Toward a theory of Yangians

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We review the main ideas underlying the emerging theory of Yangians—the new type of hidden symmetry in string-inspired models. Their classification by quivers is a far-going generalization of simple Lie algebras classification by Dynkin diagrams. However, this is still a kind of project, while a more constructive approach goes through toric Calabi-Yau spaces, related supersymmetric systems, and the Duistermaat-Heckman or equivariant integrals between the fixed points in the Atiyah-Drinfeld-Hitchin-Manin (ADHM)-like moduli spaces. These fixed points are classified by crystals (Young-type diagrams), and Yangian generators describe "instanton" transitions between them. Detailed examples will be presented elsewhere.

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I. INTRODUCTION AND DISCUSSION

One of the main goals of string theory [1] is to extend the notion of symmetries, to make them responsible for most of dynamical properties of physical systems. This requires a vast extension of the Lie-Noether concept, and such new symmetries are often referred to as *hidden*. We already know a number of examples—from *integrability* of generic nonperturbative (functional) integrals [2–4] and *superintegrability* of many of them [5] to more exotic properties like *categorical symmetries* [6–8] or *differential expansions* [9–11] and their *defects* [12–14] for Wilson loop averages in Chern-Simons theory and knot polynomials. These hidden symmetries clearly have relation to Lie algebras, but they still remain somewhat indirect and not fully understood.

Much closer to direct deformation of Lie structure are *Yangians* [15–20]—and, naturally, they have received more and more attention in recent years [21–25]. Though original motivation, which we begin from in this introduction, is very much in the spirit of other hidden symmetries, it is clear that Cartan-like formal description, similar to that of simple Lie algebras, is also available [18,19,26,27] (see Sec. IV A), where conventional Dynkin diagrams are generalized to quivers (see Sec. III A). However, even in this case there is still a big way toward understanding of representation theory of Yangians, and more constructive "physical" approaches continue to dominate [19,20]—as we discuss in Secs. III and V and summarize in Fig. 1.

In this short paper, we outlined the main ideas used in description of Yangians and their representations. These ideas are four different points of view on the same quiver Yangian, that correspond to boxes $Y_1(\mathfrak{Q})$, $Y_2(\mathfrak{Q})$, $Y_3(\mathfrak{Q})$, and $Y_4(\mathfrak{Q})$ in Fig. 1 and Secs. II–V, respectively. These structures arrive from seemingly different sources:

(i) $Y_1(\mathfrak{Q})$ is an algebra of orthogonal polynomials of an infinite set of commuting variables p_k called "times." We could form an algebra of raising and lowering operators acting on polynomials by multiplication $p_k\cdot$ and derivatives $\partial/\partial p_k$.

(ii) $Y_2(\mathfrak{Q})$ acts on equivariant fixed points of quiver varieties (in some cases corresponding to the instanton moduli spaces). Operators are explicitly constructed as matrices whose entries represent equivariant Duistermaat-Heckman integrals over homomorphism loci for pairs of fixed points.

(iii) Of $Y_3(\mathfrak{Q})$ one might think of as a bootstrap of an algebra having modules with vectors labeled by Young diagrams, or plain partitions and 3D molten crystals in more generic terms. On this route, one might aim to design a system of rational matrix coefficients for adding or subtracting boxes raising or lowering operators, so that the latter have proper zeros and poles preventing one from misplacing the box.

(iv) Finally, $Y_4(\mathfrak{Q})$ emphasizes the role of an instanton algebra in a quantum field theory (QFT). One might consider a vector space spanned by quasiclassical

