UV.

B. QCD$_3$

Further insight can be gained by going to a concrete dimension. Let us focus on $D = 5$ and type IIB supergravity. The deformation has $\alpha = \lambda^{-1}\beta$. In this case the massless scalar field can be identified with the dilaton of string theory. The glueballs dual to the operator $\mathrm{Tr} F_{\mu\nu}^2$ are in the 0^{++} channel. A remarkable qualitative feature of the spectra of the new boundary conditions is that the 2^{++} spectrum (given by the $\beta = 0$ column of Table I, or $\lambda^{-1} = \pm\infty$) is no longer degenerate with that of the 0^{++} one generated by the dilaton. This is rather desirable, since to break this degeneracy was an open problem, in the attempts to describe the glueball spectrum of QCD with holographic techniques. Furthermore, the lightest glueball in the 0^{++} channel can be seen as arising from the dilaton (previously it was from the metric, see Ref. [35]). The double trace deformation allows us to pick λ, in order to match the mass ratio of the ground states of some arbitrary desired channel. For instance, in the large N limit, lattice QCD$_3$ predicts a ratio for the ground states $M_{2^{++}}/M_{0^{++}} \approx 1.68$ [36]. This is exactly reproduced in the gravity side with a ground state of $M_{0^{++}}^2 \approx 4.1/z_0^2$; namely $\lambda \approx e$. In other words, choosing the background (with a given ℓ and z_0), fixes $M_{2^{++}}$, and fixing $\lambda \approx e$ then achieves the best fit. This is, so far, the best quantitative matching of top-down holography to lattice QCD in the literature,[7] and is achieved by a nontrivial choice of λ. This is a rather unexpected result as lattice QCD calculations are at weak t'Hooft coupling and the gravity result is at strong t'Hooft coupling [40].[8]

TABLE I. Masses^2 of the 0^{++} glueballs from the dilaton in type IIB supergravity. The $\beta = 0$ column coincides with the numerics available in the literature [35,41], the other spectra are given as possible examples of the formalism developed here and are new.

Mass^2 spectrum for different IR boundary conditions for QCD$_3$		
$\beta = 0$	$\alpha = 0$	$\alpha = 2\beta$
11.5878	5.05077	3.76515
34.537	27.4248	26.6285
68.9753	60.424	59.5767
114.987	104.961	103.975

In Fig. 1 we show how the value $\lambda^{-1} = \ln 4$ is the onset of an instability characterized by the existence of tachyonic glueballs in the region $0 < \lambda < \frac{1}{\ln 4}$. The Neumann theory ($\beta = 0$) is located at $\lim_{\lambda \to 0^-} M^2$. Remarkably enough, the spectrum shows the property that $M_n^2(\lambda^{-1} = -\infty) = M_{n+1}^2(\lambda^{-1} = \infty)$ for each eigenvalue labeled by n.

1. Conformal quantum mechanics

Conformal quantum mechanics is characterized by a potential of the form $1/u^2$, see Ref. [42]. The d'Alembertian operator in the AdS soliton background can be seen to have this conformal behavior in the IR and in the UV. If we replace the ansatz $\Phi = e^{-i(\omega t - \vec{k}\cdot\vec{x})} \frac{z^{3/2}}{F(z)^{1/4}} \Psi(u)$ with $\frac{dz}{du} = \sqrt{F(z)}$, in the wave equation $\Box\Phi = 0$ we get (through the usual procedure) the one-dimensional Schrödinger problem,

$$-\frac{d^2\Psi(u)}{du^2} + V(u)\Psi(u) = (\omega^2 - k^2)\Psi(u) \equiv M^2\Psi(u),$$
$$V(z) = -z^{5/2} F(z)^{1/4} \frac{d}{dz} \times \left(\frac{F(z)}{z^3} \frac{d}{dz} \left(\frac{z^{3/2}}{F(z)^{1/4}} \right) \right). \quad (3.2)$$

For the AdS soliton in five dimensions, one can actually find the change of coordinates from z to u explicitly, from dz/du. We have $z = z_0\, sn(\frac{u}{z_0}, i)$, which is the Jacobi elliptic sine of modulus i. This means that the interval over which the spacetime is defined is mapped as $z \in (0, z_0) \to u \in (0, z_0 K)$, where $K \approx 1.31102$ is the complete elliptic

[5] To show this, one has to put the metric given in Eq. (13) of [33] in the gauge where $g_{xx} = g_{zz}$; g_{xx} is the metric used in [33] and g_{zz} is our metric (2.1). This yields the change of coordinates $x = b \frac{2z_0^{D-1} - z^{D-1}}{z^{D-1}}$.

