Origin of higher Schwarzians

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(Received 6 February 2024; accepted 7 March 2024; published 29 March 2024)

In this paper we analyze higher Schwarzians and show that they are closely related to the nonlinear realization of the Virasoro algebra. The Goldstone fields of such a realization provide a new set of $SL(2,\mathbb{R})$ invariant higher Schwarzians that are deeply related to those of Aharonov [Duke Math. J. **36**, 599 (1969)], Tamanoi [Math. Ann. **305**, 127 (1996)], and Bonora-Matone [Nucl. Phys. **B327**, 415 (1989); Int. J. Mod. Phys. A **10**, 289 (1995); arXiv:hep-th/9306150]. A minor change of the coset space parametrization leads to a new set of $SL(2,\mathbb{R})$ noninvariant higher Schwarzians now related to the Schippers [Proc. Am. Math. Soc. **128**, 3241 (2000)] and Bertilsson [Ark. Mat. **36**, 255 (1998)] Schwarzians.

DOI: 10.1103/PhysRevD.109.065029

I. INTRODUCTION: STANDARD SCHWARZIAN WITHIN NONLINEAR REALIZATION

The Schwarzian derivatives or simply Schwarzians have been known for a long time. Their story goes back to Lagrange who introduced a version of the Schwarzians (see, e.g., [1] for a survey). Later on, this derivative appeared in projective and conformal geometry as well as in many other contexts in mathematics and mathematical physics.

The Schwarzian derivative S_t is defined by the relation

$$S_t=\frac{\dddot{t}}{\dot{t}}-\frac{3}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2=\frac{d}{d\tau}\left(\frac{\ddot{t}}{\dot{t}}\right)-\frac{1}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2=\dot{T}_t-\frac{1}{2}T_t^2,\qquad \dot{t}=\partial_\tau t. \tag{1.1}$$

Here, $T_t=\ddot{t}/\dot{t}$ is the pre-Schwarzian derivative of $t(\tau)$. Applications of the Schwarzian derivative are especially connected with problems of univalent analytic functions [2–7]. Despite its well-established role in mathematical physics, the Schwarzian derivative remains somewhat mysterious. It is not a derivative, exactly, but what is it? The most known property of the Schwarzian derivative is its invariance under $SL(2,\mathbb{R})$ transformations, acting on t, i.e.

$$\text{if } t'=\frac{at+b}{ct+d},\qquad ad-bc\neq 0\quad \text{then } S_{t'}=S_t. \tag{1.2}$$

The infinitesimal form of the condition (1.2) is

$$\delta t=a_{-1}+a_0 t+\frac{a_1}{2}t^2,\qquad S_t[t+\delta t]=S_t[t]. \tag{1.3}$$

The parameters a_{-1}, a_0, a_1 correspond to translation, dilatation, and conformal boost, respectively. The group $SL(2,\mathbb{R})$ with the commutative relations

$$\mathrm{i}[L_n,L_m]=(n-m)L_{n+m},\qquad n,m=-1,0,1 \tag{1.4}$$

is just one dimensional conformal group.

It is interesting, but the pre-Schwarzian derivative is not invariant under $SL(2,\mathbb{R})$ transformations,

$$T_t[t+\delta t]=T_t[t]+a_1\dot{t}. \tag{1.5}$$

As we will see below (1.10), the $\dot{t}$ starts from a constant, and therefore the pre-Schwarzian derivative T_t is shifted by a constant under the transformation $\delta t=a_1 t^2$. Thus, T_t is the Goldstone field accompanying the spontaneous breaking of the conformal boost. On the other hand, we have

$$\delta \mathrm{Log}[\dot{t}]=a_0+a_1 t, \tag{1.6}$$

and thus $\mathrm{Log}[\dot{t}]$ is the Goldstone field for the spontaneously broken dilatation. Finally, note that the translations $\delta t=a_{-1}$ are also spontaneously broken with $t(\tau)$ being the corresponding Goldstone field.