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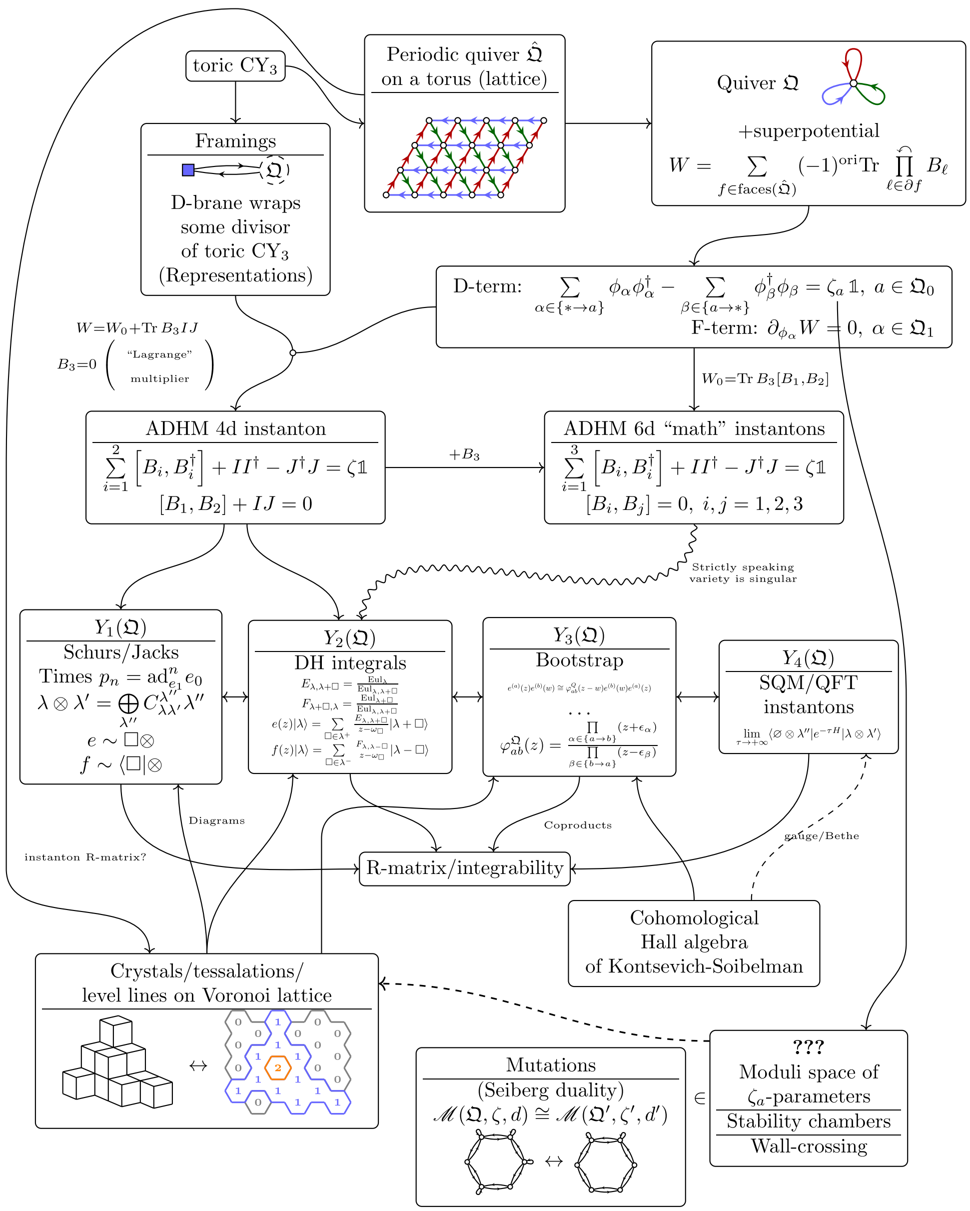


FIG. 1. Approaches to the quiver Yangians.

wave functions for a set of classical vacua as a module for the instanton algebra where the matrix elements are calculated by instanton amplitudes between corresponding pairs of vacua. We believe that $Y_4(\mathfrak{Q})$ may be reconstructed from the instanton algebra of a quiver-$\mathfrak{Q}$ (supersymmetric) gauge theory.

However, resulting algebras appear to be equivalent. And this equivalence ("quadrality") lies at the heart of a study program proposed in this note.

Detailed examples and further review of existing literature will be presented elsewhere. Further deformations to Ding-Iohara-Miki (DIM) algebras [28–30] and associated brane and network models [31] can use the same patterns, along with the matrix model [32] and conformal field theory (free field) [23,33–35] techniques.

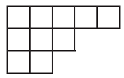
II. YANGIAN FROM SCHUR-JACK FAMILY OF POLYNOMIALS

In this section, we discuss the first appearance of quiver Yangian algebras through the family of Schur-Jack polynomials that correspond to rectangular $Y_1(\mathfrak{Q})$ in Fig. 1.

