Modular symmetry in magnetized T^{2g} torus and orbifold models

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We study the modular symmetry in magnetized T^{2g} torus and orbifold models. The T^{2g} torus has the modular symmetry $\Gamma_g = Sp(2g,\mathbb{Z})$. The magnetic flux background breaks the modular symmetry to a certain normalizer $N_g(H)$. We classify remaining modular symmetries by magnetic flux matrix types. Furthermore, we study the modular symmetry for wave functions on the magnetized T^{2g} and certain orbifolds. It is found that wave functions on magnetized T^{2g} as well as its orbifolds behave as the Siegel modular forms of weight $1/2$ and $\tilde{N}_g(H,h)$, which is the metaplectic congruence subgroup of the double covering group of $N_g(H)$, $\tilde{N}_g(H)$. Then, wave functions transform nontrivially under the quotient group, $\tilde{N}_{g,h} = \tilde{N}_g(H)/\tilde{N}_g(H,h)$, where the level h is related to the determinant of the magnetic flux matrix. Accordingly, the corresponding four-dimensional chiral fields also transform nontrivially under $\tilde{N}_{g,h}$ modular flavor transformation with modular weight $-1/2$. We also study concrete modular flavor symmetries of wave functions on magnetized T^{2g} orbifolds.

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I. INTRODUCTION

One of the significant mysteries in particle physics is the origin of the flavor structure of quarks and leptons such as their hierarchical masses, flavor mixing angles, and CP phases. Many scenarios have been studied. Among them, one of the interesting approaches is to assume certain non-Abelian discrete flavor symmetries such as S_N, A_N, $\Delta(3N^2)$, and $\Delta(6M^2)$ among generations of quarks and leptons [1–6].

As the origin of flavor symmetries, the geometrical symmetries of compact spaces predicted in higher-dimensional theories such as superstring theory have been attractive. (See, e.g., Refs. [7,8].) The modular symmetry is the geometrical symmetry of compact spaces such as tori, orbifolds, and Calabi-Yau manifolds as the transformation of cycle basis. Recently, the modular symmetry has been attractive since it includes certain non-Abelian discrete flavor symmetries such as S_3, A_4, S_4, and A_5 [9]. Thus, the modular symmetry can be a source of flavor symmetries of quarks and leptons, obtained from the compactification. Actually, various bottom-up approaches of model building have been studied [10–20], in which the assumed modular flavor symmetries play an important role to determine the flavor structure of quarks and leptons from the geometrical parameters called moduli. (See for more Ref. [21].)

In top-down approaches, on the other hand, it is important to look for what modular flavor symmetries of chiral fields such as quarks and leptons appear in an effective theory of superstring theory. For example, the ten-dimensional (10D) $\mathcal{N}=1$ supersymmetric Yang-Mills theory with nonvanishing magnetic fluxes on a torus or its orbifold is an interesting effective theory of magnetized D-brane models in superstring theory [22–25]. Magnetized D-brane models have several interesting features. Multi-generational chiral fermions [26–35] (including three generations of chiral fermions [36–38]) can be obtained by specific magnetic fluxes and boundary conditions. Their Yukawa couplings [26,32,33,39] as well as higher-order couplings [40] and also Majorana neutrino mass terms generated by D-brane instanton effects [41,42] can be calculated analytically since we can find their wave functions explicitly. (See for D-brane instanton computations Refs. [43,44].) Actually, realistic quark and lepton masses and mixing angles as well as CP phases have been realized in Refs. [39,45–52].

These magnetized torus and orbifold models have the modular symmetry.[1] Their effective field theory is controlled by the modular symmetry. The torus T^2 has a single complex structure modulus and the modular symmetry $SL(2,\mathbb{Z})$. The modular symmetry and its implications on

[1]For heterotic orbifold models, see Refs. [53–57].

the flavor structure in the magnetized T^2 and its certain orbifold models with a single complex structure modulus have been studied in Refs. [30,58] and in detail in Refs. [59,60]. (See also Refs. [61–70].)

Generic compact space has many moduli, and they have a larger modular symmetry, $Sp(2g,\mathbb{Z})$. That can lead to a rich flavor structure. Such symplectic modular symmetries were studied in Calabi-Yau compactifications [71–74] and others [75–78]. The torus T^{2g} has the modular symplectic symmetry $Sp(2g,\mathbb{Z})$. In this paper, we extend the above analysis on magnetized T^2 and orbifold models to the magnetized T^{2g} (including T^4 and T^6) and its certain orbifold models. We study the modular flavor symmetry in these higher-dimensional torus and orbifold models with magnetic fluxes.

This paper is organized as follows. We review the modular symmetry on T^{2g} in Sec. II A and modular forms in Sec. II B. In Sec. III, we review magnetized T^{2g} compactification. Then, we study the modular symmetry in magnetized T^{2g} and orbifold compactifications. Generally, the modular symmetry on T^{2g} with a magnetic flux can be smaller than that on T^{2g} without a magnetic flux. Hence, we classify the modular symmetry, which is consistent with a magnetic flux in Sec. IV A, and then we study the modular flavor symmetry of wave functions on magnetized T^{2g} as well as orbifold by general analysis and concrete examples in each class, in Sec. IV B. In Sec. V, we conclude this study. In Appendix A, we discuss the algebraic relation between the S transformation and the general T transformation. In Appendix B, we derive the Landsberg-Schaar relation. In Appendix C, we prove that the generators of $\Delta(96)\times Z_4$ satisfy the algebraic relation. In Appendix D, we discuss the modular flavor symmetry with the moduli that are proportional to the unit matrix.

II. MODULAR SYMMETRY ON T^{2g} AND MODULAR FORMS

In this section, we review the modular symmetry on T^{2g} and modular forms. (See, e.g., Refs. [77,79–87].) A $2g$-dimensional torus T^{2g} can be constructed as $\mathbb{C}^g/\Lambda$, where Λ is a lattice on $\mathbb{C}^g$ spanned by $2g$ numbers of lattice vectors α_j and β_j ($j=1,\ldots,g$). We write complex coordinates of α_j and β_j as $\alpha_j = {}^t(\alpha_{1j},\ldots,\alpha_{gj})$ and $\beta_j=(\beta_{1j},\ldots,\beta_{gj})$, respectively. Then, the ath component of the complex coordinate on $\mathbb{C}^g$, u^a, can be written as

$$u^a = \sum_{j=1}^{g} \alpha^a_j x^j + \beta^a_j y^j \qquad (x^j, y^j \in \mathbb{R}). \tag{1}$$

We also define the jth component of the complex coordinate on T^{2g}, z^j, as

$$z^j = (\alpha^{-1})^j_a u^a = x^j + \Omega^j_k y^k \qquad (0\le x^j, y^k \le 1), \tag{2}$$

where the $g\times g$ complex matrix, $\Omega=\alpha^{-1}\beta$, is called the complex structure moduli. Here, we consider that the complex structure moduli lie on the Siegel upper-half plane, defined as

$$\mathcal{H}_g = \{\Omega \in GL(g,\mathbb{C}) | {}^t\Omega = \Omega, \mathrm{Im}\Omega > 0\}. \tag{3}$$

Then, the lattice identification is written by

$$z + e_k \sim z + \Omega e_k \sim z, \tag{4}$$

for $\forall\, k$, where the jth component of e_k is $\delta_{j,k}$. The metric of T^{2g} is given by

$$ds^2 = d\bar{u}^a du^a = (\alpha^\dagger \alpha)_{ij} d\bar{z}^i dz^j = 2h_{ij} d\bar{z}^i dz^j, \tag{5}$$

and then the volume of T^{2g} can be calculated as

$$\begin{aligned}\mathrm{Vol}(T^{2g}) &= \int_{T^{2g}} d^g z d^g \bar{z}\, \sqrt{|\det(2h)|} \\ &= |\det(\alpha^\dagger\alpha)|^2 \det(2\mathrm{Im}\Omega).\end{aligned} \tag{6}$$

The gamma matrices on T^{2g}, $\Gamma^{\bar{z}^i}$, and Γ^{z^j}, satisfying $\{\Gamma^{\bar{z}^i},\Gamma^{z^j}\} = 2h^{ij}$, are given by

$$\Gamma^{\bar{z}^i} = [(\alpha^\dagger)^{-1}]^i_a \Gamma^{\bar{u}^a}, \qquad \Gamma^{z^j} = [\alpha^{-1}]^j_b \Gamma^{u^b}, \tag{7}$$

where $\Gamma^{\bar{u}^a}$ and Γ^{u^b} are the gamma matrices on $\mathbb{C}^g$, satisfying $\{\Gamma^{\bar{u}^a},\Gamma^{u^b}\} = 2\delta^{a,b}$.

