$U(N)$ torus link invariants in the large N limit from the matrix model approach

Archana Maji,[1,*] Kushal Chakraborty,[2,3,†] Suvankar Dutta,[2,‡] and P. Ramadevi[1,§]

[1]*Department of Physics, Indian Institute of Technology Bombay, Mumbai 400076, India*
[2]*Indian Institute of Science Education and Research Bhopal, Bhopal bypass, Bhopal 462066, India*
[3]*Mandelstam Institute for Theoretical Physics, School of Physics, University of the Witwatersrand, Johannesburg WITS 2050, South Africa*

(Received 1 January 2024; accepted 16 February 2024; published 18 March 2024)

In this paper we study $U(N)$ colored HOMFLY-PT polynomials of torus links in the double scaling limit (polynomial variable $q \to 1$, $N \to \infty$ keeping q^N fixed). We show that, in this limit, the colored HOMFLY-PT polynomial of any $(L\alpha, L\beta)$ torus link can be expressed in terms of the colored HOMFLY-PT polynomial of (L, L) torus link. Using the connection between matrix models and the Chern-Simons field theoretic invariants, we show that the colored torus link invariants are uniquely expressed in terms of connected correlation functions of operators in $U(N)$ matrix model. We determine the leading and subleading contribution to some of the correlators at large N from the matrix model approach and find that they match exactly with those obtained from the corresponding colored HOMFLY-PT polynomials.

DOI: 10.1103/PhysRevD.109.065021

I. INTRODUCTION

The famous discovery of Jones polynomial [1,2] by Vaughan Jones in 1984 led to a rejuvenation of the mathematical theory of knots and links. A knot $\mathcal{K}$ is an embedding of a circle S^1 inside a three-manifold M whereas a link $\mathcal{L}$ is a collection of two or more knots linked in a nontrivial way. Two knots $\mathcal{K}_1$ and $\mathcal{K}_2$ are considered to be equivalent if one can be deformed to another through any of the three Reidemeister moves. The central problem of knot theory is to classify different knots and links. To address this classification problem there comes various polynomial invariants viz. Alexander polynomial [3], Jones polynomial, HOMFLY-PT polynomial [4,5], etc., in the order of increasing sophistication.

Knot theory has become of spectacular interest to physicists ever since Witten's seminal work giving an intrinsically three-dimensional definition of Jones polynomial from the perspective of quantum field theory [6]. The Chern-Simons theory, a three-dimensional gauge theory with an underlying gauge group G, provides a natural framework to compute the topological invariants of three-manifolds and of knots and links embedded inside those three-manifolds. In particular, the correlation function of the observables of Chern-Simons theory, viz. Wilson loop operators, is nothing but the Jones polynomial when the gauge group at consideration is $SU(2)$ and the representations put on the Wilson loops are that of the fundamental representation $(R \equiv \square)$ of $SU(2)$. Likewise, with the fundamental representations of $G = SU(N)$ and $SO(N)$ we obtain HOMFLY-PT polynomial and Kauffman polynomial [7], respectively. If other higher-dimensional representations are placed on the Wilson loops then we obtain the corresponding colored polynomials.

In this work we confine ourselves to a class of links called torus links that can be embedded on the surface of a torus in such a way that they do not cross over themselves. Any L component torus link can be denoted as $(L\alpha, L\beta)$ where $L\alpha$ and $L\beta$ denote the number of times the link wraps around the meridian and the longitude of the torus, respectively. Here α and β are coprime to each other. These torus links can be obtained as a closure of $L\alpha$ strand braid with $(L\alpha - 1)L\beta$ crossings. We will study these torus link invariants in the double scaling limit of the Chern-Simons theory associated with the gauge group $U(N)$. This limit is defined as follows:

$$N, k \to \infty \quad \text{such that } \lambda = \frac{N}{N+k} = \text{fixed}, \qquad (1.1)$$

*archana_phy@iitb.ac.in
†kushal16@iiserb.ac.in
‡suvankar@iiserb.ac.in
§ramadevi@iitb.ac.in

where k is the coupling constant of the Chern-Simons theory. In terms of the variable $q = \exp(\frac{2\pi i}{k+N})$ the above limit can be rephrased as

$$q \to 1, N \to \infty, \quad \text{such that } q^N = \text{fixed}. \tag{1.2}$$

By two of the authors, it has been shown that in this limit the invariants for a specific set of torus links viz. $(2, 2m)$ torus links can be expressed in terms of (2, 2) torus link (also called the Hopf link) invariant [8]. Interestingly, this result can be generalized for any arbitrary L component torus link. In particular, we show that in the double scaling limit we can express the $(L\alpha, L\beta)$ torus link invariant in terms of (L, L) torus link invariant.

Although the partition function of Chern-Simons theory can be obtained for some three-manifolds, the calculation of correlation functions is difficult in general. However, in the double-scaling limit one can use the saddle point approximation to compute some of the correlation functions. In particular, the correlation function that corresponds to the (2, 2) torus link invariant was computed in Ref. [8] by mapping the problem to that of an incompressible fluid with initial and final fluid densities being related to the representations placed on the two component knots. In this paper we further study the correlation functions corresponding to the (L, L) torus link in the double scaling limit, for any arbitrary L. We obtain an exact expression for the leading contribution of such link invariants using the saddle point approximation [8,9]. We validate this result by explicitly computing the colored HOMFLY-PT polynomials for the (3, 3) torus link carrying same/different representations on its three components.

It is well-known in knot theory, that all knot invariants reduce to unknot invariants in the double-scaling limit. Similarly, any L component link invariant reduces to the product of L unknot invariants. The intertwining between the components is not captured in this limit. These observations are also seen in the matrix model results using saddle point approximation. Specifically, the leading-order contribution to any n-point correlator in the large-N limit is the product of n one-point correlators. Hence, we need to investigate the subleading corrections to capture the intertwining between the component knots constituting the link. The subleading corrections to the knot invariants are captured by the connected piece of correlation functions in the matrix model approach. We use the techniques developed in Ref. [10,11] to compute the connected pieces. We then validate the results for the subleading contributions of Hopf link invariants for some specific representations placed on the components of the link. Further, we show that any knot or link invariant modulo the leading contribution (i.e., subtracting the leading contribution from the link invariant) can be uniquely written in terms of connected correlation function of operators in $U(N)$ matrix model. These connected correlators are related to the reformulated invariants (which are written in terms of the colored HOMFLY-PT polynomials) [12].

The organisation of the paper is as follows: In Sec. II we review how to obtain the correlation function of links in $U(N)$ Chern-Simons theory. In Sec. III we write down explicitly the $U(N)$ invariant of $(2, 2m)$ torus link embedded inside three manifold S^3. In Sec. IV A we review the double scaling limit of $(2, 2m)$ torus link invariant [8]. Section IV B contains the generalization of this result for any L component $(L\alpha, L\beta)$ torus link invariant. In Sec. V we discuss how one can express the (L, L) torus link invariant in terms of a correlation function involving L Schur polynomials. An analytic expression for the leading contribution of any (L, L) torus link invariant is obtained in the large-N limit using the saddle point approximation. Particularly we validate this result by tabulating the colored HOMFLY-PT polynomial of the (3, 3) torus link. Subleading contributions to the link invariants are discussed in Sec. VI. In that same section we also argue how the torus knot/link invariants with arbitrary representations can be expressed in terms of connected correlation functions of operators in $U(N)$ matrix model. Finally, in Sec. VII we summarize our results and elaborate on the future outlook.

II. LINK INVARIANTS IN $U(N)$ CHERN-SIMONS THEORY

The three-dimensional Chern-Simons theory on a manifold M corresponding to the gauge group $U(N)$ can be written as a sum of $SU(N)$ and $U(1)$ Chern-Simons actions [13],

$$\begin{aligned} S &= \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) + \frac{k_1}{4\pi}\int_M \mathrm{Tr}(B\wedge dB) \\ &\equiv \frac{k}{4\pi}\int_M d^3x\epsilon^{\mu\nu\rho}\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) + \frac{k_1}{4\pi}\int_M d^3x\epsilon^{\mu\nu\rho}\mathrm{Tr}(B_\mu\partial_\nu B_\rho), \end{aligned} \tag{2.1}$$

where A, B are the gauge connections for the gauge groups $SU(N)$, $U(1)$, respectively, and k, k_1 are the respective coupling constants. Thus, the partition function of $U(N)$ Chern-Simons theory obtained, by integrating over gauge-inequivalent connections, is a product of two partition functions,

$$Z^{U(N)}[M] = \int_M [\mathcal{D}A][\mathcal{D}B]e^{iS} \equiv Z^{SU(N)}[M] \times Z^{U(1)}[M]. \tag{2.2}$$

As the action is explicitly metric independent, the corresponding partition function is a topological invariant of the three-manifold M. The other topological invariants associated with links can be described by expectation value of Wilson-loop operators in Chern-Simons theory. For an oriented knot $\mathcal{K}$ in an irreducible representation $\mathcal{R}$, the Wilson-loop operator is defined as

$$W_{\mathcal{R}}(\mathcal{K}) = \left(\mathrm{Tr}_R \mathbb{H}^{(A)}[\mathcal{K}]\right)\left(\mathrm{Tr}_n \mathbb{H}^{(B)}[\mathcal{K}]\right), \tag{2.3}$$

where

$$\mathbb{H}^{(A)}[\mathcal{K}] = \mathcal{P}\left[\exp \oint_{\mathcal{K}} A\right] \quad \text{and} \quad \mathbb{H}^{(B)}[\mathcal{K}] = \mathcal{P}\left[\exp \oint_{\mathcal{K}} B\right] \tag{2.4}$$

are the holonomies of the $SU(N)$ and $U(1)$ gauge connections, respectively. The symbol $\mathcal{P}$ denotes path ordering. Note that the representation $\mathcal{R} \in U(N)$ involves representation $R \in SU(N)$ and charge $n \in U(1)$. Given (R, n), we can define $U(N)$ representation $\mathcal{R}$ in the following way:

$$n_i = \begin{cases} l_i + s, & i = 1, \ldots, N-1 \\ s, & i = N \end{cases} \tag{2.5}$$

where l_i (n_i) is the number of boxes in the ith row corresponding to the Young diagram of $SU(N)$ $(U(N))$ representation R $(\mathcal{R})$ and s is any integer. The total number of boxes of representation $\mathcal{R}$, known as the charge of $U(1)$ theory, is given by $n = \sum_{i=1}^{N} n_i$. The relation between $SU(N)$ Chern-Simons theory to $\mathfrak{su}(N)_k$ Wess-Zumino Witten conformal field theory implies that the l_i's of the $SU(N)$ representations must obey $l_i \le k$. Such representations are known as integrable representations. With suitable choice of s, we can shift the range of n_i appropriately.

If instead we have a link $\mathcal{L}$ made up of L number of component knots $\mathcal{K}_a$ each carrying representation $\mathcal{R}_a$ of $U(N)$ gauge group, then the Wilson loop operator of $\mathcal{L}$ is

$$W_{\mathcal{R}_1, \cdots \mathcal{R}_L}[\mathcal{L}] = \prod_{a=1}^{L} W_{\mathcal{R}_a}[\mathcal{K}_a]. \tag{2.6}$$

The correlation function of this topological operator yields the link invariant of $\mathcal{L}$ corresponding to $U(N)$ Chern-Simons theory,

$$\begin{aligned} \mathcal{W}^{\mathcal{L}}_{\mathcal{R}_1,\ldots,\mathcal{R}_L}(M,k,k_1) &= \langle W_{\mathcal{R}_1,\ldots,\mathcal{R}_L}[\mathcal{L}] \rangle \\ &= \frac{1}{Z(M)} \int_M [\mathcal{D}A][\mathcal{D}B] \left(\prod_{a=1}^{L} W_{\mathcal{R}_a}[\mathcal{K}_a]\right) e^{iS} \\ &= \mathcal{W}^{\mathcal{L}}_{n^{(1)},\ldots,n^{(L)}}(M,k_1)\mathcal{W}^{\mathcal{L}}_{R_1,\ldots,R_L}(M,k). \end{aligned} \tag{2.7}$$

These link invariants can be computed using one of the two framings:

(i) Standard framing where the self-linking numbers of the component knots are zero;
(ii) Vertical framing where the self-linking number of any component knot is equal to its crossing number.

In fact, the link invariants in standard framing are unchanged under all the three Reidemeister moves and are called ambient isotopy invariants. However, the link invariants in vertical framing, referred to as framed link invariants, are unchanged only under the Reidemeister moves II and III. These framed link invariants are known in the knot theory literature as regular isotopy invariants.

In this work, we will confine ourselves to framed links $[\mathcal{L}; f]$ where the framing $f = \{f_1, \ldots, f_L\}$ is a set of L integers, each element of the set denoting the self-linking number (or framing number) of the component knot.

Let us consider the three manifold M to be S^3. The $U(1)$ invariant of $[\mathcal{L}; f]$ is given as

$$\mathcal{W}^{[\mathcal{L};f]}_{n^{(1)},\ldots,n^{(L)}}(S^3, k_1) = \exp\left(\frac{i\pi}{k_1}\sum_{a=1}^{L} f_a (n^{(a)})^2\right) \times \exp\left(\frac{i\pi}{k_1}\sum_{a\neq b}^{L} (lk_{ab}) n^{(a)} n^{(b)}\right), \tag{2.8}$$

where lk_{ab} denotes the linking number between the components $\mathcal{K}_a$ and $\mathcal{K}_b$. For the polynomial invariant of $U(N)$ Chern-Simons theory to be a function only of two variables

$$q = \exp\left(\frac{2\pi i}{k+N}\right) \quad \text{and} \quad v = q^N, \tag{2.9}$$

in the $U(1)$ invariant (2.8) we need to make the following replacement for the charge $n^{(a)}$ and the coupling constant k_1 [13,14].

$$n^{(a)} \to \frac{n^{(a)}}{\sqrt{N}}, \qquad k_1 \to k + N. \tag{2.10}$$

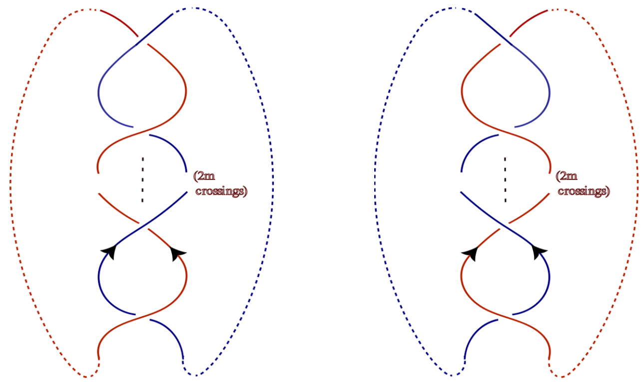
We will elaborate the salient features of computing the $(L\alpha, L\beta)$ torus link invariants in the following section.

III. TORUS LINK INVARIANTS IN $U(N)$ CHERN-SIMONS THEORY

Torus links with L components are characterised by two integers $(L\alpha, L\beta)$ where α and β are coprime to each other. These links can be obtained from the closure of a braid with $L\alpha$ strands. The braid word for a right-handed $(L\alpha, L\beta)$ torus link, in terms of the braid generators b_i, is

$$\mathcal{B}(L\alpha, L\beta) = (b_1 b_2 \ldots b_{L\alpha-1})^{L\beta}. \tag{3.1}$$

The corresponding mirror image of this link is left handed and its braid word becomes $(b_1^{-1} b_2^{-1} \ldots b_{L\alpha-1}^{-1})^{L\beta}$. As an illustration, we show in Fig. 1 a two component ($L = 2$) torus link (with $\alpha = 1, \beta = m$) and its mirror image, where m is any positive integer. The dotted lines in this figure indicate the closure of the two-strand braid. For clarity, we now discuss the colored HOMFLY-PT polynomial for these $(2, 2m)$ torus links before getting into arbitrary L component torus links.



(a) Right handed (linking number $+m$)

(b) Left handed (linking number $-m$)

FIG. 1. $(2, 2m)$ torus link; closure of two parallelly oriented strands with $2m$ crossings.