Schur polynomials S_R form a distinguished basis in the space of homogeneous polynomials of variables p_a, $a = 1, 2, 3, \ldots, \infty$. One of the crucial properties of the Schur polynomials is that they form a character ring of GL_N:

$$R\otimes R' = \sum_{R''} N_{RR'}^{R''} R'' \Rightarrow S_R\cdot S_{R'} = \sum_{R''} N_{RR'}^{R''} S_{R''}, \tag{2.1}$$

where $N_{RR'}^{R''}$ are the famous Littlewood-Richardson coefficients and R, R', and R'' are the Young diagrams:

$R =$  , $\quad R = [R_1, R_2, \ldots, R_{l(R)}],$ where $R_i \in \mathbb{Z}$ and $R_1 \geqslant R_2 \geqslant \ldots \geqslant R_{l(R)} > 0.$ (2.2)

The numbers $p(n)$ of Young diagrams with n boxes are nicely collected in the following generating function:

$$\prod_{k=1} \frac{1}{1-q^k} = \sum_{n=0} p(n) q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots. \tag{2.3}$$

The orthogonality property for Schur polynomials can be represented as the Cauchy identity:

$$\sum_R S_R\{p\} S_{R'}\{\bar{p}\} = \exp\left\{\sum_{k=1}^{\infty} \frac{p_k \bar{p}_k}{k}\right\}. \tag{2.4}$$

An avatar of Schur-Weyl duality between linear and symmetric groups [36] can be formulated in terms of commuting set of cut-and-join operators [37]:

$$\hat{W}_\Delta S_R = \phi_R(\Delta) S_R, \tag{2.5}$$

where $\phi_R(\Delta)$ are properly normalized characters of symmetric groups. The simplest nontrivial cut-and-join operator $\hat{W}_{[2]}$ actually defines the Schur functions as the set of own eigenfunctions. In other words, all the Schur polynomials are encoded in the following nice looking operator:

$$\hat{W}_{[2]} = \frac{1}{2} \sum_{a,b=1}^{\infty} \left[ab p_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} + (a+b) p_a p_b \frac{\partial}{\partial p_{a+b}} \right]. \tag{2.6}$$

An important fact for our presentation is that the commutative algebra of cut-and-join operators can be extended to the full noncommutative $W_{1+\infty}$ algebra [38],[1] that is, a special case of affine Yangian $Y(\widehat{\mathfrak{gl}}_1)$ [16,17]. Remarkably, this Yangian is generated by multiple commutators of the small set of simple operators:

(i) operator of multiplication on the first p variable p_1:

$$p_1 \cdot S_R = \sum_{\square \in R^+} S_{R+\square}, \tag{2.7}$$

(ii) derivative with respect to p_1:

$$\frac{\partial}{\partial p_1} S_R = \sum_{\square \in R^-} S_{R-\square}, \tag{2.8}$$

(iii) diagonal operator $\hat{W}_{[2]}$:

$$\hat{W}_{[2]} S_R = \left(\sum_{\square \in R} j_\square - i_\square \right) \cdot S_R, \tag{2.9}$$

where $i_\square$ and $j_\square$ are the two coordinates of the box in the Young diagram. The notation $R^\pm$ means the positions of the boxes outside (inside) the Young diagram where one can add (remove) a box in a way that $R \pm \square$ is still a Young diagram.

Miraculously, almost the whole picture lifts to the level of β deformation [44]. The Schurs S_R become Jack polynomials J_R [45]:

$$J_R \cdot J_{R'} = \sum_{R''} \mathcal{N}_{RR'}^{R''}(\beta) J_{R''}. \tag{2.10}$$

The orthogonality property survives β deformation:

$$\sum_R \frac{J_R\{p\} J_{R'}\{\bar{p}\}}{||J_R||^2} = \exp\left\{\sum_{k=1}^{\infty} \frac{\beta \cdot p_k \bar{p}_k}{k}\right\}, \tag{2.11}$$

and cut-and-join operator undergoes simple deformation:

$$\hat{W}_{[2]}^{\beta} = \frac{1}{2} \sum_{a,b=1}^{\infty} \left[ab p_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} + \beta(a+b) p_a p_b \frac{\partial}{\partial p_{a+b}} \right] + \frac{(1-\beta)}{2} \sum_{a=1}^{\infty} (a-1) a p_a \frac{\partial}{\partial p_a}. \tag{2.12}$$