A. Modular symmetry on T^{2g}

Now, let us review the Siegel modular symmetry on T^{2g}. Let us consider the following lattice transformation:

$$\begin{pmatrix} {}^t\gamma(\beta) \\ {}^t\gamma(\alpha)\end{pmatrix} = \begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} {}^t\beta \\ {}^t\alpha\end{pmatrix} = \begin{pmatrix} A{}^t\beta + B{}^t\alpha \\ C{}^t\beta + D{}^t\alpha\end{pmatrix},$$
$$\gamma = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in Sp(2g,\mathbb{Z}) \equiv \Gamma_g, \tag{8}$$

where the almost complex structure,

$$J = \begin{pmatrix} 0 & I_g \\ -I_g & 0\end{pmatrix}, \tag{9}$$

needs to be invariant, i.e.,

$${}^t\gamma J \gamma = J, \Rightarrow\ {}^tAC = {}^tCA, \quad {}^tBD = {}^tDB, \quad {}^tAD - {}^tCB = I_g. \tag{10}$$

Since the lattice spanned by $\gamma(\alpha)$ and $\gamma(\beta)$ is the same as the lattice spanned by α and β, there is $\Gamma_g = Sp(2g,\mathbb{Z})$ symmetry in T^{2g} compactification. Associated with the lattice transformations in Eq. (8), the moduli Ω and the coordinate z transform as

$$\gamma : \Omega = \alpha^{-1}\beta \to \gamma(\Omega) = \gamma(\alpha)^{-1}\gamma(\beta) = (A\Omega + B)(C\Omega + D)^{-1}, \tag{11}$$

$$\gamma : z = \alpha^{-1}u \to \gamma(z) = \gamma(\alpha)^{-1}u = {}^t(C\Omega + D)^{-1}z. \tag{12}$$

Equation (11) is called the (inhomogeneous) Siegel modular transformation. Here, we call Eqs. (11) and (12) the Siegel modular transformation.

The generators of the Siegel modular group, $\Gamma_g = Sp(2g, \mathbb{Z})$, are given by

$$S = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}, \qquad T_{ab} = \begin{pmatrix} I_g & B_{ab} \\ 0 & I_g \end{pmatrix}, \tag{13}$$

where B_{ab} are concretely written as

$$B_{11} = 1, \tag{14}$$

for $g = 1$,

$$B_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{15}$$

for $g = 2$, and

$$B_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$B_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$B_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{16}$$

for $g = 3$. The generators in Eq. (13) satisfy the following relations:

$$\begin{aligned} S^2 &= \begin{pmatrix} -I_g & 0 \\ 0 & -I_g \end{pmatrix} = -I_{2g} \equiv R, \\ S^4 &= R^2 = I_{2g}, \\ (ST_{ab})^3 &= \begin{pmatrix} B_{ab} & -I_g + B_{ab}^2 \\ I_g - B_{ab}^2 & B_{ab} \end{pmatrix}, \\ (ST_{ab})^6 &= \begin{pmatrix} -I_g + 2B_{ab}^2 & 0 \\ 0 & -I_g + 2B_{ab}^2 \end{pmatrix} = A_{-I_g + 2B_{ab}^2} \equiv U, \\ (ST_{ab})^{12} &= I_{2g}, \\ RX &= XR (X = S, T_{ab}), \end{aligned} \tag{17}$$

where we denote

$$A_X \equiv \begin{pmatrix} X & 0 \\ 0 & {}^tX^{-1} \end{pmatrix} \in Sp(2g, \mathbb{Z}), \quad X \in GL(g, \mathbb{Z}), \tag{18}$$

and then $A_{\pm I_g} = \pm I_{2g}$. Note that Eq. (17) corresponds to the S^3 transformation in the space spanned by lattice vectors α_c and β_c ($c \neq a, b$). In Appendix A, we discuss the algebraic relation between ths S transformation and the general T transformation generated by the T_{ab} transformation. Under the S and T_{ab} transformations, z and Ω transform as

$$\begin{aligned} S &: (z, \Omega) \to (\gamma(z, \Omega)) = (-\Omega^{-1}z, -\Omega^{-1}), \\ T_{ab} &: (z, \Omega) \to (\gamma(z, \Omega)) = (z, \Omega + B_{ab}). \end{aligned} \tag{19}$$

B. Modular forms

Now, let us review the Siegel modular forms. First, we introduce the principal congruence subgroup of level n,

$$\Gamma_g(n) = \left\{ \gamma' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \middle| \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \equiv \begin{pmatrix} I_g & 0 \\ 0 & I_g \end{pmatrix} \pmod{n} \right\}, \tag{20}$$

in particular, $\Gamma_g(1) = \Gamma_g$. The Siegel modular forms $f(\Omega)$ of integral weight k and level n at genus g are holomorphic functions of Ω that satisfy

$$\begin{aligned} &\gamma : f(\Omega) \to f(\gamma(\Omega)) = J_k(\gamma, \Omega)\rho(\gamma)f(\Omega), \\ &J_k(\gamma, \Omega) = [\det(C\Omega + D)]^k, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g, \end{aligned} \tag{21}$$

$$\begin{aligned} J_k(\gamma_2\gamma_1, \Omega) &= J_k(\gamma_2, \gamma_1(\Omega))J_k(\gamma_1, \Omega), \\ \rho(\gamma_2\gamma_1) &= \rho(\gamma_2)\rho(\gamma_1), \qquad \gamma_1, \gamma_2 \in \Gamma_g, \end{aligned} \tag{22}$$

with

$$\rho(\gamma') = I, \gamma' \in \Gamma_g(n). \tag{23}$$

Thus, ρ is a unitary representation of the quotient group, $\Gamma_{g,n} = \Gamma_g / \Gamma_g(n)$, so-called the finite Siegel modular group. In other words, the Siegel modular forms transform nontrivially under the finite Siegel modular transformation, $\Gamma_{g,n}$. Concretely, we obtain the relations

$$\begin{aligned} &\rho(R)^2 = \rho(S)^4 = [\rho(S)\rho(T_{ab})]^{12} = \rho(T_{ab})^n = I, \\ &(\rho(R) = \rho(S)^2 = [\rho(S)\rho(T_{ab})]^6 = I (n = 2)), \end{aligned} \tag{24}$$

and also

$$\rho(R) = \rho(S)^2 = (-1)^{gk} I$$
$$(\Leftarrow f(\Omega) = f(S^2(\Omega)) = (-1)^{gk}\rho(S)^2 f(\Omega)). \quad (25)$$

Then, we also find that

$$\rho(R)\rho(X) = \rho(X)\rho(R) \qquad (X = S, T_{ab}). \quad (26)$$

On the other hand, the overall factor $J_k(\gamma, \Omega)$ is called the automorphic factor, and it can be uniquely determined once γ is given.

Here, we comment on the stabilizer, H, and the normalizer, $N_g(H)$, mentioned in Ref. [77]. When the moduli Ω are restricted to a certain region, as shown in Sec. IV A, the modular symmetry is reduced from the Siegel modular group, Γ_g, to the normalizer $N_g(H)$, called the Siegel modular subgroup. When Ω is fixed to a certain form in the region, the stabilizer H is the unbroken group that is generated by the modular transformation such that Ω is invariant. In general, the stabilizer is a normal subgroup of $N_g(H)$. As with $\Gamma_g(n)$ and $\Gamma_{g,n} = \Gamma_g/\Gamma_g(n)$, we can consider the principal congruence subgroup of $N_g(H)$,

$$N_g(H, n) \equiv \{\gamma' \in N_g(H) | \gamma' \equiv I_{2g} \pmod{n}\}, \quad (27)$$

and the quotient group, $N_{g,n}(H) = N_g(H)/N_g(H, n)$, is called the finite Siegel modular subgroup.

We extend the above analysis for wave functions $\psi(z, \Omega)$ in Sec. IV B. To see that, however, we have to introduce the Siegel modular forms of the half-integral weight. First, we introduce the metaplectic double covering group of $\Gamma_g = Sp(2g, \mathbb{Z})$,

$$\tilde{\Gamma}_g = \widetilde{Sp}(2g, \mathbb{Z}) = \{\tilde{\gamma} = [\gamma, \epsilon] | \gamma \in \Gamma_g, \epsilon \in \{\pm 1\}\}. \quad (28)$$

The multiplication is given by

$$\tilde{\gamma}_1\tilde{\gamma}_2 = [\gamma_1, \epsilon_1][\gamma_2, \epsilon_2] = [\gamma_1\gamma_2, A(\gamma_1, \gamma_2)\epsilon_1\epsilon_2]$$
$$\equiv [\gamma_{1,2}, \epsilon_{1,2}] = \tilde{\gamma}_{1,2} \in \tilde{\Gamma}_g, \quad (29)$$

where $A(\gamma_1, \gamma_2)$ denotes the Rao's 2-cocycle [88],[2] satisfying the following relation:

$$(\tilde{\gamma}_1\tilde{\gamma}_2)\tilde{\gamma}_3 = \tilde{\gamma}_1(\tilde{\gamma}_2\tilde{\gamma}_3),$$
$$\Leftrightarrow A(\gamma_1, \gamma_2)A(\gamma_1\gamma_2, \gamma_3) = A(\gamma_1, \gamma_2\gamma_3)A(\gamma_2, \gamma_3). \quad (30)$$