A. $(2,2m)$ torus-link invariant

Given a braid word (3.1), there is a systematic procedure of directly writing down the invariants of links that are obtained from the closure of such braids [15]. Basically, we use the basis where the braiding generators are diagonal. As any two consecutive generators b_i and $b_{i\pm1}$ do not commute, we need to perform a suitable unitary transformation to go from one basis to another. The invariant of the right-handed $(2, 2m)$ torus link (refer to Fig. 1a), closure of a simple two-strand braid with braid word b_1^{2m}, involves the eigenvalues only of the braiding generator b_1. In vertical framing, the eigenvalues for the right-handed braiding generator b_1 on two parallelly oriented strands carrying representations R_1 and R_2 of $SU(N)$ gauge group is given by [16]

$$\lambda_{R_t}^{(+)}(R_1, R_2) = (-1)^{\epsilon_{R_1}+\epsilon_{R_2}-\epsilon_{R_t}} q^{-\left(\frac{C_{R_1}+C_{R_2}}{2}\right)+\frac{C_{R_t}}{2}}, \quad R_t \in R_1 \otimes R_2, \tag{3.2}$$

where $\epsilon_{R_1}, \epsilon_{R_2}, \epsilon_{R_t} = \pm 1$, q is defined in Eq. (2.9) and C_{R_a}, the quadratic Casimir of representation R_a is defined as

$$C_{R_a} = -\frac{(l^{(a)})^2}{2N} + \frac{1}{2}\sum_j l_j^{(a)}(l_j^{(a)} - 2j + N + 1). \tag{3.3}$$

Here $l_i^{(a)}$ denotes the number of boxes in the ith row of representation R_a and $l^{(a)} = \sum_i l_i^{(a)}$. Using the above braiding eigenvalue (3.2), we can write down the colored HOMFLY-PT polynomial of the $(2, 2m)$ torus link with framing $f = \{f_1, f_2\}$ of the component knots and linking number m [13],

$$\begin{aligned} \mathcal{W}_{R_1,R_2}^{[(2,2m);f]}(S^3, k) &= q^{f_1 C_{R_1} + f_2 C_{R_2}} \sum_{R_t \in R_1 \otimes R_2} (\dim_q R_t)(\lambda_{R_t}^{(+)})^{2m} \\ &= q^{f_1 C_{R_1} + f_2 C_{R_2}} \sum_{R_t \in R_1 \otimes R_2} (\dim_q R_t) \\ &\quad \times \left(q^{-\frac{C_{R_1}+C_{R_2}}{2}+\frac{C_{R_t}}{2}}\right)^{2m}, \end{aligned} \tag{3.4}$$

where the q-dimension of a representation is defined as

$$\dim_q R_a = \prod_{1\le i<j\le N} \frac{[l_i^{(a)} - l_j^{(a)} + j - i]_q}{[j-i]_q} \quad \text{and} \quad [x]_q = \frac{q^{\frac{x}{2}} - q^{-\frac{x}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \tag{3.5}$$

The $U(N)$ invariant of framed $(2, 2m)$ torus link can be written using (2.7), (2.8), and (3.4) as

$$\begin{aligned} \mathcal{W}_{\mathcal{R}_1,\mathcal{R}_2}^{[(2,2m);f]}(S^3,k) &= \mathcal{W}_{n^{(1)},n^{(2)}}^{[(2,2m);f]}(S^3,k) \times \mathcal{W}_{R_1,R_2}^{[(2,2m);f]}(S^3,k) \\ &= \exp\left(\frac{i\pi}{k+N}\sum_{a=1}^{2}\frac{f_a(n^{(a)})^2}{N}\right)\exp\left(\frac{i\pi}{k+N}\frac{2(+m)n^{(1)}n^{(2)}}{N}\right) q^{f_1 C_{\mathcal{R}_1}+f_2 C_{\mathcal{R}_2}} \sum_{\mathcal{R}_t \in \mathcal{R}_1\otimes\mathcal{R}_2} (\dim_q \mathcal{R}_t)(q^{-\frac{C_{\mathcal{R}_1}+C_{\mathcal{R}_2}}{2}+\frac{C_{\mathcal{R}_t}}{2}})^{2m}. \end{aligned} \tag{3.6}$$

Note that, in writing the above expression we have used the fact that the quadratic Casimir and q-dimension of a $U(N)$ representation $\mathcal{R}_a$ is same as that of the quadratic Casimir and q-dimension of $SU(N)$ representation R_a,

$$C_{\mathcal{R}_a} = -\frac{(n^{(a)})^2}{2N} + \frac{1}{2}\sum_{i=1}^{N} n_i^{(a)}(n_i^{(a)} - 2i + N + 1) = C_{R_a}, \tag{3.7}$$

$$\dim_q \mathcal{R}_a = \prod_{1\le i<j\le N} \frac{[n_i^{(a)} - n_j^{(a)} + j - i]_q}{[j-i]_q} = \dim_q R_a. \tag{3.8}$$

We introduce $\kappa_{\mathcal{R}_a}$ in terms of the number of boxes $n_i^{(a)}$ in the ith row of the Young diagram of representation $\mathcal{R}_a$,

$$\kappa_{\mathcal{R}_a} = \frac{1}{2}\sum_{i=1}^{N} n_i^{(a)}(n_i^{(a)} - 2i + N + 1). \tag{3.9}$$

In terms of these variables, we express the $U(N)$ invariant of the framed $(2, 2m)$ torus link as

$$\mathcal{W}^{[(2,2m);f]}_{\mathcal{R}_1,\mathcal{R}_2}(S^3, k) = q^{f_1\kappa_{\mathcal{R}_1} + f_2\kappa_{\mathcal{R}_2}} \sum_{\mathcal{R}_t} (\dim_q \mathcal{R}_t) \times (q^{-\kappa_{\mathcal{R}_1} - \kappa_{\mathcal{R}_2} + \kappa_{\mathcal{R}_t}})^m. \tag{3.10}$$

Hereafter, we refer $\kappa_{\mathcal{R}_a}$ to be the quadratic Casimir of $U(N)$ representation $\mathcal{R}_a$. The invariants for arbitrary L-component torus links can be compactly written using these quadratic casimirs $\kappa_{\mathcal{R}_a}$ and q-dimension of representations [17]. We will briefly present the explicit form of these link invariants in the following subsection.

B. $(L\alpha, L\beta)$ torus link invariant

For a general two-component right-handed torus link $(2\alpha, 2\beta)$ carrying $U(N)$ representations $\mathcal{R}_1$ and $\mathcal{R}_2$ on the component knots, the polynomial invariant is [17,18]

$$\mathcal{W}^{(2\alpha,2\beta)}_{\mathcal{R}_1,\mathcal{R}_2}(S^3, k) = \prod_{i=1}^{2} \left(q^{-\frac{\alpha\beta}{2}\varkappa_{\mathcal{R}_i}} v^{-\frac{\beta(\alpha-1)}{2}n^{(i)}}\right) \sum_{\mathcal{R}_t \vdash \alpha(n^{(1)}+n^{(2)})} \zeta^{\mathcal{R}_t}_{\mathcal{R}_1,\mathcal{R}_2} q^{\frac{\beta}{2\alpha}\varkappa_{\mathcal{R}_t}} s^{\star}_{\mathcal{R}_t}(q, v), \tag{3.11}$$

where

$$\zeta^{\mathcal{R}_t}_{\mathcal{R}_1,\mathcal{R}_2} = \sum_{\Lambda^{(1)}\vdash n^{(1)}} \frac{|c_{\Lambda^{(1)}}|}{(n^{(1)})!} \chi_{\mathcal{R}_1}(c_{\Lambda^{(1)}}) \sum_{\Lambda^{(2)}\vdash n^{(2)}} \frac{|c_{\Lambda^{(2)}}|}{(n^{(2)})!} \chi_{\mathcal{R}_2}(c_{\Lambda^{(2)}}) \chi_{\mathcal{R}_t}\left(c_{(\Lambda^{(1)}+\Lambda^{(2)})_{(\alpha)}}\right),$$

$$s^{\star}_{\mathcal{R}_t}(q, v) = \sum_{\Lambda\vdash n^{(t)}} \frac{|c_\Lambda|}{(n^{(t)})!} \chi_{\mathcal{R}_t}(c_\Lambda) \prod_{i=1}^{\ell(\Lambda)} \frac{v^{\Lambda_i/2} - v^{-\Lambda_i/2}}{q^{\Lambda_i/2} - q^{-\Lambda_i/2}} \equiv \dim_q \mathcal{R}_t, \qquad v = q^N. \tag{3.12}$$

Here $\varkappa_{\mathcal{R}}$ in terms of $\kappa_{\mathcal{R}}$ (3.9) can be written as

$$\varkappa_{\mathcal{R}} = \sum_i n_i(n_i - 2i + 1) = 2\kappa_{\mathcal{R}} - N\sum_i n_i. \tag{3.13}$$

The notation $\Lambda^{(a)} \vdash n^{(a)}$ means that $\Lambda^{(a)} \equiv (\Lambda_1^{(a)}, \Lambda_2^{(a)}, \Lambda_3^{(a)}, \ldots)$ is a partition, in other words a sequence of positive integers with $\Lambda_1^{(a)} \ge \Lambda_2^{(a)} \ge \Lambda_3^{(a)} \ge \cdots$, satisfying

$$\sum_{i=1}^{\ell(\Lambda^{(a)})} \Lambda_i^{(a)} = n^{(a)}, \tag{3.14}$$

where $\ell(\Lambda^{(a)})$ is the highest value of i for which $\Lambda_i^{(a)}$ is nonzero,

$$\ell(\Lambda^{(a)}) = \max\left\{i | \Lambda_i^{(a)} > 0\right\}. \tag{3.15}$$

The conjugacy class associated to the partition $\Lambda^{(a)}$ is denoted as $c_{\Lambda^{(a)}}$ and it consists of one $\Lambda_1^{(a)}$-cycle, one $\Lambda_2^{(a)}$-cycle, one $\Lambda_3^{(a)}$-cycle, and so on. $|c_{\Lambda^{(a)}}|$ denotes the total number of elements belonging to the conjugacy class $c_{\Lambda^{(a)}}$ having the same cycle structure. $\chi_{\mathcal{R}_a}(c_{\Lambda^{(a)}})$ is the corresponding character labeled by representation $\mathcal{R}_a$. The sum of two partitions is the sum of each of the elements of the respective partitions,

$$\Lambda^{(1)} + \Lambda^{(2)} \equiv \left(\Lambda_1^{(1)} + \Lambda_1^{(2)}, \Lambda_2^{(1)} + \Lambda_2^{(2)}, \Lambda_3^{(1)} + \Lambda_3^{(2)}, \cdots\right), \tag{3.16}$$

and $(\Lambda)_{(\alpha)}$ implies the partition $(\alpha\Lambda_1, \alpha\Lambda_2, \alpha\Lambda_3, \ldots)$.

Rewriting the torus link invariant (3.11) in terms of variable $\kappa_{\mathcal{R}}$ we get,

$$\mathcal{W}^{(2\alpha,2\beta)}_{\mathcal{R}_1,\mathcal{R}_2}(S^3, k) = q^{-\alpha\beta(\kappa_{\mathcal{R}_1} + \kappa_{\mathcal{R}_2})} \times \sum_{\mathcal{R}_t\vdash\alpha(n^{(1)}+n^{(2)})} \zeta^{\mathcal{R}_t}_{\mathcal{R}_1,\mathcal{R}_2} q^{\frac{\beta}{\alpha}\kappa_{\mathcal{R}_t}} s^{\star}_{\mathcal{R}_t}(q, v). \tag{3.17}$$

If the component knots are having f_1 and f_2 self-linking numbers, which we denote as $f = \{f_1, f_2\}$, then the polynomial invariant will have an additional framing factor

$$\mathcal{W}^{[(2\alpha,2\beta);f]}_{\mathcal{R}_1,\mathcal{R}_2}(S^3,k) = q^{f_1\kappa_{\mathcal{R}_1}+f_2\kappa_{\mathcal{R}_2}}\mathcal{W}^{(2\alpha,2\beta)}_{\mathcal{R}_1,\mathcal{R}_2}(S^3,k). \quad (3.18)$$

Now, we can generalize the colored HOMFLY-PT polynomial for an arbitrary L component torus link $(L\alpha, L\beta)$, with self-linking number $f = \{f_1, \ldots f_L\}$ of the component knots, as

$$\mathcal{W}^{[(L\alpha,L\beta);f]}_{\mathcal{R}_1,\ldots,\mathcal{R}_L}(S^3,k) = \left(\prod_{a=1}^{L} q^{(f_a-\alpha\beta)\kappa_{\mathcal{R}_a}}\right) \times \sum_{\mathcal{R}_t\vdash\alpha(n^{(1)}+\cdots+n^{(L)})} \zeta^{\mathcal{R}_t}_{\mathcal{R}_1,\ldots,\mathcal{R}_L} q^{\frac{\beta}{\alpha}\kappa_{\mathcal{R}_t}} s^{\star}_{\mathcal{R}_t}(q,v), \quad (3.19)$$

where

$$\zeta^{\mathcal{R}_t}_{\mathcal{R}_1,\ldots,\mathcal{R}_L} = \prod_{a=1}^{L}\left(\sum_{\Lambda^{(a)}\vdash n^{(a)}} \frac{|c_{\Lambda^{(a)}}|}{(n^{(a)})!}\chi_{\mathcal{R}_a}(c_{\Lambda^{(a)}})\right) \times \chi_{\mathcal{R}_t}(c_{(\Lambda^{(1)}+\cdots+\Lambda^{(L)})_{(\alpha)}}). \quad (3.20)$$

In Ref. [8], two of the authors have analyzed the double-scaling limit (1.1) and (1.2) of the two-component $(2, 2m)$ torus link invariants. Particularly, these invariants were shown to be expressed in terms of (2, 2) torus link invariants. We believe that this must be generalizable for an arbitrary L-component torus link $(L\alpha, L\beta)$. In the following section we present our result after briefly reviewing the works on $(2, 2m)$ torus links.

IV. TORUS LINK INVARIANTS IN THE DOUBLE-SCALING LIMIT OF $U(N)$ CHERN-SIMONS THEORY

As discussed in the introduction, the double-scaling limit (1.1) of $U(N)$ Chern-Simons theory involves the Chern-Simons coupling $k \to \infty$. The range of the $n_i^{(a)}$'s (2.5) for the integrable representations $\mathcal{R}_a$ at large k can be chosen as

$$-\frac{k}{2} \leq n_N^{(a)} \leq \cdots \leq n_1^{(a)} \leq \frac{k}{2}. \quad (4.1)$$

For the representation $\mathcal{R}_a$ we define a set of N variables $\{\theta_1^{(a)}, \ldots, \theta_N^{(a)}\}$,

$$\theta_i^{(a)} = \frac{2\pi}{N+k}\left(h_i^{(a)} - \frac{N-1}{2}\right), \quad (4.2)$$

$$\text{where,} \quad h_i^{(a)} = n_i^{(a)} + N - i. \quad (4.3)$$

It is easy to check that in the double-scaling limit the variables $\theta_i^{(a)}$ range from $-\pi$ to π. The quadratic Casimir of representation $\mathcal{R}_a$ (3.9) can be expressed in terms of these variables as

$$\kappa_{\mathcal{R}_a} = \frac{(N+k)^2}{8\pi^2}\sum_i \theta_i^{(a)2} - \frac{N(N^2-1)}{24}. \quad (4.4)$$

In the large N limit, the set of discrete variables θ_i becomes a continuous function

$$\theta_i^{(a)} \to \theta^{(a)}(x), \quad \text{where } x = \frac{i}{N} \in [0,1]. \quad (4.5)$$

All the discrete summations over i are then replaced by integration over x,

$$\frac{1}{N}\sum_i = \int_0^1 dx. \quad (4.6)$$

Moreover the distribution of θ_i, in the large N limit is captured by a distribution function

$$\sigma(\theta) = \frac{1}{N}\sum_{i=1}^{N}\delta(\theta-\theta_i) = \left|\frac{\partial x}{\partial\theta}\right|, \quad (4.7)$$

such that $\int d\theta\sigma(\theta) = 1$. Hence, for any arbitrary functions of θ_i we can write

$$\frac{1}{N}\sum_i f(\theta_i) \to \int_0^1 dx f(\theta(x)) = \int d\theta\sigma(\theta)f(\theta). \quad (4.8)$$

Now we will incorporate the continuum limit discussed above for the $(2, 2m)$ torus link invariant.