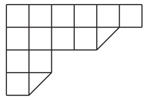
In the β-deformed case, this operator corresponds to the integrable Hamiltonian of a quantum many-body system [46,47]. Triple of operators generate the algebra affine Yangian $Y(\widehat{\mathfrak{gl}}_1)$:

$$e_0 = p_1, \qquad \psi_3 = \hat{W}_{[2]}^{\beta}, \qquad f_0 = -\frac{\partial}{\partial p_1}. \tag{2.13}$$

The names of these operators (e_n, ψ_n, f_n) are borrowed from the theory of simple Lie algebras where we have a triple of (rising operator, Cartan generator, lowering operator) for any node in the Dynkin diagram. Namely, e_n, f_n operators are rising and lowering operators, since they add or remove boxes of the Young diagram. Operators ψ_n are similar to Cartan generators and act diagonally in representations.

The same story about cut-and-join operators and Young diagrams translates to other Yangians. We provide here an example of affine super-Yangian $Y(\widehat{\mathfrak{gl}}_{1,1})$ that possesses a semi-Fock representation, where vectors are enumerated by super-Young diagrams [48]:

[1]Another avatar for these algebras are so-called vertex operator algebras. On recent developments, see [39–43], and references therein.

$R =$  $, \qquad R = [R_1, R_2, \ldots, R_{l(R)}], \text{ where } R_i \in \frac{1}{2}\mathbb{Z} \text{ and } R_1 \geqslant R_2 \geqslant \ldots \geqslant R_{l(R)} > 0. \qquad (2.14)$

In case R_i and R_{i+1} are both half-integers, then the condition $R_i > R_{i+1}$ is satisfied. The numbers $p_s(n, m)$ of super-Young diagrams with n boxes and m half-boxes are collected in the following generating function:

$$\prod_{k=1} \frac{1 + \eta \cdot q^{k-1}}{1 - q^k} = \sum_{n=0} p_s(n, m) q^n \eta^m = 1 + \eta + q + 2\eta q + (2q^2 + \eta^2 q) + 4\eta q^2 + \cdots. \tag{2.15}$$

As one can see from the generating function, in the case of super-Yangian we need two sets of variables: bosonic p variables p_a (denominator) and fermionic or Grassmann variables θ_a (numerator). If one assigns degrees $\deg(p_a) = a$ and $\deg(\theta_a) = a - \frac{1}{2}$, then the numbers of homogeneous polynomials of fixed degree are described by the above generating function.

In the case of super-Yangian, the corresponding quiver has two nodes [$(+)$ node and $(-)$ node]; therefore, the minimal set of operators is twice bigger than in the previous case of $Y(\widehat{\mathfrak{gl}}_1)$. Rising and lowering operators:

$$\begin{aligned} e_0^+ &= \theta_1, \qquad f_0^+ = \frac{\partial}{\partial \theta_1}, \\ e_0^- &= \sum_k p_k \frac{\partial}{\partial \theta_k}, \qquad f_0^- = \epsilon_1 \epsilon_2 \sum_k k \theta_k \frac{\partial}{\partial p_k}, \end{aligned} \tag{2.16}$$

and two super-cut-and-join operators $\hat{W}^+$ and $\hat{W}^-$ [48]:

$$\begin{aligned} \hat{W}^\pm = &\frac{1}{2} \sum_{a,b=1}^{\infty} \left[ab p_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} - \epsilon_1 \epsilon_2 (a+b) p_a p_b \frac{\partial}{\partial p_{a+b}} \right] \\ &+ \sum_{a,b=1}^{\infty} \left(b - \frac{1}{2} \pm \frac{1}{2} \right) \cdot \left[a \theta_{a+b} \frac{\partial^2}{\partial p_a \partial \theta_b} - \epsilon_1 \epsilon_2 p_a \theta_b \frac{\partial}{\partial \theta_{a+b}} \right] \\ &+ \frac{\epsilon_1 + \epsilon_2}{2} \cdot \sum_{a=1}^{\infty} \left[a \left(a - \frac{1}{2} \mp \frac{1}{2} \right) p_a \frac{\partial}{\partial p_a} \right. \\ &\left. + (a-1) \left(a - \frac{1}{2} \pm \frac{1}{2} \right) \theta_a \frac{\partial}{\partial \theta_a} \right]. \end{aligned} \tag{2.17}$$