In particular, the generators of $\tilde{\Gamma}_g = \widetilde{Sp}(2, \mathbb{Z})$ are given by

$$\tilde{S} \equiv [S, (-1)^g], \qquad \tilde{T}_{ab} \equiv [T_{ab}, 1], \quad (31)$$

and they satisfy the following relations:

$$\begin{aligned}
&\tilde{S}^2 = [-I_{2g}, 1] \equiv \tilde{R}, \qquad \tilde{R}^2 = \tilde{S}^4 = [I_{2g}, (-1)^g], \qquad \tilde{R}^4 = \tilde{S}^8 = [I_{2g}, 1] \equiv \tilde{I}_{2g},\\
&(\tilde{S}\tilde{T}_{ab})^3 = [(ST_{ab})^3, (-1)^{g'-1}], \qquad (\tilde{S}\tilde{T}_{ab})^6 = [A_{-I_g+2B_{ab}^2}, (-1)^{g-1}],\\
&(\tilde{S}\tilde{T}_{ab})^{12} = [I_{2g}, (-1)^{g-g'}] \equiv \tilde{U}, \qquad \tilde{U}^2 = (\tilde{S}\tilde{T}_{ab})^{24} = [I_{2g}, 1] = \tilde{I}_{2g},\\
&\tilde{R}\,\tilde{X} = \tilde{X}\,\tilde{R}, \qquad \tilde{U}\,\tilde{X} = \tilde{X}\,\tilde{U}, \qquad (\tilde{X} = \tilde{S}, \tilde{T}_{ab}),
\end{aligned} \quad (32)$$

with

$$g' = \begin{cases} 1 & (a = b) \\ 2 & (a \neq b) \end{cases}. \quad (33)$$

Similarly, we can introduce the metaplectic congruence subgroup of level $n (\in 4\mathbb{Z})$,

$$\tilde{\Gamma}_g(n) = \{\tilde{\gamma}' = [\gamma', \epsilon] \in \tilde{\Gamma}_g | \gamma' \in \Gamma_g(n), \epsilon = 1\}. \quad (34)$$

Then, the Siegel modular forms $f(\Omega)$ of half-integral weight $k/2$ and level n at genus g are holomorphic functions of Ω that satisfy

$$\tilde{\Gamma}_g \ni \tilde{\gamma} : f(\Omega) \to f(\tilde{\gamma}(\Omega)) = \tilde{J}_{k/2}(\tilde{\gamma}, \Omega)\rho(\tilde{\gamma})f(\Omega), \qquad \tilde{J}_{k/2}(\tilde{\gamma}, \Omega) = \epsilon^k J_{k/2}(\gamma, \Omega) = \epsilon^k [\det(C\Omega + D)]^{k/2}, \quad (35)$$

$$\tilde{J}_{k/2}(\tilde{\gamma}_2\tilde{\gamma}_1, \Omega) = [A(\gamma_1, \gamma_2)]^k \tilde{J}_{k/2}(\tilde{\gamma}_2, \tilde{\gamma}_1(\Omega))\tilde{J}_{k/2}(\tilde{\gamma}_1, \Omega), \qquad \rho(\tilde{\gamma}_2\tilde{\gamma}_1) = \rho(\tilde{\gamma}_2)\rho(\tilde{\gamma}_1), \qquad \tilde{\gamma}_1, \tilde{\gamma}_2 \in \tilde{\Gamma}_g, \quad (36)$$

with

$$\rho(\tilde{\gamma}') = I, \qquad \tilde{\gamma}' \in \tilde{\Gamma}_g(n). \quad (37)$$

[2]See also Ref. [89].

Here, we note that $\tilde{\gamma}(\Omega) = \gamma(\Omega)$, and we choose $(-1)^{gk/2} = e^{-\pi i g k/2}$. Thus, ρ is a unitary representation of the quotient group, $\tilde{\Gamma}_{g,n} = \tilde{\Gamma}_g/\tilde{\Gamma}_g(n)$, called the metaplectic finite Siegel modular group. In other words, the Siegel modular forms transform nontrivially under the metaplectic finite Siegel modular transformation, $\tilde{\Gamma}_{g,n}$. Concretely, we obtain the relations

$$\rho(\tilde{R})^4 = \rho(\tilde{S})^8 = I, \qquad \rho(\tilde{U})^2 = [\rho(\tilde{S})\rho(\tilde{T}_{ab})]^{24} = I, \qquad \rho(\tilde{T}_{ab})^n = I,$$
$$(\rho(\tilde{R}) = \rho(\tilde{S})^2 = I(n=2)), \tag{38}$$

and also

$$\rho(\tilde{R}) = \rho(\tilde{S})^2 = e^{\pi i g k/2} I \qquad (\Leftarrow f(\Omega) = f(\tilde{S}^2(\Omega)) = e^{-\pi i g k/2}\rho(\tilde{S})^2 f(\Omega)), \tag{39}$$

$$\rho(\tilde{R})^2 = \rho(\tilde{S})^4 = (-1)^{gk} I \qquad (\Leftarrow f(\Omega) = f(\tilde{S}^4(\Omega)) = (-1)^{gk}\rho(\tilde{S})^4 f(\Omega)), \tag{40}$$

$$\rho(\tilde{U}) = \rho(\tilde{S}\tilde{T}_{ab})^{12} = (-1)^{(g-g')k} I \qquad (\Leftarrow f(\Omega) = f((\tilde{S}\tilde{T}_{ab})^{12}(\Omega)) = (-1)^{(g-g')k}\rho(\tilde{S}\tilde{T}_{ab})^{12} f(\Omega)). \tag{41}$$

Then, we also find that

$$\rho(\tilde{R})\rho(\tilde{X}) = \rho(\tilde{X})\rho(\tilde{R}), \qquad \rho(\tilde{U})\rho(\tilde{X}) = \rho(\tilde{X})\rho(\tilde{U}), \qquad (\tilde{X} = \tilde{S}, \tilde{T}_{ab}). \tag{42}$$

On the other hand, the overall factor $J_k(\gamma, \Omega)$, called the automorphic factor, can be uniquely determined once $\tilde{\gamma}$ is given.

Similarly, we can consider the double covering group of $N_g(H)$, $\tilde{N}_g(H)$, and the metaplectic congruence subgroup of level $n(\in 4\mathbb{Z})$, $\tilde{N}_g(H,n)$, by replacing $\gamma \in \Gamma_g$ in Eq. (28) with $\gamma \in N_g(H)$ and replacing $\gamma' \in \Gamma_g(n)$ in Eq. (34) with $\gamma' \in N_g(H,n)$, respectively. In that case, ρ is a unitary representation of the quotient group, $\tilde{N}_{g,n}(H) = \tilde{N}_g(H)/\tilde{N}_g(H,n)$, called the metaplectic finite Siegel modular subgroup.

III. MAGNETIZED T^{2g} COMPACTIFICATION

In this section, we review a T^{2g} compactification with a background magnetic flux, so-called a magnetized T^{2g} compactification [26,32]. We introduce the following $U(1)$ magnetic flux:

$$F = \pi[{}^t N(\mathrm{Im}\Omega)^{-1}]_{ij}(i dz^i \wedge d\bar{z}^j), \tag{43}$$

with

$${}^t(N\Omega) = N\Omega, \tag{44}$$

which is needed for the magnetic flux F to be the (1,1) form (F-flat condition). Here, N denotes the $g \times g$ flux matrix and the flux must be quantized, i.e., $N_{ij} \in \mathbb{Z}$. It is induced by the gauge potential,

$$\begin{aligned} A(z) &= \pi \mathrm{Im}[{}^t(N\bar{z})(\mathrm{Im}\Omega)^{-1} dz] \\ &= -\frac{\pi i}{2}[{}^t(N\bar{z})(\mathrm{Im}\Omega)^{-1}]_k dz^k + \frac{\pi i}{2}[{}^t(Nz)(\mathrm{Im}\Omega)^{-1}]_k d\bar{z}^k \\ &= A_{z^k} dz^k + A_{\bar{z}^k} d\bar{z}^k. \end{aligned} \tag{45}$$

Here, we do not consider Wilson lines. Under the lattice translations, it transforms as

$$\begin{aligned} A(z + e_k) &= A(z) + d[\pi {}^t N(\mathrm{Im}\Omega)^{-1}\mathrm{Im} z]_k \\ &= A(z) + d\chi_{e_k}(z), \end{aligned} \tag{46}$$

$$\begin{aligned} A(z + \Omega e_k) &= A(z) + d[\pi \mathrm{Im}\{(N\bar{\Omega})(\mathrm{Im}\Omega)^{-1} z\}]_k \\ &= A(z) + d\chi_{\Omega e_k}(z). \end{aligned} \tag{47}$$

