Boundary terms in the Witten-Sakai-Sugimoto model at finite density

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We consider the Witten-Sakai-Sugimoto model in the approximation of smeared instantons at finite density via a homogeneous ansatz, which is known to be discontinuous in order to be able to contain a nonvanishing baryon density. The discontinuity at the infrared tip of the bulk spacetime gives rise to subtleties of discarding boundary terms that are normally discarded in the literature. We propose a reason for discarding this boundary term, by scrutinizing the currents and topological properties of the model. Along the way, we find a very effective and simple condition to compute the point of thermodynamic equilibrium.

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I. INTRODUCTION

Chern-Simons (CS) terms exist in field theories of odd (spacetime) dimensions, most famously perhaps in three dimensions. The CS theory by itself seems at first glance to be quite uninteresting, since it is a topological theory and possesses no dynamics. This property changes, however, once the theory is coupled to Yang-Mills (YM) (or Maxwell) theory [1,2] or even just to a scalar field theory [3]. The coupling of CS theory induces immediately new behavior into the theory with which one couples it to; for example, the traditional Gauss law is modified and if CS and YM theories are both present, the gauge-field propagator will have a richer pole structure, yielding a topologically massive theory [1,2].

Another aspect of CS terms is that they are not manifestly gauge invariant, unlike the YM ($\mathrm{tr}F \wedge *F$) or Maxwell (F and hence $F \wedge *F$) counterparts, where F is $\mathfrak{g}$-valued with $\mathfrak{g}$ being the algebra corresponding to the Lie group G [for Maxwell the generator of $U(1)$ is just a real number]. This is immediately clear from the fact that the CS term contains the gauge field A in addition to the field strength $F = \mathrm{d}A + \mathrm{i}A \wedge A$, and since A transforms like $A \to A + \mathrm{d}_A\eta$ under an infinitesimal gauge transform η, where $\mathrm{d}_A\eta = \mathrm{d}\eta + \mathrm{i}[A, \eta]$ is the gauge covariant derivative and η is also $\mathfrak{g}$-valued. The field strength transforms covariantly and hence the trace of any power of the field strength is invariant under gauge transformations. In particular, YM theory is manifestly gauge invariant. The CS term, on the other hand, transforms into (itself plus) a total derivative and plus a winding number of the gauge fields. For suitably chosen integer coefficients (when the CS term is appropriately normalized), the latter winding number term yields a contribution of $2\pi k$ to the action, under which $e^{\mathrm{i}S}$ does not change. The total derivative term is normally not causing any trouble in field theories for two reasons; physicists often work on infinite Cartesian spaces like $\mathbb{R}^3$ or $\mathbb{R}^5$ and the fields are almost always assumed to be continuous and differentiable.

A counterexample to the first reason, leads to beautiful results in condensed matter theory when the CS term is utilized for the fractional quantum Hall effect [4] and there exists a boundary, where the fields are not necessarily pure gauge, the boundary effects give rise to a phenomenon of edge modes living on the boundary (circle) of the material [5].

A counterexample to the latter reason, on the other hand is what we are concerned with in this paper. We are interested in holographic nuclear matter, which is the situation in which we have large densities of matter in holographic QCD [6]. To be specific, we are considering the popular top-down holographic QCD model, namely that of Witten-Sakai-Sugimoto (WSS) [7,8].[1] This model at low energies is indeed described by five-dimensional YM and CS terms coupled together in a curved anti–de Sitter-like spacetime. This particular theory has a conformal boundary with a finite curvature, which is known as the UV

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[1]See also Refs. [9,10] for a derivation of the chiral effective action from the WSS model, as well as an estimate of the axial coupling, magnetic dipole moments, electromagnetic form factors and vector dominance.

boundary where the UV degrees of freedom live. The holographic principle states that the theory in the bulk is dual to the field theory living on the conformal boundary and observables of the bulk fields can be read off of the tails of the fields near the conformal boundary. This should already raise concern for the astute reader, since nontrivial or nonpure gauge behavior on a boundary could spell trouble for the theory. It turns out that this gauge variance is welcome, as it reproduces the chiral anomaly of QCD [8].[2] The above-mentioned second issue with CS, is when the fields are not continuous. This issue arises only in the limit of finite density baryonic matter in the approximation of homogeneous matter in the bulk, where it has been proved that no such continuous configurations can exist [12]. It is possible, however, to describe homogeneous nuclear matter at finite/large densities if we allow the gauge field configurations to be discontinuous at the IR tip of the cigar-shaped spacetime of the WSS model [13].[3] This would correspond to a smeared configuration of baryons/instantons.

This is where we meet the issue with the CS term, in particular because the most convenient form of the CS term for use in the baryonic sector, as written down in Ref. [15] as the Abelian "electric" field multiplied by $\mathrm{tr} F \wedge F$. This formulation of the five-dimensional CS term turns out to be natural for homogeneous nuclear matter, but differs with the full CS term $\omega_5(A)$ by a boundary term. In this paper, we explore the difference of the WSS model when taking this boundary term into account or discarding it. It turns out that the holographic dictionary and the thermodynamical laws are well-defined in either case, but that there is a preferred choice if we scrutinize the currents of the theory. In particular, matching the baryon charge and the behavior of the fields at the conformal boundary provides a way to choose which boundary terms to discard. We additionally find that the same choice of the CS term, makes this term invariant under the $\mathrm{SU}(2)$ gauge transformation that is needed to show the equivalence between isospin realized by isorotations of the fields and turning on a chemical potential at infinity.

This paper is organized as follows. Section II reviews the derivation of the CS term from the point of view of the WSS model in string theory. In Sec. III, we set up the notation for the homogeneous ansatz for nuclear matter in the model at hand. In Sec. IV, we present a systematic way to derive the thermodynamic equilibrium conditions, which turns out to be very useful for numerics. In Sec. V, we show via the energy momentum tensor that the standard thermodynamic relations work, regardless of whether the boundary term in CS is included or not. In Sec. VI, we illustrate the difference between taking the boundary term in the CS term into account or not, by computing a range of observables. We conclude the paper in Sec. VII with a discussion. Details of the equivalence between $\mathrm{SU}(2)$-isospin rotation of the fields and the introduction of an external chemical potential at infinity are shown in Appendix A, whereas the details on how the chiral anomaly of QCD is unchanged by our proposal are delegated to Appendix B.

II. CHERN-SIMONS TERM FROM STRING THEORY

Dp branes are described by the Dirac-Born-Infeld action,

$$S_{\mathrm{DBI}} = -T_p \int \mathrm{d}^{p+1}\xi e^{-\Phi} \sqrt{-\det(\gamma_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}, \tag{1}$$

where F, B, γ, are respectively the gauge field strength, the Kalb-Ramond form and the induced metric on the $(p+1)$-dimensional world volume. The overall brane tension T_p can be given terms of in string parameters as $T_p = (2\pi)^{-p}\alpha'^{-\frac{p+1}{2}}$. We are interested in the $B_{ab} = 0$ scenario, and we will employ the Yang-Mills approximation of this action, obtained by expanding the square root and keeping the quadratic order in the field strength. Following Refs. [8,16,15], we write the resulting Yang-Mills action in the form

$$S_{\mathrm{YM}} = -\kappa \,\mathrm{Tr} \int \mathrm{d}^4x \mathrm{d}z \left[\frac{1}{2}h(z)\mathcal{F}_{\mu\nu}^2 + k(z)\mathcal{F}_{\mu z}^2\right],$$
$$\kappa = \frac{\lambda N_c}{216\pi^3}. \tag{2}$$

The other coupling present in the case of a stack of Dp branes is given by world volume coupling to the Ramond-Ramond (RR) forms. As argued in Ref. [17], in the presence of a D brane background inducing a nontrivial flux of an RR form, the correct expression of this coupling to be considered, is the one after integration by parts, where the RR field strength appears explicitly. Since we will work in a setup with only the flux of F_4 turned on, we take the coupling to be

$$S_{\mathrm{CS}} = \frac{1}{48\pi^3} \int_{D8} F_4 \wedge \omega_5(\mathcal{A}),$$
$$\omega_5(\mathcal{A}) = \mathrm{Tr}\left(\mathcal{A} \wedge \mathcal{F}^2 - \frac{\mathrm{i}}{2}\mathcal{A}^3 \wedge \mathcal{F} - \frac{1}{10}\mathcal{A}^5\right), \tag{3}$$

where powers of forms are understood with the wedge product. Assuming now dependence of the gauge fields only on coordinates transverse to S^4, we can integrate out F_4 using its flux to obtain

[2] There are subtleties for gauge invariance of the CS term when topologically nontrivial gauge configurations are considered and when the gauge group is $\mathrm{SU}(N)$ with $N > 2$, see Ref. [11].