These super-cut-and-join operators define a new set of polynomials $\mathcal{S}_R$, that we call super-Schur polynomials, as the set of own eigenfunctions:

$$\hat{W}^\pm \mathcal{S}_R = w_R^\pm \mathcal{S}_R. \tag{2.18}$$

These polynomials have two free parameters $\epsilon_{1,2}$ as in the case of usual Jack polynomials, where β is a free parameter.

Remarkably, super-Schur polynomials also form a orthogonal basis and obey the modified Cauchy identity:

$$\sum_R \frac{\mathcal{S}_R\{p, \theta\} \cdot \mathcal{S}_R\{\bar{p}, \bar{\theta}\}}{||\mathcal{S}_R||^2} = \exp\left\{ \sum_{k=1}^{\infty} \frac{p_k \bar{p}_k}{k} + \theta_k \bar{\theta}_k \right\}. \tag{2.19}$$

III. YANGIAN FROM QUIVER REPRESENTATIONS

Yangians arise as the special algebras that act on the space of Bogomol'nyi-Prasad-Sommerfield (BPS) states in type IIA string theory with a system of D-branes compactified on the toric Calabi-Yau threefold [15]. BPS moduli space can be described in terms of the *quiver and superpotential*—they are extracted directly from the toric Calabi-Yau [19,49]. What is important for our presentation is that the Yangian acts between different fixed points (of torus action) in the space of quiver representations (= BPS moduli space). The transition amplitudes are given by the equivariant integrals.

A. Definitions: Quiver

Generic quiver data $\mathfrak{Q}$:

(1) Quiver—an oriented graph—a collection of nodes $\mathfrak{Q}_0+$ a collection of arrows $\mathfrak{Q}_1$.

(2) Superpotential W. We could denote it as $\mathfrak{Q}_2$, since it is made of closed loops in quiver.

(3) Equivariant parameters $\epsilon_{\alpha \in \mathfrak{Q}_1} \in \mathbb{C}$ assigned to each arrow in $\mathfrak{Q}$ and constrained by a condition that sums of ϵ_α's over any arrow loop contributing to W is zero. So that the equivariant weight of W is zero.

Also we adopt the following notations:

(i) $\{a \to b\}$—a set of arrows in $\mathfrak{Q}$ flowing from node a to node b;

(ii) $|a \to b|$—a number of arrows in $\mathfrak{Q}$ flowing from node a to node b.

B. Quiver representation and torus fixed points

(i) Quiver representation assigns a vector space to each quiver node and a matrix B_i to each quiver arrow. The dimensions of different vector spaces in quiver nodes could be different; therefore, in general, B_i are rectangular matrices. The important point is that the matrices B_i are considered up to the change of basis in the quiver nodes.

(ii) Superpotential is defined from the periodic lattice corresponding to torus CY_3 by the following rule:

$$W = \mathrm{Tr}\sum_{\text{faces}}(-)^{\text{orientation}}\prod_{\text{arrows}} B_i. \tag{3.1}$$

Then we impose so-called F-term equations on matrices B_i:

$$\frac{\partial W}{\partial B_i} = 0. \tag{3.2}$$

These equations correspond to part of the famous ADHM equations in the case of $\mathrm{CY}_3 = \mathbb{C}^3$. The other part of ADHM equations (so-called D-term equations) involve additional Fayet-Illiopoulos parameters ζ_a, on which the moduli spaces of solutions can depend in a nontrivial way. In this paper, we consider only the case $\zeta_a > 0$, and these D-term equations are irrelevant for our presentation of the simplest Yangian representations. However, for the other choices of parameters ζ_a the corresponding Yangians can be different but related by mutations.

(iii) Framing is an additional data, which can be represented by additional nodes added to the quiver, and it defines a representation of Yangian. In particular, it can distinguish between $2d$ Fock and $3d$ MacMahon representations.