It corresponds to the $U(1)$ gauge transformation. Through the covariant derivative with $U(1)$ unit charge, $q = 1$,

$$\begin{aligned} D &= d - iA(z) \\ &= \left(\partial_{z^k} - \frac{\pi}{2}[{}^t(N\bar{z})(\mathrm{Im}\Omega)^{-1}]_k\right) dz^k \\ &\quad + \left(\partial_{\bar{z}^k} + \frac{\pi}{2}[{}^t(Nz)(\mathrm{Im}\Omega)^{-1}]_k\right) d\bar{z}^k \\ &= D_{z^k} dz^k + D_{\bar{z}^k} d\bar{z}^k, \end{aligned}$$

wave functions on the magnetized T^{2g} with $q = 1$ satisfy the following boundary conditions:

$$\Psi(z + e_k, \Omega) = e^{2\pi i {}^t\alpha^S e_k} e^{i\chi_{e_k}(z)} \Psi(z), \tag{48}$$

$$\Psi(z + \Omega e_k, \Omega) = e^{2\pi i {}^t\beta^S e_k} e^{i\chi_{\Omega e_k}(z)} \Psi(z), \tag{49}$$

where $\alpha^S = {}^t(\alpha_1^S, \ldots, \alpha_g^S)$ and $\beta^S = {}^t(\beta_1^S, \ldots, \beta_g^S)$ with $0 \le \alpha_k^S, \beta_k^S \le 1$ are called as the Scherk-Schwarz (SS) phases. Note that the Wilson line phases can be converted into the SS phases through a proper redefinition of fields [28]. We consider solutions of the zero-mode Dirac equation,

$$i\not{D}\Psi(z,\Omega) = i(\Gamma^{z^k}D_{z^k} + \Gamma^{\bar{z}^k}D_{\bar{z}^k})\Psi(z,\Omega) = 0, \tag{50}$$

with the boundary conditions in Eqs. (48) and (49). When all eigenvalues of N are positive in addition to Eq. (3), only the component of $\Psi(z,\Omega)$ whose chirality on $\forall\, z^k (k = 1,\ldots,g)$ is positive has the $\det N$ number of degenerated zero modes,

$$\begin{aligned}\psi_{T^{2g}}^{(J+\alpha^S,\beta^S),N}(z,\Omega) &= [\mathrm{Vol}(T^{2g})]^{-1/2}(\det N)^{1/4} \\ &\quad\times e^{-2\pi i {}^t(J+\alpha^S)N^{-1}\beta^S} e^{\pi i {}^t(Nz)(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z}\vartheta \\ &\quad\times \begin{bmatrix} {}^t(J+\alpha^S)N^{-1} \\ -{}^t\beta \end{bmatrix}(Nz, N\Omega),\end{aligned} \tag{51}$$

for $\forall\, J \in \Lambda_N$, where ϑ denotes the Riemann-theta function defined by

$$\begin{aligned}\vartheta\begin{bmatrix} {}^t a \\ {}^t b \end{bmatrix}(\nu,\Omega) &= \sum_{l\in\mathbb{Z}^g} e^{\pi i {}^t(l+a)\Omega(l+a)} e^{2\pi i {}^t(l+a)(\nu+b)}, \\ & a,b\in\mathbb{R}^g, \nu\in\mathbb{C}^g,\end{aligned} \tag{52}$$

and Λ_N denotes the lattice cell spanned by

$${}^tNe_k \quad (k = 1,\ldots,g). \tag{53}$$

This means that

$$\psi_{T^{2g}}^{(J+{}^tNe_k+\alpha^S,\beta^S),N}(z,\Omega) = \psi_{T^{2g}}^{(J+\alpha^S,\beta^S),N}(z,\Omega). \tag{54}$$

The normalization condition is given by

$$\begin{aligned}&\int_{T^{2g}} d^g z d^g \bar{z}\, \sqrt{|\det(2h)|}\left(\psi_{T^{2g}}^{(J+\alpha^S,\beta^S),N}(z,\Omega)\right)^* \psi_{T^{2g}}^{(K+\alpha^S,\beta^S),N} \\ &\quad\times (z,\Omega) = [\det(2\mathrm{Im}\Omega)]^{-1/2}\delta_{\bar{J},K}.\end{aligned} \tag{55}$$

Finally, we give a comment on the four-dimensional (4D) low-energy effective field theory. We assume that 4D $\mathcal{N} = 1$ supersymmetry remains, although our results on modular flavor symmetries in the following sections are independent of whether supersymmetry remains or not. Higher-dimensional fields $\Phi(x,z)$ are decomposed as follows:

$$\Phi(x,z) = \sum_{n,I}\varphi_n^I(x)\psi_n^I(z), \tag{56}$$

i.e., the Kaluza-Klein decomposition. Here, I denotes the degeneracy index for a fixed mass level n. The lowest modes with $n = 0$ are relevant to the 4D effective field theory, although they may be degenerate. We naturally assume the canonical kinetic term of $\Phi(x,z)$. Then, we integrate the extra dimension so as to obtain the Kähler potential of 4D low-energy effective field theory,

$$K(\varphi,\bar{\varphi}) = Z_{\bar{J}K}\varphi_0^J(x)^\dagger\varphi_0^K(x), \quad Z_{\bar{J}K} = [\det(2\mathrm{Im}\Omega)]^{-1/2}\delta_{\bar{J},K}. \tag{57}$$

In the following section, we study the modular symmetry in the magnetized T^{2g} and its orbifold compactifications. Since the field $\Phi(x,z)$ is invariant under the modular symmetry, the modular transformation of the 4D fields $\varphi_0^I(x)$ is inverse to one of $\psi_0^I(z)$ [90].[3] Hereafter, we consider the modular transformation for the zero-mode wave functions in Eq. (51).

IV. MODULAR SYMMETRY IN MAGNETIZED T^{2g} AND ORBIFOLD COMPACTIFICATIONS

In this section, we study the modular symmetry in magnetized T^{2g} and orbifold compactifications.

A. Modular symmetry consistent with magnetized fluxes

First, in this subsection, let us see what kind of the modular symmetry is consistent with a magnetic flux matrix N. By using Eqs. (11), (12), and (44), we can find that Eq. (43) is invariant and Eq. (44) is consistent for the modular transformation by $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, when the following conditions:

$$\begin{aligned}(C\Omega + D)^{-1}\, {}^tN(C\Omega + D) = N \Rightarrow {}^tCN = CN, \\ {}^tD = D{}^tN,\end{aligned} \tag{58}$$

$$\begin{aligned}{}^t[N(A\Omega+B)(C\Omega+D)^{-1}] = N(A\Omega+B)(C\Omega+D)^{-1} \\ \Rightarrow AN = NA, \quad B{}^tN = NB,\end{aligned} \tag{59}$$

are satisfied. In particular, for the N matrix to be consistent with generators in Eq. (13), the following relations:

$${}^tN = N, \tag{60}$$

$$B_{ab}{}^tN = NB_{ab}, \tag{61}$$

have to be satisfied.[4] The N matrix that satisfies Eqs. (60) and (61) for $\forall\, a, b$ is just $N = nI$. However, it is too restrictive for the N matrix, and then we consider other compactifications with a more relaxed N matrix while the restricted modular symmetry than Γ_g appears. In particular, to study nontrivial modular symmetry, we consider N matrices that are consistent with S and certain T transformations generated by combinations of some of T_{ab}. Thus, we consider the case that Eqs. (3), (44), and (60), i.e.,

[3] Note that the modular transformation corresponds to the change of basis $\psi_0^I(z)$.

[4] See also Ref. [34].

TABLE I. The complex structure modulus (in the third column) and the modular symmetry (in the fourth column) are consistent with the magnetic flux (matrix) (in the second column) in the class.

Class	Magnetic flux (matrix) N	Complex structure moduli Ω	Modular symmetry $N_1^{(\mathrm{class})}(H)$ (generators)
(1-1-a)	n	τ	$\Gamma_1\ (S, T_{(11)})$

$$\Omega_{ij} = \Omega_{ji}, \tag{62}$$

$$N_{ij} = N_{ji}, \tag{63}$$

$$\sum_k N_{ik}\Omega_{kj} = \sum_k \Omega_{ik}N_{kj}, \tag{64}$$

$$\Leftrightarrow \sum_{k\neq i,j}(N_{ik}\Omega_{jk} - N_{jk}\Omega_{ik}) + [(N_{ii} - N_{jj})\Omega_{ij} - N_{ij}(\Omega_{ii} - \Omega_{jj})] = 0, \tag{65}$$

are satisfied. Hereafter, we denote $\Delta N_{ij} \equiv N_{ii} - N_{jj}$ and $\Delta\Omega_{ij} \equiv \Omega_{ii} - \Omega_{jj}$. Obviously, we find $\Delta N_{ji} = -\Delta N_{ij}$ and $\Delta\Omega_{ji} = -\Delta\Omega_{ij}$. Equation (65) is concretely written as