[6] The logarithmic branch was previously excluded in its entirety, without taking into account these subtleties; see, e.g., [34].

[7] See, for instance, [37,38]. Note also [39] for a top-down construction for QCD$_4$.

[8] We thank Carlos Núñez for pointing out this important detail to us.

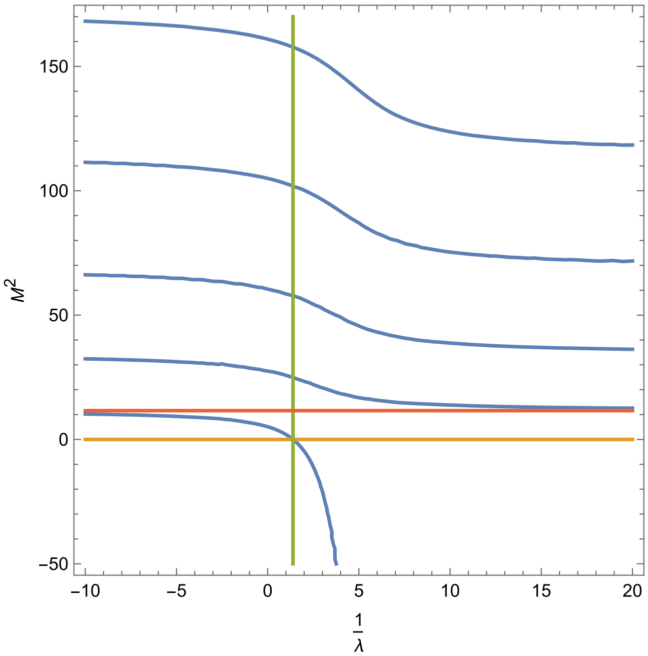

FIG. 1. M^2 vs λ^{-1} for the multitrace deformation $\alpha = \lambda^{-1}\beta$ for QCD_3. Glueball mass[2] as a function of the inverse of the coupling constant λ. The green line is $\lambda^{-1} = \ln 4$. For $\lambda^{-1} > \ln 4$ there is a tachyon. At $\lim_{\lambda\to 0^-} M^2$ the Neumann theory is recovered. The Dirichlet theory is located at $\lambda^{-1} = 0$. We observe that $M_n^2(\lambda^{-1} = -\infty) = M_{n+1}^2(\lambda^{-1} = \infty)$ for each eigenvalue labeled by n. For the ease of visualization of this property, we include the red line which shows the matching of the asymptotic values of the lowest eigenvalue.

integral at modulus i. We observe the same power-law behavior in the UV and the IR of $V(u)$, namely,

$$V(u \simeq 0) \approx \frac{15}{4u^2} + O(u^6), \tag{3.3}$$

$$V(u \simeq Kz_0) \approx \frac{-1}{4(u - Kz_0)^2} + \frac{4}{z_0^2} + O\big((u - Kz_0)^2\big). \tag{3.4}$$

The behavior around these two points is that of conformal quantum mechanics, but with different coupling constants, namely $15/14$ in the UV and $-1/4$ in the IR. The value of $-1/4$ is the *analog of the saturation of the Breitenlohner-Freedman bound* for the conformal quantum mechanics describing the gravity dual by reduction to $0 + 1$ dimensions, and generates the log branch in the IR. The IR solution, using only the divergent part of the potential, is

$$\Psi = \sqrt{Kz_0 - u}\,[C_1 J_0((u - Kz_0)M) + C_2 Y_0((u - Kz_0)M)], \tag{3.5}$$