(iv) Torus action scales the matrices B_i:

$$B_i \to e^{\epsilon_i} \cdot B_i. \tag{3.3}$$

Vectors of the Yangian representation correspond to fixed points of torus action in the space of quiver representations. In other words, vectors of Yangian representation correspond to a set of matrices B_i (up to change of basis in quiver nodes) obeying F-term equations + fixed point constraints.

These fixed points of the torus action are labeled by crystals = Young-type diagrams. We further denote these diagrams λ, and the vector space of the Yangian representation will be space of $|\lambda\rangle$. These diagrams will be constructed from building blocks—atoms $\square_a$ of different types, that are enumerated by the quiver nodes $a \in \mathfrak{Q}_0$.

C. Yangian algebra

To construct the Yangian algebra we go through the following algorithm.

(1) First, for each quiver node a we introduce two fields $e^{(a)}(z)$ and $f^{(a)}(z)$. These fields will have a simple interpretation in terms of the diagrams—they add and remove atoms $\square_a$ by the following rule:

$$e^{(a)}(z)|\lambda\rangle = \sum_{\square_a \in \lambda^+} \frac{E_{\lambda,\lambda+\square_a}}{z - \omega(\square_a)} |\lambda + \square_a\rangle, \tag{3.4}$$

$$f^{(a)}(z)|\lambda\rangle = \sum_{\square_a \in \lambda^-} \frac{F_{\lambda,\lambda-\square_a}}{z - \omega(\square_a)} |\lambda - \square_a\rangle. \tag{3.5}$$

There are several comments on the above formulas. As was mentioned in the previous section on Schur polynomials, the notation $\lambda^\pm$ means the set of atoms outside (inside) the diagram λ where we can add (remove) the atom $\square_a$ of type a. The function $\omega(\square_a)$ is a weighted coordinate of the $\square_a$. The weight of the x_i coordinate is ϵ_i; therefore,

$$\omega(\square_a) = \sum_i x_i(\square_a) \cdot \epsilon_i. \tag{3.6}$$

According to this procedure, the field actions $e^{(a)}(z)|\lambda\rangle$ and $f^{(a)}(z)|\lambda\rangle$ can have only simple poles and the residues exactly match the vectors $|\lambda \pm \square_a\rangle$ with one atom added or removed.

(2) Second, we should choose the coefficients $E_{\lambda,\lambda+\square_a}$ and $F_{\lambda,\lambda-\square_a}$. This choice is important, because it controls the commutation relations of the resulting fields $e^{(a)}(z)$ and $f^{(a)}(z)$. In our presentation, *the coefficients are extracted from the quiver representation* described in the previous Sec. III B. The extraction procedure goes in four steps.

(a) For diagrams λ and $\lambda + \square_a$, we find sets of matrices B_i and B'_i satisfying F-term equations + fixed point constraints.

(b) We find a subspace in the space of small perturbations ΔB_i and $\Delta B'_i$ around fixed point solutions B_i and B'_i that obey F-term relations up to the change of basis in the quiver nodes. From this operation, we derive the normalization rules:

$$\langle\lambda||\lambda\rangle = \mathrm{Eul}_\lambda, \tag{3.7}$$

where the new object Eul_λ is the product of all the weights of all matrix coefficients of ΔB_i. The terminology Eul comes from the fact that the above prescription implicitly corresponds to equivariant (Duistermaat-Heckman) integration. In that formalism, the answer for the integral is given by the sum of the different fixed points of the torus action, and each contribution is the Euler class of the tangent space at the fixed point.

(c) At the final step, we compute the last ingredients $\mathrm{Eul}_{\lambda,\lambda+\square_a}$ to compute coefficients $E_{\lambda,\lambda+\square_a}$:

$$E_{\lambda,\lambda+\square_a} = \frac{\mathrm{Eul}_\lambda}{\mathrm{Eul}_{\lambda,\lambda+\square_a}}. \tag{3.8}$$

To compute coefficients $\mathrm{Eul}_{\lambda,\lambda+\square_a}$, we should connect two solutions corresponding to diagrams λ and $\lambda + \square_a$ by a linear (generally irreversible) matrix τ (a zero-dimensional analog of the singular Hecke modification [50] shifting the Chern classes of a bundle):

$$B_i' \cdot \tau = \tau \cdot B_i. \tag{3.9}$$

The map τ is continued to the level of small perturbations and selects a smaller subspace of ΔB_i and $\Delta B_i'$ that obey (3.9) up to the first order. The products of all weights in this smaller subspace defines $\mathrm{Eul}_{\lambda,\lambda+\square_a}$.