$$\Delta N_{12}\Omega_{12} - N_{12}\Delta\Omega_{12} = 0, \tag{66}$$

for $g = 2$ and

$$\begin{cases} (N_{31}\Omega_{23} - N_{23}\Omega_{31}) + (\Delta N_{12}\Omega_{12} - N_{12}\Delta\Omega_{12}) = 0 \\ (N_{12}\Omega_{31} - N_{31}\Omega_{12}) + (\Delta N_{23}\Omega_{23} - N_{23}\Delta\Omega_{23}) = 0\,, \\ (N_{12}\Omega_{23} - N_{23}\Omega_{12}) + (\Delta N_{31}\Omega_{31} - N_{31}\Delta\Omega_{31}) = 0 \end{cases} \tag{67}$$

$$\Leftrightarrow \begin{pmatrix} \Delta N_{12} & N_{31} & -N_{23} \\ -N_{31} & \Delta N_{23} & N_{12} \\ N_{23} & -N_{12} & \Delta N_{31} \end{pmatrix}\begin{pmatrix} \Omega_{12} \\ \Omega_{23} \\ \Omega_{31} \end{pmatrix} - \begin{pmatrix} N_{12} & & \\ & N_{23} & \\ & & N_{31} \end{pmatrix}\begin{pmatrix} \Delta\Omega_{12} \\ \Delta\Omega_{23} \\ \Delta\Omega_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{68}$$

for $g = 3$. In the case of $g = 1$, obviously, any moduli Ω are consistent with an arbitrary magnetic flux (matrix) N. In the following, we classify the modular symmetry, which is consistent with a magnetic flux matrix N, for $g = 1$, 2, 3.

First, let us see the case of $g = 1$. The complex structure modulus and the modular symmetry, which are consistent with the magnetic flux (matrix), are summarized in Table I.

(i) *Class (1-1-a)* There is only one case for $g = 1$. Since any moduli $\Omega = \tau$ are consistent with a magnetic flux (matrix) $N = n$, we can consider S and $T_{(11)}$ transformations, which generate a $\Gamma_1 = Sp(2, \mathbb{Z}) = SL(2, \mathbb{Z})$ transformation. In particular, between S and $T_{(11)}$, the following relation:

$$(ST_{(11)})^3 = I_2, \tag{69}$$

is satisfied. The stabilizer is $H = \{\pm I_2\} = \mathbb{Z}_2^t$, which acts on $z = z^1$ as

$$\pm I_2\colon\ z^1 \to \pm z^1. \tag{70}$$

Hence, the $T^2/\mathbb{Z}_2^t$ twisted orbifold also has the same modular symmetry.

Next, let us see the case of $g = 2$ in the following classes. The complex structure moduli and the modular symmetry, which are consistent with the magnetic flux matrix in each class, are summarized in Table II.

Class (2-1) In this class, we consider the case with $N_{12} = 0$. Equation (66) is written by

$$\Delta N_{12}\Omega_{12} = 0, \tag{71}$$

TABLE II. The complex structure moduli (in the third column) and the modular symmetry (in the fourth column) are consistent with the magnetic flux matrix (in the second column) in each class.

Class	Magnetic flux matrix N	Complex structure moduli Ω	Modular symmetry $N_2^{(\mathrm{class})}(H)$ (generators)
(2-1-*a*)	$\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$	$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} = \sum_{i,j=1,2}\tau_{ij}B_{ij}$	$\Gamma_2\ (S, T_{ab}(\forall\, a, b))$
(2-1-*b*)	$\begin{pmatrix} n_{11} & 0 \\ 0 & n_{22} \end{pmatrix}$	$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} = \sum_{k=1,2}\tau_{kk}B_{kk}$	$\otimes_{k=1}^2\Gamma_{1_k}$ (generators of Γ_{1_k})
(2-2-*a*)	$\begin{pmatrix} n & n_{12} \\ n_{12} & n \end{pmatrix}$	$\begin{pmatrix} \tau & \tau_{12} \\ \tau_{12} & \tau \end{pmatrix} = \tau B_{I_2} + \tau_{12}B_{12}$	$N_2^{(2-2-a)}(H)\,(S, T_{I_2}, T_{12})$
(2-2-*b*)	$\begin{pmatrix} n_{11} & n_{12} \\ n_{12} & n_{22} \end{pmatrix}$	$\begin{pmatrix} \tau + \tau_N N_{12}/p & \tau_N N_{12}/p \\ \tau_N N_{12}/p & \tau \end{pmatrix} = \tau B_{I_2} + \tau_N B_{N_2}$	$N_2^{(2-2-b)}(H)\,(S, T_{I_2}, T_{N_2})$

and it is satisfied for $\forall\, \Delta\Omega_{12}$. This class is further classified as follows.

Class (2-1-*a*) In this class, we also consider the case with $\Delta N_{12} = 0$, i.e.,

$$N = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} = nI_2. \tag{72}$$

Equation (71) is also satisfied for $\forall\, \Omega_{12}$, i.e.,

$$\Omega = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} = \sum_{i,j=1,2} \tau_{ij} B_{ij}. \tag{73}$$

Then, we can consider S and $T_{ab}(\forall\, a, b)$ transformations, which generate the $\Gamma_2 = Sp(4,\mathbb{Z})$ transformation. In particular, among S and T_{ab}, the following relations:

$$\begin{aligned} (ST_{11})^6 &= A_{B_{11}-B_{22}}, & (ST_{11})^{12} &= I_4, \\ (ST_{22})^6 &= A_{B_{22}-B_{11}}, & (ST_{22})^{12} &= I_4, \\ (ST_{12})^3 &= A_{B_{12}}, & (ST_{12})^6 &= I_4, \end{aligned} \tag{74}$$

are satisfied. The stabilizer is $H = \{\pm I_4\} = \mathbb{Z}_2^t$, which acts on $z = {}^t(z^1, z^2)$ as

$$\pm I_4\colon \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \to \begin{pmatrix} \pm z^1 \\ \pm z^2 \end{pmatrix}. \tag{75}$$

Hence, the $T^4/\mathbb{Z}_2^t$ twisted orbifold also has the same modular symmetry.

Class (2-1-*b*) In this class, we consider the case with $\Delta N_{12} \neq 0$, i.e.,

$$N = \begin{pmatrix} n_{11} & 0 \\ 0 & n_{22} \end{pmatrix}. \tag{76}$$

Equation (71) is satisfied for $\Omega_{12} = 0$, i.e.,

$$\Omega = \begin{pmatrix} \tau_{11} & 0 \\ 0 & \tau_{22} \end{pmatrix} = \sum_{i=1,2} \tau_{ii} B_{ii}. \tag{77}$$

This is nothing but direct products of magnetized T^2 compactification. Then, we can consider S_{kk} and T_{kk} $(k = 1, 2)$ transformations, which generate the $N_2^{(2-1-b)}(H) = \otimes_{k=1}^2 \Gamma_{1_k} = \otimes_{k=1}^2 SL(2,\mathbb{Z})_k$ transformation. Between S_{kk} and T_{kk}, the following relation:

$$(S_{kk} T_{kk})^3 = I_2, \tag{78}$$

is satisfied. In particular, the stabilizer is $H = \{\pm I_4, \pm A_{B_{11}-B_{22}}\} = \otimes_{k=1}^2 \mathbb{Z}_2^{t_k}$, which acts on $z = {}^t(z^1, z^2)$ as

$$\pm A_{B_{11}-B_{22}}\colon \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \to \begin{pmatrix} \pm z^1 \\ \mp z^2 \end{pmatrix}, \tag{79}$$

in addition to Eq. (75). Hence, the $\otimes_{k=1}^2 T_k^2/\mathbb{Z}_2^{t_k}$ twisted orbifold also has the same modular symmetry.

Class (2-2) In this class, we consider the case with $N_{12} \neq 0$. Equation (66) is written by

$$\Delta N_{12} \Omega_{12} = N_{12} \Delta\Omega_{12}. \tag{80}$$

This class is further classified as follows.

Class (2-2-*a*) In this class, we also consider the case with $\Delta N_{12} = 0$, i.e.,

$$N = \begin{pmatrix} n & n_{12} \\ n_{12} & n \end{pmatrix}. \tag{81}$$

Equation (80) is satisfied for $\Delta\Omega_{12} = 0$ and $\forall\, \Omega_{12}$, i.e.,

$$\begin{aligned} \Omega &= \begin{pmatrix} \tau & \tau_{12} \\ \tau_{12} & \tau \end{pmatrix} \\ &= \tau \sum_{i=1,2} B_{ii} + \tau_{12} B_{12} \\ &\equiv \tau B_{I_2} + \tau_{12} B_{12}. \end{aligned} \tag{82}$$

Then, we can consider S, T_{12}, and T_{I_2} with $B = B_{I_2}$ transformations, which generate the $N_2^{(2-2-a)}(H)$ transformation. In particular, between the S and T_{I_2} transformations, the following relation:

$$(ST_{I_2})^3 = I_4, \tag{83}$$

is satisfied. In particular, the stabilizer is $H = \{\pm I_4, \pm A_{B_{12}}\} = \mathbb{Z}_2^t \times \mathbb{Z}_2^p$, which acts on $z = {}^t(z^1, z^2)$ as

$$\pm A_{B_{12}}\colon \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \to \begin{pmatrix} \pm z^2 \\ \pm z^1 \end{pmatrix}, \tag{84}$$

in addition to Eq. (75). Hence, the $T^4/(\mathbb{Z}_2^t \times \mathbb{Z}_2^p)$ twisted and permutation orbifold also has the same modular symmetry.