[3] Instead of employing a discontinuity in the fields, it is possible to impose asymmetric boundary conditions on the fields [14].

$$S_{\rm CS} = \frac{N_c}{24\pi^2}\int_{5D}\omega_5(\mathcal{A}). \tag{4}$$

In the general case, we will have an arbitrary number N_f of flavor branes (although for $N_f > N_c$ it would be appropriate to include the backreaction of the branes onto the geometry), hence it is possible to write $S_{\rm CS}$ separating out the $\mathrm{SU}(N_f)$ part,

$$S_{\rm CS} = \frac{N_c}{24\pi^2}\int_{5D}\mathrm{Tr}\bigg[\omega_5^{\mathrm{SU}(N_f)} + 3\hat{A}\wedge F^2 + \hat{A}\wedge\hat{F}^2 + \mathrm{d}\bigg(\hat{A}\wedge\bigg(2F\wedge A - \frac{\mathrm{i}}{2}A^3\bigg)\bigg)\bigg], \tag{5}$$

where $\mathcal{F} = F + \hat{F}$ splits the field strength into the non-Abelian and Abelian part, respectively, and similarly for the gauge field $\mathcal{A}$. Since we consider the $N_f = 2$ case, accounting only for the existence of two light flavors (of quarks), the first term proportional to $\omega_5^{\mathrm{SU}(2)}$ vanishes, leaving us with the simpler expression,

$$S_{\rm CS} = \frac{N_c}{24\pi^2}\int_{5D}\mathrm{Tr}\bigg[3\hat{A}\wedge F^2 + \hat{A}\wedge\hat{F}^2 + \mathrm{d}\bigg(\hat{A}\wedge\bigg(2F\wedge A - \frac{\mathrm{i}}{2}A^3\bigg)\bigg)\bigg]. \tag{6}$$

If we assume the $\mathrm{SU}(2)$ fields, A^a, to be continuous functions vanishing at spatial infinity fast enough for the configuration to have a finite energy, then the total-derivative term vanishes and we are left with the commonly used expression for the CS action in the Witten-Sakai-Sugimoto model. If we employ the field expansion,

$$A = A^a_\alpha T^a \mathrm{d}x^\alpha, \qquad \hat{A} = \hat{A}_\alpha \frac{\mathbb{1}}{2}\mathrm{d}x^\alpha,$$
$$\alpha,\beta,\ldots = 0, M, \qquad M, N, \ldots = i, z, \qquad i, j, \ldots = 1,2,3, \tag{7}$$

then the resulting action term reads,

$$S_{\rm CS} = \frac{N_c}{384\pi^2}\epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} \times \int \mathrm{d}^4x\,\mathrm{d}z\hat{A}_{\alpha_1}[3F^a_{\alpha_2\alpha_3}F^a_{\alpha_4\alpha_5} + \hat{F}_{\alpha_2\alpha_3}\hat{F}_{\alpha_4\alpha_5}]. \tag{8}$$

The main focus of this article is to study a particular situation, the homogeneous ansatz, in which the assumptions for reducing the CS term as above, are not justified, leading to a nonvanishing contribution from the total-derivative term in Eq. (6).

III. THE HOMOGENEOUS ANSATZ

In holographic QCD, baryons are realized as topological solitons of the bulk theory, whose holonomy produces skyrmions of the boundary theory [18]. A system of baryons like a nucleus or in general nuclear matter is then realized as a many-soliton configuration in a five-dimensional curved spacetime; this problem is in general very difficult to solve even using numerical methods, and a variety of approximations are usually employed to make it more treatable. Note that topological solitons in this model are indeed smooth configurations of the gauge fields, with the exception of a singularity at $\xi = 0$, with $\xi^2 \equiv (\vec{x} - \vec{X})^2 + (z - Z)^2$, which is however a gauge artifact. Hence, dealing with this description of baryons we can safely drop the total derivative term in the CS action.

The homogeneous ansatz is an approximation employed in many holographic setups to describe matter at high density. We employ the approximation of forming a homogeneous fluid, where the single baryons are no longer identifiable, smeared along the three dimensions of space of the boundary theory. To realize this, we assume the fields to depend only on z, and in the static scenario only the following fields are turned on [see Ref. [19] for a generalization including time dependence in the form of a slow $\mathrm{SU}(2)$ rotation][4]:

$$A_i = -\frac{H(z)}{2}\tau^i, \qquad \hat{A}_0 = \hat{a}_0(z), \tag{9}$$

where we use the $A_z = 0$ gauge as done in Ref. [13].[5] The complicated many-soliton problem in four-dimensional space is now substituted by the simpler physics of continuous matter in one dimension. The price we pay for this great simplification is the loss of any information on the properties of the individual baryons, and their configuration in space (e.g., we lose information about the favored lattice configuration at a certain density). In particular, we compute the baryon number, which will be infinite given the homogeneity of the system, but most of all will not be quantized in integers. The only meaningful quantity within this ansatz is then the baryon density; usually this quantity is encoded in the field strengths, F_{MN}, whereas now it will be encoded in the function $H(z)$ as

$$\begin{aligned} d &= \frac{1}{32\pi^2}\int \mathrm{d}z\epsilon^{MNPQ}\mathrm{Tr}F_{MN}F_{PQ} \\ &= -\frac{1}{8\pi^2}\int \mathrm{d}z\partial_z(H^3) \\ &= -\frac{1}{8\pi^2}[H^3]^{z=+\infty}_{z=0^+} - \frac{1}{8\pi^2}[H^3]^{z=0^-}_{z=-\infty}. \end{aligned} \tag{10}$$

[4]The homogeneous ansatz has also been employed in bottom-up holographic QCD models, such as VQCD [20,21] and the hard-wall model [22].

[5]A general gauge transformation consistent with the requirement of homogeneity $A_M \to U(z)(A_M - \mathrm{i}\partial_M)U(z)^{-1}$ alters only A_z whereas the field strengths transform covariantly. The topological charge and the YM action remain invariant, but so does the Chern-Simon action since F_{i0} vanishes in the static homogeneous ansatz.

For the energy density to be finite, the function $H(z)$ has to vanish at $z \to \pm\infty$, so that we remain with

$$d = \frac{1}{8\pi^2}[H^3(z \to 0^+) - H^3(z \to 0^-)]. \tag{11}$$

We see that the only way to have a finite density, and so to describe nuclear matter with this ansatz, is for the function $H(z)$ to have a discontinuity; we choose the value $z = 0$ for the location of this discontinuity because it is the energetically favored position at low densities, as can be inferred by the semiclassical value of the pseudomodulus $Z_{cl} = 0$ of the single baryon configuration [15], but in principle the location of the discontinuity should be determined by minimization of the free energy.

Requiring $H(z)$ to be odd, thus leading to a continuous field strength, we finally obtain that the density is given by its infrared boundary condition as

$$H(z \to 0^\pm) = \pm(4\pi^2 d)^{\frac{1}{3}}. \tag{12}$$

From this result, we are brought to the conclusion that we cannot drop the total-derivative term in Eq. (6), which will instead in principle contribute to the energy and free energy of the system. Before moving to the discussion regarding the physical effects of this term, we write the full action of the flavor fields when the static homogeneous ansatz is employed:

$$S = S_{\mathrm{YM}} + S_{\mathrm{CS}}^{\mathrm{bulk}} + S_{\mathrm{CS}}^{\partial}, \tag{13}$$

$$S_{\mathrm{YM}} = -\kappa \int \mathrm{d}^4x \int_0^\infty \mathrm{d}z [3hH^4 + 3k(H')^2 - k(\hat{a}_0')^2], \tag{14}$$

$$S_{\mathrm{CS}}^{\mathrm{bulk}} = -\frac{3N_c}{8\pi^2} \int \mathrm{d}^4x \int_0^\infty \mathrm{d}z\, \hat{a}_0 H' H^2, \tag{15}$$

$$\begin{aligned} S_{\mathrm{CS}}^{\partial} &= \frac{3N_c}{32\pi^2} \int \mathrm{d}^4x \int_0^\infty \mathrm{d}z\, \partial_z(\hat{a}_0 H^3) \\ &= -\frac{3N_c}{32\pi^2} \int \mathrm{d}^4x\, \hat{a}_0(0) H^3(0), \end{aligned} \tag{16}$$

where we have turned every integration over $z \in [-\infty, \infty]$ into one over $z \in [0, \infty]$ with a factor of two coming from the other halves of the branes. Adopting this "folding", on top of making it more convenient to deal with integrations since we can avoid handling discontinuous functions, also makes the necessity of $S_{\mathrm{CS}}^{\partial}$ more manifest as we effectively have a "boundary" at $z = 0$ which will give a nonvanishing contribution [while the UV contribution will still vanish due to the boundary condition $H(\infty) = 0$]. The equations of motion derived from this action are

$$\partial_z(k(z)H') - 2h(z)H^3 + \frac{N_c}{16\pi^2\kappa} H^2 \hat{a}_0' = 0, \tag{17}$$

$$\partial_z(k(z)\hat{a}_0') + \frac{3N_c}{16\pi^2\kappa} H^2 H' = 0. \tag{18}$$

Note that $S_{\mathrm{CS}}^{\partial}$, being a boundary term, does not contribute to the equations of motion, but as we will see it enters physics via a different mechanism.

IV. THERMODYNAMIC EQUILIBRIUM

Solving the equations of motion by itself does not guarantee that the system is in its equilibrium configuration. Boundary conditions encode physical quantities; quark chemical potential μ and baryon density d are introduced as

$$\hat{a}_0(\infty) = \mu, \qquad H(0) = (4\pi^2 d)^{\frac{1}{3}}. \tag{19}$$

We are left with two more boundary conditions to impose; for the field $H(z)$ we require $H(\infty) = 0$ to have finite energy density, while for the field $\hat{a}_0(z)$ usually a Neumann condition is employed at $z = 0$ since it is an even and continuous function. However, this choice is not always justified; we will show that the new term introduced in the CS action can change this prescription. We start by requiring our field configuration to extremize the action; this procedure includes imposing the equations of motion, but is not limited to it, as boundary term arise when we integrate by parts to make the equations of motion manifest. These boundary terms are in many cases vanishing when requiring Dirichlet or Neumann conditions for the fields, but our new effective boundary at $z = 0$ introduces a nontriviality in this process.

When we vary the fields as $H \to H + \delta H$ and $\hat{a}_0 \to \hat{a}_0 + \delta\hat{a}_0$, the action is varied as $S \to S = \delta S$ with

$$\delta S = \int \mathrm{d}^4x\,\mathrm{d}z[(\mathrm{E.o.M.})_H \delta H + (\mathrm{E.o.M.})_{\hat{a}_0} \delta\hat{a}_0] + \delta S_{\mathrm{boundary}}, \tag{20}$$

where the first term (in the bracket) vanishes upon imposition of the equations of motion. The explicit form of the second term is for now omitted as we will review first the case in which we neglect the presence of $S_{\mathrm{CS}}^{\partial}$, to then move to the analysis of the full CS.