with J_0 and Y_0 the standard Bessel functions of degree zero. The Neumann boundary condition sets $C_2 = 0$ (eliminating the log-divergent mode Y_0), and the UV boundary condition, *imposed assuming that the IR solution is valid throughout*, an approximation that could be questioned, but will be justified *a posteriori* shortly, implies then the quantization condition, giving the masses of the 0^{++} glueballs,

$$J_0(Kz_0 M_n) = 0. \tag{3.6}$$

The matching of the exact numerical solution with Dirichlet boundary condition to the IR solution (3.5), exact for small argument of J_0, is not too good, but for its zeros it is quite good; the match to the Bessel function $J_{3/2}$ is actually almost perfect as we see in Fig. 2.

Note that in the UV, where the space is just AdS, we have the same solution for Ψ, just replacing J_0, Y_0 with J_2, Y_2 (and this corresponds to large argument for J_2, Y_2). But since $J_2(t) + J_0(t) = \frac{2}{t} J_1(t)$, $Y_2(t) + Y_0(t) = \frac{2}{t} Y_1(t)$, and at $|t| \to \infty$, $J_{3/2}(t) \simeq \frac{J_1(t)+Y_1(t)}{\sqrt{2}} = \frac{t}{2\sqrt{2}}(J_0(t) + Y_0(t) + J_2(t) + Y_0(t))$, this suggests that the average of the IR and UV solutions, with the IR given by the sum of the constant mode J_0 and the log divergent, yet normalizable one, Y_0, provides a better fit to the exact numerical solution with Neumann boundary conditions, whose zeros give the quantization condition, replacing (3.6); thus the zeros of $J_{3/2}$, or $J_1 + Y_1$ replace the zeros of J_0.

We note that the application of the AdS_5 soliton to the holographic superconductor in [43] gave similar results to our glueball spectrum, for the nontrivial deformation of the conductivity $\sigma(\omega)$ by the analog of the λ deformation we considered (see, for instance, their Fig. 3 for the behavior in the UV), since $1/\lambda$ takes the role of the conductivity σ, and M^2 of ω. The holographic superconductor is a scalar deformation of the horizon (the IR theory), just that one usually calculates transport coefficients in the UV; but via Iqbal and Liu's membrane paradigm [21], that is equivalent with an IR calculation at the horizon under certain conditions.

C. AdS_7 and four-dimensional pure glue theory

In this case the fluctuations of the different supergravity fields have to be combined with other metric fluctuations to yield the dual of the 0^{++} glueball spectra of the pure glue theory. However, by comparing with [35] we find that the operator associated to $T_4(r)$ in [35] is the same as that of the d'Alembertian of the massless scalar. To make the logarithmic branch relevant for the glueballs in this case, one should verify that it is normalizable with the norm associated to metric fluctuations. Hence, we study this scalar fluctuation, normalizable with respect to the Klein-Gordon norm, but we postpone its connection to glueballs for a future work. Here the relevant background is the soliton in $AdS_7 \times S^4$, corresponding to near-horizon near-extremal M5-branes, after we compactify two directions to get effectively $D = 4$ on the boundary, and thus obtaining the four-dimensional pure glue (QCD_4) theory.