Class (2-2-*b*) In this class, we consider the case with $\Delta N_{12} \neq 0$, i.e.,

$$N = \begin{pmatrix} n_{11} & n_{12} \\ n_{12} & n_{22} \end{pmatrix}. \tag{85}$$

Equation (80) is satisfied if the following condition:

TABLE III. The complex structure moduli (in the third column) and the modular symmetry (in the fourth column) are consistent with the magnetic flux matrix (in the second column) in each class.

Class	Magnetic flux matrix N	Complex structure moduli Ω	Modular symmetry $N_3^{(\mathrm{class})}(H)$ (generators)
(3-1-a)	$\begin{pmatrix} n & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n \end{pmatrix}$	$\sum_{i,j=1,2,3} \tau_{ij} B_{ij}$	$\Gamma_3\,(S, T_{ab}(\forall\, a,b))$
(3-1-b)	$\begin{pmatrix} N^{(2-1)} & 0 \\ 0 & n_{33} \end{pmatrix}$	$\sum_{kk} \tau_{kk} B_{kk}\,(+\tau_{12}B_{12})$	$N_2^{(2-1)}(H)\times\Gamma_1$ (generators of $N_2^{(2-1)}(H)$ and Γ_1)
(3-2-a)	$\begin{pmatrix} N^{(2-2)} & 0 \\ 0 & n_{33} \end{pmatrix}$ with Eq. (99)	$\tau B_{I_2} + \tau_N B_{N_2} + \tau_{33} B_{33}$	$N_2^{(2-2)}(H)\times\Gamma_1$ (generators of $N_2^{(2-2)}(H)$ and Γ_1)
(3-2-b)	$\begin{pmatrix} N^{(2-2)} & 0 \\ 0 & n_{33} \end{pmatrix}$ with Eq. (103)	$\tau B_{I_2} + \tau_N B_{N_2} + \tau_{33} B_{33} + \tau' B'_{\pm}$	$N_3^{(3-2-b)}(H)$ (generators of $N_2^{(2-1)}(H), T'_{\pm}$)
(3-3)	$\begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{13} & n_{22} & 0 \\ n_{13} & 0 & n_{33} \end{pmatrix}$	$\tau B_{I_3} + \tau_N B_{N_3} + \tau_{N^{-1}} B_{N_3^{-1}}$	$N_3^{(3-3)}(H)\,(S, T_{I_3}, T_{N_3}, T_{N_3^{-1}})$
(3-4-a)	$\begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{13} & n_{22} & n_{23} \\ n_{13} & n_{23} & n_{33} \end{pmatrix}$ with Eq. (119)	$\tau B_{I_3} + \tau_N B_{N_3} + \tau_{N^{-1}} B_{N_3^{-1}}$	$N_3^{(3-4-a)}(H)\,(S, T_{I_3}, T_{N_3}, T_{N_3^{-1}})$
(3-4-b)	$\begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{13} & n_{22} & n_{23} \\ n_{13} & n_{23} & n_{33} \end{pmatrix}$ with Eq. (124)	$\tau B_{I_3} + \tau'_{12} B'_{12} + \tau'_{23} B'_{23} + \tau'_{31} B'_{31}$	$N_3^{(3-4-b)}(H)\,(S, T_{I_3}, T'_{12}, T'_{23}, T'_{31})$

$$\Omega_{12} :\, N_{12} = \Delta\Omega_{12} :\Delta N_{12}, \tag{86}$$

is satisfied, i.e.,

$$\begin{aligned} \Omega &= \begin{pmatrix} \tau + \tau_N \Delta N_{12}/p & \tau_N N_{12}/p \\ \tau_N N_{12}/p & \tau \end{pmatrix} \\ &= \tau \sum_{i=1,2} B_{ii} + \tau_N (\Delta N_{12}/p B_{11} + N_{12}/p B_{12}) \\ &\equiv \tau B_{I_2} + \tau_N B_{N_2}, \end{aligned} \tag{87}$$

where $p = \gcd(\Delta N_{12}, N_{12})$. Note that the classes (2-1-b) and (2-2-a) are specific cases of the class (2-2-b). Then, we can consider S, T_{I_2}, and T_{N_2} with the $B = B_{N_2}$ transformations, which generate the $N_2^{(2-2-b)}(H)$ transformation. In particular, when $\Delta N_{12} = N_{12}$, between S and T_{N_2}, the following relation:

$$(S T_{N_2})^5 = I_4, \tag{88}$$

is satisfied, as shown in Appendix A. The stabilizer is $H = \{\pm I_4\} = \mathbb{Z}_2^t$, which acts on $z = {}^t(z^1, z^2)$ as Eq. (75). Hence, the $T^4/\mathbb{Z}_2^t$ twisted orbifold also has the same modular symmetry.

Finally, let us see the case of $g = 3$ in the following classes. The complex structure moduli and the modular symmetry, which are consistent with the magnetic flux matrix in each class, are summarized in Table III.

Class (3-1) In this class, we consider the case with $N_{12} = N_{23} = N_{31} = 0$. Equation (68) is written by

$$\begin{pmatrix} \Delta N_{12} & 0 & 0 \\ 0 & \Delta N_{23} & 0 \\ 0 & 0 & \Delta N_{31} \end{pmatrix} \begin{pmatrix} \Omega_{12} \\ \Omega_{23} \\ \Omega_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{89}$$

and it is satisfied for $\forall\, \Delta\Omega_{12}$, $\forall\, \Omega_{23}$, and $\forall\, \Omega_{31}$. This class is further classified as follows.

Class (3-1-a) In this class, we also consider the case with $\Delta N_{12} = \Delta N_{23} = \Delta N_{31} = 0$, i.e.,

$$N = \begin{pmatrix} n & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n \end{pmatrix} = n I_3. \tag{90}$$

Equation (89) is satisfied for $\forall\Omega_{12}$, $\forall\Omega_{23}$, and $\forall\Omega_{31}$, i.e.,

$$\begin{aligned} \Omega &= \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \tau_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{pmatrix} \\ &= \sum_{i,j=1,2,3} \tau_{ij} B_{ij}. \end{aligned} \tag{91}$$

Then, we can consider the S and $T_{ab}(\forall\, a, b)$ transformations, which generate the $\Gamma_3 = Sp(6, \mathbb{Z})$ transformation. In particular, among S and T_{ab}, the following relations:

$$\begin{aligned}
(ST_{11})^6 &= A_{B_{11}-B_{22}-B_{33}}, & (ST_{11})^{12} &= I_6,\\
(ST_{22})^6 &= A_{B_{22}-B_{33}-B_{11}}, & (ST_{22})^{12} &= I_6,\\
(ST_{33})^6 &= A_{B_{33}-B_{11}-B_{22}}, & (ST_{33})^{12} &= I_6,\\
(ST_{12})^6 &= A_{B_{11}+B_{22}-B_{33}}, & (ST_{12})^{12} &= I_6,\\
(ST_{23})^6 &= A_{B_{22}+B_{33}-B_{11}}, & (ST_{23})^{12} &= I_6,\\
(ST_{31})^6 &= A_{B_{33}+B_{11}-B_{22}}, & (ST_{31})^{12} &= I_6,
\end{aligned} \tag{92}$$

are satisfied. The stabilizer is $H = \{\pm I_6\} = \mathbb{Z}_2^t$, which acts on $z = {}^t(z^1, z^2, z^3)$ as

$$\pm I_6\colon \begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix} \to \begin{pmatrix} \pm z^1 \\ \pm z^2 \\ \pm z^3 \end{pmatrix}. \tag{93}$$

Hence, the $T^6/\mathbb{Z}_2^t$ twisted orbifold also has the same modular symmetry.