A. (2,2m) torus link invariant in the double-scaling limit

The invariant $\mathcal{W}^{[(2,2m);f]}_{\mathcal{R}_1,\mathcal{R}_2}(S^3,k)$ (3.10) can be expressed using the $\{\theta_i\}$ variables (4.2) as

$$\mathcal{W}^{[(2,2m);f]}_{\mathcal{R}_1,\mathcal{R}_2}(S^3,k) = \exp\left[\frac{i(N+k)}{4\pi}\left((f_1-m)\sum_i\theta_i^{(1)2} + (f_2-m)\sum_i\theta_i^{(2)2}\right)\right]\exp\left[-\frac{i\pi(N^3-N)}{12(N+k)}(f_1+f_2-m)\right] \times \sum_{\{\theta_i^{(t)}\}} \frac{\exp[\frac{1}{2}\sum_{i\neq j}^{N}\log|\sin(\frac{\theta_i^{(t)}-\theta_j^{(t)}}{2})|]}{\exp[\frac{1}{2}\sum_{i\neq j}^{N}\log|\sin(\frac{\pi(j-i)}{N+k})|]}\exp\left[\frac{im(N+k)}{4\pi}\sum_i\theta_i^{(t)2}\right]. \quad (4.9)$$

Here $\theta_i^{(1)}$, $\theta_i^{(2)}$, and $\theta_i^{(t)}$ are the set of variables corresponding to representations $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_t$, respectively. In the double scaling limit with appropriate replacement of the discrete variables in terms of the continuous functions (4.5), (4.6), and (4.7), the above invariant becomes

$$\mathcal{W}^{[(2,2m);f]}(S^3,\lambda) = \mathcal{G}^{(f)}(m,\lambda,\sigma_1,\sigma_2)\mathcal{F}(m,\lambda), \tag{4.10}$$

where

$$\begin{aligned}\mathcal{G}^{(f)}(m,\lambda,\sigma_1,\sigma_2) = \exp&\left[\frac{iN^2}{4\pi\lambda}\left(\int d\theta\theta^2((f_1-m)\sigma_1(\theta)+(f_2-m)\sigma_2(\theta))\right)\right]\\ &\times\exp\left[-\frac{i\pi\lambda N^2}{12}(f_1+f_2-m)-\frac{N^2}{2}\int dx \fint dy \log|\sin(\pi\lambda(y-x))|\right]\end{aligned} \tag{4.11}$$

is dependent on framing f and

$$\begin{aligned}\mathcal{F}(m,\lambda) = \int[\mathcal{D}\theta^{(t)}(x)]\exp&\left[\frac{N^2}{2}\int dx\fint dy\right.\\ &\times\left.\log\left|\sin\left(\frac{\theta^{(t)}(x)-\theta^{(t)}(y)}{2}\right)\right|\right]\\ &\times\exp\left[\frac{imN^2}{4\pi\lambda}\int dx\theta^{(t)}(x)^2\right]\end{aligned} \tag{4.12}$$

is independent of framing. Observe that the function $\mathcal{G}^{(f)}(m,\lambda,\sigma_1,\sigma_2)$ depends only on the given representations $\mathcal{R}_1$ and $\mathcal{R}_2$ and is independent of $\mathcal{R}_t$. However, $\mathcal{F}(m,\lambda)$ involves the $\theta^{(t)}$ integration which is constrained, since it runs only over the irreducible representations $\mathcal{R}_t$ appearing in the tensor product of $\mathcal{R}_1$ and $\mathcal{R}_2$. Therefore, this function $\mathcal{F}(m,\lambda)$ is difficult to evaluate even in the large-N limit. Note that, the space of functional integration over $\theta^{(t)}(x)$ in $\mathcal{F}(m,\lambda)$ remains the same for any value of m. As a result, the function $\mathcal{F}(m,\lambda)$ has the following property:

$$\mathcal{F}(m,\lambda) = \mathcal{F}(1,\lambda/m). \tag{4.13}$$

With $m=1$, $\mathcal{W}^{[(2,2m);f]}_{\mathcal{R}_1,\mathcal{R}_2}(S^3,k)$ becomes the invariant of framed Hopf link. Therefore, by exploiting the property (4.13), we can see that a framed $(2,2m)$ torus link invariant can be expressed in terms of a framed Hopf link invariant as follows:

$$\begin{aligned}\mathcal{W}^{[(2,2m);f]}_{\sigma_1,\sigma_2}(S^3,\lambda) = &\left(\frac{\mathcal{G}^{(f)}(m,\lambda,\sigma_1,\sigma_2)}{\mathcal{G}^{(f)}(1,\lambda/m,\sigma_1,\sigma_2)}\right)\\ &\times\mathcal{W}^{[(2,2);f]}_{\sigma_1,\sigma_2}(S^3,\lambda/m).\end{aligned} \tag{4.14}$$

That is, we first find the distribution functions σ_1, σ_2 corresponding to the representations $\mathcal{R}_1$, $\mathcal{R}_2$ placed on the two component knots. After which we can write the Hopf link invariant as a function of σ_1, σ_2, and λ. Then the link invariant for $(2,2m)$ torus link as a function of λ becomes proportional to the Hopf link invariant with λ replaced by λ/m but importantly, keeping σ_1 and σ_2 fixed while performing such a replacement.

In the following subsection we generalize (4.14) for any L-component torus link $(L\alpha,L\beta)$ embedded inside S^3.

B. $(L\alpha,L\beta)$ torus link invariant in the double-scaling limit

Let us first focus on arbitrary 2 component torus links $(2\alpha,2\beta)$. Using the expression of $\kappa_{\mathcal{R}_a}$ in terms of the set of N variables $\theta_i^{(a)}$ (4.4) we can express the $(2\alpha,2\beta)$ torus link invariant (3.18) as

$$\begin{aligned}\mathcal{W}^{[(2\alpha,2\beta);f]}_{\mathcal{R}_1,\mathcal{R}_2}(S^3,k) = \exp&\left[\frac{i(N+k)}{4\pi}\left((f_1-\alpha\beta)\sum_i\theta_i^{(1)2}+(f_2-\alpha\beta)\sum_i\theta_i^{(2)2}\right)\right]\\ &\times\exp\left[-\frac{i\pi(N^3-N)}{12(N+k)}\left(f_1+f_2-2\alpha\beta+\frac{\beta}{\alpha}\right)\right]\\ &\times\sum_{\{\theta_i^{(t)}\}}\zeta^{\theta^{(t)}}_{\theta^{(1)},\theta^{(2)}}\exp\left[\frac{i\beta(N+k)}{4\pi\alpha}\sum_i\theta_i^{(t)2}\right]\frac{\exp\left[\frac{1}{2}\sum_{i\neq j}^N\log\left|\sin\left(\frac{\theta_i^{(t)}-\theta_j^{(t)}}{2}\right)\right|\right]}{\exp\left[\frac{1}{2}\sum_{i\neq j}^N\log\left|\sin\left(\frac{\pi(j-i)}{N+k}\right)\right|\right]}.\end{aligned} \tag{4.15}$$

We now take the double-scaling limit with the appropriate replacement of the discrete variables in terms of the continuous functions (4.5), (4.6), and (4.7),

$$\mathcal{W}^{[(2\alpha,2\beta);f]}_{\sigma_1,\sigma_2}(S^3,\lambda) = \exp\left[\frac{iN^2}{4\pi\lambda}\left(\int d\theta\theta^2((f_1-\alpha\beta)\sigma_1(\theta)+(f_2-\alpha\beta)\sigma_2(\theta))\right)\right]\exp\left[-\frac{i\pi\lambda N^2}{12}\left(f_1+f_2-2\alpha\beta+\frac{\beta}{\alpha}\right)\right] \tag{4.16}$$

$$\times\int[\mathcal{D}\theta^{(t)}]\zeta^{\theta^{(t)}}_{\theta^{(1)},\theta^{(2)}}\exp\left[\frac{i\beta N^2}{4\pi\alpha\lambda}\int d\theta\theta^2\sigma_t(\theta)\right]\frac{\exp\left[\frac{N^2}{2}\int dx ⨍ dy\log\left|\sin(\frac{\theta^{(t)}(x)-\theta^{(t)}(y)}{2})\right|\right]}{\exp\left[\frac{N^2}{2}\int dx ⨍ dy\log|\sin(\pi\lambda(y-x))|\right]}$$
$$\equiv\mathcal{G}^{(f)}(\alpha,\beta,\lambda,\sigma_1,\sigma_2)\mathcal{F}\left(\frac{\beta}{\alpha},\lambda\right), \tag{4.17}$$

where

$$\mathcal{G}^{(f)}(\alpha,\beta,\lambda,\sigma_1,\sigma_2) = \exp\left[\frac{iN^2}{4\pi\lambda}\left(\int d\theta\theta^2((f_1-\alpha\beta)\sigma_1(\theta)+(f_2-\alpha\beta)\sigma_2(\theta))\right)\right]$$
$$\times\exp\left[-\frac{i\pi\lambda N^2}{12}\left(f_1+f_2-2\alpha\beta+\frac{\beta}{\alpha}\right)-\frac{N^2}{2}\int dx ⨍ dy\log|\sin(\pi\lambda(y-x))|\right], \tag{4.18}$$

and

$$\mathcal{F}\left(\frac{\beta}{\alpha},\lambda\right) = \int[\mathcal{D}\theta^{(t)}]\zeta^{\theta^{(t)}}_{\theta^{(1)},\theta^{(2)}}\exp\left[\frac{N^2}{2}\int dx ⨍ dy\log\left|\sin\left(\frac{\theta^{(t)}(x)-\theta^{(t)}(y)}{2}\right)\right|\right]\exp\left[\frac{i\beta N^2}{4\pi\alpha\lambda}\int d\theta\theta^2\sigma_t(\theta)\right]. \tag{4.19}$$

When $\alpha=\beta=1$, we get the double scaling limit of Hopf link invariant. We exploit the symmetry of $\mathcal{F}(\frac{\beta}{\alpha},\lambda)$ under

$$\frac{\beta}{\alpha}\to 1 \quad\text{and}\quad \lambda\to\frac{\lambda}{\beta/\alpha}, \tag{4.20}$$

to write down the invariant of any 2 component torus link in terms of Hopf link invariant as

$$\mathcal{W}^{[(2\alpha,2\beta);f]}_{\sigma_1,\sigma_2}(S^3,\lambda) = \frac{\tilde{\mathcal{G}}^{(f)}(\alpha,\beta,\lambda,\sigma_1,\sigma_2)}{\tilde{\mathcal{G}}^{(f)}(1,1,\frac{\lambda}{\beta/\alpha},\sigma_1,\sigma_2)}\times\mathcal{W}^{[(2,2);f]}_{\sigma_1,\sigma_2}\left(S^3,\frac{\lambda}{\beta/\alpha}\right). \tag{4.21}$$

Similar exercise for any arbitrary L-component torus link (3.19) is straightforward and we find that $(L\alpha, L\beta)$ torus link invariant can be expressed in terms of (L,L) torus link invariant as follows:

$$\mathcal{W}^{[(L\alpha,L\beta);\{f_1,\ldots,f_L\}]}_{\sigma_1,\ldots,\sigma_L}(S^3,\lambda) = \frac{\tilde{\mathcal{G}}^{(f)}(\alpha,\beta,\lambda,\sigma_1,\ldots,\sigma_L)}{\tilde{\mathcal{G}}^{(f)}(1,1,\frac{\lambda}{\beta/\alpha},\sigma_1,\ldots,\sigma_L)}\times\mathcal{W}^{[(L,L);\{f_1,\ldots,f_L\}]}_{\sigma_1,\ldots\sigma_L}\left(S^3,\frac{\lambda}{\beta/\alpha}\right). \tag{4.22}$$

Thus, given a (L,L) torus link invariant in variable λ with $\sigma_1,\ldots,\sigma_L$ being the eigenvalue densities of the component knots, the $(L\alpha,L\beta)$ torus link invariant (carrying the same eigenvalue densities) in variable λ becomes proportional to the (L,L) torus link invariant with λ replaced by $\frac{\lambda}{\beta/\alpha}$ while keeping $\sigma_1,\ldots,\sigma_L$ unaltered.

We can explicitly work out the (L,L) link invariants (3.19) with components carrying different representations whose Young diagrams have small number of boxes. For these representations, we can infer the leading and subleading terms for a general $(L\alpha,L\beta)$ link invariant in the double-scaling limit (4.22). Extracting such leading and subleading contributions appear to be practically difficult for representations with large number of boxes. In Ref. [8], large-N contributions for the two component Hopf link invariant was determined by mapping it to a one-dimensional fluid equation, where the initial and final fluid densities correspond to two of the representations placed on the two component knots. However, such a procedure is not generalizable for a (L,L) torus link with $L>2$. The matrix model method appears to be an efficient approach to handle such difficulties. In the following section, we will briefly review $U(N)$ matrix model and the computation of the correlators corresponding to the torus link invariants in the double-scaling limit.

V. TORUS LINK INVARIANTS AND MATRIX MODELS AT LARGE N

Study of the partition function and the correlators in $U(N)$ matrix model at large N will allow us to deduce the leading and subleading contributions of torus link invariants. Within the matrix model approach, we obtain analytic results for these contributions at the following two instances:

(i) Representations having small number of boxes (small as compared to $N, k \to \infty$); these are validated with those obtained from explicit colored HOMLFLY-PT polynomials;

(ii) Large symmetric representations placed on the component knots.

We now focus on the invariant of a framed Hopf link, embedded on a three-manifold $S^3/\mathbb{Z}_p$, which can equivalently be written in terms of the modular transformation matrices ($\mathcal{S}$, $\mathcal{T}$) of $\mathfrak{u}(N)_k$ Wess-Zumino conformal field theory. The explicit form of Hopf link invariant on $S^3/\mathbb{Z}_p$ with representations $\mathcal{R}_1$ and $\mathcal{R}_2$ on the two components is given by

$$\begin{aligned}\tilde{\mathcal{V}}^{(2,2)}_{\mathcal{R}_1,\mathcal{R}_2}(S^3/\mathbb{Z}_p,k) &= \sum_{\mathcal{R}} \mathcal{S}_{\mathcal{R}_1\mathcal{R}}\mathcal{S}_{\mathcal{R}_2\mathcal{R}}\mathcal{T}^{-p}_{\mathcal{R}\mathcal{R}} \\ &= \sum_{\mathcal{R}} \frac{\mathcal{S}_{\mathcal{R}_1\mathcal{R}}}{\mathcal{S}_{0\mathcal{R}}}\frac{\mathcal{S}_{\mathcal{R}_2\mathcal{R}}}{\mathcal{S}_{0\mathcal{R}}}\mathcal{S}^2_{0\mathcal{R}}\mathcal{T}^{-p}_{\mathcal{R}\mathcal{R}}. \qquad (5.1)\end{aligned}$$

The summation runs over all the irreducible integrable representations $\mathcal{R}$ of $\mathfrak{u}(N)_k$. Considering $\mathcal{R}_1$ and $\mathcal{R}_2$ to be trivial representations we obtain the partition function of Chern-Simons theory on $S^3/\mathbb{Z}_p$,

$$\mathcal{Z}(S^3/\mathbb{Z}_p,k) = \sum_{\mathcal{R}} \mathcal{S}^2_{0\mathcal{R}}\mathcal{T}^{-p}_{\mathcal{R}\mathcal{R}}. \qquad (5.2)$$

In the double-scaling limit (1.1) the partition function is dominated by a single representation. Contributions due to other representations are suppressed by $\mathcal{O}(1/N^2)$. This is the standard saddle point approximation. The Hopf link invariant defined in (5.1) is unnormalized. We can define the normalized invariant as

$$\mathcal{V}^{(2,2)}_{\mathcal{R}_1,\mathcal{R}_2}(S^3/\mathbb{Z}_p,k) = \frac{\tilde{\mathcal{V}}^{(2,2)}_{\mathcal{R}_1,\mathcal{R}_2}(S^3/\mathbb{Z}_p,k)}{\mathcal{Z}(S^3/\mathbb{Z}_p,k)}, \qquad (5.3)$$

such that $\mathcal{V}^{(2,2)}_{\mathcal{R}_1,\mathcal{R}_2}(S^3/\mathbb{Z}_p,k)$ with the representations $\mathcal{R}_1$ and $\mathcal{R}_2$ being trivial becomes identically equal to one. Note that, with $p=1$, $\mathcal{V}^{(2,2)}_{\mathcal{R}_1,\mathcal{R}_2}(S^3/\mathbb{Z}_p,k)$ becomes a Hopf link (3.10) embedded inside three manifold S^3 with framing of component knots $f=\{1,1\}$.