Within a nonlinear realization of spontaneously broken $sl(2,\mathbb{R})$ symmetry, the natural place for the Schwarzian is

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to be associated with one of the $sl(2,\mathbb{R})$ Cartan forms $\omega_{-1}, \omega_0, \omega_1$ defined in the standard way as

$$g^{-1}dg = \mathrm{i}\omega_{-1}L_{-1} + \mathrm{i}\omega_0 L_0 + \mathrm{i}\omega_1 L_1, \quad g = e^{\mathrm{i}tL_{-1}} e^{\mathrm{i}uL_0} e^{\frac{\mathrm{i}}{2}z_1 L_1}. \tag{1.7}$$

Explicitly, these forms read

$$\omega_{-1} = e^{-u}dt, \qquad \omega_0 = du - e^{-u}z_1 dt,$$
$$\omega_1 = \frac{1}{2}\left[dz_1 - z_1 du + \frac{1}{2}e^{-u}z_1^2 dt\right]. \tag{1.8}$$

All of these forms are invariant with respect to $sl(2,\mathbb{R})$ symmetry (1.3). Thus, one can introduce invariant time τ and reduce the number of independent fields (inverse Higgs phenomenon [8]) by imposing the following constraints:

$$e^{-u}dt = d\tau, \qquad \omega_0 = du - e^{-u}z_1 dt = 0. \tag{1.9}$$

As a result of these constraints (1.9) and treating all fields as dependent on the invariant time, τ, one can get

$$\dot{t} = e^{u(\tau)} \ (a), \qquad z_1 = \dot{u} = \frac{\ddot{t}}{\dot{t}} = T_t \ (b). \tag{1.10}$$

Substituting expressions in (1.10) into form ω_1, we have

$$\omega_1 = \frac{1}{2}d\tau\left[\frac{d}{d\tau}\left(\frac{\ddot{t}}{\dot{t}}\right) - \frac{1}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2\right] = \frac{1}{2}S_t d\tau. \tag{1.11}$$

Thus, as was expected, the Schwarzian S_t appears as the $d\tau$ projection of the Cartan form ω_1 associated with the generator of the conformal boost L_1.

This approach to Schwarzians was initiated by Galajinsky in [9]. Later on, it was generalized to the cases of supersymmetric Schwarzians in [10–15]. Being very productive, this approach can tell nothing about higher Schwarzians: $sl(2,\mathbb{R})$ invariant higher Schwarzians of Aharonov [2] as well as higher Schwarzians of Tamanoi [3] were completely out of game. The nature of $sl(2,\mathbb{R})$ noninvariant higher Schwarzians of Schippers [4] and Bertilsson [5] was also unclear. The goal of the present paper is to provide a unified description of $sl(2,\mathbb{R})$ invariant higher Schwarzians as Goldstone fields for spontaneously broken Virasoro symmetry spanned by the generators $\{L_n, n \geq -1\}$. From this point of view, noninvariant higher Schwarzians are just specific deformations of invariant ones by T_t dependent terms.

II. HIGHER SCHWARZIANS

In 1969, Aharonov gave definitions of higher-order analogues of the Schwarzian derivative [2]. Later on, Tamanoi introduced another set of higher order Schwarzian derivatives [3]. Finally, Kim and Sugawa derived relations between the Aharonov invariants and Tamanoi's Scwarzian derivatives [7]. Another, more physical definition of the higher Schwarzians has been formulated by Bonora and Matone [16].

All these definitions lead to higher order Schwarzian derivatives invariant with respect to $sl(2,R)$ transformations [(1.2) and (1.3)] and both definitions are nongeometric ones. Using the results from the previous section, one can propose a purely geometric definition of higher Schwarzians. The basic idea comes from the transformation of the Schwarzian under transformation

$$\delta t = \frac{a_2}{6}t^3 \to \delta S_t = a_2 t'[\tau]^2 = a_2 + \cdots. \tag{2.1}$$

Thus, we see that the Schwarzian itself behaves like a Goldstone boson for partially broken symmetry (2.1). It is not too difficult to verify that transformation (2.1) together with (1.3) form the centerless subalgebra of the Virasoro algebra

$$\mathrm{i}[L_n, L_m] = (n-m)L_{n+n}, \qquad n, m \geq -1. \tag{2.2}$$

It is a natural guess to associate the Schwarzian with the Goldstone fields for transformation (2.1) generated by the operator L_2. With such identification, one can expect that higher Schwarzians will appear as the Goldstone fields associated with the higher generators $L_n, n \geq 3$. Technically, the realization of this idea consists in three steps:

(i) Step 1

One has to choose a proper parametrization of the group element corresponding to the algebra (2.2)

$$g = e^{\mathrm{i}tL_{-1}} e^{\mathrm{i}uL_0} e^{\frac{\mathrm{i}}{2}z_1 L_1} \prod_{i=2}^{\infty} e^{\frac{\mathrm{i}}{(i+1)!}z_i L_i}. \tag{2.3}$$

The order of the first three exponents is fixed by the invariance of the Goldstone field[1] z_1 under the transformation generated by L_0 (dilatations) (1.5).

(ii) Step 2

Next, one has to calculate the Cartan forms

$$\Omega = g^{-1}dg = \mathrm{i}\sum_{n=-1}\omega_n L_n. \tag{2.4}$$

[1]We remember that the Goldstone z_1 has to be identified with the pre-Schwarzian T_t.

Let us list several first Cartan forms:

$$\begin{aligned}
\omega_{-1} &= e^{-u}dt,\\
\omega_0 &= du - z_1 e^{-u}dt,\\
\omega_1 &= \frac{1}{2}\left[dz_1 - z_1 du - e^{-u}\left(z_2 - \frac{1}{2}z_1^2\right)dt\right],\\
\omega_2 &= \frac{1}{6}[dz_2 - 2z_2 du - e^{-u}(z_3 - 2z_1 z_2)dt,],\\
\omega_3 &= \frac{1}{24}[dz_3 - 2z_2 dz_1 + 2du z_1 z_2 - 3du z_3 - e^{-u}(z_4 - 3z_1 z_3 - z_2^2 + z_1^2 z_2)dt],\\
\omega_4 &= \frac{1}{120}\left[dz_4 - 5z_1 z_3 + 5du z_1 z_3 - 4du z_4 - e^{-u}\left(z_5 - 4z_1 z_4 - 5z_2 z_3 + \frac{5}{2}z_1^2 z_3\right)dt\right], \qquad \text{etc.}
\end{aligned} \tag{2.5}$$

(iii) Step 3

Finally, one has to introduce the invariant time τ and express all the Goldstone fields u, z_n in terms of $t[\tau]$ and its derivatives. Moreover, these expressions have to be invariant with respect to the whole subgroup of the Virasoro group (2.2). All of these can be achieved by imposing the following constraints that generalize the constraints (1.9):

$$\omega_{-1} = d\tau, \qquad \omega_n = 0, \qquad n \geq 0. \tag{2.6}$$

As a result of constraints (2.6), we obtain the following expressions for the parameters u, z_n:

$$\begin{aligned}
u &= \mathrm{Log}(t'),\\
z_1 &= \frac{t''}{t'} = T_t,\\
z_2 &= \frac{t'''}{t'} - \frac{3}{2}\left(\frac{t''}{t'}\right)^2 = S_t,\\
z_3 &= S_t',\\
z_4 &= S_t'' - S_t^2,\\
z_5 &= S_t^{(3)} - 2S_t S_t',\\
z_6 &= S_t^{(4)} - 2S_t S_t'' - \frac{9}{2}(S_t')^2 - \frac{10}{3}S_t^3,\\
z_7 &= S_t^{(5)} - 2S_t S_t^{(3)} - 11 S_t' S_t'' - 10 S_t^2 S_t', \qquad \text{etc.}
\end{aligned} \tag{2.7}$$

One can compare our set of higher Schwarzians in (2.7) with Aharonov's invariants [2] and Tamanoi Schwarzians [3]