A. Without $S_{\mathrm{CS}}^{\partial}$

If we assume that the part of the CS term, $\delta S_{\mathrm{CS}}^{\partial}$, vanishes then the boundary term in the variation of the action is given by

$$\delta S_{\text{boundary}} = 2\kappa \int \mathrm{d}^4x \left[k(z)\hat{a}_0' \delta\hat{a}_0 - 3\left(k(z)H' + \frac{N_c}{16\pi^2\kappa}\hat{a}_0 H^2 \right)\delta H \right]_{z=0}^{z=\infty}. \tag{21}$$

We now proceed, as with the equations of motion terms, to require these two terms (proportional to $\delta\hat{a}_0$ and δH) to vanish separately. As a first simplification we note that the contribution at $z=\infty$ vanishes: In fact $\delta\hat{a}_0(\infty)=0$ because we work at a fixed $\hat{a}_0(\infty)=\mu$, and $\delta H(\infty)=0$ for the energy density to be finite. We are thus left with two additional equations to satisfy on top of the equations of motion to truly extremize the action,

$$\hat{a}_0'(0)\delta\hat{a}_0(0) = 0, \tag{22}$$

$$\left(H'(0) + \frac{N_c}{16\pi^2\kappa}\hat{a}_0(0)H^2(0) \right)\delta H(0) = 0. \tag{23}$$

To solve Eq. (22) we can simply enforce Neumann boundary conditions $\hat{a}_0'(0)=0$, consistently with common practice. At first sight it would also seem that Eq. (23) is satisfied with our choice $H(0)=(4\pi^2 d)^{\frac{1}{3}}$ which seems to imply $\delta H(0)=0$. However, we have to keep in mind that we are not working at fixed density, but rather at fixed chemical potential, hence $d(\mu)$ is a dynamical quantity that should be determined from the minimization of the action itself. This leads us to enforce the condition

$$H'(0) + \frac{N_c}{16\pi^2\kappa}\hat{a}_0(0)H^2(0) = 0. \tag{24}$$

We recognize the above equation as the condition that was only numerically verified in Appendix D of Ref. [22] as part of (D.16); we now see that indeed the condition is exactly satisfied upon extremization of the action, which corresponds to the equilibrium configuration, at least for every $\mu \geq \mu_{\text{onset}}$.[6] Since the field $\hat{a}_0$ carries information on μ, while H carries information about the density, we can see this equation as giving us the relation between these two quantities at equilibrium. It is easier to see it as defining $\mu(d)$; to make it explicit, we write $\hat{a}_0$ as

$$\hat{a}_0(z) = \mu - \int_z^\infty \mathrm{d}z'\, \hat{a}_0'(z'),$$
$$\hat{a}_0' = -\frac{N_c}{16\pi^2\kappa}\frac{1}{k(z)}(H^3(z) - H^3(0)), \tag{25}$$

where the second relation comes from the integration of the equation of motion. After rearranging to isolate μ [which does not enter the equations of motion, so it does not affect the values of $\hat{a}_0'(0)$ or $H'(0)$], we find

$$\mu(d) = -\frac{4\kappa}{N_c}\left(\frac{4\pi^2}{d^2}\right)^{\frac{1}{3}} H'(0) + \frac{N_c d}{4\kappa}\int_0^\infty \mathrm{d}z \frac{1}{k(z)}\left(1 - \frac{H^3(z)}{H^3(0)}\right), \tag{26}$$

which allows us to compute $\mu(d)$ for every given field configuration satisfying the equations of motion. In the baryonic phase ($\mu \geq \mu_{\text{onset}}$) the relation is invertible (the inversion has to be performed numerically) giving us a simple way to compute $d(\mu)$. What we found is that requiring not only the equations of motion to be satisfied, but the more general extremization of the action including nontrivial boundary terms, we obtain the thermodynamic equilibrium condition on top of the field configuration. Note that we have no information about what phase is favored, so the equilibrium found this way may very well be unstable; the baryonic phase will be the stable only when $\Omega_B < 0$, with Ω_B being the grand canonical potential in the baryonic phase (and we made use of the trivial $\Omega_{\text{vacuum}} = 0$).

One more detail we need to pay attention to, is the identification of the parameters we choose to describe the chemical potential, the baryonic density, and the physical quantities; from the holographic dictionary we can read off the physical baryon density d_B from the expansion of $\hat{a}_0$ near the boundary. This task is straightforward since we have an explicit expression for $\hat{a}_0'(z)$, so that we can directly use the formula derived in Ref. [23],

$$d_B = \frac{2}{N_c}\kappa[k(z)\hat{a}_0']_{-\infty}^{\infty} = \frac{4}{N_c}\kappa[k(z)\hat{a}_0']_{z=\infty} = d, \tag{27}$$

so in this case the parameter d, introduced as a boundary condition, really coincides with the physical baryon density. With this result, we now extract the physical expression of the baryon chemical potential μ_B in terms of μ; we do so by looking at the action terms linear in μ. Since μ appears only as an overall shift of $\hat{a}_0$, only the CS action contains such a term, which reads

$$\mu_B d_B = -\frac{N_c}{8\pi^2}\mu\int_0^\infty \mathrm{d}z\, \partial_z(H^3) = \frac{N_c}{2}\mu d, \tag{28}$$

from which follows the identification:

$$\mu_B = \frac{N_c}{2}\mu. \tag{29}$$

[6] In Ref. [22] the relation is obtained for the hard-wall model. It can be mapped to relation (24) in the Witten-Sakai-Sugimoto via $M_5 \to \kappa$, $h(z) \to a(z)$, $k(z) \to a(z)$ and a change in sign due to different conventions adopted and the different orientation of the integration domain. All considerations we make for the Witten-Sakai-Sugimoto model translate directly to the hard-wall model, where however the formalities concerning the choice of the CS term are less stringent given the bottom-up nature of the model.

B. Including $S^{\partial}_{\rm CS}$

We now wish to include the presence of $S^{\partial}_{\rm CS}$ in our considerations. We again wish to extremize the action; the equations of motion (unchanged from the previous section) will take care of the bulk contribution, leaving us again with the need to satisfy a pair of equations on the IR boundary. The novelty with respect to the previous section is that now $S^{\partial}_{\rm CS}$ will provide additional boundary terms, modifying Eqs. (22) and (23),

$$\left(\hat{a}_0'(0) + \frac{3N_c}{64\pi^2\kappa}H^3(0)\right)\delta\hat{a}_0(0) = 0, \qquad (30)$$

$$\left(H'(0) + \frac{N_c}{16\pi^2\kappa}\hat{a}_0(0)H^2(0) - \frac{3N_c}{64\pi^2\kappa}\hat{a}_0(0)H^2(0)\right) \times \delta H(0) = 0. \qquad (31)$$

Both equations acquire a new term, but most notably the new boundary equation for $\hat{a}_0$ is no longer satisfied by a Neumann boundary condition, which has to be modified to

$$\hat{a}_0'(0) = -\frac{3N_c}{64\pi^2\kappa}H^3(0), \qquad (32)$$

which in turn leads to a different solution to the integrated equation of motion (18),

$$\hat{a}_0' = -\frac{1}{k(z)}\frac{N_c}{16\pi^2\kappa}\left(H^3(z) - \frac{H^3(0)}{4}\right). \qquad (33)$$

As a consequence, also the thermodynamic equilibrium relation $\mu(d)$ gets modified, both by the new term in Eq. (31) and the new solution (33),

$$\mu(d) = -\frac{16\kappa}{N_c}\left(\frac{4\pi^2}{d^2}\right)^{\frac{1}{3}}H'(0) + \frac{N_c d}{4\kappa}\int_0^\infty {\rm d}z\frac{1}{k(z)}\left(\frac{1}{4} - \frac{H^3(z)}{H^3(0)}\right). \qquad (34)$$

Repeating the argument from the previous section, we now want to map the parameters d, μ to the physical quantities μ_B, d_B, again we can compute the baryon density from the asymptotics of the $\hat{a}_0$ field according to the formula for the current,[7] which however now yields a different result because of the new expression (33),

$$d_B = \frac{2}{N_c}\kappa[k(z)\hat{a}_0']_{-\infty}^{\infty} = \frac{4}{N_c}\kappa[k(z)\hat{a}_0']_{z=\infty} = \frac{d}{4}. \qquad (35)$$

Since the asymptotic leading order for $\hat{a}_0$ is unchanged, we expect μ_B to be unchanged, and this is confirmed by looking at the action terms proportional to μ, now coming both from $S^{\rm bulk}_{\rm CS}$ and $S^{\partial}_{\rm CS}$, amounting to a contribution,

$$S_\mu = -\frac{N_c}{8}\mu d = -\frac{N_c}{2}\mu d_B, \qquad (36)$$

hence, we can still rely on the identification (29).

V. FREE ENERGY, ENERGY, AND PRESSURE

In Appendix D of Ref. [22] the computation of the energy density and pressure from the stress-energy tensor of the flavor fields was presented; the computation illustrated a number of nontrivialities, including contributions of boundary terms for both quantities, and the use of the newly formulated boundary equations (23) and (31). Here, we repeat the calculation in both the cases with $S^{\partial}_{\rm CS} = 0$ and with $S^{\partial}_{\rm CS} \neq 0$, showing how everything comes consistently together and keeping the holographic dictionary entry $\Omega = -L^{\text{on-shell}}$.

It starts with the definition of the stress-energy tensor,

$$T^\nu_\mu = -2g^{\nu\rho}\frac{\partial\mathcal{L}^m}{\partial g^{\mu\rho}} + \delta^\nu_\mu\mathcal{L}^m, \qquad (37)$$

$$\mathcal{L}^m = -\kappa{\rm Tr}\left[\frac{1}{2}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma} + \mathcal{F}_{\mu z}\mathcal{F}_{\nu z}g^{\mu\nu}g^{zz}\right], \qquad (38)$$

where the metric $g_{\mu\nu}$ is the full metric of the Witten-Sakai-Sugimoto model. We note that only the Yang-Mills part of the action appears, since the CS term is independent of the metric.