In order to proceed further we explicitly write down the modular transformation matrix $\mathcal{S}$ for $\mathfrak{u}(N)_k$ [19],

$$\mathcal{S}_{\mathcal{R}_a\mathcal{R}_b} = (-i)^{N(N-1)/2}\frac{N^{-1/2}}{(k+N)^{(N-1)/2}}e^{\frac{2\pi i n^{(a)}n^{(b)}}{N(N+k)}}\det M(\mathcal{R}_a,\mathcal{R}_b), \qquad (5.4)$$

where for any integrable representations $\mathcal{R}_a$ and $\mathcal{R}_b$,

$$M_{ij}(\mathcal{R}_a,\mathcal{R}_b) = \exp\left[\frac{2\pi i}{k+N}\phi_i(\mathcal{R}_a)\phi_j(\mathcal{R}_b)\right],\quad i,j=1,\ldots,N \qquad (5.5)$$

$$\phi_i(\mathcal{R}_a) = n_i^{(a)} - \frac{n^{(a)}}{N} - i + \frac{N+1}{2}. \qquad (5.6)$$

Here, as before, $n^{(a/b)}$ and $n_i^{(a/b)}$ denote the total number of boxes and the number of boxes in the ith row corresponding to the Young diagram of $\mathcal{R}_{a/b}$, respectively.

The other modular transformation matrix $\mathcal{T}$ is given by

$$\mathcal{T}_{\mathcal{R}_a\mathcal{R}_b} = \exp\left[2\pi i\left(\mathcal{Q}_{\mathcal{R}_a} - \frac{c}{24}\right)\right]\delta_{\mathcal{R}_a\mathcal{R}_b}. \qquad (5.7)$$

$\mathcal{Q}_{\mathcal{R}}$ and c denote the conformal weight of representation $\mathcal{R}$ and central charge, respectively

$$\mathcal{Q}_{\mathcal{R}} = \frac{\kappa_{\mathcal{R}}}{(k+N)} \quad \text{and} \quad c = \frac{N(Nk+1)}{N+k}. \qquad (5.8)$$

Next we want to calculate the ratio of modular transformation matrices $\frac{\mathcal{S}_{\mathcal{R}\mathcal{R}_a}}{\mathcal{S}_{0\mathcal{R}}}$ appearing in Eq. (5.1). Instead of representing a Young diagram by the set of box numbers $\{n_i^{(a)}\}$ we use the variables θ_i defined in Eq. (4.2). Since in the double scaling limit θ_i ranges between $-\pi$ and π, we can arrange them as eigenvalues of an $N\times N$ unitary matrix,

$$U = \mathrm{diag}(e^{i\theta_1},\ldots,e^{i\theta_N}). \qquad (5.9)$$

After some calculation one can show that the ratio $\frac{\mathcal{S}_{\mathcal{R}\mathcal{R}_a}}{\mathcal{S}_{0\mathcal{R}}}$ can be expressed as

$$\frac{\mathcal{S}_{\mathcal{R}\mathcal{R}_a}}{\mathcal{S}_{0\mathcal{R}}} = \mathbf{s}_{\mathcal{R}_a}(U), \qquad (5.10)$$

where the Schur polynomial $\mathbf{s}_{\mathcal{R}_a}(U)$ of the unitary group element U in representation $\mathcal{R}_a$ is

$$\mathbf{s}_{\mathcal{R}_a}(U) = \frac{\det[e^{i\theta_i h_j^{(a)}}]}{\det[e^{i\theta_i(N-j)}]}. \qquad (5.11)$$

Note that the notation U used in Eq. (5.10) corresponds to representation $\mathcal{R}$ on its left-hand side i.e., the diagonal elements of U correspond to h_i (4.3) of $\mathcal{R}$. The unnormalized Hopf link invariant, therefore can be expressed as[1]

[1]Note the difference between the expressions of Hopf link here and in [8].

$$\tilde{\mathcal{V}}^{(2,2)}_{\mathcal{R}_1,\mathcal{R}_2}(S^3/\mathbb{Z}_p,k) = q^{\frac{pN(Nk+1)}{24}}\sum_{\mathcal{R}} q^{-p\kappa_{\mathcal{R}}}\mathcal{S}^2_{0\mathcal{R}}\mathbf{s}_{\mathcal{R}_1}(U)\mathbf{s}_{\mathcal{R}_2}(U), \tag{5.12}$$

where q is the $(k+N)$th root of unity (2.9). In Eq. (5.12) each $\mathcal{R}$ corresponds to a unitary matrix U in an ensemble of unitary matrices. The summation over $\mathcal{R}$ therefore corresponds to summation over unitary matrices. The factor $\mathcal{S}^2_{0\mathcal{R}}$ when written in terms of θ_i variables, reduces to Vandermonde measure for unitary matrices. Thus in the large N limit we can write

$$\sum_{\mathcal{R}}\mathcal{S}^2_{0\mathcal{R}} = \int\prod_i d\theta_i\prod_{i<j}\sin^2\left(\frac{\theta_i-\theta_j}{2}\right) \equiv \int[DU], \tag{5.13}$$

where we have ignored few θ independent overall factors which does not affect our analysis. Hence, the partition function (5.2) can be written as a zero-dimensional unitary matrix model with weight factor $e^{-(N+k)^2S[\theta]}$ as[2]

$$\mathcal{Z}(S^3/\mathbb{Z}_p,k) = \int\prod_i d\theta_i\prod_{i<j}\sin^2\left(\frac{\theta_i-\theta_j}{2}\right)e^{-(N+k)^2S[\theta]}, \tag{5.14}$$

where

$$S[\theta] = \frac{p\lambda}{N\pi}\sum_{i=1}^{N}\left(\frac{\theta_i^2}{4}-\frac{\pi^2}{12}\right)+\frac{\pi p\lambda(1-\lambda)}{12}. \tag{5.15}$$

Note that, in order to arrive at the above result we have replaced p in Eq. (5.2) by $-ip$. The normalized Hopf link invariant (5.3) is therefore given as the integral of two Schur polynomials, $\mathbf{s}_{\mathcal{R}_1}(U)$ and $\mathbf{s}_{\mathcal{R}_2}(U)$, over the unitary ensemble with the aforementioned weight factor,

$$\begin{aligned}\mathcal{V}^{(2,2)}_{\mathcal{R}_1,\mathcal{R}_2}&(S^3/\mathbb{Z}_p,k)\\ &= \frac{1}{\mathcal{Z}(S^3/\mathbb{Z}_p,k)}\int[DU]e^{-(N+k)^2S[\theta]}\mathbf{s}_{\mathcal{R}_1}(U)\mathbf{s}_{\mathcal{R}_2}(U)\\ &= \langle\mathbf{s}_{\mathcal{R}_1}(U)\mathbf{s}_{\mathcal{R}_2}(U)\rangle.\end{aligned} \tag{5.16}$$

[2]Such a matrix model appears in different contexts in Chern-Simons theory. For example a similar matrix model was considered in [20]. In [20] the matrix model was expressed in terms of weights of a simply-laced group G. One can use the relation between the weight and the number of boxes in a Young diagram (see Appendix B of [21]) and using the relations (4.2), (4.3) express the matrix model in terms of eigenvalues of unitary matrices (up to some overall constant factor) as in Eq. (5.14). Also, the spectral curve (which is the leading contribution to the expectation value of resolvent) for (P,Q) torus knot was computed in [20]. Here we compute leading as well as subleading contributions to different torus link invariants in arbitrary representations in the double-scaling limit and explicitly show that those are in agreement with the respective colored HOMFLY-PT polynomials.

For large $\mathcal{R}_1$ and $\mathcal{R}_2$ [i.e., when the number of boxes $n_i^{(1)}$ and $n_i^{(2)}$ become of order N or k, in the double-scaling limit] it is difficult to calculate the correlation function even using the saddle point approximation. However, if $\mathcal{R}_1$ and $\mathcal{R}_2$ are small, one can find the leading-order contribution to $\mathcal{V}^{(2,2)}_{\mathcal{R}_1,\mathcal{R}_2}(S^3/\mathbb{Z}_p,k)$ by computing $\mathbf{s}_{\mathcal{R}_1}(U)$ and $\mathbf{s}_{\mathcal{R}_2}(U)$ on the classical solution that extremizes the partition function (5.2).

A. Leading contribution to (L,L) torus link invariant for small representations

For a (2,2) torus link invariant when the "size" of $\mathcal{R}_1$ and $\mathcal{R}_2$ placed on the component knots are much smaller than N, they do not back react on the classical solution extremizing the partition function. Hence, we can consider the operators $\mathbf{s}_{\mathcal{R}_1}(U)$ and $\mathbf{s}_{\mathcal{R}_2}(U)$ as probe and evaluate them on the background solution (5.18).

Let us define a distribution function for the variables $\{\theta_i\}$ in the large N limit as

$$\rho(\theta) = \frac{1}{N}\sum_{i=1}^{N}\delta(\theta-\theta_i), \quad \text{such that} \int_{-\pi}^{\pi}d\theta\rho(\theta) = 1. \tag{5.17}$$

It was shown in Ref. [9] that in the double scaling limit (1.1) the partition function (5.2) is dominated by the following distribution function[3]:

$$\begin{aligned}\rho(\theta) &= \frac{p}{2\pi^2\lambda}\tanh^{-1}\sqrt{1-\frac{e^{-2\pi\lambda/p}}{\cos^2(\theta/2)}}, \quad \text{with}\\ &-2\ \sec^{-1}e^{\pi\lambda/p} < \theta < 2\ \sec^{-1}e^{\pi\lambda/p},\end{aligned} \tag{5.18}$$

which is a solution to the following saddle point equation obtained from the partition function (5.14)

$$⨍\rho(\theta')\cot\left(\frac{\theta-\theta'}{2}\right)d\theta' = \frac{p}{2\pi\lambda}\theta. \tag{5.19}$$

Therefore, in the probe approximation the Hopf link invariant (5.16) can be computed by evaluating the Schur polynomials on the solution (5.18),

$$\mathcal{V}^{(2,2)}_{\mathcal{R}_1,\mathcal{R}_2}(S^3/\mathbb{Z}_p,\lambda) = \mathbf{s}_{\mathcal{R}_1}(\bar{U})\mathbf{s}_{\mathcal{R}_2}(\bar{U}), \tag{5.20}$$

where $\bar{U}$ corresponds to the distribution (5.18).

[3]In this paper we have used the no-cap solution. In the double scaling limit Chern-Simons theory admits a phase transition at some critical value of λ. Beyond that critical value the dominant distribution is given by a capped eigenvalue density. As a result the behavior of correlation functions and hence knot/link invariants will change. We have commented on this in the conclusion section.

The Schur polynomial admits a character expansion [22],

$$\mathbf{s}_{\mathcal{R}_a}(U) = \sum_{\vec{k}^{(a)}} \frac{\chi_{\mathcal{R}_a}\left(c(\vec{k}^{(a)})\right)}{z_{\vec{k}^{(a)}}} \Upsilon_{\vec{k}^{(a)}}(U), \tag{5.21}$$

where

$$\vec{k}^{(a)} = (k_1^{(a)}, k_2^{(a)}, k_3^{(a)}, \ldots) \text{ such that} \sum_r r k_r^{(a)} = n^{(a)}, \tag{5.22}$$

and

$$\Upsilon_{\vec{k}^{(a)}}(U) = \prod_r (\mathrm{Tr}U^r)^{k_r^{(a)}}, \qquad z_{\vec{k}^{(a)}} = \prod_r k_r^{(a)}! r^{k_r^{(a)}}, \tag{5.23}$$

where $n^{(a)}$ is the total number of boxes in representation $\mathcal{R}_a$. $c(\vec{k}^{(a)}) \equiv 1^{k_1^{(a)}} 2^{k_2^{(a)}} 3^{k_3^{(a)}} \cdots$ is the conjugacy class of the symmetric group $\mathbb{G}_{n^{(a)}}$ of degree $n^{(a)}$, consisting of $k_1^{(a)}$ cycles of length 1, $k_2^{(a)}$ cycles of length 2, and so on. In other words, $k_r^{(a)}$ denotes the number of r-cycle of the conjugacy class $c(\vec{k}^{(a)})$. $\chi_{\mathcal{R}_a}(c(\vec{k}^{(a)}))$ is the corresponding character, in representation $\mathcal{R}_a$.

Since to each vector $\vec{k}^{(a)}$ we can associate a conjugacy class $c(\vec{k}^{(a)})$, the sum over $\vec{k}^{(a)}$ in (5.21) represents a sum over all possible conjugacy classes subjected to the condition (5.22). Thus, in the probe approximation, computation of the Hopf link invariant reduces to computation of the quantities $\Upsilon_{\vec{k}^{(a)}}(U)$ on the solution (5.18).

In the large-N limit the expectation value of $\mathrm{Tr}U^r$ is dominated by the background solution i.e.,

$$\begin{aligned}\langle \mathrm{Tr}U^r \rangle \to \mathrm{Tr}\bar{U}^r &= N \int e^{ir\theta}\rho(\theta) d\theta \\ &= N \int \cos r\theta \rho(\theta) d\theta \equiv N\rho_r,\end{aligned} \tag{5.24}$$

where

$$\rho_r = \int \cos r\theta \rho(\theta) d\theta, \tag{5.25}$$

with $\rho(\theta)$ as given in (5.18). Therefore, in the large-N limit $\mathbf{s}_{\mathcal{R}_a}(\bar{U})$ is given by

$$\begin{aligned}\mathbf{s}_{\mathcal{R}_a}(\bar{U}) &= \sum_{\vec{k}^{(a)}} \frac{\chi_{\mathcal{R}_a}(c(\vec{k}^{(a)}))}{z_{\vec{k}^{(a)}}} \prod_r N^{k_r^{(a)}} \rho_r^{k_r^{(a)}} \\ &= \sum_{\vec{k}^{(a)}} N^{K^{(a)}} \frac{\chi_{\mathcal{R}_a}(c(\vec{k}^{(a)}))}{z_{\vec{k}^{(a)}}} \prod_r \rho_r^{k_r^{(a)}},\end{aligned}$$

$$\text{where } K^{(a)} = \sum_r k_r^{(a)}. \tag{5.26}$$

For small representations $k_r^{(a)}$'s are $\mathcal{O}(1)$ numbers. Hence $z_{\vec{k}^{(a)}}$'s are also $\mathcal{O}(1)$ numbers. As a result the N dependence of $\mathbf{s}_{\mathcal{R}_a}(\bar{U})$ is given by $N^{K^{(a)}}$. Therefore, in the large-N limit $\mathbf{s}_{\mathcal{R}_a}(\bar{U})$ is dominated by a conjugacy class for which $K^{(a)}$ is maximum subjected to (5.22). Using (5.22) we can replace $k_1^{(a)}$ in $K^{(a)}$ and find

$$K^{(a)} = n^{(a)} - k_2^{(a)} - 2k_3^{(a)} - \cdots. \tag{5.27}$$

Hence, the maximum value of $K^{(a)}$ is given by $k_1^{(a)} = n^{(a)}$ and $k_2^{(a)} = k_3^{(a)} = \cdots = 0$. This choice reduces Eq. (5.22) to $\vec{k}^{(a)} = (n^{(a)}, 0, 0, \ldots)$, as a result the Schur polynomial (5.26) evaluated on the classical solution becomes

$$\mathbf{s}_{\mathcal{R}_a}(\bar{U}) = N^{n^{(a)}} \rho_1^{n^{(a)}} \frac{\chi_{\mathcal{R}_a}(1^{n^{(a)}})}{(n^{(a)})!}, \tag{5.28}$$

where ρ_1 can be evaluated using (5.25) and is given by

$$\rho_1 = p\left(\frac{1 - e^{-\frac{2\pi\lambda}{p}}}{2\pi\lambda}\right). \tag{5.29}$$