$$\begin{array}{rl}
\text{Aharonov's invariants}\, \psi_k & \text{Tamanoi's Schwarzians}\, s_k\\
\frac{t'[\tau]}{(t[\tau+w]-t[\tau])} = \frac{1}{w} - \sum\limits_{k=0}\psi_{k+1}\frac{w^k}{(k+2)!} & \frac{t'[\tau](t[\tau+w]-t[\tau])}{\frac{1}{2}t''[\tau](t[\tau+w]-t[\tau])+t'[\tau]^2} = \sum\limits_{k=0} s_k \frac{w^{k+1}}{(k+1)!}\\
\psi_1 = \frac{t''}{t'} = T_t & s_0 = 1, s_1 = 0\\
\psi_2 = S_t & s_2 = S_t\\
\psi_3 = S_t' & s_3 = S_t'\\
\psi_4 = S_t'' + \frac{2}{3}S_t^2 & s_4 = S_t'' + 4S_t^2\\
\psi_5 = S_t''' + 3S_t S_t' & s_5 = S_t''' + 13 S_t S_t'\\
\psi_6 = S_t^{(4)} + 5S_t S_t'' + \frac{17}{4}(S_t')^2 + \frac{4}{3}S_t^3 & s_6 = S_t^{(4)} + 19 S_t'' S_t + 13(S_t')^2 + 34 S_t^3\\
\psi_7 = S_t^{(5)} + \frac{22}{3}S_t S_t^{(3)} + 17 S_t' S_t'' + \frac{40}{3}S_t^2 S_t' & s_7 = S_t^{(5)} + 26 S_t^{(3)} S_t + 45 S_t' S_t'' + 228 S_t^2 S_t'.
\end{array} \tag{2.8}$$

Thus, we see that the difference in Schwarzians appears already at the fourth order. The same story happened in the case of Bonora-Matone Scwarzians [16]:

$$
\begin{aligned}
&\text{Bonora-Matone Schwarzians } S_{2k+1}\\
S_{2k+1} &= \frac{1}{k}(t'[\tau])^k \partial_\tau\left(\frac{1}{t'[\tau]}\partial_\tau\left(\frac{1}{t'[\tau]}\cdots\partial_\tau(t'[\tau])^k\right)\right)\\
S_2 &= S_t\\
S_3 &= S_t'\\
S_4 &= S_t'' + \frac{3}{2}S_t^2\\
S_5 &= S_t''' + 8S_tS_t'\\
S_6 &= S_t^{(4)} + \frac{31}{2}S_tS_t'' + 13(S_t')^2 + \frac{45}{4}S_t^3\\
S_7 &= S_t^{(5)} + 26S_tS_t''' + 59S_t'S_t'' + 144S_t^2S_t', \qquad \text{etc.}
\end{aligned} \tag{2.9}
$$

Despite the fact that we did not know the generic expressions for the Goldstone bosons z_n, we can relate z_n with ψ_n, s_n and/or S_n. Several first such relations read

$$
\begin{aligned}
z_2 &= \psi_2 = s_2 = S_2,\\
z_3 &= \psi_3 = s_3 = S_3,\\
z_4 &= \psi_4 - \frac{5}{3}\psi_2^2 = s_4 - 5s_2^2 = S_4 - \frac{5}{2}S_2^2,\\
z_5 &= \psi_5 - 5\psi_2\psi_3 = s_5 - 15s_2s_3 = S_5 - 10S_2S_3,\\
z_6 &= \psi_6 - \frac{35}{4}\psi_3^2 - 7\psi_2\psi_4 = s_6 - \frac{35}{2}s_3^2 - 21s_2s_4 + \frac{140}{3}s_2^2 = S_6 - \frac{35}{2}S_2S_4 - \frac{35}{2}S_3^2 + \frac{35}{3}S_2^3,\\
z_7 &= \psi_7 - 28\psi_3\psi_4 - \frac{28}{3}\psi_2\psi_5 + \frac{70}{3}\psi_2^2\psi_3 = s_7 - 56s_3s_4 - 28s_2s_5 + 350s_2^2s_3\\
&= S_7 - 28S_2S_5 - 70S_3S_4 + 175S_3S_2^2, \qquad \text{etc.}
\end{aligned} \tag{2.10}
$$

This means that if we insert in the parametrization of the group element g (2.3) the expressions for z_n in terms of ψ_n, s_n and/or S_n, then the same constraints (2.4) lead to proper expressions (2.8). One should note that nonlinear expressions in (2.10) include only ψ_n, ψ_n and/or S_n but not their derivatives. So, in some sense, these are canonical transformations of variables, although nonlinear.