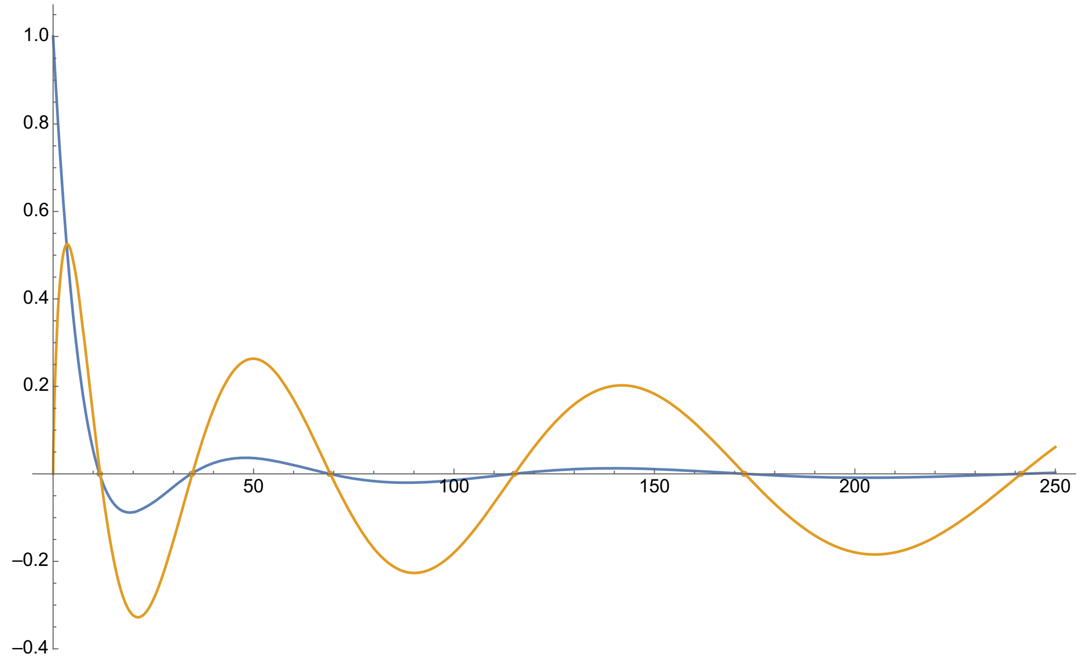

FIG. 2. The numerical solution for the d'Alembertian operator on the AdS soliton in blue vs $J_{3/2}(KM)$ in orange. One can see that the zeros of both functions are in very good agreement. The x axis is M^2.

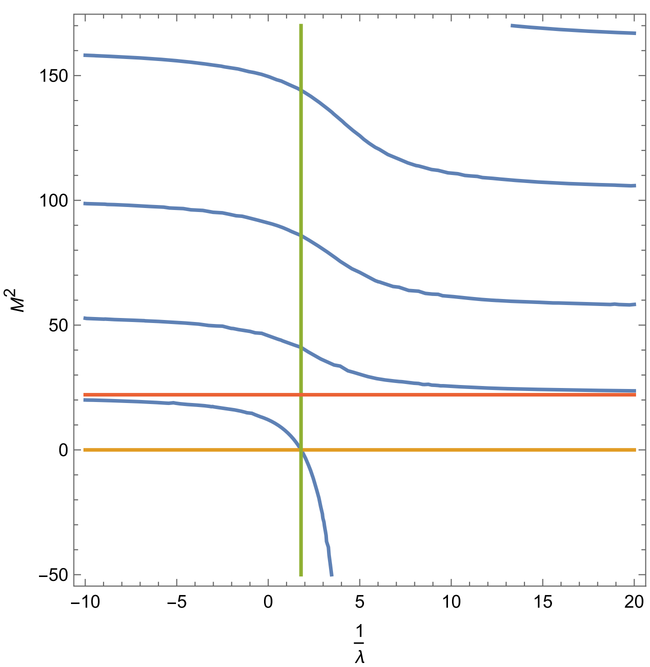

FIG. 3. M^2 vs λ^{-1} for the multitrace deformation $\alpha = \lambda^{-1}\beta$ for the AdS soliton in $D = 7$. The green line is $\lambda = \ln 6$. For $\lambda^{-1} > \ln 6$ there are tachyons. At $\lambda^{-1} = 0$ the Neumann theory. We observe again that $M_n^2(\lambda^{-1} = -\infty) = M_{n+1}^2(\lambda^{-1} = \infty)$ for each eigenvalue labeled by n. The red line eases the visualization of this property.

Equivalently, consider the Witten-Sakai-Sugimoto setup (1.2), of near-horizon near-extremal D4-branes, with background metric conformal to $AdS_6 \times S^4$. The discussion is qualitatively the same. The spectrum can be found in Table II, and Fig. 3 shows the masses^2 and deformations with λ.