$\chi_{\mathcal{R}_a}(1^{n^{(a)}})$ is the character of the conjugacy class consisting of $n^{(a)}$ number of 1-cycles in representation $\mathcal{R}_a$. Thus, the leading contribution (denoted by $\boldsymbol{\ell.c.}$) to the unknot invariant in fundamental representation is given by

$$\boldsymbol{\ell.c.}[\mathcal{V}_{\square}^{(2,1)}(S^3/\mathbb{Z}_p, \lambda)] = \mathbf{s}_{\square}(\bar{U}) = N\rho_1, \tag{5.30}$$

the superscript (2, 1) refers to an unknot obtained as closure of a two-strand braid with one crossing. Therefore, the leading contribution to the Hopf link invariant in the double-scaling limit can be expressed as (5.20)

$$\begin{aligned}\boldsymbol{\ell.c.}\left[\mathcal{V}_{\mathcal{R}_1,\mathcal{R}_2}^{(2,2)}(S^3/\mathbb{Z}_p, \lambda)\right] &= N^{n^{(1)}+n^{(2)}} \rho_1^{n^{(1)}+n^{(2)}} \frac{\chi_{\mathcal{R}_1}\left(1^{n^{(1)}}\right)\chi_{\mathcal{R}_2}\left(1^{n^{(2)}}\right)}{(n^{(1)})!(n^{(2)})!} \\ &= \left(\boldsymbol{\ell.c.}\left[\mathcal{V}_{\square}^{(2,1)}(S^3/\mathbb{Z}_p, \lambda)\right]\right)^{n^{(1)}+n^{(2)}} \frac{\chi_{\mathcal{R}_1}\left(1^{n^{(1)}}\right)\chi_{\mathcal{R}_2}\left(1^{n^{(2)}}\right)}{\left(n^{(1)}\right)!\left(n^{(2)}\right)!}.\end{aligned} \tag{5.31}$$

It shows that the leading contribution of the Hopf link invariant with small representation placed on the component knots is proportional to appropriate powers of the leading contribution of unknot invariant in fundamental representation. Likewise, the leading contribution of unknot carrying arbitrary representation is

$$\begin{aligned}&\boldsymbol{\ell.c.}\left[\mathcal{V}^{(2,1)}_{\mathcal{R}_1}(S^3/\mathbb{Z}_p,\lambda)\right]\\&=\left(\boldsymbol{\ell.c.}\left[\mathcal{V}^{(2,1)}_{\square}(S^3/\mathbb{Z}_p,\lambda)\right]\right)^{n^{(1)}}\frac{\chi_{\mathcal{R}_1}(1^{n^{(1)}})}{(n^{(1)})!}.\end{aligned}\tag{5.32}$$

We can generalize (5.31) for (L,L) torus link invariant in the probe approximation and it is given by

$$\begin{aligned}\boldsymbol{\ell.c.}\left[\mathcal{V}^{(L,L)}_{\mathcal{R}_1,\cdots\mathcal{R}_L}(S^3/\mathbb{Z}_p,\lambda)\right]&=\prod_{a=1}^{L}N^{n^{(a)}}\rho_1^{n^{(a)}}\frac{\chi_{\mathcal{R}_a}(1^{n^{(a)}})}{(n^{(a)})!}\\&\equiv\prod_{a=1}^{L}\boldsymbol{\ell.c.}\left[\mathcal{V}^{(2,1)}_{\mathcal{R}_a}(S^3/\mathbb{Z}_p,\lambda)\right].\end{aligned}\tag{5.33}$$

For the (3, 3) torus link, with the three component knots carrying some specific representations with a small number of boxes in the corresponding Young diagrams, we tabulate the colored HOMFLY-PT polynomials and the leading contributions in Table 1. Note that in order to match the matrix model result (5.33) with the tabulated ones we need to follow the prescription as mentioned below:

(i) First, replace $p\to ip$ in the expressions obtained by matrix model calculations to compensate the replacement $p\to -ip$ done earlier [see below Eq. (5.15)];

(ii) then set $p=1$, since the colored HOMFLY-PT polynomials are obtained for framed links embedded inside S^3.

Explicitly, it implies that

$$\mathcal{V}^{(L,L)}_{\mathcal{R}_1,\ldots,\mathcal{R}_L}(S^3/\mathbb{Z}_p,\lambda)\xrightarrow[p=1]{p\to ip,}\mathcal{W}^{[(L,L),f=\{1,\ldots,1\}]}_{\mathcal{R}_1,\ldots,\mathcal{R}_L}(S^3,\lambda).\tag{5.34}$$

When these modifications are taken into account, the expression of ρ_1 (5.29) becomes identical to the leading-order contribution of the vertical framing $U(N)$ invariant $\mathcal{W}^{(2,1)}_{\square}(S^3,\lambda)$ of an unknot with fundamental representation placed on its component.

The leading contributions as obtained from the colored HOMFLY-PT polynomials of (3, 3) torus links (refer to Table 1) are in exact agreement with those (5.33) obtained using the eigenvalue density (5.18) extremizing the partition function.

It is to be noted that, in the large-N limit the evaluation of leading contribution using the eigenvalue density of the dominant representation is only possible when the representations placed on the component knots of the link are much smaller as compared to the rank N of the gauge group at consideration.

For a large representation $\mathcal{R}_a$, the total number of boxes $n^{(a)}$ is of $\mathcal{O}(N)$ which implies some of the $k_r^{(a)}$'s (5.22) can also become of $\mathcal{O}(N)$. As a result, the denominator $z_{\vec{k}}$ in Eq. (5.21) will no longer be of $\mathcal{O}(1)$. Hence the above saddle point analysis will not give correct answer in such scenarios. However, we have a way to find the leading contribution for a class of large representations which will be addressed in the following subsection.

B. Leading contribution to (L,L) torus link invariant for large symmetric representations

In this section, we provide a large-N expression for the (L,L) link invariant when all the representations are large symmetric representations. The $U(N)$ invariant for the (L,L) torus link, embedded inside three manifold $S^3/\mathbb{Z}_p$, carrying integrable representations $\mathcal{R}_1,\ldots,\mathcal{R}_L$ on the component knots is given by

$$\mathcal{V}^{(L,L)}_{\mathcal{R}_1,\ldots,\mathcal{R}_L}(S^3/\mathbb{Z}_p,k)=\frac{1}{\mathcal{Z}(S^3/\mathbb{Z}_p,k)}\sum_{\mathcal{R}}\mathcal{S}^2_{0\mathcal{R}}\mathcal{T}^{-p}_{\mathcal{R}\mathcal{R}}\prod_{a=1}^{L}\mathbf{s}_{\mathcal{R}_a}(U)\equiv\left\langle\prod_{a=1}^{L}\mathbf{s}_{\mathcal{R}_a}(U)\right\rangle.\tag{5.35}$$

Using the character expansion for Schur polynomial (5.21) and (5.13), in the double-scaling limit we can express the above invariant as

$$\mathcal{V}^{(L,L)}_{\mathcal{R}_1,\cdots\mathcal{R}_L}(S^3/\mathbb{Z}_p,\lambda)=\frac{1}{\mathcal{Z}(S^3/\mathbb{Z}_p,\lambda)}\sum_{\vec{k}^{(1)},\cdots\vec{k}^{(L)}}\prod_{a=1}^{L}\frac{\chi_{\mathcal{R}_a}(c(\vec{k}^{(a)}))}{z_{\vec{k}^{(a)}}}\int[D\theta]e^{-N^2S_{\mathrm{eff}}^{(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})}[\rho(\theta)]},\tag{5.36}$$

where

$$\begin{aligned}S_{\mathrm{eff}}^{(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})}[\rho(\theta)]=&-\frac{1}{2}\int d\theta\rho(\theta)⨍d\theta'\rho(\theta')\log\left[4\sin^2\left(\frac{\theta-\theta'}{2}\right)\right]+\frac{p}{\pi\lambda}\int\rho(\theta)\left(\frac{\theta^2}{4}-\frac{\pi^2}{12}\right)d\theta+\frac{\pi p(1-\lambda)}{12\lambda}\\&-\frac{1}{N^2}\sum_r(k_r^{(1)}+\cdots+k_r^{(L)})\log\left[N\int d\theta\rho(\theta)e^{ir\theta}\right].\end{aligned}\tag{5.37}$$

TABLE I. (3, 3) torus link invariants and their leading contributions.

$\mathcal{W}^{[(3,3);f=\{1,1,1\}]}_{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3}(S^3,k)$	The colored HOMFLY-PT polynomial in variables q and v	Leading contribution
$\mathcal{W}^{[(3,3);f]}_{\square,\square,\square}(S^3,k)$	$(-1+q)^{-3}q^{-3/2}(-1+2q-2q^2+q^3-2q^4+2q^5-q^6+v-qv+q^2v+q^3v+q^4v-q^5v+q^6v-qv^2+q^2v^2-3q^3v^2+q^4v^2-q^5v^2+q^3v^3)$	$\frac{i(-1+e^{2\pi i\lambda})^3N^3}{8\pi^3\lambda^3}$
$\mathcal{W}^{[(3,3);f]}_{\square\square,\square,\square}(S^3,k)$	$(-1+q)^{-4}q^{-2}(1+q)^{-1}(1-2q+q^2+q^3-q^4+q^5-q^7+q^8-v+qv-q^3v-q^4v-q^6v-q^9v+qv^2-q^2v^2+q^3v^2+q^4v^2+q^5v^2+q^6v^2+q^7v^2+q^9v^2-q^3v^3+q^4v^3-2q^5v^3-q^6v^3-q^8v^3+q^6v^4)$	$\frac{(-1+e^{2\pi i\lambda})^4N^4}{32\pi^4\lambda^4}$
$\mathcal{W}^{[(3,3);f]}_{\square\square,\square\square,\square}(S^3,k)$	$(-1+q)^{-5}q^{-7/2}(1+q)^{-2}(-1+2q-3q^3+2q^4+q^5-3q^6+2q^7+q^8-3q^9+q^{10}+q^{11}-q^{12}+v-qv-q^2v+2q^3v+q^6v-q^7v+q^8v+2q^9v-q^{11}v+q^{12}v+q^{13}v-qv^2+q^2v^2-q^4v^2-q^5v^2-q^7v^2-q^8v^2-2q^9v^2-2q^{10}v^2+q^{11}v^2-q^{12}v^2-2q^{13}v^2+q^3v^3-q^4v^3+3q^6v^3-q^7v^3+5q^9v^3+2q^{12}v^3+q^{13}v^3-q^6v^4+q^7v^4-q^8v^4-2q^9v^4-q^{11}v^4-q^{12}v^4+q^{10}v^5)$	$-\frac{i(-1+e^{2\pi i\lambda})^5N^5}{128\pi^5\lambda^5}$
$\mathcal{W}^{[(3,3);f]}_{\square\square,\square\square,\square\square}(S^3,k)$	$(-1+q)^6q^{-6}(1+q)^{-3}(1-2q-q^2+5q^3-2q^4-4q^5+4q^6+q^7-3q^8+2q^{10}+q^{12}-2q^{13}-2q^{14}+4q^{15}-2q^{17}+q^{18}-v+qv+2q^2v-3q^3v-q^4v+3q^5v-2q^6v-q^7v+3q^8v-2q^9v-2q^{10}v+q^{11}v-3q^{12}v+q^{13}v+4q^{14}v-4q^{15}v-3q^{16}v+2q^{17}v-q^{19}v+qv^2-q^2v^2-q^3v^2+2q^4v^2+q^7v^2-q^8v^2+q^9v^2+2q^{10}v^2+2q^{11}v^2+q^{12}v^2+2q^{13}v^2-q^{14}v^2+5q^{16}v^2+q^{17}v^2-q^{18}v^2+2q^{19}v^2-q^3v^3+q^4v^3+q^5v^3-3q^6v^3+2q^8v^3-3q^9v^3-q^{10}v^3-5q^{12}v^3-2q^{13}v^3+q^{14}v^3-3q^{15}v^3-3q^{16}v^3-q^{17}v^3-2q^{18}v^3-q^{19}v^3+q^6v^4-q^7v^4-q^8v^4+4q^9v^4-4q^{11}v^4+6q^{12}v^4+4q^{13}v^4-2q^{14}v^4+2q^{15}v^4+3q^{16}v^4+q^{17}v^4+2q^{18}v^4-q^{10}v^5+q^{11}v^5-3q^{13}v^5-2q^{16}v^5-q^{17}v^5+q^{15}v^6)$	$-\frac{(-1+e^{2\pi i\lambda})^6N^6}{512\pi^6\lambda^6}$
$\mathcal{W}^{[(3,3);f]}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square,\square}(S^3,k)$	$(-1+q)^{-5}q^{-5/2}(1+q+q^2)^{-1}(-q+2q^2-2q^3+2q^4-4q^5+5q^6-4q^7+2q^8-2q^9+2q^{10}-q^{11}+v-qv+q^2v-q^3v+3q^4v-q^5v+q^6v-q^7v+3q^8v-q^9v+q^{10}v-q^{11}v+q^{12}v-v^2+qv^2-2q^2v^2+2q^3v^2-6q^4v^2+4q^5v^2-6q^6v^2+4q^7v^2-6q^8v^2+2q^9v^2-2q^{10}v^2+q^{11}v^2-q^{12}v^2+qv^3-q^2v^3+2q^3v^3+4q^5v^3-2q^6v^3+4q^7v^3+2q^9v^3-q^{10}v^3+q^{11}v^3-q^3v^4+q^4v^4-3q^5v^4+q^6v^4-3q^7v^4+q^8v^4-q^9v^4+q^6v^5)$	$-\frac{i(-1+e^{2\pi i\lambda})^5N^5}{96\pi^5\lambda^5}$
$\mathcal{W}^{[(3,3);f]}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square\square,\square}(S^3,k)$	$(-1+q)^6q^{-4}(1+q)^{-1}(1+q+q^2)^{-1}(q-2q^2+q^3+q^5-2q^7+2q^8-q^9+q^{10}-q^{12}+2q^{13}-2q^{14}+q^{15}-v+qv-q^4v-q^5v-2q^{10}v-q^{11}v-q^{16}v+v^2-qv^2+q^2v^2-q^3v^2+2q^4v^2+q^5v^2+q^6v^2+q^7v^2+2q^9v^2+2q^{10}v^2+2q^{11}v^2+2q^{12}v^2-q^{13}v^2+2q^{14}v^2-q^{15}v^2+2q^{16}v^2-qv^3+q^2v^3-q^3v^3-2q^5v^3-q^6v^3-2q^7v^3-q^8v^3-2q^9v^3-3q^{10}v^3-3q^{11}v^3-q^{12}v^3-2q^{13}v^3-q^{15}v^3-q^{16}v^3+q^3v^4-q^4v^4+q^5v^4+q^6v^4+2q^7v^4+q^8v^4+q^9v^4+3q^{10}v^4+3q^{11}v^4+2q^{13}v^4+q^{15}v^4-q^6v^5+q^7v^5-2q^8v^5-2q^{10}v^5-q^{11}v^5-q^{13}v^5+q^{10}v^6)$	$-\frac{(-1+e^{2\pi i\lambda})^6N^6}{384\pi^6\lambda^6}$

For a given $\vec{k}^{(1)},\ldots,\vec{k}^{(L)}$, $\int[D\theta]e^{-N^2 S_{\text{eff}}^{(\vec{k}^{(1)},\ldots,\vec{k}^{(L)})}[\rho(\theta)]}$ is dominated by a classical configuration satisfying the saddle point equation,

$$\fint \rho(\theta')\cot\left(\frac{\theta-\theta'}{2}\right)d\theta' = \frac{p}{2\pi\lambda}\theta - \frac{i}{N^2}\sum_r \frac{r(k_r^{(1)}+\cdots+k_r^{(L)})}{\rho_r}e^{ir\theta}. \quad (5.38)$$

This is a coupled equation because the moments of the distribution $\rho_r = \int d\theta\rho(\theta)e^{ir\theta}$ appear in the last term and hence difficult to solve. The last term is proportional to $1/N^2$. Apparently, it seems that this term will not be important to find $\rho(\theta)$ in the large-N limit but if any of the $\mathcal{R}_a$ is large such that $n^{(a)}$ is of $\mathcal{O}(N^2)$, then following (5.22) some of the $k_r^{(a)}$ can also be of $\mathcal{O}(N^2)$. In such cases we cannot neglect the last term to solve for $\rho(\theta)$.