To complete this section, note that our constraints (2.6) are invariant with respect to the whole Virasoro algebra (2.2). If we consider the left multiplication of our group element g (2.3) by the element $g_n = e^{a_n L_n}$,

$$
g_n g = g', \tag{2.11}
$$

we will obtain

$$
\delta_n t = a_n t^{n+1}, \qquad \delta_n S_t = a_n n(n^2-1)Y_n, \qquad \delta Y_k = a_n(k+2n)Y_{k+n}, \tag{2.12}
$$

where

$$
Y_k = t^{k-2}(t')^2. \tag{2.13}
$$

It is clear that the variation of the higher Schwarzians will provide us with a new set of expressions that will contain both S and Y^n and their derivatives. The usefulness of this full Virasoro symmetry is not clear for us yet.

III. $SL(2,\mathbb{R})$ NONINVARIANT SCHWARZIANS

In addition to the Aharonov [2], Tamanoi [3] and Bonora-Mattone [16] versions of higher Schwarzians, Schippers [4] and Bertilsson [5] proposed another definition of Schwarzian derivatives of higher order. They have nice properties, but however they do not possess $SL(2,\mathbb{R})$ invariance. These new higher Schwarzians are defined as follows.

Schippers set

The Schippers set of higher Schwarzians is defined as follows [4]:

$$\sigma_n := \sigma'_{n-1} - (n-2)\frac{t''}{t'}\sigma_{n-1}, \qquad \sigma_3 = S_t. \tag{3.1}$$

A few nontrivial Schwarzians have the following form:

$$\begin{aligned}
\sigma_3 &= S_t,\\
\sigma_4 &= S'_t - 2T_tS_t,\\
\sigma_5 &= S''_t - 2S_t^2 - 5T_tS'_t + 5T_t^2S_t,\\
\sigma_6 &= S_t^{(3)} - 9S'_tS_t - 9T_tS''_t + 18T_tS_t^2 + \frac{45}{2}T_t^2S'_t - 15T_t^3S_t,\\
\sigma_7 &= S_t^{(4)} - 18S''_tS_t - 9(S'_t)^2 + 18S_t^3 - 14T_tS_t^{(3)} + 126T_tS_tS'_t + 63T_t^2S''_t - 126T_t^2S_t^2 - 105T_t^3S'_t + \frac{105}{2}T_t^4S_t,\\
\sigma_8 &= S_t^{(5)} - 32S'''_tS_t - 36S'_tS''_t + 180S_t^2S'_t - 20T_tS^{(4)} + 360T_tS_tS''_t + 180T(S'_t)^2 - 360T_tS_t^3\\
&\quad + 140T_t^2S^{(3)} - 1260T_t^2S_tS'_t - 420T_t^3S''_t + 840T_t^3S_t^2 + 525T_t^4S'_t - 210T_t^5S_t, \qquad \text{etc.}
\end{aligned} \tag{3.2}$$

Bertilsson set

The Bertilsson variant of higher Schwarzians is defined as follows [5]:

$$\mathcal{S}_n := -\frac{2}{n}(t')^{\frac{n}{2}}\frac{\partial^{n+1}}{\partial\tau^{n+1}}(t')^{-\frac{n}{2}}. \tag{3.3}$$

Several first members of this set read

$$\begin{aligned}
\mathcal{S}_0 &= T_t,\\
\mathcal{S}_1 &= S_t,\\
\mathcal{S}_2 &= S'_t - 2T_tS_t,\\
\mathcal{S}_3 &= S''_t - \frac{7}{2}S_t^2 - 5T_tS'_t + 5T_t^2S_t,\\
\mathcal{S}_4 &= S_t^{(3)} - 17S'_tS_t - 9T_tS''_t + \frac{45}{2}T_t^2S'_t + 34T_tS_t^2 - 15T_t^3S_t,\\
\mathcal{S}_5 &= S_t^{(4)} - \frac{67}{2}S''_tS_t - 22(S'_t)^2 + \frac{241}{4}S_t^3 - 14T_tS_t^{(3)} + \frac{511}{2}T_tS_tS'_t + 63T_t^2S''_t - \frac{511}{2}T_t^2S_t^2 - 105T_t^3S'_t + \frac{105}{2}T_t^4S_t,\\
\mathcal{S}_6 &= S_t^{(5)} - 58S_t^{(3)}S_t - 95S'_tS''_t + 676S_t^2S'_t - 20T_tS^{(4)} + 712T_tS_tS''_t + 475T_t(S'_t)^2 - 1352T_tS_t^3\\
&\quad + 140T_t^2S^{(3)} - 2730T_t^2S_tS'_t - 420T_t^3S''_t + 1820T_t^3S_t^2 + 525T_t^4S'_t - 210T_t^5S_t, \qquad \text{etc.}
\end{aligned} \tag{3.4}$$