For the solution of the $\Box\Phi = 0$, in a general dimension D of the AdS soliton, we have the ansatz $\Phi = e^{-i(\omega t - \vec{k}\cdot\vec{x})} \frac{z^{(D-2)/2}}{F(z)^{1/4}} \Psi(u)$, again with $\frac{dz}{du} = \sqrt{F(z)}$, and for the Schrödinger problem we get

$$V(z) = -z^{\frac{D-2}{2}} F(z)^{1/4} \frac{d}{dz}\left(\frac{F(z)}{z^{D-2}} \frac{d}{dz}\left(\frac{z^{\frac{D-2}{2}}}{F(z)^{1/4}}\right)\right), \quad (3.7)$$

TABLE II. Masses^2 for the scalar fluctuations on the AdS soliton in $D = 7$, corresponding to four-dimensional pure-glue theory. The $\beta = 0$ column coincides with the numerics available in the literature for the T4 in [35], the other spectra are new.

Mass^2 spectrum for different IR boundary conditions for AdS_7		
$\beta = 0$	$\alpha = 0$	$\alpha = -\beta$
22.1007	11.9994	14.5728
55.5851	45.5793	47.3567
102.452	90.9102	92.7819
162.708	149.552	151.554

which gives the same IR potential (3.4), so the same IR solution (3.5), but the UV potential is now $V(u \simeq 0) \approx \frac{D(D-2)}{4u^2} + O(u^6)$, so the UV solution is

$$\Psi = \sqrt{u}\left[C_1 J_{\frac{D-1}{2}}(uM) + C_2 Y_{\frac{D-1}{2}}(uM)\right]. \tag{3.8}$$

In the case relevant here, of $D = 7$, we get the functions J_3 and Y_3 in the UV, instead of J_2 and Y_2 as in the QCD_3 case. But since $J_3(t) = (4/t)J_2(t) - J_1(t)$, $Y_3(t) = (4/t)Y_2(t) - Y_1(t)$, at least at large t (large $z_0 M$), if we take a combination of the IR and UV solutions, again a linear combination of the J_1 and Y_1 gives the correct quantization condition.

D. Universal quantization relations

We have seen that the quantization conditions for the mass of the QCD_3 glueballs were, in a first approximation, written in terms of the zeros of J_0, $j_{0,n}$, as in (3.6). In a better approximation, this was given by the zeros of J_1, or rather of $J_1 + Y_1$. These M_ns for glueball states were obtained from the $\Box$ operator eigenstates in the AdS soliton, with an essential contribution from the IR of the soliton solution.

But there are two other cases that result in masses for glueball solutions. One is the case of high-energy hadron scattering in the Polchinski-Strassler scenario, considered in the limiting case that saturates the Froissart unitarity bound, analyzed in [12–15]. There it was argued that the gravity dual matches exactly the Heisenberg model for high-energy scattering in field theory, when at sufficiently high energies the scattering looks like a collision of gravitational shock waves in the gravity dual, creating a black hole. In the limiting (Froissart saturated) case, the gravity scattering happens effectively on the IR cutoff (IR brane), and produces a black hole entirely on this IR cutoff and completely classical (with negligible fluctuations). The gravitational shock wave profile in AdS with this IR cutoff was calculated in [12] and found to be, at the IR cutoff position $y = 0$,

$$f(r) = \Phi(r, y=0) = R_s \sqrt{\frac{2\pi\ell}{r}} \sum_{n\geq 1} \frac{j_{1,n}^{-1/2} J_2(j_{1,n})}{a_{1,n}} e^{-M_n r},$$
$$M_n = \frac{j_{1,n}}{\ell}, \tag{3.9}$$

where M_n are the glueball states, $j_{1,n}$ are the zeros of J_1, ℓ is the AdS radius, and $R_s \sim G_4\sqrt{s}$ the Schwarzschild radius of the shockwave collision. Thus, as usual, the states in the IR dominate the high-energy scattering giving the Froissart bound: keeping only M_1, then the black hole horizon radius r_H is reached when $f(r) \sim 1$, giving a Froissart cross section $\sigma(s) \sim \pi r_H^2 \sim \frac{\pi}{M_1^2} \ln^2(\sqrt{s} G_4 M_1)$ [13].

So the glueball states in this case are given by the quantization condition $J_1(\ell M_n) = 0$.