From the stress-energy tensor we can extract the pressure, P, and energy density $\mathcal{E}$ as

$$\mathcal{E} = -\int_{-\infty}^\infty {\rm d}z\sqrt{-g}T^0_0, \qquad P = \frac{1}{3}\int_{-\infty}^\infty {\rm d}z\sqrt{-g}T^i_i, \qquad (39)$$

which when used together with the holographic prescription $\Omega = -L^{\text{on-shell}}$ should give the familiar thermodynamic relations for homogeneous systems,

$$\mathcal{E} = -P + \mu_B d_B, \qquad PV = -\Omega. \qquad (40)$$

A. Without $S^{\partial}_{\rm CS}$

As before, we start by considering the situation in which $S^{\partial}_{\rm CS} = 0$. We can compute the pressure P by dividing it into two contributions, corresponding to the two terms in the stress-energy tensor (37), $P = P^{(1)} + P^{(2)}$, with

[7] A naive approach to the calculation of this charge is to trade the evaluation at the boundary for the integral over z of the derivative of the expression, and then use Eq. (18) to obtain exactly the instanton number density d. However, since $k\hat{a}_0'$ is not continuous in this scenario, we cannot exchange the function evaluated at the UV boundary for the bulk integral of the derivative, as it would pick up IR contributions that do not belong to the definition of the current.

$$P^{(1)} = \frac{2\kappa}{3}\int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{Tr}(h(z)F_{ij}^2 + k(z)F_{iz}^2),$$

$$P^{(2)} = L_{\mathrm{YM}} = \int_0^{\infty} \mathrm{d}z\, \mathcal{L}_{\mathrm{YM}}, \tag{41}$$

where L_{YM} indicates the integrand of S_{YM} in Eq. (14) and the terms displayed in $P^{(1)}$ are the only nonvanishing terms upon insertion of the homogeneous ansatz (9).

Since $P^{(2)}$ is trivial and already gives manifestly a part of the Lagrangian density, we only need to compute $P^{(1)}$; substituting the homogeneous ansatz, performing the traces, and accounting for a factor of two after trading the whole integration domain for only half of the brane, we end up with

$$P^{(1)} = 2\kappa\int_0^{\infty} \mathrm{d}z\,[2h(z)H^4 + k(z)H'^2], \tag{42}$$

whose second term we can integrate by parts in order to make the equation of motion (17) manifest, leaving us with a boundary term as a result,

$$P^{(1)} = \frac{N_c}{8\pi^2}\int_0^{\infty} \mathrm{d}z\, \hat{a}_0' H^3 + 2\kappa[(k(z)H'H)]_0^{\infty}. \tag{43}$$

We can integrate by parts again to make the (bulk) CS term (15) appear, at the price of obtaining a second boundary term,

$$P^{(1)} = \int_0^{\infty} \mathrm{d}z\mathcal{L}_{\mathrm{CS}}^{\mathrm{bulk}} - 2\kappa\left[\left(k(z)H' + \frac{N_c}{16\pi^2\kappa}\hat{a}_0 H^2\right)H\right]\Big|_{z=0}, \tag{44}$$

where we made use of the fact that the boundary term only contributes in the IR ($z = 0$). In the boundary term, we now recognize Eq. (23), so that it vanishes on-shell at equilibrium (since $k(0) = 1$). In the end we obtain the expected result,

$$P = L_{\mathrm{YM}} + L_{\mathrm{CS}}^{\mathrm{bulk}} = L, \tag{45}$$

and since from holography at $T = 0$, we identify the on-shell Lagrangian with the grand-canonical potential, we obtain $PV = -\Omega$.

The next quantity we need to compute is the energy density $\mathcal{E}$. Again, we can divide the expression into two contributions, corresponding to the two terms of Eq. (37) and again the latter will trivially give minus the Yang-Mills Lagrangian,

$$\mathcal{E}^{(2)} = -\int_{-\infty}^{\infty} \mathrm{d}z\sqrt{-g}\mathcal{L}^m = -L_{\mathrm{YM}}, \tag{46}$$

while the first term requires more attention as it will also produce boundary terms in the effort to make the presence of L_{CS} manifest. We start by computing the derivative with respect to g^{00} to obtain,

$$\mathcal{E}^{(1)} = 2\kappa\int_0^{\infty} \mathrm{d}z\, k(z)\hat{F}_{0z}^2 = 2\kappa\int_0^{\infty} \mathrm{d}z\, k(z)(\hat{a}_0')^2, \tag{47}$$

where the term after the first equality is the only nonvanishing one upon insertion of the homogeneous ansatz (9). Again we can proceed to integrate by parts obtaining the kinetic term of the equation of motion in exchange for a boundary term,

$$\mathcal{E}^{(1)} = 2\kappa[k(z)\hat{a}_0\hat{a}_0']_0^{\infty} - 2\kappa\int_0^{\infty} \mathrm{d}z\, \partial_z(k(z)\hat{a}_0')\hat{a}_0. \tag{48}$$

This time the only contribution from the boundary term comes from $z = \infty$, as also noted in the hard-wall model in Ref. [22]. Making use of Eqs. (18) and (25), the bulk CS term and the chemical potential coupled to the baryon density appear,

$$\mathcal{E}^{(1)} = -L_{\mathrm{CS}}^{\mathrm{bulk}} + \frac{N_c}{2}\mu d. \tag{49}$$

In the end, for the total energy density we obtain

$$\mathcal{E} = -L_{\mathrm{YM}} - L_{\mathrm{CS}}^{\mathrm{bulk}} + \frac{N_c}{2}\mu d. \tag{50}$$

By making use of Eqs. (45), (29), and (27) we obtain the correct thermodynamic relation,

$$\mathcal{E} = -P + \mu_B d_B. \tag{51}$$

B. Including $S_{\mathrm{CS}}^{\partial}$

We now want to include the effects of the boundary term $S_{\mathrm{CS}}^{\partial}$. Of course, the holographic dictionary still has to be valid, and the system is still a homogeneous one, so consistency requires us to still find the relation $PV = -\Omega = L$ on-shell at equilibrium.

To obtain the result (44), we only used the definition of $T_{\mu\nu}$, which is sensitive only to the Yang-Mills action, and the equations of motion, which are insensitive to boundary terms. Hence, the entire derivation is not altered by the presence of a boundary term in the CS action and Eq. (44) still holds. However, this time the boundary term in Eq. (44) does not vanish, but gives a contribution according to Eq. (31),

$$P^{(1)} = \int_0^{\infty} \mathrm{d}z\, \mathcal{L}_{\mathrm{CS}}^{\mathrm{bulk}} + \frac{3N_c}{32\pi^2}[\hat{a}_0 H^3]|_{z=0} = L_{\mathrm{CS}}^{\mathrm{bulk}} + L_{\mathrm{CS}}^{\partial}, \tag{52}$$

so that in the end we again end up with the pressure equating the full on-shell Lagrangian density at equilibrium, complying with the holographic dictionary and the homogeneity of the system,

$$P = L_{\rm YM} + L_{\rm CS}^{\rm bulk} + L_{\rm CS}^{\partial} = L, \qquad PV = -\Omega. \tag{53}$$

Let us now turn to the energy density; here too the derivation from the previous section holds up to the evaluation of the boundary terms, since again only the Yang-Mills action and the equations of motion are involved, so we only need to take a better look at

$$\mathcal{E}^{(1)} = -2\kappa \int_0^\infty {\rm d}z \partial_z(k(z)\hat{a}_0')\hat{a}_0 + 2\kappa[k(z)\hat{a}_0\hat{a}_0']_0^\infty. \tag{54}$$

The integral still provides $L_{\rm CS}^{\rm bulk}$ upon insertion of the equations of motion, but we note that $\hat{a}_0'(z=0)=0$ no longer holds; hence, both boundaries will contribute now. At both $z=\infty$ and $z=0$ we use of the explicit expression (33) to obtain,

$$\mathcal{E}^{(1)} = -L_{\rm CS}^{\rm bulk} + \frac{N_c}{8}\mu d + \frac{3N_c}{32\pi^2}H^3(0)\hat{a}_0(0), \tag{55}$$

where we notice that the infrared contribution amounts exactly to $-L_{\rm CS}^{\partial}$. The term proportional to μd instead provides again the correct quantity $\mu_B d_B$ once we make use of the dictionary entries (35) and (29). Combining $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$, we obtain the correct thermodynamic formula accounting for the presence of $S_{\rm CS}^{\partial}$,

$$\mathcal{E} = -L_{\rm YM} - L_{\rm CS}^{\rm bulk} - L_{\rm CS}^{\partial} + \frac{N_c}{8}\mu d = -P + \mu_B d_B. \tag{56}$$

VI. EFFECTS ON OBSERVABLES

In this section, see how much impact the boundary term $S_{\rm CS}^{\partial}$ has on a few selected observables relevant for physics at finite densities; i.e. we will compare the observables with and without the presence of the boundary term, recalling that the top-down model with the definition from string theory, should contain this boundary term as part of the CS term.