Class (3-1-b) In this class, we consider the case with $\Delta N_{23} \neq 0$, $\Delta N_{31} \neq 0$, and also $\Delta N_{12} = 0$, i.e.,

$$N = \begin{pmatrix} n & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n_{33} \end{pmatrix}. \tag{94}$$

Equation (89) is satisfied for $\Omega_{23} = \Omega_{31} = 0$ and $\forall \Omega_{12}$, i.e.,

$$\begin{aligned}
\Omega &= \begin{pmatrix} \tau_{11} & \tau_{12} & 0 \\ \tau_{12} & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix} \\
&= \sum_{i,j=1,2} \tau_{ij} B_{ij} + \tau_{33} B_{33}.
\end{aligned} \tag{95}$$

This is nothing but direct products of magnetized T^4 with the class (2-1-a) and T^2 compactifications. Thus, we can consider the $N_3^{(3-1-b)}(H) = \Gamma_2 \times \Gamma_1 = Sp(4, \mathbb{Z}) \times SL(2, \mathbb{Z})$ transformation. The $T^4/\mathbb{Z}_2^t \times T^2/\mathbb{Z}_2^t$ twisted orbifold also has the same modular symmetry. On the other hand, when we have $\Delta N_{12} \neq 0$, $\Delta N_{23} \neq 0$, and also $\Delta N_{31} \neq 0$, i.e.,

$$N = \begin{pmatrix} n_{11} & 0 & 0 \\ 0 & n_{22} & 0 \\ 0 & 0 & n_{33} \end{pmatrix}, \tag{96}$$

Equation (89) is satisfied for $\Omega_{12} = \Omega_{23} = \Omega_{31} = 0$, i.e.,

$$\begin{aligned}
\Omega &= \begin{pmatrix} \tau_{11} & 0 & 0 \\ 0 & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix} \\
&= \sum_{i=1,2,3} \tau_{ii} B_{ii}.
\end{aligned} \tag{97}$$

This is nothing but direct products of magnetized $T^4 = T^2 \times T^2$ with the class (2-1-b) and T^2 compactifications. Thus, we can consider the $N_3^{(3-1-c)}(H) = \otimes_{k=1}^3 \Gamma_{1_k} = \otimes_{k=1}^3 SL(2, \mathbb{Z})_k$ transformation. The $\otimes_{k=1}^3 T_k^2/\mathbb{Z}_2^{t_k}$ twisted orbifold also has the same modular symmetry.

Class (3-2) In this class, we consider the case with $N_{23} = N_{31} = 0$ and $N_{12} \neq 0$. Equation (68) is written by

$$\begin{pmatrix} \Delta N_{12} & 0 & 0 \\ 0 & \Delta N_{23} & N_{12} \\ 0 & -N_{12} & \Delta N_{31} \end{pmatrix} \begin{pmatrix} \Omega_{12} \\ \Omega_{23} \\ \Omega_{31} \end{pmatrix} = \begin{pmatrix} N_{12} \Delta \Omega_{12} \\ 0 \\ 0 \end{pmatrix}, \tag{98}$$

and it is satisfied for $\forall \Delta\Omega_{23}$ and $\forall \Delta\Omega_{31}$. This class is further classified as follows.

Class (3-2-a) In this class, we also consider the case satisfying

$$\begin{aligned}
&\Delta N_{23} \Delta N_{31} + N_{12}^2 \neq 0, \\
&\Leftrightarrow N_{33} \neq \frac{(N_{11} + N_{22}) \pm \sqrt{\Delta N_{12}^2 + (2N_{12})^2}}{2},
\end{aligned} \tag{99}$$

that is,

$$\begin{aligned}
N &= \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{12} & n_{22} & 0 \\ 0 & 0 & n_{33} \end{pmatrix}, \\
n_{33} &\neq \frac{(n_{11} + n_{22}) \pm \sqrt{(n_{11} - n_{22})^2 + (2n_{12})^2}}{2}.
\end{aligned} \tag{100}$$

In particular, when $\Delta N_{12} = 0 \Leftrightarrow n_{11} = n_{22} = n$ and $n_{33} \neq n \pm n_{12}$, Eq. (98) is satisfied for $\Delta\Omega_{12} = \Omega_{23} = \Omega_{31} = 0$ and $\forall\, \Omega_{12}$, i.e.,

$$\begin{aligned}
\Omega &= \begin{pmatrix} \tau & \tau_{12} & 0 \\ \tau_{12} & \tau & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix} \\
&= \tau \sum_{i=1,2} B_{ii} + \tau_{12} B_{12} + \tau_{33} B_{33} \\
&= \tau B_{I_2} + \tau_{12} B_{12} + \tau_{33} B_{33}.
\end{aligned} \tag{101}$$

This is nothing but products of magnetized T^4 with the class (2-2-a) and T^2 compactifications. Thus, we can consider the $N_3^{(3-2-a)}(H) = N_2^{(2-2-a)}(H) \times \Gamma_1$ transformation. The $T^4/(\mathbb{Z}_2^t \times \mathbb{Z}_2^p) \times T^2/\mathbb{Z}_2^t$ orbifold also has the same modular symmetry. On the other hand, when $\Delta N_{12} \neq 0$, Eq. (98) is satisfied if $\Omega_{23} = \Omega_{31} = 0$ and Eq. (86) is satisfied, i.e.,

$$\Omega = \begin{pmatrix} \tau + \tau_N \Delta N_{12}/p & \tau_N N_{12}/p & 0 \\ \tau_N N_{12}/p & \tau & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix} = \tau B_{I_2} + \tau_N B_{N_2} + \tau_{33} B_{33}. \quad (102)$$

This is nothing but products of magnetized T^4 with the class (2-2-b) and T^2 compactifications. Thus, we can consider the $N_3^{(3-2-b)}(H) = N_2^{(2-2-b)}(H) \times \Gamma_1$ transformation. The $T^4/\mathbb{Z}_2^t \times T^2/\mathbb{Z}_2^t$ twisted orbifold also has the same modular symmetry.

Class (3-2-b) On the other hand, in this class, we consider the case satisfying

$$\Delta N_{23} \Delta N_{31} + N_{12}^2 = 0, \quad \Leftrightarrow N_{33} = \frac{(N_{11} + N_{22}) \pm \sqrt{\Delta N_{12}^2 + (2N_{12})^2}}{2}, \quad (103)$$

that is,

$$N = \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{12} & n_{22} & 0 \\ 0 & 0 & n_{33} \end{pmatrix}, \quad n_{33} = \frac{(n_{11} + n_{22}) \pm \sqrt{(n_{11} - n_{22})^2 + (2n_{12})^2}}{2} \in \mathbb{Z}. \quad (104)$$

In particular, when $\Delta N_{12} = 0 \Leftrightarrow n_{11} = n_{22} = n$ and $n_{33} = n \pm n_{12}$, Eq. (98) is satisfied for $\Delta \Omega_{12} = 0$, $\forall\, \Omega_{12}$, and $\Omega_{31} = \pm \Omega_{23}$, i.e.,

$$\Omega = \begin{pmatrix} \tau & \tau_{12} & \tau' \\ \tau_{12} & \tau & \pm\tau' \\ \tau' & \pm\tau' & \tau_{33} \end{pmatrix} = \tau B_{I_2} + \tau_{12} B_{12} + \tau_{33} B_{33} + \tau' (B_{13} \pm B_{23}), \equiv \tau B_{I_2} + \tau_{12} B_{12} + \tau_{33} B_{33} + \tau' B'_{\pm}. \quad (105)$$

On the other hand, when $\Delta N_{12} \neq 0$, Eq. (98) is satisfied if Eq. (86) and

$$\Omega_{23} : \Omega_{31} = N_{12} : \Delta N_{32} = \Delta N_{31} : N_{12} = r_{23}^{\pm} : r_{13}^{\pm} \quad (\gcd(r_{23} : r_{13}) = 1) \quad (106)$$

are satisfied, i.e.,

$$\Omega = \begin{pmatrix} \tau + \tau_N \Delta N_{12}/p & \tau_N N_{12}/p & \tau' r_{13}^{\pm} \\ \tau_N N_{12}/p & \tau & \tau' r_{23}^{\pm} \\ \tau' r_{13}^{\pm} & \tau' r_{23}^{\pm} & \tau_{33} \end{pmatrix} = \tau B_{I_2} + \tau_N B_{N_2} + \tau_{33} B_{33} + \tau' \sum_{i=1,2} r_{i3}^{\pm} B_{i3} \equiv \tau B_{I_2} + \tau_N B_{N_2} + \tau_{33} B_{33} + \tau' B'_{\pm}. \quad (107)$$

Note that the former case is the specific case of the latter case. Then, we can consider $T'_{\pm}$ with the $B'_{\pm}$ transformation in addition to transformations by $N_3^{(3-2-a)}$, which generate the $N_3^{(3-2-b)}(H)$ transformation. In particular, when $n_{11} = n_{22} = n$ and $n_{33} = n \pm n_{12}$, between S and $T'_{\pm}$, the following relations:

$$(ST'_{\pm})^4 = A_{\mp B_{12} - B_{33}}, \qquad (ST'_{\pm})^8 = I_6, \quad (108)$$

are satisfied, as shown in Appendix A. Except for the case with $n_{11} = n_{22} = n$ and $n_{33} = n + n_{12}$, the stabilizer is $H = \{\pm I_6\} = \mathbb{Z}_2^t$, which acts on $z = {}^t(z^1, z^2, z^3)$ as

$$\pm I_6 : \begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix} \to \begin{pmatrix} \pm z^1 \\ \pm z^2 \\ \pm z^3 \end{pmatrix}. \quad (109)$$

Hence, the $T^6/\mathbb{Z}_2^t$ twisted orbifold also has the same modular symmetry. When $n_{11} = n_{22} = n$ and $n_{33} = n + n_{12}$, on the other hand, the stabilizer is $H = \{\pm I_6, \pm A_{B_{12}+B_{33}}\} = \mathbb{Z}_2^t \times \mathbb{Z}_2^p$, which acts on $z = {}^t(z^1, z^2, z^3)$ as

$$\pm A_{B_{12}+B_{33}} : \begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix} \to \begin{pmatrix} \pm z^2 \\ \pm z^1 \\ \pm z^3 \end{pmatrix}, \quad (110)$$

in addition to Eq. (109). Hence, the $T^6/(\mathbb{Z}_2^t \times \mathbb{Z}_2^p)$ twisted and permutation orbifold also has the same modular symmetry.