The coefficients $\mathrm{Eul}_{\lambda,\lambda+\square_a}$ needed for $F_{\lambda,\lambda-\square_a}$ are computed in the same way but for another pair of diagrams:

$$F_{\lambda,\lambda-\square_a} = \frac{\mathrm{Eul}_\lambda}{\mathrm{Eul}_{\lambda,\lambda-\square_a}}. \tag{3.10}$$

According to the above procedure, the representation of the Yangian is constructed via simple calculations with matrices. Some details and examples of calculations can be found in [51–53] and Appendix C in [20].

IV. THE FORMAL DEFINITION OF YANGIAN DIRECTLY FROM QUIVER

A. Yangian algebra

Yangian $Y_{\mathfrak{Q}}$ is entirely and directly defined by quiver $\mathfrak{Q}$.

For each vertex a in $\mathfrak{Q}$ we associate the generators: positive and negative "simple roots" $e^{(a)}(z)$ and $f^{(a)}(z)$ and related "Cartan elements" $\psi^{(a)}(z)$:

$$e^{(a)}(z) = \sum_{n=0}^{\infty} \frac{e_n^{(a)}}{z^{n+1}}, \qquad \psi^{(a)}(z) = \sum_{n=-\infty}^{\infty} \frac{\psi_n^{(a)}}{z^{n+1}},$$
$$f^{(a)}(z) = \sum_{n=0}^{\infty} \frac{f_n^{(a)}}{z^{n+1}}. \tag{4.1}$$

$Y_{\mathfrak{Q}}$ is a superalgebra. Therefore, the root generators acquire definite $\mathbb{Z}_2$-parity P: bosonic $P = 0$ or fermionic $P = 1$. Parity of node a is defined according to the following formula:

$$P_a = (|a \to a| + 1) \mod 2. \tag{4.2}$$

Cartan generators are all bosonic.

Supercommutation relations are defined by the maps and arrows of $\mathfrak{Q}$:

$$\begin{aligned}
&\psi^{(a)}(z)\psi^{(b)}(w) \cong \psi^{(b)}(w)\psi^{(a)}(z),\\
&e^{(a)}(z)e^{(b)}(w) \cong (-1)^{P_aP_b}\varphi_{ab}^{\mathfrak{Q}}(z-w)e^{(b)}(w)e^{(a)}(z),\\
&f^{(a)}(z)f^{(b)}(w) \cong (-1)^{P_aP_b}\varphi_{ab}^{\mathfrak{Q}}(z-w)^{-1}f^{(b)}(w)f^{(a)}(z),\\
&\psi^{(a)}(z)e^{(b)}(w) \cong \varphi_{ab}^{\mathfrak{Q}}(z-w)e^{(b)}(w)\psi^{(a)}(z),\\
&\psi^{(a)}(z)f^{(b)}(w) \cong \varphi_{ab}^{\mathfrak{Q}}(z-w)^{-1}f^{(b)}(w)\psi^{(a)}(z),\\
&[e^{(a)}(z), f^{(b)}(w)\} \cong -\delta_{ab}\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w},
\end{aligned} \tag{4.3}$$

where

(i) $[x, y\} \coloneqq xy - (-1)^{P_xP_y}yx$ is a supercommutator;
(ii) sign $\cong$ equates Taylor series expansion at the points $z = \infty$, $w = \infty$ on both sides up to the terms of the form $z^{n\geq 0}w^m$ and $z^n w^{m\geq 0}$;
(iii) quiver bond factors:

$$\varphi_{ab}^{\mathfrak{Q}}(u) \coloneqq \frac{\prod_{\alpha\in\{a\to b\}}(u+\epsilon_\alpha)}{\prod_{\beta\in\{b\to a\}}(u-\epsilon_\beta)}. \tag{4.4}$$

Remark 1. Cubic and higher order Serre relations require additional consideration, left beyond the scope of the present text. A conjecture for Serre relations in the case of a generic quiver is given in [54]. Cases $Y(\widehat{\mathfrak{gl}}_{m|n})$ are described in [55]. Some suggestions for $Y(K_{\mathbb{P}^2})$ and $Y(K_{\mathbb{P}^1\times\mathbb{P}^1})$ (not based on Lie algebras) are given in Appendix D in [56].