A. Saturation density

Let us begin with evaluating the saturation density at the phenomenological value (as derived from fitting the ρ meson mass and the pion-decay constant [8]) of the 't Hooft coupling, i.e., $\lambda = 16.63$. We simply vary the density d and determine μ by the thermodynamic equilibrium [i.e., Eq. (26) or (34)], until we find the same value of the canonical potential Ω for the baryon phase, as for the vacuum (which is $\Omega = 0$); this is the onset of the baryon phase and we define the corresponding density as the nuclear saturation density, d_0. We find

$$d_0^{\rm bulk} = 0.436\left(\frac{M_{\rm KK}}{949~{\rm MeV}}\right)^3~{\rm fm}^{-3}, \tag{57}$$

$$d_0^{{\rm bulk}+\partial} = 0.601\left(\frac{M_{\rm KK}}{949~{\rm MeV}}\right)^3~{\rm fm}^{-3}, \tag{58}$$

where $d_0^{\rm bulk}$ is computed with only the bulk CS term, whereas $d_0^{{\rm bulk}+\partial}$ is computed with the full CS term. The mesonic fit of the model sets $M_{\rm KK} = 949$ MeV, hence we can immediately see that the result closer to the phenomenological $d_0^{\rm ph} = 0.15~{\rm fm}^{-3}$ is the one in which we neglect the presence of $S_{\rm CS}^{\partial}$. In order to obtain the nuclear saturation density of experiments [24], the Kaluza-Klein scale would have to be adjusted as

$$\begin{aligned} d_0^{\rm bulk} &= 0.15~{\rm fm}^{-3} \Rightarrow M_{\rm KK} = 665.0~{\rm MeV},\\ d_0^{{\rm bulk}+\partial} &= 0.15~{\rm fm}^{-3} \Rightarrow M_{\rm KK} = 597.6~{\rm MeV}, \end{aligned} \tag{59}$$

for the bulk and full CS terms, respectively.

B. Speed of sound

We will now compute the speed of sound for the two cases, i.e., with and without the boundary term in the CS action taken into account, which is given by

$$c_s^2 = \frac{d_B}{\mu_B}\frac{\partial \mu_B}{\partial d_B} = \frac{d}{\mu}\frac{\partial \mu}{\partial d}, \tag{60}$$

and the only relation needed is $\mu(d)$ given by Eqs. (26) and (34), for the bulk CS and the bulk + boundary CS term, respectively. The result is shown in Fig. 1 for both cases. Notice that the speed of sound does not depend on the calibration of the KK scale.

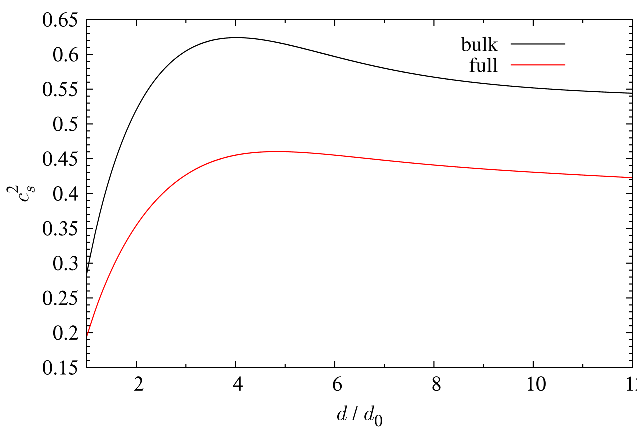

FIG. 1. Sound speed squared for the two cases of the CS term without the boundary contribution (bulk) displayed with a solid black line and with it (full) displayed with a red solid line. The results are independent of $M_{\rm KK}$.

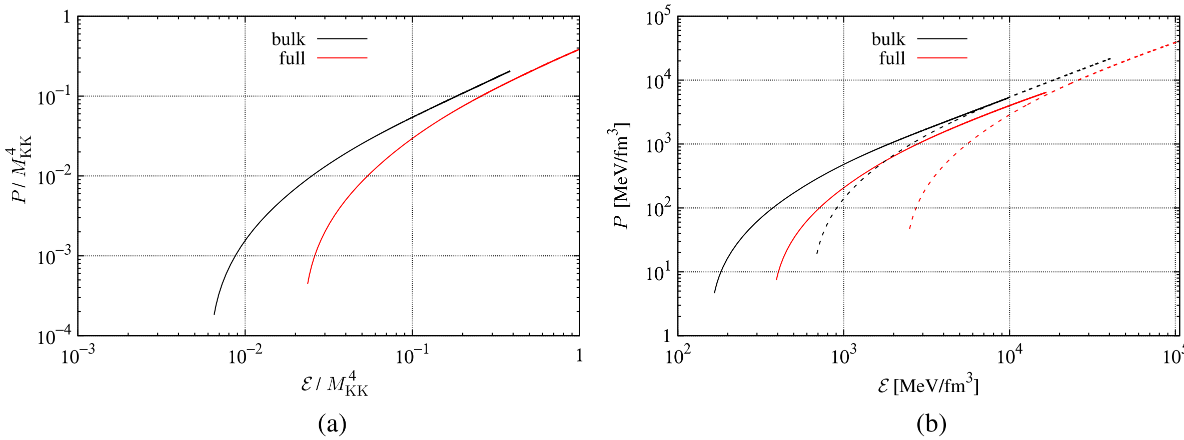

FIG. 2. (a) The EOS in dimensionless units and (b) in calibrated units, for densities $d \in [1.1, 20]d_0$ in the two cases of not including the boundary contribution in the CS term (black line) and with including it (red line). In (b) the solid lines correspond to the case that $M_{\rm KK}$ is calibrated after Eq. (59) such that the saturation density is physical, whereas the dashed lines correspond to the meson calibrated value of $M_{\rm KK} \sim 949$ MeV [8].

C. Equation of state

Next, we will turn to the equation of state, which is a fundamental relation for physics at finite density, with applications ranging from heavy ion physics to neutron stars. The equation of state (EOS) is a relation between the energy and the pressure of a physical system and by the familiar thermodynamic relations, $P = -\mathcal{E} + \mu_B d_B$, where the chemical potential can again be computed from Eqs. (26) and (34), for the bulk CS and the bulk + boundary CS term, respectively. For the thermodynamic relation, we have to recall the conversion between the model parameters μ, d and the physical quantities μ_B, d_B, by Eqs. (27) and (35) for the density for the bulk CS and the bulk + boundary CS term, respectively, and Eq. (29) for the chemical potential.

The results for the EOS for symmetric nuclear matter are shown in Fig. 2. The panel (a) shows the dimensionless pressure and energy relation (EOM) both normalized by the KK scale to the fourth power. By using the calibration of having physical values of the saturation density (59), the EOM is shown in physical units with solid lines in panel (b) of the figure. Using instead the mesonic fit of Ref. [8], viz. $M_{\rm KK} \sim 949$ MeV, the EOM is shown with dashed lines.

D. Symmetry energy

Next, we consider the nuclear symmetry energy, which measures the energy cost in having more neutrons than protons, or vice versa. For computing the symmetry energy, we follow Ref. [19], but we show the equivalence between using a chemical isospin potential according to the traditional holographic dictionary in Appendix A. Rotating all gauge fields in isospin space $\mathcal{A} \to a\mathcal{A}a^\dagger$ with a time-dependent SU(2) rotation, $a = a(t)$, and angular velocity $\boldsymbol{\chi} = -\mathrm{i}\mathrm{Tr}(a^\dagger \dot{a}\boldsymbol{\tau})$, we obtain the new time-dependent ansatz for the gauge fields,

$$\mathcal{A}_0 = Ga\boldsymbol{\chi}\cdot\boldsymbol{\tau}a^\dagger + \frac{1}{2}\hat{a}_0, \qquad \mathcal{A}_i = -\frac{1}{2}(Ha\tau^i a^\dagger + L\chi^i), \qquad \mathcal{A}_z = 0, \tag{61}$$

where the field L has been turned on in order to satisfy the equations of motion [19]. Notice that $\boldsymbol{\chi}\cdot\boldsymbol{\tau}$ is the angular velocity, whereas $a\boldsymbol{\chi}\cdot\boldsymbol{\tau}a^\dagger$ is the isospin angular velocity. We want to study the theory without external fields turned on, while the effects of isospin chemical potential have been already accounted for by the rotating ansatz as shown in Ref. [19], so the UV boundary conditions for the new fields, G and L are

$$G(\infty) = 0, \tag{62}$$

$$L(\infty) = 0. \tag{63}$$

Computing the energy contribution arising from the new terms and the isospin rotation, we find that they correspond to a kinetic term for the Hamiltonian,

$$H_{\rm kin} = \frac{1}{2}V\Lambda\boldsymbol{\chi}\cdot\boldsymbol{\chi} = \frac{I(I+1)}{2V\Lambda}, \qquad \Lambda = \kappa\int \mathrm{d}z\,(2hH^2(2G+1)^2 + k[(L')^2 + 4(G')^2]), \tag{64}$$

where in the second equality, zeromode quantization has been performed on the coordinates $a = a_0 + \mathrm{i}a_i\tau^i$ on the 3-sphere, yielding $\pi_m = \frac{\partial H}{\partial \dot{a}_m} = 4V\Lambda\dot{a}_m$, $\pi_m^2 = 4I(I+1)$, with $I \in \frac{1}{2}\mathbb{Z}_+$ being the isospin quantum number. Using the relation between isospin and the difference between the number of neutrons (N) and protons (Z),

$$2I = Z - N = -\beta B, \tag{65}$$

with B being the total number of nucleons, the symmetry energy can readily be extracted from the Hamiltonian (64) as

$$S(d_B) = \frac{d_B}{8\Lambda}, \tag{66}$$

where d_B is the physical nucleon density, given by Eqs. (27) and (35) for the bulk CS and the bulk + boundary CS term, respectively.