A. Our modified set

It is completely clear from the explicit form of the Schippers (3.2) and Bertilsson (3.4) Schwarzians that $SL(2,\mathbb{R})$ symmetry breaking is caused by the presence of the pre-Schwarzian derivative $T_t = \ddot{t}/\dot{t}$ in these versions of Schwarzians. Within our approach, we can simulate such behavior as follows:

(i) Step 1

We choose the following parametrization of the Virasoro group element:

$$\hat{g} = e^{\mathrm{i}tL_{-1}}e^{\mathrm{i}uL_0}\prod_{i=2}^{\infty}e^{\frac{\mathrm{i}}{(i+1)!}\hat{z}_iL_i}e^{\frac{\mathrm{i}}{2}\hat{z}_1L_1}. \tag{3.5}$$

The order of the exponents forced breaking of the invariance of higher Goldstone fields $\hat{z}_k, k \geq 2$ under transformations generated by L_1 and, therefore, under $SL(2,\mathbb{R})$.

(ii) Step 2

Next, one has to calculate the Cartan forms

$$\Omega = \hat{g}^{-1} d\hat{g} = \mathrm{i} \sum_{n=-1} \omega_n L_n. \tag{3.6}$$

(iii) Step 3

Finally, one has to introduce invariant time τ and impose constraints on the Cartan forms (2.6)

$$\hat{\omega}_{-1} = d\tau, \qquad \hat{\omega}_n = 0, \qquad n \geq 0. \tag{3.7}$$

As a result, we will obtain the following expressions for the parameters $u, \hat{z}_n$:

$$\begin{aligned}
u &= \mathrm{Log}(t'),\\
\hat{z}_1 &= T_t,\\
\hat{z}_2 &= S_t,\\
\hat{z}_3 &= S'_t - 2T_t S_t,\\
\hat{z}_4 &= S''_t - S_t^2 - 5T_t S'_t + 5T_t^2 S_t,\\
\hat{z}_5 &= S_t^{(3)} - 2S'_t S_t - 9T_t S''_t + 4T_t S_t^2 + \frac{45}{2} T_t^2 S'_t - 15 T_t^3 S_t,\\
\hat{z}_6 &= S_t^{(4)} - 2S''_t S_t - \frac{9}{2}(S'_t)^2 - \frac{10}{3} S_t^3 - 14 T_t S_t^{(3)} + 28 T_t S_t S'_t + 63 T_t^2 S''_t - 28 T_t^2 S_t^2 - 105 T_t^3 S'_t + \frac{105}{2} T_t^4 S_t,\\
\hat{z}_7 &= S_t^{(5)} - 10 S_t^2 S'_t - 11 S'_t S''_t - 2 S_t^{(3)} S'_t - 20 T_t S^{(4)} + 40 T_t S_t S''_t + 55 T_t (S'_t)^2 + 20 T_t S_t^3\\
&\quad + 140 T_t^2 S^{(3)} - 210 T_t^2 S_t S'_t - 420 T_t^3 S''_t + 140 T_t^3 S_t^2 + 525 T_t^4 S'_t - 210 T_t^5 S_t, \qquad \text{etc.}
\end{aligned} \tag{3.8}$$

Thus, we see that changing the parametrization of the same coset (3.5) leads to the transformation of $SL(2,\mathbb{R})$ invariant Schwarzians (2.7) into noninvariant Schwarzians (3.8). It should be noted here that these three sets of Schwarzians, (3.2), (3.4), and (3.8), are related quite similarly to those in (2.10):