We can also obtain the glueball states by considering the Randall-Sundrum model [10,11], understood as AdS_5/CFT_4 with both an IR and an UV cutoff. Then the solution for the graviton wave function at small M is of the $J_2 + Y_2$ type,

$$\psi = N_M(|z| + 1/k)^{1/2}\left[Y_2(M(|z| + 1/k)) + \frac{4k^2}{\pi M^2} J_2(M(|z| + 1/k))\right], \tag{3.10}$$

where $z = \pm(e^{ky} - 1)/k \gg 1/k$, $k = 1/\ell$, and $M/k \ll 1$, and the IR brane is situated at a large $z = z_c$ (and the UV brane is at $z = 0$). There is a zero mode wave function, the massless graviton, obtained from the Y_2 solution in a certain $M \to 0$ limit, and KK modes, all corresponding to states in the boundary field theory. The quantization condition for KK modes (QFT states) is obtained by imposing a boundary condition at z_c, which amounts to

$$Y_1(t+a) + \frac{4}{\pi^2 a^2} J_1(t+a) = 0, \quad t = M|z_c|,$$
$$a = M/k = M\ell \ll 1, \quad t + a = M_n \ell e^{y_c/\ell}, \tag{3.11}$$

so it is approximately again $J_1(t) = J_1(M_n z_c) = 0$, but otherwise again involves both J_1 and Y_1. We note that both of these cases give 0^{++} QFT states coming from the graviton spectrum in the dual, and in the latter case there is also a massless mode.

In conclusion, we can say that the most general quantization condition is of the type

$$J_1(M_n \ell K) + c Y_1(M_n \ell K) = 0, \tag{3.12}$$

for some constants K and c, and K can be absorbed in the overall rescaling of M_n.

IV. CONCLUSIONS

In this paper we have considered the logarithmic (log-divergent) branch of the d'Alembertian operator in AdS soliton and related confining holographic backgrounds, we have shown that it is normalizable, and it has the interpretation of giving multitrace deformations of the theory, but otherwise not changing the RG flow at the UV, which is a universal flow between the UV theory and a fixed IR.

The one thing that does change is the glueball spectrum, for which we can have different results, depending on the deformation. We have considered the deformation with $\alpha = \lambda^{-1}\beta$, which we have found corresponds to a renormalization of the same functional form as the one-loop non-Abelian case (2.9), but now in the IR instead of the UV, and we have found that we can use the parameter λ to lift the degeneracy of the 0^{++} and 2^{++} spectra and match desired lattice data better than previous ones in the literature for QCD_3.

We have also found that we can express to a good degree of accuracy the quantization conditions of the spectra through a combination of the functions J_1 and Y_1, as in (3.12), which matches both the results of spectra from holographic high-energy hadron scattering in QCD, and of spectra from the Randall-Sundrum model, understood as AdS/CFT with UV and IR cutoffs.

The coupling constant that appears due to the IR deformation, λ, is such that the eigenvalues seem to make a continuous curve on a cylinder. This follows from the fact that $M_n^2(\lambda^{-1} = -\infty) = M_{n+1}^2(\lambda^{-1} = \infty)$, and therefore one can identify these points.

The spectrum is free of tachyons provided that the coupling of the deformation satisfies $\lambda^{-1} < \lambda_c^{-1}$. At the onset of this instability there is a massless mode and the probe limit is no longer valid. This seems to indicate that the scale at which the confining gauge theories confine is important, and that the value of the coupling constant there can be constrained by purely theoretical reasoning.

ACKNOWLEDGMENTS

We thank Carlos Núñez for many detailed comments on the manuscript, and Antal Jevicki, Simon Ross, Guillermo Silva, and Jacob Sonnenchein for useful comments. The work of H. N. is supported in part by CNPq Grant No. 301491/2019-4 and FAPESP Grant No. 2019/21281-4. H. N. would also like to thank the ICTP-SAIFR for their support through FAPESP Grant No. 2021/14335-0. The work of A. A. is supported in part by the FAPESP visiting researcher Award No. 2022/11765-7 and the FONDECYT Grants No. 1200986, No. 1210635, No. 1221504, and No. 1230853.

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