To consistently compute the IR boundary conditions we again follow the method of vanishing IR boundary terms in the variation of the action, the same as imposing thermodynamic equilibrium as previously shown. The additional terms that appear in the Yang-Mills and CS actions because of the rotation in SU(2) (ignoring second-order time derivatives) are given by

$$S_\chi = S_{\chi\mathrm{YM}} + S_{\chi\mathrm{CS}}^{\mathrm{bulk}} + S_{\chi\mathrm{CS}}^{\partial}, \tag{67}$$

$$S_{\chi\mathrm{YM}} = -\kappa\chi^2 \int \mathrm{d}^4x \int_0^\infty \mathrm{d}z \left[k(L')^2 - 4k(G')^2 - 8hH^2\left(G + \frac{1}{2}\right)^2\right], \tag{68}$$

$$S_{\chi\mathrm{CS}}^{\mathrm{bulk}} = \frac{N_c}{4\pi^2}\chi^2 \int \mathrm{d}^4x \int_0^\infty \mathrm{d}z \left[H^2LG' + 2\left(G + \frac{1}{2}\right)LHH'\right], \tag{69}$$

$$S_{\chi\mathrm{CS}}^{\partial} = \frac{3N_c}{16\pi^2}\chi^2 \int \mathrm{d}^4x\, H^2(0)L(0)\left(G(0) + \frac{4}{9}\right). \tag{70}$$

The equations of motion for all the fields do not depend on boundary terms in the action, and they read

$$hH^3 - \frac{1}{2}\partial_z(kH') - \frac{N_c}{32\pi^2\kappa}H^2\hat{a}_0' = 0, \tag{71}$$

$$\partial_z(k\hat{a}_0') + \frac{3N_c}{16\pi^2\kappa}H^2H' = 0, \tag{72}$$

$$\partial_z(kG') - 2hH^2\left(G + \frac{1}{2}\right) + \frac{N_c}{32\pi^2\kappa}H^2L' = 0, \tag{73}$$

$$\partial_z(kL') + \frac{N_c}{8\pi^2\kappa}H[HG' + (1 + 2G)H'] = 0. \tag{74}$$

To determine the IR boundary conditions, we perform a variation of the action and look at the IR boundary terms, imposing them to vanish. It is now that we have to take into account whether $S_{\chi\mathrm{CS}}^{\partial}$ is present or not. As before, we consider first the situation in which we get rid of it, so that the action at order χ^2 is given only by $S_\chi = S_{\chi\mathrm{YM}} + S_{\chi\mathrm{CS}}^{\mathrm{bulk}}$. The IR boundary terms that have to vanish are then given by

$$\left[G'(0) + \frac{N_c}{32\pi^2\kappa}H^2(0)L(0)\right]\delta G(0) = 0, \tag{75}$$

$$L'(0)\delta L(0) = 0. \tag{76}$$

The second equation of this set is trivially solved by imposing Neumann boundary conditions for L, exactly as it happened for $\hat{a}_0$. Since we expect L to have odd parity with respect to z, it will be a discontinuous function if $L(0) \neq 0$. If in turn $L(0) \neq 0$, $G'(0) \neq 0$ will be a nonvanishing derivative at the IR tip and hence its derivative will not be continuous due to the positive parity in z.

We can integrate the equation of motion (74) once obtaining,

$$kL' + \frac{N_c}{8\pi^2\kappa}H^2\left(G + \frac{1}{2}\right) = \mathrm{const}, \tag{77}$$

where the right-hand side constant is a constant of motion, which can be determined by using Eq. (76), yielding

$$kL' + \frac{N_c}{8\pi^2\kappa}H^2\left(G + \frac{1}{2}\right) = \frac{N_c}{8\pi^2\kappa}H^2(0)\left(G(0) + \frac{1}{2}\right), \tag{78}$$

which when evaluated at $z \to \infty$ yields a nonvanishing axial U(1) current

$$\kappa[kL']_{z=\infty} = \frac{N_c}{8\pi^2}H^2(0)\left(G(0) + \frac{1}{2}\right). \tag{79}$$

Let us now include the presence of $S_{\chi\mathrm{CS}}^{\partial}$; in this case the IR boundary terms that have to vanish are

$$\left[G'(0) + \frac{N_c}{128\pi^2\kappa}H^2(0)L(0)\right]\delta G(0) = 0, \tag{80}$$

$$\left[L'(0) + \frac{3N_c}{32\pi^2\kappa}H^2(0)\left(G(0) + \frac{4}{9}\right)\right]\delta L(0) = 0. \tag{81}$$

As can be seen, the IR boundary condition for L is no longer a Neumann condition. Evaluating the constant of motion in the integrated equation of motion for L, (77), we obtain now

$$kL' + \frac{N_c}{8\pi^2\kappa}H^2\left(G + \frac{1}{2}\right) = \frac{N_c}{32\pi^2\kappa}H^2(0)\left(G(0) + \frac{2}{3}\right), \tag{82}$$

which when evaluated at $z \to \infty$ yields a different but still nonvanishing axial U(1) current

$$\kappa[kL']_{z=\infty} = \frac{N_c}{32\pi^2} H^2(0)\left(G(0) + \frac{2}{3}\right). \quad (83)$$

We note that both in the case of discarding the boundary term in Eq. (6) to arrive at the CS action (8) and in the case of keeping it, the axial U(1) current is turned on. We have turned on isospin by isorotating the baryons and it is in fact equivalent to using a chemical potential, see Appendix A. We expect on general grounds that isorotation will induce a nonvanishing U(1) current. If, however, we would like to switch it off we can perform another integration by parts at the level of χ^2 in the action, writing the CS bulk and boundary terms at this order as

$$S_{\chi\mathrm{CS}}^{\mathrm{bulk,no\text{-}current}} = -\frac{N_c}{4\pi^2}\chi^2 \int \mathrm{d}^4x \int_0^\infty \mathrm{d}z\, H^2 L' \left(G + \frac{1}{2}\right), \quad (84)$$

$$S_{\chi\mathrm{CS}}^{\partial,\mathrm{no\text{-}current}} = \frac{N_c}{16\pi^2}\chi^2 \int \mathrm{d}^4x H^2(0) L(0) \left(G(0) + \frac{2}{3}\right). \quad (85)$$

The integration by parts has been performed such that the field $L(z)$ is removed in favor of its derivative. In this case, the IR boundary terms that have to vanish are then given by

$$G'(0)\delta G(0) = 0, \quad (86)$$

$$\left[L'(0) + \frac{N_c}{8\pi^2\kappa} H^2(0)\left(G(0) + \frac{1}{2}\right)\right]\delta L(0) = 0. \quad (87)$$

The first equation of this set is trivially solved by imposing a Neumann boundary condition for G, exactly as it happened for $\hat{a}_0$. We note that in this case the IR boundary equation for $L'(0)$ reduces to the same form of the integrated equation of motion (77), with the constant of integration to be determined by the boundary conditions. The condition (87) forces the constant to be zero; as done with $\hat{a}_0$ and the associated charge, we can now use Eq. (77) to compute the current associated with $\hat{A}_i$, [an axial current since $\hat{A}_i(z)$ is an odd function of z]. To do so, we evaluate Eq. (77) at the UV boundary to find

$$\kappa[kL']_{z=\infty} = 0, \quad (88)$$

and since the left-hand side of the equation above is proportional to the current, we conclude that we are working at zero axial U(1) current. Note that by fixing the derivative $L'(0)$ we no longer have the freedom of choosing the function $L(z)$ to be continuous at $z = 0$; the value $L(0)$ is now determined by the equations of motion and the two boundary conditions, so $L(z)$, just like $H(z)$ is in general discontinuous and odd.

We note that in all cases, the boundary conditions $L(0) = 0$ and $G'(0) = 0$ are consistent with the variational principle, as the Dirichlet boundary condition $L(0) = 0$ eliminates the possibility of the variation. $G'(0) = 0$ is also consistent in this case, since it is always proportional to $L(0)$, which when vanishing implies a Neumann condition for G in the IR. The boundary conditions found here, however, are expected to lower the free energy slightly with respect to the simplistic, but consistent boundary conditions $L(0) = G'(0) = 0$.

In Appendix A, we have shown the result of Ref. [19] adapted to the three cases of different CS terms utilized in this section; the bulk CS, the full (bulk + boundary) CS and the last *ad hoc* construction that eliminates the U(1) axial current. The summary of the analysis in the appendix, is that only the bulk CS term, $S_{\mathrm{CS}}^{\mathrm{bulk}}$, remains invariant under the specific gauge transformation that connects the rotation of the isospin moduli to the introduction of a finite isospin chemical potential as the UV boundary value of the field $A_0^{a=3}$, as prescribed per usual in the holographic dictionary. We are then led to conclude that, also in this case, it is preferable to neglect $S_{\mathrm{CS}}^{\partial}$, in order to preserve the equivalence between angular velocity in isospin space and the isospin chemical potential.

The results for the symmetry energy in the two cases of including the boundary term in the CS action and not including it, are shown in Fig. 3 with red and black colors, respectively. Additionally, the ad-hoc choice of setting the U(1) axial current to zero, according to the boundary conditions (86) and (87) is shown with orange curves. In Fig. 3(b) two calibrations are shown; viz. that corresponding to Eq. (59) such that the saturation density is physical and that for which the standard rho meson calibrated value of $M_{\mathrm{KK}} \sim 949$ MeV [8]. Finally, the phenomenologically expected region extracted from many experiments via a fit [25], is shown with a light-blue shaded area.

E. Neutron stars

Our final observable to consider here, are the masses and radii of neutron stars, ignoring fine details as the crust—which however are crucially important to obtain correct radii, that are approximately 1 or more kms larger than predicted by dense neutron matter at masses around 1.4 solar masses—and neglecting also isospin asymmetry, whose contribution in the gauge fields (the same we presented in the previous section) is suppressed as N_c^{-1}.