$$\begin{aligned}
\hat{z}_2 &= \sigma_3 = \mathcal{S}_1,\\
\hat{z}_3 &= \sigma_4 = \mathcal{S}_2,\\
\hat{z}_4 &= \sigma_5 + \sigma_3^2 = \mathcal{S}_3 + \frac{5}{2}\mathcal{S}_1^2,\\
\hat{z}_5 &= \sigma_6 + 7\sigma_3\sigma_4 = \mathcal{S}_4 + 15\mathcal{S}_1\mathcal{S}_2,\\
\hat{z}_6 &= \sigma_7 + 16\sigma_3\sigma_5 + \frac{9}{2}\sigma_4^2 + \frac{32}{3}\sigma_3^3 = \mathcal{S}_5 + \frac{63}{2}\mathcal{S}_1\mathcal{S}_3 + \frac{35}{2}\mathcal{S}_2^2 + \frac{140}{3}\mathcal{S}_1^3,\\
\hat{z}_7 &= \sigma_8 + 25\sigma_4\sigma_5 + 30\sigma_3\sigma_6 + 130\sigma_3^2\sigma_4 = \mathcal{S}_6 + 84\mathcal{S}_2\mathcal{S}_3 + 56\mathcal{S}_1\mathcal{S}_4 + 650\mathcal{S}_1^2\mathcal{S}_2, \qquad \text{etc.}
\end{aligned} \tag{3.9}$$

It is important that nonlinear transformations from one set to another include only Schwarzians themselves, without any derivative. Thus, one can claim that the three series of Schwarzians generate the same ring of differential operators.

IV. CONCLUSION

In this paper, a physical view was proposed on the origin of higher Schwarzians, treating them as Goldstone fields associated with the generators of the Virasoro algebra. The motivation for such an association comes from the transformation properties of higher Schwarzians under Virasoro symmetry. Using this fact, the standard nonlinear realization of the Virasoro symmetry was constructed, equipped with the constraints that did the following:

(i) The first constraint

$$\omega_{-1} = d\tau$$

introduced new time τ, completely inert with respect to Virasoro symmetry (this step is mainly the same as in the papers [9–11,13–15]).

(ii) Next we imposed the constraints that nullified all Cartan forms of the Virasoro group,

$$\omega_n = 0, \qquad n \geq 1.$$

These constraints express all Goldstone fields of our nonlinear realization in terms of the unique field $t(\tau)$ (associated with the generator L_{-1}) and its derivatives. The expressions for higher Goldstone fields provide a new set of higher Schwarzians.

We still have no generic expressions for our higher Schwarzians. Instead, we presented the relations of our few first Schwarzians (2.7) with those from the Aharonov, Tamanoi, and Bonora-Matone sets (2.10).

We also explicitly demonstrated that minor change in the coset space parametrization:

$$e^{\mathrm{i}tL_{-1}} e^{\mathrm{i}uL_0} e^{\frac{\mathrm{i}}{2}z_1 L_1} \prod_{i=2}^{\infty} e^{\frac{\mathrm{i}}{(i+1)!} z_i L_i} \Rightarrow e^{\mathrm{i}tL_{-1}} e^{\mathrm{i}uL_0} \prod_{i=2}^{\infty} e^{\frac{\mathrm{i}}{(i+1)!} z_i L_i} e^{\frac{\mathrm{i}}{2}z_1 L_1}$$

together with the same constraints (1.9) leads to $SL(2,\mathbb{R})$ noninvariant Schwarzians (3.8). We established the relations of a few first such Schwarzians with Schippers's and Bertilsson's Schwarzians (3.9).

Thus, all basic variants of higher Schwarzians have deep relations with a set of Schwarzians that follow from a nonlinear realization of Virasoro symmetry. It is clear that the construction of supersymmetric generalizations of higher Schwarzians becomes now almost straightforward along the line similar to that considered in [10–15]. First, one has to solve the purely technical task of finding generic expressions for our higher Schwarzians.

Finally, note that in the recent paper [17] two possibility to retrieve the Bertilsson higher Schwarzians [5] from the physical systems have been discussed. However, there is no direct relation between this paper and ours, because the ℓ-conformal Galilei symmetry considered there acts, in our formulation, on the variable τ leaving all Schwarzians invariant.

ACKNOWLEDGMENTS

This research was partially funded by the Ministry of Science and Higher Education of Russia, Government Order for 2023–2025, Project No. FEWM-2023-0015 (TUSUR). The Author is grateful to M. Matone for drawing his attention to the papers [16].

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