However, in the case of all representations $(\mathcal{R}_a)$ large but completely symmetric, there can be at most k boxes in the first row and hence $n^{(a)} \le k$. As a result all the $k_r^{(a)}$ s are less than (or equal to) k. Therefore in the large-N limit we can still neglect the last term. This suggest us that in order to compute $\mathcal{V}^{(L,L)}_{\mathcal{R}_1,\cdots\mathcal{R}_L}(S^3/\mathbb{Z}_p,\lambda)$ for large symmetric representations we can still use the probe approximation and calculate the Schur polynomials on the background solution (5.18). Thus, the (L,L) link invariant (5.35) becomes

$$\mathcal{V}^{(L,L)}_{\mathcal{R}_1,\cdots\mathcal{R}_L}(S^3/\mathbb{Z}_p,\lambda) = \prod_{a=1}^{L}\mathbf{s}_{\mathcal{R}_a}(\bar{U}), \quad (5.39)$$

where $\bar{U}$ corresponds to the density distribution solving (5.19). Note that, unlike (5.28), $\mathbf{s}_{\mathcal{R}_a}(\bar{U})$ will no longer be dominated by the conjugacy class having $n^{(a)}$ number of 1-cycles.

Since the character of any conjugacy class of the permutation group for a completely symmetric representation is 1, the Schur polynomial (5.26) can be written as

$$\mathbf{s}_{\mathcal{R}_a}(\bar{U}) = \sum_{\vec{k}^{(a)}}\prod_r \frac{(N\rho_r)^{k_r^{(a)}}}{k_r^{(a)}!\,r^{k_r^{(a)}}} = \sum_{\vec{k}^{(a)}}\exp\left[\sum_r\left(k_r^{(a)}\log\left(\frac{N\rho_r}{r}\right) - k_r^{(a)}\log k_r^{(a)} + k_r^{(a)}\right)\right], \quad (5.40)$$

subjected to the condition $\sum_r rk_r^{(a)} = n^{(a)}$. We use the Lagrange multiplier $\alpha^{(a)}$ to extremize the above summation. Suppose

$$\mathcal{H}^{(a)} = \sum_r\left(k_r^{(a)}\log\left(\frac{N\rho_r}{r}\right) - k_r^{(a)}\log k_r^{(a)} + k_r^{(a)}\right) + \alpha^{(a)}\left(\sum_r rk_r^{(a)} - n^{(a)}\right). \quad (5.41)$$

Extremizing $\mathcal{H}^{(a)}$ with respect to $k_r^{(a)}$ we find,

$$k_r^{(a)} = \frac{N\rho_r}{r}e^{r\alpha^{(a)}}. \quad (5.42)$$

The Lagrange multiplier $\alpha^{(a)}$ can be obtained by substituting $k_r^{(a)}$ in the equation $\sum_r rk_r^{(a)} = n^{(a)}$,

$$\sum_r \rho_r e^{r\alpha^{(a)}} = \frac{n^{(a)}}{N}. \quad (5.43)$$

Finally, evaluating $\mathcal{H}^{(a)}$ on the solution we find

$$\bar{\mathcal{H}}^{(a)} = \sum_r \frac{N\rho_r}{r}e^{r\alpha^{(a)}} - \alpha^{(a)}n^{(a)}. \quad (5.44)$$

Hence, we have

$$\mathbf{s}_{\mathcal{R}_a}(\bar{U}) = e^{\bar{\mathcal{H}}^{(a)}}. \quad (5.45)$$

We can use this result to find the Schur polynomial evaluated on the classical solution. Analytically it is challenging to find the Schur polynomial for any arbitrary values of λ. However, we can find it for small λ.

For small λ the eigenvalue distribution (5.18) is given by

$$\rho(\theta) = \frac{p}{2\pi^2\lambda}\sqrt{\frac{2\pi\lambda}{p} - \frac{\theta^2}{4}}. \quad (5.46)$$

We can compute ρ_r by using the formula (5.25) with $\rho(\theta)$ as defined above and range of θ lying from $-2\sqrt{2\pi\lambda/p}$ to $2\sqrt{2\pi\lambda/p}$. The result is as follows:

$$\rho_r = 1 - \frac{r^2\pi\lambda}{p}. \quad (5.47)$$

Using (5.43) we can find that the Lagrange multiplier is given by

$$\alpha^{(a)} = \log\left(\frac{\kappa^{(a)}}{1+\kappa^{(a)}}\right) + \frac{(1+2\kappa^{(a)})\pi\lambda}{p}, \quad (5.48)$$

where $\kappa^{(a)} = \frac{n^{(a)}}{N}$. Hence $\bar{\mathcal{H}}^{(a)}$ can be evaluated to obtain,

$$\bar{\mathcal{H}}^{(a)} = \log\mathbf{s}_{\mathcal{R}_a} = (1+\kappa^{(a)})\log(1+\kappa^{(a)}) - \kappa^{(a)}\log\kappa^{(a)} - \frac{\kappa^{(a)}(1+\kappa^{(a)})\pi\lambda}{p}. \quad (5.49)$$

Substituting $\bar{\mathcal{H}}^{(a)}$ in Eq. (5.45) and then using (5.39) we can deduce the leading-order contribution for (L,L) torus link invariant whose components carry arbitrary large symmetric representations.

To understand how different components are intertwined in a link we have to go beyond the leading contribution. From the matrix model point of view, the subleading correction is captured by the connected piece of the correlator. The computation of the connected piece becomes tedious involving resolvent and contour integration for a general (L,L) torus link. However, one can extract the subleading contributions from the colored HOMFLY-PT polynomials (as in Table 1) when small representations are placed on the component knots of the torus link.

In order to confirm that the subleading contribution from the colored HOMFLY-PT matches with the matrix model approach, we will explicitly present the computation of connected correlator for the Hopf link invariant in the following section.

VI. SUBLEADING CORRECTIONS TO THE HOPF LINK INVARIANT

In this section we compute the subleading contribution to the invariant of Hopf link. From (5.16) we note that the Hopf link invariant is equal to correlation of two Schur polynomials. Let us consider that both the component knots of the link are in fundamental representation i.e., $\mathcal{R}_1 = \mathcal{R}_2 = \square$. The character expansion of the Schur polynomial (5.21) tells us that $\mathbf{s}_\square(U) = \mathrm{Tr}U$. Thus the Hopf link invariant with both the components in fundamental representations can be expressed as

$$\mathcal{V}^{(2,2)}_{\square,\square}(S^3/\mathbb{Z}_p, k) = \langle \mathrm{Tr}U\mathrm{Tr}U\rangle. \tag{6.1}$$

Likewise, an unknot invariant in fundamental representation is given by

$$\mathcal{V}^{(2,1)}_{\square}(S^3/\mathbb{Z}_p, k) = \langle \mathrm{Tr}U\rangle. \tag{6.2}$$

We introduce a rescaled trace operator,

$$\mathcal{T}\mathrm{r}U^m = \frac{1}{N}\mathrm{Tr}U^m, \qquad m \in \mathbb{Z} \tag{6.3}$$

such that $\langle \mathcal{T}\mathrm{r}U^m\rangle$, when evaluated on saddle point gives a $\mathcal{O}(1)$ number, i.e., the leading contribution to $\langle \mathcal{T}\mathrm{r}U^m\rangle$ becomes of $\mathcal{O}(1)$. The corrections to $\langle \mathcal{T}\mathrm{r}U^m\rangle$ are suppressed by higher powers of N. In terms of rescaled trace operators, the Hopf link and unknot invariants can be written as

$$\begin{aligned}\mathcal{V}^{(2,2)}_{\square,\square}(S^3/\mathbb{Z}_p, k) &= N^2\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle,\\ \mathcal{V}^{(2,1)}_{\square}(S^3/\mathbb{Z}_p, k) &= N\langle \mathcal{T}\mathrm{r}U\rangle.\end{aligned} \tag{6.4}$$

We define the connected piece of a two-point function in the following way [10]:

$$\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c \equiv N^2(\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle - \langle \mathcal{T}\mathrm{r}U\rangle^2), \tag{6.5}$$

where we have put an overall factor of N^2 on the right-hand side such that the leading contribution of $\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c$ is again of $\mathcal{O}(1)$. This connected piece can be expressed in terms of the Hopf link and unknot invariants as

$$\mathcal{V}^{(2,2)}_{\square,\square}(S^3/\mathbb{Z}_p, k) - \left(\mathcal{V}^{(2,1)}_{\square}(S^3/\mathbb{Z}_p, k)\right)^2 = \langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c. \tag{6.6}$$

In the large-N limit, $\langle \mathcal{T}\mathrm{r}U\rangle$ admits the expansion

$$\langle \mathcal{T}\mathrm{r}U\rangle = \langle \mathcal{T}\mathrm{r}U\rangle_0 + \frac{1}{N^2}\langle \mathcal{T}\mathrm{r}U\rangle_1 + \mathcal{O}\left(\frac{1}{N^4}\right), \tag{6.7}$$

where $\langle \mathcal{T}\mathrm{r}U\rangle_0 \equiv \rho_1$ (5.29) is the expectation value of $\mathcal{T}\mathrm{r}U$ evaluated on the saddle point (5.18) and $\langle \mathcal{T}\mathrm{r}U\rangle_1$ is the subleading contribution. Therefore, from (6.5) we see that the subleading contribution to the two-point correlator,

$$\begin{aligned}\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle = \langle \mathcal{T}\mathrm{r}U\rangle_0^2 &+ \frac{1}{N^2}(2\langle \mathcal{T}\mathrm{r}U\rangle_0\langle \mathcal{T}\mathrm{r}U\rangle_1\\ &+ \langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c) + \mathcal{O}\left(\frac{1}{N^4}\right),\end{aligned} \tag{6.8}$$

is the bracketed term multiplying $1/N^2$. It suggests that at the large-N perturbation, we can write the Hopf link and unknot invariants (6.4) as

$$\begin{aligned}\mathcal{V}^{(2,2)}_{\square,\square}(S^3/\mathbb{Z}_p, \lambda) &= N^2\langle \mathcal{T}\mathrm{r}U\rangle_0^2 + (2\langle \mathcal{T}\mathrm{r}U\rangle_0\langle \mathcal{T}\mathrm{r}U\rangle_1 + \langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c) + \mathcal{O}\left(\frac{1}{N^2}\right),\\ \mathcal{V}^{(2,1)}_{\square}(S^3/\mathbb{Z}_p, \lambda) &= N\langle \mathcal{T}\mathrm{r}U\rangle_0 + \frac{1}{N}\langle \mathcal{T}\mathrm{r}U\rangle_1 + \mathcal{O}\left(\frac{1}{N^3}\right).\end{aligned} \tag{6.9}$$

The subleading correction to Hopf link invariant has two parts. $\langle \mathcal{T}\mathrm{r}U\rangle_0\langle \mathcal{T}\mathrm{r}U\rangle_1$, which being a product of the leading and subleading contributions to the unknot invariant is insensitive to the intertwining between the two component knots constituting

Hopf link. The other contribution $\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c$, viz. the connected piece of the two-point correlator, contains information about how the two unknots are intertwined in a Hopf link when all the representations are fundamental. That is, it contains the information about the topological nature of the Hopf link. Instead of fundamental, if we consider the Hopf link in some other representations, the structure of the right-hand side of (6.6) will be very different. We shall discuss this at the end of this section.

Our next goal is to compute $\langle \mathcal{T}\mathrm{r}U\rangle_1$ and $\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c$ using the methods developed in [10,11][4] and show that the matrix model results are consistent to those obtained from the HOMFLY-PT polynomials.

A. Calculation of $\langle \mathcal{T}\mathrm{r}U\rangle_1$ and $\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c$

Introducing a set of complex variables $z_i = e^{i\theta_i}$, the partition function (5.14) can be written as

$$\mathcal{Z}(S^3/\mathbb{Z}_p) = \int \prod_i dz_i \Delta(z_i) e^{-N\sum_{i=1}^N V(z_i)}, \tag{6.10}$$

where $\Delta(z_i) = \prod_{i<j}(z_i - z_j)^2$ and the potential $V(z_i)$ is given by

$$V(z_i) = -\frac{p}{4\pi\lambda}(\log z_i)^2 + \log z_i - \frac{\pi p}{12}. \tag{6.11}$$

In the large-N limit with the continuous variables $x = i/N$, $z_i \to z(x)$ and $\sum_i = N\int_0^1 dx$, we can write

$$-N\sum_{i=1}^N V(z_i) = -N^2 \int_0^1 dx V(z(x)), \tag{6.12}$$

where

$$V(z) = -\frac{p}{4\pi\lambda}(\log z)^2 + \log z - \frac{\pi p}{12}. \tag{6.13}$$

Following (5.17) we define the eigenvalue density in the complex z-plane as

$$\rho(z) = \left\langle \frac{1}{N}\sum_{i=1}^N \delta(z - z_i)\right\rangle. \tag{6.14}$$

In the large-N limit the above density satisfies the saddle point equation,

$$2 ⨍ dw \frac{\rho(w)}{z - w} = V'(z). \tag{6.15}$$

[4]The techniques to compute subleading corrections developed in Refs. [10,11] are for Hermitian matrix models. Here we develop the similar techniques for unitary matrix models.

1. One-loop correlator

We introduce the one-loop correlator (also known as resolvent) $R(z)$ as

$$R(z) = \frac{1}{N}\left\langle \mathrm{Tr}\frac{1}{z - U}\right\rangle \equiv \frac{1}{N}\sum_{m=0}^{\infty} z^{-1-m}\langle \mathrm{Tr}U^m\rangle. \tag{6.16}$$

Since $\langle \mathrm{Tr}U^m\rangle/N \leq 1$ for any m, $R(z)$ is an analytic function in the complex z plane both inside and outside the circle $|z| = 1$. In diagonal basis the eigenvalues of U lie on $|z| = 1$ circle. Hence the function $R(z)$ has poles at the eigenvalues of U. In the large N limit, the eigenvalues get accumulated and form a distribution on disconnected finite arcs on the unit circle. As a result $R(z)$ develops branch cut singularities on these arcs. Using (6.14) we can write

$$R(z) = \int dw \frac{\rho(w)}{z - w}. \tag{6.17}$$

Using the Sokhotski-Plemelj theorem,

$$\lim_{\epsilon\to 0}\int dx \frac{f(x)}{x_0 - x \mp i\epsilon} = ⨍ dx \frac{f(x)}{x_0 - x} \pm i\pi f(x_0), \tag{6.18}$$

we find that the resolvent satisfies

$$\lim_{\epsilon\to 0}(R(z + i\epsilon) + R(z - i\epsilon)) = V'(z) \tag{6.19}$$

and

$$\lim_{\epsilon\to 0}(R(z + i\epsilon) - R(z - i\epsilon)) = -2\pi i\rho(z). \tag{6.20}$$

Moreover $R(z)$ has the following asymptotic $(z \to \infty)$ structure:

$$R(z\to\infty) = \frac{1}{z} + \frac{1}{z^2}\langle \mathcal{T}\mathrm{r}U\rangle + \frac{1}{z^3}\langle \mathcal{T}\mathrm{r}U^2\rangle + \mathcal{O}(1/z^4). \tag{6.21}$$

Thus, $R(z)$ is the generating function for $\langle \mathcal{T}\mathrm{r}U^m\rangle$ for different $m > 0$. The resolvent $R(z)$ admits a perturbative large-N expansion

$$R(z) = R_0(z) + \frac{1}{N^2}R_1(z) + \cdots. \tag{6.22}$$

The leading contribution of the resolvent, $R_0(z)$ can be obtained by solving the Dyson-Schwinger equation [23,24] in the $N \to \infty$ limit. $R_0(z)$ also has a branch cut on the unit circle. The eigenvalue density $\rho(z)$ can be determined from the discontinuity of $R_0(z)$ on the branch cut. The higher-genus contributions $R_1(z)$, $R_2(z)$, etc., then can be calculated using loop equations order-by-order [11]. For a given potential $V(w)$, if the eigenvalue density has support between a and b on the unit circle then $R_0(z)$ can be expressed as [25]

$$R_0(z) = \frac{1}{2}\oint_{\mathcal{C}} \frac{dw}{2\pi i}\frac{V'(w)}{z-w}\sqrt{\frac{(z-a)(z-b)}{(w-a)(w-b)}}, \tag{6.23}$$

where $\mathcal{C}$ is a counterclockwise contour around the branch cut between a and b. The spectral edges (a, b) can be found using the properties of the asymptotic expansion of $R_0(z)$. We can do the contour integral (6.23) for the potential $V(w)$ given in Eq. (6.13). Since the potential (5.15) is an even function of θ, the eigenvalue density is symmetric about $\theta = 0$. As a result, in the complex z-plane the branch cuts of $R(z)$ are symmetrically located on the unit circle about the real axis. Hence, we can take

$$a = b^{-1} = e^{i\theta}. \tag{6.24}$$

The final answer is given by

$$R_0(z) = \frac{1}{2z} - \frac{p}{4\pi\lambda z}\log\left[\frac{z^2(\cos\theta - z + \sqrt{z^2-2z\cos\theta+1})}{1-z\cos\theta+\sqrt{z^2-2z\cos\theta+1}}\right]. \tag{6.25}$$

It is easy to check that $R_0(z)$ has a branch cut singularity between $\theta = \pm 2\ \sec^{-1} e^{\pi\lambda/p}$ on the unit circle in the complex z plane. This implies the spectral edges are given by

$$a = b^{-1} = \exp(2i \sec^{-1} e^{\pi\lambda/p}). \tag{6.26}$$

Using an expression analogous to (6.20), we can calculate the discontinuity of $R_0(z)$ and find the leading eigenvalue density. The answer is given in (5.18). The coefficient of $1/z^2$ in the asymptotic expansion of $R_0(z)$ can be computed and that matches with leading contribution of the expectation value of $\mathcal{T}\mathrm{r}U$ given in (5.29) for $p = 1$.