The mass and radius of a single neutron star is obtain by solving the Tolman-Oppenheimer-Volkoff equations, which are given by

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -G(\mathcal{E} + P)\frac{m + 4\pi r^3 P}{r(r - 2Gm)}, \quad (89)$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \mathcal{E}, \quad (90)$$

where the nuclear physics input is in the form of the equation of state, or more precisely, the inverse is needed; $\mathcal{E}(P)$. Due to

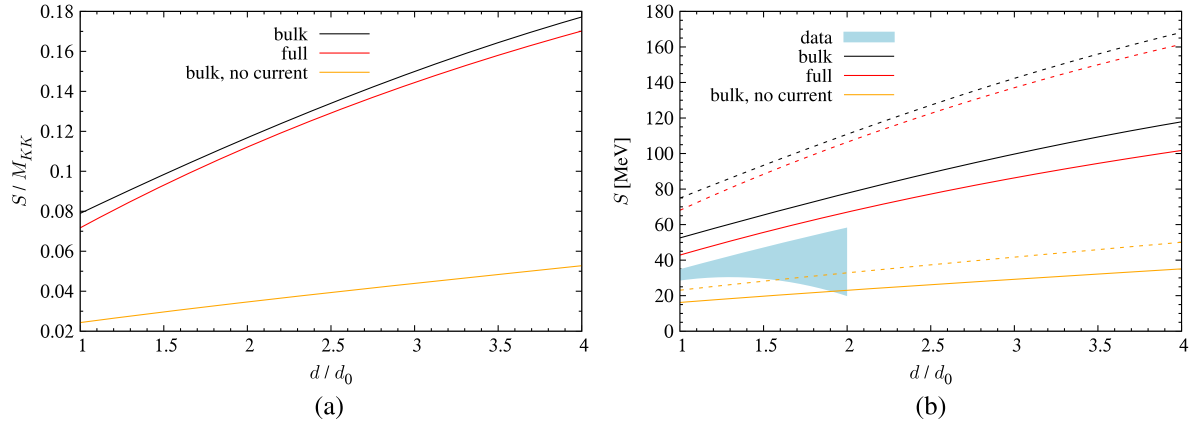

FIG. 3. Symmetry energy in (a) dimensionless and (b) physical units, in the cases of not including the boundary contribution in the CS term (black line) and with including it (red line), as well as the choice of setting the external U(1) axial current to zero (orange line). In (b) the solid lines correspond to the case that M_{KK} is calibrated after Eq. (59) such that the saturation density is physical, whereas the dashed lines correspond to the meson calibrated value of $M_{\mathrm{KK}} \sim 949$ MeV [8]. In (b) fitted data from the survey [25] is shown with a light-blue shaded area. In this figure, $\lambda = 16.63$.

the astronomical units in this system of equations, it will prove useful to rescale the variables to dimensionless quantities, for which the equations read

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}\tilde{r}} = (\tilde{\mathcal{E}} + \tilde{P}) \frac{\tilde{m} + A\tilde{r}^3\tilde{P}}{\tilde{r}(2\tilde{m} - \tilde{r}/B)}, \tag{91}$$

$$\frac{\mathrm{d}\tilde{m}}{\mathrm{d}\tilde{r}} = A\tilde{r}^2\tilde{\mathcal{E}}, \tag{92}$$

with the dimensionless conversion quantities,

$$A = \frac{4\pi r_0^2 \mathcal{E}_0}{m_0} \simeq 1.188911 \left(\frac{M_{\mathrm{KK}}}{949\ \mathrm{MeV}}\right)^4, \tag{93}$$

$$B = \frac{G m_0}{r_0} \simeq 1.477063, \tag{94}$$

where the physical and the dimensionless quantities are related as $P = \mathcal{E}_0\tilde{P}$, $r = r_0\tilde{r}$, $\mathcal{E} = \mathcal{E}_0\tilde{\mathcal{E}}$, and $m = m_0\tilde{m}$, and for convenience, we have chosen $\mathcal{E}_0 = M_{\mathrm{KK}}^4$, $r_0 = 1$ km, and $m_0 = M_\odot$ (1 solar mass). Finally, we denote by R the radius $r(P)$ with $P = 0$ and correspondingly M the mass $m(P)$ with $P = 0$.

In Fig. 4, we show the results of the masses and radii for the two cases of including the boundary term in the CS action and not including it. In the figure, two calibrations are shown; viz. that corresponding to Eq. (59) such that the saturation density is physical and the standard meson calibrated value of $M_{\mathrm{KK}} \sim 949$ MeV [8]. Since there is not taken any crust (softer matter at the surface of the star) into account, the radius must be at least about 1 km smaller than the upper bound of the constraint from J0740 + 6620 (violet bar). The maximum mass should be around $\sim 2.35 M_\odot$ but probably smaller than $\sim 2.5 M_\odot$. The sensitivity of the mass/radius curves to the 't Hooft coupling is quite large, so even an order one change in the coupling from $\lambda = 16.63$, could make viable neutron star phenomenology. In Ref. [26], it was found that indeed it is possible to fit the model to properties of nuclear matter at saturation density, and obtain realistic neutron stars; there the Dirichlet boundary condition $L(0) = 0$ was used, so the problem of choosing the correct CS term does not arise, as any boundary term that we can choose is proportional to $L(0)$. Imposing the Neumann boundary condition $L'(0) = 0$ and neglecting $S_{\mathrm{CS}}^{\partial}$ is expected to provide corrections to the

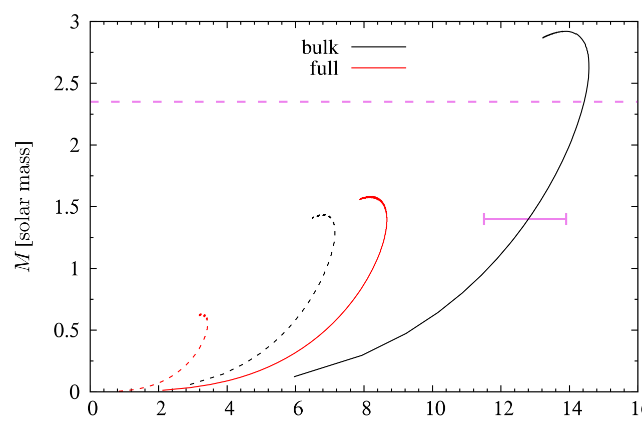

FIG. 4. Neutron star mass and radius, in the two cases of not including the boundary contribution in the CS term (black line) and with including it (red line). For the solid lines, M_{KK} is calibrated after Eq. (59) such that the saturation density is physical, whereas the dashed lines correspond to the meson calibrated value of $M_{\mathrm{KK}} \sim 949$ MeV [8]. The constraints on the radius at $1.4 M_\odot$ come from J0740 + 6620, whereas the maximum observed stable mass at $\sim 2.35 M_\odot$ is measured from PSR J0952-0607.

results obtained; here we restricted ourselves to the analysis of the general problem created by the presence of boundary terms, so we leave the detailed computation of more realistic neutron stars to a future work.

VII. CONCLUSION AND DISCUSSION

In this paper, we have considered the subtle issue of a boundary term in the CS action, that is usually discarded in the literature although the reason for discarding/ignoring it has not been clear. We propose a reason for discarding it, so that the baryon density defined by the topological integral matches with the density that is read off of the tails of the fields at the conformal boundary and enters the thermodynamic relations of energy and pressure. This method can straightforwardly be extended to other observable quantities, at least in principle. We have also shown that when the above matching holds, the CS action is invariant under a gauge transformation that relates the method of describing isospin via rotation of the moduli being the orientation in $\mathrm{SU}(2) \subset \mathrm{SU}(N_f)$, and that of introducing a chemical potential as dictated by the standard holographic dictionary.

As a byproduct of our analysis, we find a very precise way of computing the conditions for the thermodynamic equilibrium, without even evaluating the action on field configurations. We find that the thermodynamics relations can be made sense of both with and without including the mentioned boundary term. Nevertheless, for the above-mentioned formal reasons, we argue that a single form of the CS action is preferred over the others possible; namely, the one dubbed CS bulk.

We have shortly mentioned that the CS term is known in the WSS model to be subtle in the literature already. Elaborating a little on this, Hata and Murata made a proposal for changing the CS term in order to reproduce a constraint on wave functions coming from a Wess-Zumino-Witten term for $\mathrm{SU}(N_f)$ with $N_f > 2$ [27], but their proposal was pointed out by Lau and Sugimoto to be well defined only on a compact 5-manifold and furthermore not being able to reproduce the chiral anomaly of QCD [11] (see Appendix B for an explanation on why the anomaly is safe in our proposal). The proposal of Lau and Sugimoto fixed this issue with a more complicated expression for an alternative CS term. A natural question would be whether the proposal of Lau and Sugimoto would cancel the boundary term, that we are proposing to cancel in the context of the homogeneous nuclear matter in the WSS model. The answer is negative. The proposal of Lau and Sugimoto is very simply put to split the holographic direction up into two parts and add a total derivative term that would correspond to what would come from a gauge transformation of transforming the left-hand gauge fields into the right-hand gauge fields, as well as an integral over a 5-cycle that gives rise to the WZW term. Since we consider only SU(2), the integral of the 5-cycle vanishes and it is straightforward to show that the difference between our CS terms at $z > 0$ and $z < 0$ is not a total derivative. This means that there is no unitary gauge transformation that connects the field configurations, which is not unexpected since the fields are by construction made discontinuous.

Finally, let us observe that the ambiguity arises from the lack of a rigorous derivation of the homogeneous ansatz itself; one can imagine that in the rigorous setup, where an infinite multi-instanton configuration is built, no IR boundary term would be generated due to the smoothness of the fields, the extra dimensions of $\mathbb{R}^3$ would provide the winding number and the limit of very high density would not break this topological feature. When formulating the homogeneous ansatz, the opposite is done: The fields are taken to be in the high density regime, where some kind of "spatial average" has been performed, and the instanton number is then restored by hand by means of a discontinuity. Despite intuitive arguments in favor of the presence of the discontinuity [21], no rigorous derivation with true nonlinear solutions in the large density limit has been performed. We leave such a laborious task for future work.