Remark 2. Canonically, Yangians with different shifts (when one allows for nonzero negative Cartan element modes $\psi^{(a)}_{-\mathfrak{s}\leq n<0} \neq 0$) are considered to be different algebras [57,58]. However, physically, one might try to classify them as simply different representations. Both treatments have their own peculiarities. On one hand, mere framing modification leaving the unframed quiver diagram intact introduces shifts [26,30]. And the naive nth tensor power on the Yangians could be also installed in the physical picture as a framing modification; thus, mixing together Yangians with different shifts in a single object of a tensor category is not forbidden. Moreover, for the quiver Yangians not based on the affine Lie superalgebras, the shift $\mathfrak{s}$ is infinity; formally speaking, it cannot be bounded above for infinite MacMahon-like crystal representations [59]. On the other hand, the negative Cartan element modes seem somewhat decoupled from the algebra, since in the Yangian relation $[e_n^{(a)}, f_m^{(b)}\} = \delta_{ab}\psi_{n+m}^{(a)}$, n, $m \geq 0$ only non-negative modes are generated. This induces certain obstacles in a search for the universal (representation-independent) coproduct structure that is a homomorphism of algebras.

V. YANGIAN AND QUANTUM FIELD THEORY

Finally, we could name another seemingly different source of the Yangian algebra as a multidimensional supersymmetric QFT. The crucial difference of this approach with that mentioned above is that we do incorporate an effective theory of D-branes in the type IIA string theory compactified on the toric Calabi-Yau. The effective QFT in this picture is related to the ADHM construction and instanton equations in diverse dimensions only implicitly after a careful analysis of D-brane charges and theory moduli [60].

Nevertheless, this physical approach to a mathematical problem of finding algebraic structures reveals richness and fruitfulness repeating in some aspects the story about the physical avatar of the knot invariant construction problem—the 3D Chern-Simons theory and the boundary conformal Wess-Zumino-Witten theory [61].

Here, we mention this story in a seemingly inverse order. The Yangian algebra admits a Hopf algebra structure; in particular, one could define the coproduct even for the affine cases [22,62,63]. Having the coproduct, Δ we can restore the R matrix—the intertwining operator permuting the order of multipliers in a tensor square:

$$\Delta_{12}R = R\Delta_{21}. \tag{5.1}$$

Both the coproduct and the R matrix depend on the complex spectral parameter having the same nature as the generating parameter in the series (4.1) or the equivariant weight in (3.4). R matrix is a solution of the Yang-Baxter equation ensuring that the representation tensoring structure is indeed associative.

On the other hand, the R matrix, a transfer matrix, may be derived independently of the Yangian structure starting with spin chains and other integrable models like the Calogero model [46]. The relation between the QFTs and integrable models is well known in the literature [64,65] as the gauge or Bethe correspondence.

The physical approach allows one to argue the appearance of such nontrivial relations as the Yang-Baxter equation in a quite elegant way: Two sides of the equality are just different yet homotopic paths in the parameter (moduli) space of the QFT; then, the equality of two sides is solely an absence of hysteresis in the theory. The absence of hysteresis for theories in question is a rather natural property induced by supersymmetry. On the other hand, this approach makes the other avatars of the Yangians mentioned above rather obscure. This story becomes even more intriguing [59] when a naive attempt to restore the whole chain of relations underlying the gauge or Bethe correspondence between QFT vacua and solutions to the Bethe equations including the construction of the coproduct and the R matrix fails.

Nevertheless, the R-matrix evolution (actually, in both cases of the Yangian and quantum algebras and their mixtures like quantum toroidal algebras) in the QFT can be expanded in a series [66–71] of transitions through instanton or soliton jumps between overlapping levels of effective QFTs. This observation allows us to expect that the algebra of BPS solitons and instantons reproduces the Yangian in an inexplicit way; in other words, we expect to have the following relations:

$$E_{\lambda,\lambda+\square} = \lim_{T\to\infty} \langle \lambda + \square | e^{\frac{\mathrm{i}}{\hbar}TH} | \lambda \rangle, \tag{5.2}$$

for some effective Hamiltonian H.

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