Our next goal is to compute $R_1(z)$. The coefficient of $1/z^2$ in the asymptotic expansion of $R_1(z)$ gives $1/N^2$ correction of $\langle \mathcal{T}\mathrm{r}U\rangle$ viz. $\langle \mathcal{T}\mathrm{r}U\rangle_1$. We follow the prescription given in [11] to write $R_1(z)$ in terms of a series of functions. We shall not elaborate on the method in this paper. $R_1(z)$ is given by

$$R_1(z) = \frac{1}{8(a-b)}(\psi^{(1)}(z) - \chi^{(1)}(z)) + \frac{1}{16}(\psi^{(2)}(z) + \chi^{(2)}(z)), \tag{6.27}$$

where

$$\begin{aligned}
\chi^{(1)}(z) &= \frac{1}{M_1}\Phi_a^{(1)}(z),\\
\psi^{(1)}(z) &= \frac{1}{J_1}\Phi_b^{(1)}(z),\\
\chi^{(2)}(z) &= \frac{1}{M_1}(\Phi_a^{(2)}(z) - \chi^{(1)}(z)M_2),\\
\psi^{(2)}(z) &= \frac{1}{J_1}(\Phi_b^{(2)}(z) - \psi^{(1)}(z)J_2).
\end{aligned} \tag{6.28}$$

The functions $\Phi_a^{(n)}(z)$ and $\Phi_b^{(n)}(z)$ are given by

$$\begin{aligned}
\Phi_a^{(n)}(z) &= \frac{1}{(z-a)^n\sqrt{(z-a)(z-b)}},\\
\Phi_b^{(n)}(z) &= \frac{1}{(z-b)^n\sqrt{(z-a)(z-b)}}.
\end{aligned} \tag{6.29}$$

M_n and J_n are given by the following contour integrals:

$$\begin{aligned}
M_n &= \oint_{\mathcal{C}} \frac{d\omega}{2\pi i}\frac{V'(\omega)}{(\omega-a)^n\sqrt{(\omega-a)(\omega-b)}},\\
J_n &= \oint_{\mathcal{C}} \frac{d\omega}{2\pi i}\frac{V'(\omega)}{(\omega-b)^n\sqrt{(\omega-a)(\omega-b)}}.
\end{aligned} \tag{6.30}$$

Evaluating the contour integrals (6.30) we find,

$$\begin{aligned}
M_1 &= \frac{4A}{a^{3/2}(\sqrt{a}+\sqrt{b})}, \quad J_1 = \frac{4A}{b^{3/2}(\sqrt{a}+\sqrt{b})},\\
M_2 &= -\frac{4A(5\sqrt{a}+4\sqrt{b})}{3a^{5/2}(\sqrt{a}+\sqrt{b})^2}, \quad J_2 = -\frac{4A(4\sqrt{a}+5\sqrt{b})}{3b^{5/2}(\sqrt{a}+\sqrt{b})^2},
\end{aligned} \tag{6.31}$$

where $A = -\frac{p}{4\pi\lambda}$. Plugging all these expressions in (6.27) we finally arrive at

$$R_1(z) = -\frac{\pi\lambda(1+z)(1-8z\cos\theta-6z+z^2)\sin^2\frac{\theta}{2}}{12p(1-2z\cos\theta+z^2)^{5/2}}. \tag{6.32}$$

We expand $R_1(z)$ asymptotically to obtain

$$R_1(z) = -\frac{\pi\lambda\sin^2\frac{\theta}{2}}{12pz^2} + \frac{\pi\lambda\sin^2\frac{\theta}{2}(3\cos\theta+5)}{12pz^3} + \mathcal{O}\left(\frac{1}{z^4}\right). \tag{6.33}$$

Thus, from the asymptotic expansion we collect the coefficient of $1/z^2$ and get

$$\langle \mathcal{T}\mathrm{r}U\rangle_1 = -\frac{\pi\lambda\sin^2\frac{\theta}{2}}{12p} = -\frac{\pi\lambda(1-e^{-\frac{2\pi\lambda}{p}})}{12p}. \tag{6.34}$$

Following the prescription (5.34) we find

$$\langle \mathcal{T}\mathrm{r}U\rangle_1 = \frac{i\pi\lambda(1-e^{2i\pi\lambda})}{12}. \tag{6.35}$$

The result (6.35) matches exactly with the subleading contribution of the vertical framing $U(N)$ invariant of an unknot $\mathcal{W}_{\square}^{(2,1)}(S^3, \lambda)$ carrying fundamental representation on its component.

2. Two-loop correlator

We next compute the leading connected piece of two-point function $\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle$. We define two-loop correlator as

$$R(z,w) = \frac{1}{N^2}\left\langle \mathrm{Tr}\frac{1}{z-U}\mathrm{Tr}\frac{1}{w-U}\right\rangle. \tag{6.36}$$

In the large-N limit the two-loop correlator is equal to the product of two one-loop correlators $R(z)$ and $R(w)$. There is a connected subleading piece, denoted by $R^{(c)}(z,w)$ and suppressed by a factor of N^2,

$$R(z,w) = R(z)R(w) + \frac{1}{N^2}R^{(c)}(z,w). \tag{6.37}$$

Our goal is to find the connected piece of the two-point correlator. For the partition function (6.10), the eigenvalue density can be written as

$$\rho(z) = \left\langle \frac{1}{N}\sum_{i=1}^{N}\delta(z-z_i)\right\rangle = -\frac{1}{N^2}\frac{\delta}{\delta V(z)}\log\mathcal{Z}. \tag{6.38}$$

Here we have used the fact that $\frac{\delta V(x)}{\delta V(y)} = \delta(x-y)$. Repeating a similar analysis we can write

$$\frac{\delta\rho(z)}{\delta V(w)} = -N^2(\rho(z,w) - \rho(z)\rho(w)), \tag{6.39}$$

where

$$\rho(z,w) = \frac{1}{N^2}\left\langle \sum_{i,j=1}^{N}\delta(z-z_i)\delta(w-z_i)\right\rangle. \tag{6.40}$$

In the large-N limit, $\rho(z,w)$ is also decomposed into a product of two eigenvalue densities $\rho(z)$ and $\rho(w)$. The connected subleading contribution $\rho_c(z,w)$ is suppressed by the $1/N^2$ factor,

$$\rho(z,w) = \rho(z)\rho(w) + \frac{1}{N^2}\rho_c(z,w). \tag{6.41}$$

Hence, we find

$$\frac{\delta\rho(z)}{\delta V(w)} = -\rho_c(z,w). \tag{6.42}$$

We know that using the definition of eigenvalue density, the one-loop correlator $R(z)$ can be expressed as (6.17). After a little algebra we find that

$$\int \frac{dv}{w-v}\frac{\delta R(z)}{\delta V(v)} = -\int du\int dv\frac{\rho_c(u,v)}{(z-u)(w-v)}. \tag{6.43}$$

Using the definition (6.40) we can express two-point correlator as

$$R(z,w) = \int du\int dv\frac{\rho(u,v)}{(z-u)(w-v)}. \tag{6.44}$$

Likewise, for the connected components of $R(z,w)$ and $\rho(u,v)$ we can write

$$R^{(c)}(z,w) = \int du\int dv\frac{\rho_c(u,v)}{(z-u)(w-v)}. \tag{6.45}$$

Thus, we have

$$\int \frac{dv}{w-v}\frac{\delta R(z)}{\delta V(v)} = -R^{(c)}(z,w). \tag{6.46}$$

Using the analytic properties (6.19) of $R(z)$ about the cut we find that

$$R^{(c)}(z+i\epsilon,w) + R^{(c)}(z-i\epsilon,w) = -\frac{1}{(z-w)^2}. \tag{6.47}$$

Since $R(z,w)$ is symmetric in z and w a similar property holds for the w variable also. The connected piece of the resolvent, $R^{(c)}(z,w)$ being discontinuous about the cut, $R^{(c)}(z,w) + \frac{1}{2(z-w)^2}$ must change sign as we cross the cut for both the arguments. Therefore, we take an ansatz for $R^{(c)}(z,w)$ as follows:

$$R^{(c)}(z,w) = -\frac{1}{2(z-w)^2}\left(1 - \frac{Q(z,w)}{\sqrt{\sigma(z)}\sqrt{\sigma(w)}}\right), \quad \text{where}$$
$$\sigma(z) = (z-a)(z-b). \tag{6.48}$$

$Q(z,w)$ is symmetric meromorphic in its arguments. $Q(z,w)$ can be found for a one-gap solution from the properties of $R^{(c)}(z,w)$. From the definition of $R(z,w)$ we see that it is regular as $z\to w$. Expanding $R^{(c)}(z,w)$ near $w=z$ we arrive at

$$R^{(c)}(z,w) = -\frac{1}{2(z-w)^2}\left(1 - \frac{Q(z,z)}{\sigma(z)} - \frac{(w-z)}{\sigma(z)}\left(\partial_w Q(z,w) - \frac{\sigma'(w)Q(z,w)}{2\sigma(z)}\right)\bigg|_{w=z}\right). \tag{6.49}$$

Therefore, to make $R^{(c)}(z,w)$ finite at $z=w$ we have

$$\lim_{w\to z} Q(z,w)=\sigma(z) \quad \text{and} \quad \lim_{w\to z}\partial_w Q(z,w)=\frac{\sigma'(z)}{2}. \tag{6.50}$$

$Q(z,w)$ must be regular at the spectral edges; hence, $Q(z,w)$ must be a polynomial. Since $R(z,z)$ goes as $1/z^2$ asymptotically, for the one-cut solution $Q(z,z)$ must be a polynomial of order 2. The above conditions can uniquely determine Q for a one-cut solution and it is given by

$$Q(z,w)=zw-\frac{1}{2}(a+b)(z+w)+ab. \tag{6.51}$$

The leading contribution of $\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c$ can be obtained from the $\frac{1}{z^2w^2}$ coefficient of $R^{(c)}(z,w)$ in its asymptotic expansion. After a careful computation we find that,

$$\boldsymbol{\ell.c.}[\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c]=\frac{(a-b)^2}{16}=e^{-\frac{2\pi\lambda}{p}}(e^{-\frac{2\pi\lambda}{p}}-1), \tag{6.52}$$

where we have used the expressions of the spectral edges (6.26). With the prescription (5.34) of analytic continuation $p\to ip$, followed by setting $p=1$ we arrive at

$$\boldsymbol{\ell.c.}[\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c]=e^{2i\pi\lambda}(e^{2i\pi\lambda}-1). \tag{6.53}$$

The result (6.53) matches exactly with the following result from the HOMFLY-PT polynomials:

$$\mathcal{W}^{[(2,2),f=\{1,1\}]}_{\square,\square}(S^3,\lambda)-(\mathcal{W}^{(2,1)}_{\square}(S^3,\lambda))^2. \tag{6.54}$$

This suggests that $\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c$ evaluated in (6.52) is exact in N. Any further large-N corrections of $\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c$ from the matrix model approach will vanish. Thus, we have

$$\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c=e^{-\frac{2\pi\lambda}{p}}(e^{-\frac{2\pi\lambda}{p}}-1). \tag{6.55}$$

B. Knot/link invariants and connected correlators

So far we have confined ourselves to Hopf link invariant with fundamental representation placed on its components. Now we look at the connected piece of two-point correlator carrying some low-dimensional representations apart from the fundamental ones. Using the fact that Schur polynomials for any arbitrary representation can be written as a sum over different powers of $\mathrm{Tr}U^m$ (5.21), we can express the large-N results of any (torus) knot or link invariant modulo its leading contribution in terms of connected correlation functions of $\mathrm{Tr}U^m$. For example, consider the unknot invariant in representation $\square\square$, i.e., $\mathcal{V}^{(2,1)}_{\square\square}(S^3/\mathbb{Z}_p,\lambda)$. Using (5.32) the leading contribution can be read off to be

$$\boldsymbol{\ell.c.}\left[\mathcal{V}^{(2,1)}_{\square\square}(S^3/\mathbb{Z}_p,\lambda)\right]=\frac{1}{2}\left(\boldsymbol{\ell.c.}\left[\mathcal{V}^{(2,1)}_{\square}(S^3/\mathbb{Z}_p,\lambda)\right]\right)^2. \tag{6.56}$$

We know that [(5.35) and (5.21)]

$$\mathcal{V}^{(2,1)}_{\square\square}(S^3/\mathbb{Z}_p,k)=\langle \mathbf{s}_{\square\square}\rangle, \quad \text{where} \quad \mathbf{s}_{\square\square}=\frac{N}{2}\left(\mathcal{T}\mathrm{r}U^2\right)+\frac{N^2}{2}\left(\mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\right). \tag{6.57}$$

Hence, we have

$$\mathcal{V}^{(2,1)}_{\square\square}(S^3/\mathbb{Z}_p,k)-\frac{1}{2}\left(\mathcal{V}^{(2,1)}_{\square}(S^3/\mathbb{Z}_p,k)\right)^2=\frac{N}{2}\left\langle\mathcal{T}\mathrm{r}U^2\right\rangle+\frac{1}{2}\left\langle\mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\right\rangle_c. \tag{6.58}$$

This expression can also seen to be consistent using cabling prescription [26] and Frobenius relation in group theory [12,14,27]. In a similar fashion $\mathcal{V}^{(2,1)}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(S^3/\mathbb{Z}_p,k)$ admits a relation

$$\mathcal{V}^{(2,1)}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(S^3/\mathbb{Z}_p,k)-\frac{1}{2}\left(\mathcal{V}^{(2,1)}_{\square}(S^3/\mathbb{Z}_p,k)\right)^2=-\frac{N}{2}\langle\mathcal{T}\mathrm{r}U^2\rangle+\frac{1}{2}\langle\mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c. \tag{6.59}$$

Thus, we see that when we change the representations, the unknot invariants (modulo the leading contribution) have different expansions in terms of correlation functions of $\mathcal{T}\mathrm{r}U^m$ operators.