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APPENDIX A: EQUIVALENCE BETWEEN SU(2) ISOSPIN ROTATION AND EXTERNAL CHEMICAL POTENTIAL

We start with the ansatz (61) and perform a time-dependent gauge transformation in SU(2),

$$\mathcal{A} \to b\mathcal{A}b^{\dagger} - \mathrm{i}b\mathrm{d}b^{\dagger}, \qquad b = b(t) \in \mathrm{SU}(2), \tag{A1}$$

and since we want to eliminate the rotation matrices a from the ansatz, we choose $b = a^{-1}$. Using that $-\mathrm{i}a^{\dagger}\dot{a} = \frac{1}{2}\boldsymbol{\chi}\cdot\boldsymbol{\tau}$, we obtain the gauge fields

$$\mathcal{A}_0 = \left(G + \frac{1}{2}\right)\boldsymbol{\chi}\cdot\boldsymbol{\tau} + \frac{1}{2}\hat{a}_0, \qquad \mathcal{A}_i = -\frac{1}{2}(H\tau^i + L\chi^i),$$
$$\mathcal{A}_z = 0. \tag{A2}$$

The function G still has the UV boundary condition $G(\infty) = 0$, but a simple change of variables is convenient

$$\tilde{G} = G + \frac{1}{2}, \qquad \tilde{G}(\infty) = \frac{1}{2}, \tag{A3}$$

which leads to the theory with an isospin chemical potential turned on,

$$\mathcal{A}_0(\infty) = \frac{1}{2}\boldsymbol{\chi}\cdot\boldsymbol{\tau} = \frac{1}{2}\mu_I\tau^3, \tag{A4}$$

where in the last equality, we have identified the isospin angular velocity as $\boldsymbol{\chi} = (0, 0, \mu_I)$. The τ^3 direction of a spin (or isospin) system is conventional.

Since the difference between using the isospin angular rotation and the isospin chemical potential dictated by the holographic dictionary is merely a gauge transformation; this changes nothing for the Yang-Mills part of the action nor for the equations of motion as they remain unchanged.

Now the CS term is not gauge invariant and therefore the different versions of the CS terms will change differently when performing this gauge transformation. Ideally we want the CS term not to change under this gauge transformation so as to keep the free energy and boundary conditions of the system invariant.

We start with the case of the bulk CS term (69) and find by explicit computations that it does not change under the above-described gauge transformation. This can readily be inferred from Eq. (15), where we can see that the trace of $F_{\alpha_2\alpha_3}F_{\alpha_4\alpha_5}$ is invariant since the non-Abelian field strength transforms covariantly as $F \to bFb^\dagger = a^\dagger F a$, the Abelian field strength $\hat{F}$ does not transform, and the only dependence on $\mathcal{A}_\mu$ is on the Abelian part, which is untouched by an SU(2) gauge transformation. Hence, the action is unchanged, the equations of motion are unchanged, and the IR boundary conditions remain exactly those of Eqs. (75) and (76).

Moving to the case of including the boundary term in the CS action, i.e., using the full CS term, it is clear from Eq. (15) that A dependence is unavoidable. An explicit computation reveals that the full CS term changes by

$$\frac{N_c}{96\pi^2}\int \mathrm{d}^4x \int_0^\infty \mathrm{d}z\,\partial_z(LH^2)\boldsymbol{\chi}^2, \tag{A5}$$

which in turn changes the IR boundary condition for L from Eq. (81) to

$$\left[L'(0) + \frac{3N_c}{32\pi^2\kappa}H^2(0)\left(G(0)+\frac{1}{2}\right)\right]\delta L(0) = 0. \tag{A6}$$

Since the bulk CS term is invariant under the gauge transformation between the isospin rotation and the isospin chemical potential interpretation of the theory, it is in that sense preferred compared to the full CS term.

Finally, let us note that the CS term (84) is also not invariant under the gauge transformation that is necessary for switching from the isospin rotation to the isospin chemical potential realization of the theory. This can be seen from writing down the CS term, in the gauge $\mathcal{A}_z = 0$, in the form

$$\frac{N_c}{32\pi^2}\epsilon^{MNKL}[\mathrm{Tr}(\mathcal{A}_0\mathcal{F}_{MN}\mathcal{F}_{KL}) - \hat{A}_0\mathrm{Tr}(\mathcal{F}_{MN}\mathcal{F}_{KL})]. \tag{A7}$$

Clearly, the first term changes under the gauge transformation as it contains the non-Abelian gauge potential. We find that it gives rise to the CS action (84) for the gauge transformed fields, but changes to

$$-\frac{N_c}{4\pi^2}\boldsymbol{\chi}^2\int \mathrm{d}^4x\int_0^\infty \mathrm{d}z\, H^2L'G, \tag{A8}$$

when transforming back to the original gauge and hence differs from Eq. (84).

APPENDIX B: QCD ANOMALY AND CHERN-SIMONS FORMS

Here we want to show that each of the CS terms we presented in this work reproduces the QCD global anomaly: The discussion follows Appendix C of Ref. [28] closely, but specialized to our notation. Since it can be thought that correctly reproducing the anomaly could be used as a criterion for determining the "physical" CS term, it is useful to show that this is not the case.

First of all, we start by recalling the full CS term,

$$S_{\mathrm{CS}} = \frac{N_c}{24\pi^2}\int_{5D}\omega_5(\mathcal{A}). \tag{B1}$$

with ω_5 being the standard CS five-form given by

$$\omega_5 = \mathrm{Tr}\left[3\hat{A}\wedge F^2 + \hat{A}\wedge\hat{F}^2 + \mathrm{d}\left(\hat{A}\wedge\left(2F\wedge A - \frac{\mathrm{i}}{2}A^3\right)\right)\right], \qquad \omega_5^{\mathrm{SU}(2)} = 0. \tag{B2}$$

A variation of (B1) with gauge function $\alpha(z)$ whose boundary values reduce to the parameters of a chiral transformation (α_L, α_R) is given by the formula,

$$\delta_\alpha S_{\mathrm{CS}} = \frac{N_c}{24\pi^2}\int d^4x\{[\omega_4(\alpha(z), A)]_{0^+}^{+\infty} - [\omega_4(\alpha(z), A)]_{0^-}^{-\infty}\}, \tag{B3}$$

where we introduced the four-form

$$\omega_4(\alpha(z), A(z)) = \mathrm{Tr}\left[\alpha \mathrm{d}\left(A\mathrm{d}A - \frac{\mathrm{i}}{2}A^3\right)\right]. \qquad \text{(B4)}$$

We can rearrange the terms into UV and IR contributions, keeping in mind that the UV values of the gauge fields are holographically mapped to the sources for the four-dimensional theory as $A(+\infty) = l, A(-\infty) = r$. The resulting expression for the variation of the CS term is

$$\delta_\alpha S_{\mathrm{CS}} = \frac{N_c}{24\pi^2}\int d^4x\{[\omega_4(\alpha_L, l) - \omega_4(\alpha_R, r)] - [\omega_4(\alpha(z), A(z))]_{z=0^-}^{z=0^+}\}. \qquad \text{(B5)}$$

The vanishing of the $z = 0$ term together with the IR variation of the Yang-Mills action will determine the IR boundary conditions as we have explained, while the UV boundary term correctly reproduces the QCD anomaly in its symmetric form.

We can now divide the standard five-form ω_5 in two terms as before, a bulk and a boundary term $\omega_5 = \bar{\omega}_5 + \mathrm{d}X$. This separation is completely arbitrary, but to illustrate the procedure we will use the choice we employed throughout the paper,

$$\bar{\omega}_5 \equiv \mathrm{Tr}[3\hat{A} \wedge F^2 + \hat{A} \wedge \hat{F}^2], \qquad \text{(B6)}$$

$$\mathrm{d}X \equiv \mathrm{dTr}\left[\left(\hat{A} \wedge \left(2F \wedge A - \frac{\mathrm{i}}{2}A^3\right)\right)\right]. \qquad \text{(B7)}$$

If we discard the boundary term $\mathrm{d}X$, we are choosing a nonstandard form of the CS term; if the fields were continuous this would not affect any physics, but for our discontinuous fields, we argued that this choice has observable consequences. However, as far as the anomaly is concerned, we can see that everything proceeds in the same fashion; we can perform a variation of $S_{\mathrm{CS}}^{\mathrm{bulk}}$ to obtain,

$$\delta_\alpha S_{\mathrm{CS}}^{\mathrm{bulk}} = \frac{N_c}{24\pi^2}\int d^4x\{[\bar{\omega}_4(\alpha_L, l) - \bar{\omega}_4(\alpha_R, r)] - [\bar{\omega}_4(\alpha, A)]_{z=0^-}^{z=0^+}\}, \qquad \text{(B8)}$$

where we have introduced $\bar{\omega}_4$ that reads

$$\bar{\omega}_4(\alpha, A) = \mathrm{Tr}[3\hat{\alpha} \wedge F^2 + \hat{\alpha} \wedge \hat{F}^2]. \qquad \text{(B9)}$$

Once again, the IR terms will determine the boundary conditions, which will be different than the ones determined before (hence all the different physics), while the UV term reproduces again the anomaly in another form; to see that the two forms are equivalent it is sufficient to note that they only differ by the addition of a local counterterm, the variation $\delta_\alpha X$ of the four-form X. The difference between the two forms of the anomaly is then given by

$$\frac{N_c}{24\pi^2}\int d^4x\{[\omega_4(\alpha_L, l) - \omega_4(\alpha_R, r)] - [\bar{\omega}_4(\alpha_L, l) - \bar{\omega}_4(\alpha_R, r)]\} = \frac{N_c}{24\pi^2}\int d^4x[\delta_{\alpha_L}X(l) - \delta_{\alpha_R}X(r)]. \qquad \text{(B10)}$$

Since X only depends on the sources l, r, which are taken to have vanishing physical value, the form in which we cast the anomaly has no consequences.

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