A similar analysis can be done for a two-component Hopf link whose invariant $\mathcal{V}^{(2,2)}_{\square\square,\square}(S^3/\mathbb{Z}_p,k) = \langle \mathbf{s}_{\square\square}\mathbf{s}_{\square}\rangle$ satisfies,

$$\begin{aligned}\mathcal{V}^{(2,2)}_{\square\square,\square}(S^3/\mathbb{Z}_p,k) - \mathcal{V}^{(2,1)}_{\square\square}(S^3/\mathbb{Z}_p,k)\mathcal{V}^{(2,1)}_{\square}(S^3/\mathbb{Z}_p,k) &= N\langle \mathrm{Tr}U\rangle\langle \mathrm{Tr}U\mathrm{Tr}U\rangle_c \\ &\quad + \frac{1}{2}\langle \mathrm{Tr}U\mathrm{Tr}U\mathrm{Tr}U\rangle_c + \frac{1}{2}\langle \mathrm{Tr}U\mathrm{Tr}U^2\rangle_c.\end{aligned} \tag{6.60}$$

In fact any n-point correlator can be expressed in terms of the connected correlators as

$$\begin{aligned}\langle\underbrace{\mathcal{T}\mathrm{r}U\cdots\mathcal{T}\mathrm{r}U}_{n\ \text{times}}\rangle &= \langle\mathcal{T}\mathrm{r}U\rangle^n + \frac{{}^nC_2}{N^2}\langle\mathcal{T}\mathrm{r}U\rangle^{n-2}\langle\mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c + \frac{{}^nC_3}{N^3}\langle\mathcal{T}\mathrm{r}U\rangle^{n-3}\langle\mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c \\ &\quad + \frac{{}^nC_4}{N^4}\langle\mathcal{T}\mathrm{r}U\rangle^{n-4}({}^4C_2\langle\mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c^2 + \langle\mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c) \\ &\quad + \frac{{}^nC_5}{N^5}\langle\mathcal{T}\mathrm{r}U\rangle^{n-5}({}^5C_2\langle\mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c\langle\mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c + \langle(\mathcal{T}\mathrm{r}U)^5\rangle_c) \\ &\quad + \frac{{}^nC_6}{N^6}\langle\mathcal{T}\mathrm{r}U\rangle^{n-6}({}^6C_2\langle(\mathcal{T}\mathrm{r}U)^2\rangle_c\langle(\mathcal{T}\mathrm{r}U)^4\rangle_c + {}^6C_3\langle(\mathcal{T}\mathrm{r}U)^3\rangle_c^2 + \langle(\mathcal{T}\mathrm{r}U)^6\rangle_c) \\ &\quad + \cdots.\end{aligned} \tag{6.61}$$

In order to validate the result with the colored HOMFLY-PT polynomial we compute the leading contribution of (6.60) in the matrix model side. It is given by [(5.29) and (6.52)]

$$\begin{aligned}\boldsymbol{\ell.c.}\left[\mathcal{V}^{(2,2)}_{\square\square,\square}(S^3/\mathbb{Z}_p,\lambda) - \mathcal{V}^{(2,1)}_{\square\square}(S^3/\mathbb{Z}_p,\lambda)\mathcal{V}^{(2,1)}_{\square}(S^3/\mathbb{Z}_p,\lambda)\right] &= N\langle\mathrm{Tr}U\rangle_0\langle\mathrm{Tr}U\mathrm{Tr}U\rangle_c \\ &= Np\left(\frac{1-e^{-\frac{2\pi\lambda}{p}}}{2\pi\lambda}\right)e^{-\frac{2\pi\lambda}{p}}\left(e^{-\frac{2\pi\lambda}{p}}-1\right).\end{aligned} \tag{6.62}$$

Performing the analytic continuation $p\to ip$ followed by setting $p=1$ (5.34) the above expression becomes identical to the one obtained from the corresponding colored HOMFLY-PT polynomials,

$$\boldsymbol{\ell.c.}\left[\mathcal{W}^{[(2,2);f=\{1,1\}]}_{\square\square,\square}(S^3,\lambda) - \mathcal{W}^{(2,1)}_{\square\square}(S^3,\lambda)\mathcal{W}^{(2,1)}_{\square}(S^3,\lambda)\right] = \frac{2iN}{\pi\lambda}e^{4\pi i\lambda}\sin^2\pi\lambda. \tag{6.63}$$

We provide one more example before concluding this section,

$$\begin{aligned}\mathcal{V}^{(2,2)}_{\square\square,\square\square}(S^3/\mathbb{Z}_p,k) - \left(\mathcal{V}^{(2,1)}_{\square\square}(S^3/\mathbb{Z}_p,k)\right)^2 =&\ N^2\langle\mathrm{Tr}U\rangle^2\langle\mathrm{Tr}U\mathrm{Tr}U\rangle_c \\ &+N\big(\langle\mathrm{Tr}U\rangle\langle\mathrm{Tr}U\mathrm{Tr}U\mathrm{Tr}U\rangle_c + \langle\mathrm{Tr}U\rangle\langle\mathrm{Tr}U\mathrm{Tr}U^2\rangle_c\big) \\ &+\frac{5}{4}\langle\mathrm{Tr}U\mathrm{Tr}U\rangle_c^2 + \frac{1}{4}\langle\mathrm{Tr}U\mathrm{Tr}U\mathrm{Tr}U\mathrm{Tr}U\rangle_c \\ &+\frac{1}{4}\langle(\mathrm{Tr}U^2)^2\rangle_c + \frac{1}{2}\langle(\mathrm{Tr}U)^2\mathrm{Tr}U^2\rangle_c.\end{aligned} \tag{6.64}$$

Again we can compute the leading contribution of (6.64) in the double-scaling limit using the matrix model results (5.29) and (6.52),

$$\begin{aligned}\boldsymbol{\ell.c.}\left[\mathcal{V}^{(2,2)}_{\square\square,\square\square}(S^3/\mathbb{Z}_p,\lambda) - \left(\mathcal{V}^{(2,1)}_{\square\square}(S^3/\mathbb{Z}_p,\lambda)\right)^2\right] &= N^2\langle\mathrm{Tr}U\rangle_0^2\langle\mathrm{Tr}U\mathrm{Tr}U\rangle_c \\ &= N^2p^2\left(\frac{1-e^{-\frac{2\pi\lambda}{p}}}{2\pi\lambda}\right)^2 e^{-\frac{2\pi\lambda}{p}}\left(e^{-\frac{2\pi\lambda}{p}}-1\right).\end{aligned} \tag{6.65}$$

Now we analytically continue $p \to ip$ and set $p = 1$ (5.34) to get

$$\boldsymbol{\ell.c.}\left[\mathcal{W}^{[(2,2);f=\{1,1\}]}_{\square\square,\square\square}(S^3,\lambda) - \left(\mathcal{W}^{(2,1)}_{\square\square}(S^3,\lambda)\right)^2\right] = \frac{2iN^2}{\pi^2\lambda^2}e^{5\pi i\lambda}\sin^3\pi\lambda. \tag{6.66}$$

We see from Eqs. (6.6), (6.60), and (6.64) that as we change the representations associated with two of the components of Hopf link, the expansion in terms of correlation functions also get modified. Therefore, any unknot or Hopf link invariant (modulo the leading contribution) with arbitrary representations placed on the components can be uniquely expressed in terms of different connected correlation functions. For example, from Eqs. (6.58) and (6.59) we see that unknots in representations $\square\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ are expressed in terms of $\langle \mathcal{T}\mathrm{r}U\rangle$ and $\langle \mathcal{T}\mathrm{r}U\mathcal{T}\mathrm{r}U\rangle_c$. However, they differ by the sign of $\langle \mathcal{T}\mathrm{r}U\rangle$ term. Using the character expansion for Schur polynomial (5.21) and Eq. (6.61) one can similarly express any (L,L) torus link invariant uniquely in terms of connected correlation functions.

It is also to be noted that the evaluation of the corresponding leading contributions as in Eqs. (6.62) and (6.64) using the matrix model results (5.29) and (6.52) was possible because we have placed low-dimensional representations on the two component knots of the Hopf link. If we put arbitrary large representations it will be difficult to evaluate the explicit expressions of leading contributions with the matrix model techniques. This is because the results (5.29) and (6.52) involve the density distribution which extremizes the partition function of the manifold and any link carrying large representation will backreact on that solution.

Moreover, for any arbitrary L component torus link the computation of the connected correlator seems technically challenging from the matrix model side, even with low dimensional representations placed on the component knots. Because such invariants will involve higher point connected correlators, which are difficult to handle. Nonetheless, for low representations the colored HOMFLY-PT polynomial (3.19) allows us to obtain the subleading contribution for any $L > 2$. For example the polynomials tabulated in Table 1 will help us to get the subleading contribution to the (3, 3) torus link invariant.

VII. CONCLUSION

In this paper, we have discussed the torus link invariants in $U(N)$ Chern-Simons theory embedded inside three manifold S^3 as well as $S^3/\mathbb{Z}_p$. In the double-scaling limit (1.1), it has been shown that any $(L\alpha, L\beta)$ torus link invariant in S^3 (3.19) can be expressed in terms of an (L,L) torus link invariant where α and β are coprime to each other. Our result is depicted in Eq. (4.22). Using the explicit expression of the $(L\alpha, L\beta)$ invariant (3.19) we can evaluate the polynomial for some low-dimensional representations placed on the L-component knots of the link (e.g., see Table 1). Consequently, in the large-N limit, we can deduce their leading and subleading contributions. However, for links carrying higher-dimensional representations this procedure seems practically challenging. Matrix model techniques seem to provide some handle of such difficulties at certain instances. For example, when the representations placed on the component knots of any (L,L) torus link (5.35) is large (i.e., comparable to rank N of the gauge group in the limit $N \to \infty$) but completely symmetric we can determine the leading contribution (5.49) to the invariant by using the eigenvalue density (5.18) that dominates partition function (5.2) of the manifold $S^3/\mathbb{Z}_p$. Moreover, for low-dimensional representations, using the aforementioned density distribution, we have written down an analytic expression of the leading contribution of any (L,L) torus link invariant (5.33). The leading contribution of the torus link invariant being proportional to the leading contribution of the unknot invariant, it does not capture the intertwining information of the component knots constituting the link. To capture such an information the study of connected piece of the correlators in Chern-Simons theory become necessary. From the matrix model approach, we analyze in detail the connected piece of a two-point correlator (6.5), which is basically the subleading contribution to the Hopf link invariant when both the component knots carry fundamental representations (6.6). Our detailed analysis lead to the result in Eq. (6.52) and it matches exactly with the one obtained using the HOMFLY-PT polynomials (6.53). We also obtain the subleading contribution to the unknot invariant carrying fundamental representation (6.34), using the techniques of matrix model. It is interesting to see that if we place low-dimensional representations, other than the fundamental ones, then the Hopf link or unknot invariant modulo the corresponding leading contribution take different structures expressed in terms of correlators of $\mathcal{T}\mathrm{r}U^m$ operators as shown in Eqs. (6.58), (6.59), (6.60), and (6.64). As a validation, we specifically evaluate the leading contribution of (6.60) and (6.64) using our matrix model computations (5.29), (6.52) and find the result to agree with the respective colored-HOMFLY-PT polynomials.

Note that, in the double scaling limit Chern-Simons theory admits a third-order phase transition similar to Douglas-Kazakov phase transition [9]. This means the integrable representation (5.18) that dominates the partition function (5.2) changes at some critical value of λ. As at result the correlation functions when evaluated on the leading solution must change its behavior at the critical value. However, the origin of such behavior is not clear from the perspective of HOMFLY-PT polynomials.

ACKNOWLEDGMENTS

The work of A. M. is supported in part by the Prime Minister's Research Fellowship provided by the Ministry of Education, Government of India. The work of K. C. is supported in part by the South African Research Chairs Initiative (SARChI) of the Department of Science and Innovation and the National Research Foundation. P. R. would like to acknowledge Ministry of Human Resource Development, Government of India, SPARC/2019-2020/P2116/ project funding. P. R. would also like to acknowledge the ICTP's Associate program where some work on this project had happened during her visit as a senior associate.

[1] V. F. R. Jones, A polynomial invariant for knots via von Neumann algebras, Bull. Am. Math. Soc. **12**, 103 (1985).

[2] Vaughan F. R. Jones, Hecke algebra representations of braid groups and link polynomials, in *New Developments in the Theory of Knots* (World Scientific, Singapore, 1987), pp. 20–73.

[3] James W. Alexander, Topological invariants of knots and links, Trans. Am. Math. Soc. **30**, 275 (1928).

[4] Peter Freyd, David Yetter, Jim Hoste, W. B. R. Lickorish, Kenneth Millett, and Adrian Ocneanu, A new polynomial invariant of knots and links, Bull. Am. Math. Soc. **12**, 239 (1985).

[5] Józef H. Przytycki and Pawel Traczyk, Invariants of links of Conway type, Kobe J. Math. **4**, 115 (1987), https://doi.org/10.48550/arXiv.1610.06679.

[6] Edward Witten, Quantum field theory and the Jones polynomial, Commun. Math. Phys. **121**, 351 (1989).

[7] Louis H. Kauffman, An invariant of regular isotopy, Trans. Am. Math. Soc. **318**, 417 (1990).

[8] Kushal Chakraborty and Suvankar Dutta, Large *n* correlators of Chern-Simons theory in lens spaces, Phys. Rev. D **106**, 025009 (2022).

[9] Kushal Chakraborty and Suvankar Dutta, New phase in Chern-Simons theory on lens space, Phys. Rev. D **104**, 026010 (2021).

[10] Bertrand Eynard, Taro Kimura, and Sylvain Ribault, Random matrices, arXiv:1510.04430.

[11] Jan Ambjørn, L. Chekhov, C. F. Kristjansen, and Yu. Makeenko, Matrix model calculations beyond the spherical limit, Nucl. Phys. **B404**, 127 (1993); **B449**, 681(E) (1995).

[12] Zodinmawia and P. Ramadevi, Reformulated invariants for non-torus knots and links, arXiv:1209.1346.

[13] Pravina Borhade, P. Ramadevi, and Tapobrata Sarkar, U (n) framed links, three-manifold invariants, and topological strings, Nucl. Phys. **B678**, 656 (2004).

[14] Marcos Marino and Cumrun Vafa, Framed knots at large n, Contemp. Math. **310**, 185 (2002).

[15] P. Rama Devi, T. R. Govindarajan, and R. K. Kaul, Three-dimensional Chern-Simons theory as a theory of knots and links. 3. Compact semisimple group, Nucl. Phys. **B402**, 548 (1993).

[16] Romesh K. Kaul and P. Ramadevi, Three-manifold invariants from Chern-Simons field theory with arbitrary semi-simple gauge groups, Commun. Math. Phys. **217**, 295 (2001).

[17] Xiao-Song Lin and Hao Zheng, On the hecke algebras and the colored homfly polynomial, Trans. Am. Math. Soc. **362**, 1 (2010), https://doi.org/10.48550/arXiv.math/0601267.

[18] Marc Rosso and Vaughan Jones, On the invariants of torus knots derived from quantum groups, J. Knot Theor. Ramif. **2**, 97 (1993).

[19] Eli J. Mlawer, Stephen G. Naculich, Harold A. Riggs, and Howard Schnitzer, Group-level duality of WZW fusion coefficients and Chern-Simons link observables, Nucl. Phys. **352**, 863 (1991).

[20] Andrea Brini, Bertrand Eynard, and Marcos Marino, Torus knots and mirror symmetry, Ann. Henri Poincare **13**, 1873 (2012).

[21] Siddharth Dwivedi, Vivek Kumar Singh, Saswati Dhara, P. Ramadevi, Yang Zhou, and Lata Kh Joshi, Entanglement on linked boundaries in Chern-Simons theory with generic gauge groups, J. High Energy Phys. 02 (2018) 163.

[22] Marcos Marino, Chern-Simons theory and topological strings, Rev. Mod. Phys. **77**, 675 (2005).

[23] D. Friedan, Some non-Abelian toy models in the large N limit, Commun. Math. Phys. **78**, 353 (1981).

[24] Parikshit Dutta and Suvankar Dutta, Phase space distribution of Riemann zeros, J. Math. Phys. (N.Y.) **58**, 053504 (2017).

[25] Alexander A. Migdal, Loop equations and 1/N expansion, Phys. Rep. **102**, 199 (1983).

[26] P. Ramadevi and Swatee Naik, Computation of Lickorish's three manifold invariant using Chern–Simons theory, Commun. Math. Phys. **209**, 29 (2000).

[27] José M. F. Labastida, Marcos Marino, and Cumrun Vafa, Knots, links and branes at large n, J. High Energy Phys. 11 (2001) 007.