Class (3-3) In this class, we consider the case with $N_{12} \neq 0$, $N_{13} \neq 0$, and $N_{23} = 0$ case, i.e.,

$$N = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{12} & n_{22} & 0 \\ n_{13} & 0 & n_{33} \end{pmatrix}. \quad (111)$$

Equation (68) is written by

$$\begin{pmatrix} \Delta N_{12} & N_{31} & 0 \\ -N_{31} & \Delta N_{23} & N_{12} \\ 0 & -N_{12} & \Delta N_{31} \end{pmatrix} \begin{pmatrix} \Omega_{12} \\ \Omega_{23} \\ \Omega_{31} \end{pmatrix} = \begin{pmatrix} N_{12} \Delta \Omega_{12} \\ 0 \\ N_{31} \Delta \Omega_{31} \end{pmatrix}. \quad (112)$$

Then, it is satisfied when the following relations,

$$\Omega_{12} = \tau_N N_{12}/p + \frac{\Delta N_{23}}{N_{31}} \Omega_{23}, \quad \Omega_{31} = \tau_N N_{31}/p, \quad \Delta \Omega_{12} = \frac{\Delta N_{12}}{N_{12}} \Omega_{12} + \frac{N_{31}}{N_{12}} \Omega_{23}, \quad \Delta \Omega_{31} = \frac{\Delta N_{31}}{N_{31}} \Omega_{31} - \frac{N_{12}}{N_{31}} \Omega_{23}, \quad \Delta \Omega_{23} = -\frac{\Delta N_{12}}{N_{12}} \Omega_{12} - \frac{\Delta N_{31}}{N_{31}} \Omega_{31} + \frac{N_{12}^2 - N_{13}^2}{N_{12} N_{13}} \Omega_{23}, \quad (113)$$

are satisfied, i.e.,

$$
\begin{aligned}
\Omega &= \tau\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \tau_N\begin{pmatrix} \Delta N_{12}/p & N_{12}/p & N_{13}/p \\ N_{12}/p & 0 & 0 \\ N_{13}/p & 0 & -\Delta N_{23}/p \end{pmatrix} \\
&\quad + \tau_{N^{-1}}\begin{pmatrix} (\Delta N_{12}\Delta N_{23}+N_{13}^2)/p^2 & N_{12}\Delta N_{23}/p^2 & 0 \\ N_{12}\Delta N_{23}/p^2 & 0 & N_{12}N_{13}/p^2 \\ 0 & N_{12}N_{13}/p^2 & (\Delta N_{12}\Delta N_{23}+N_{13}^2-N_{12}^2)/p^2 \end{pmatrix} \\
&= \tau\sum_{i=1,2,3} B_{ii} + \tau_N\left(\Delta N_{12}/pB_{11} - \Delta N_{31}/pB_{33} + \sum_{i=2,3} N_{1i}/pB_{1i}\right) \\
&\quad + \tau_{N^{-1}}[(\Delta N_{12}\Delta N_{23}+N_{13}^2)/p^2B_{11} - (\Delta N_{12}\Delta N_{23}+N_{13}^2-N_{12}^2)/p^2B_{33} + \Delta N_{12}\Delta N_{23}/p^2 + N_{12}N_{13}/p^2B_{23}] \\
&\equiv \tau B_{I_3} + \tau_N B_{N_3} + \tau_{N^{-1}}B_{N_3^{-1}},
\end{aligned}
\tag{114}
$$

where $p = \gcd(\Delta N_{12}, \Delta N_{23}, N_{12}, N_{13})$. Then, we can consider S, T_{I_3}, T_{N_3} with $B = B_{N_3}$, and $T_{N_3^{-1}}$ with $B = B_{N_3^{-1}}$ transformations, which generate the $N_3^{(3-3)}(H)$ transformation. In particular, when $\Delta N_{12} = \Delta N_{23} = \Delta N_{31} = 0$ and $N_{12} = N_{13}$, among T_{I_3}, T_{N_3}, $T_{N_3^{-1}}$, and S, the following relations:

$$
\begin{aligned}
&(ST_{I_3})^3 = I_6, \\
&(ST_{N_3})^4 = -A_{B_{11}+B_{23}}, \qquad (ST_{N_3})^8 = I_6, \\
&(ST_{N_3^{-1}})^3 = A_{B_{11}+B_{23}}, \qquad (ST_{N_3^{-1}})^6 = I_6,
\end{aligned}
\tag{115}
$$

are satisfied, as shown in Appendix A. Except for the case with $\Delta N_{23} = 0$ and $N_{12} = N_{13}$, the stabilizer is $H = \{\pm I_6\} = \mathbb{Z}_2^t$, which acts on $z = {}^t(z^1, z^2, z^3)$ as Eq. (109). Hence, the $T^6/\mathbb{Z}_2^t$ twisted orbifold also has the same modular symmetry. When $\Delta N_{23} = 0$ and $N_{12} = N_{13}$, on the other hand, the stabilizer is $H = \{\pm I_6, \pm A_{B_{11}+B_{23}}\} = \mathbb{Z}_2^t \times \mathbb{Z}_2^p$, which acts on $z = {}^t(z^1, z^2, z^3)$ as

$$
\pm A_{B_{11}+B_{23}}: \begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix} \to \begin{pmatrix} \pm z^1 \\ \pm z^3 \\ \pm z^2 \end{pmatrix},
\tag{116}
$$

in addition to Eq. (109). Hence, the $T^6/(\mathbb{Z}_2^t \times \mathbb{Z}_2^p)$ twisted and permutation orbifold also has the same modular symmetry.

Class (3-4) In this class, we consider the case with $N_{12} \neq 0$, $N_{23} \neq 0$, and $N_{31} \neq 0$. Equation (68) is written by

$$
\begin{aligned}
\begin{pmatrix} \Delta\Omega_{12} \\ \Delta\Omega_{23} \\ \Delta\Omega_{31} \end{pmatrix} &= \begin{pmatrix} N_{12}^{-1} & 0 & 0 \\ 0 & N_{23}^{-1} & 0 \\ 0 & 0 & N_{31}^{-1} \end{pmatrix}\begin{pmatrix} \Delta N_{12} & N_{31} & -N_{23} \\ -N_{31} & \Delta N_{23} & N_{12} \\ N_{23} & -N_{12} & \Delta N_{31} \end{pmatrix}\begin{pmatrix} \Omega_{12} \\ \Omega_{23} \\ \Omega_{31} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\Delta N_{12}}{N_{12}}\Omega_{12} + \frac{N_{31}}{N_{12}}\Omega_{23} - \frac{N_{23}}{N_{12}}\Omega_{31} \\ -\frac{N_{31}}{N_{23}}\Omega_{12} + \frac{\Delta N_{23}}{N_{23}}\Omega_{23} + \frac{N_{12}}{N_{23}}\Omega_{31} \\ \frac{N_{23}}{N_{31}}\Omega_{12} - \frac{N_{12}}{N_{31}}\Omega_{23} + \frac{\Delta N_{31}}{N_{31}}\Omega_{31} \end{pmatrix}.
\end{aligned}
\tag{117}
$$

Note that it is required that

$$
\begin{aligned}
&\Delta\Omega_{12} + \Delta\Omega_{23} + \Delta\Omega_{31} = 0, \\
&\Leftrightarrow \begin{pmatrix} \frac{\Delta N_{12}}{N_{12}} - \frac{N_{31}^2 - N_{23}^2}{N_{23}N_{31}} & \frac{\Delta N_{23}}{N_{23}} - \frac{N_{12}^2 - N_{31}^2}{N_{31}N_{12}} & \frac{\Delta N_{31}}{N_{31}} - \frac{N_{23}^2 - N_{12}^2}{N_{12}N_{23}} \end{pmatrix}\begin{pmatrix} \Omega_{12} \\ \Omega_{23} \\ \Omega_{31} \end{pmatrix} = 0.
\end{aligned}
\tag{118}
$$

This class is further classified as follows.

Class (3-4-a) In this class, we consider the case with the vector,

$$\begin{pmatrix} \frac{\Delta N_{12}}{N_{12}} - \frac{N_{31}^2 - N_{23}^2}{N_{23}N_{31}} \\ \frac{\Delta N_{23}}{N_{23}} - \frac{N_{12}^2 - N_{31}^2}{N_{31}N_{12}} \\ \frac{\Delta N_{31}}{N_{31}} - \frac{N_{23}^2 - N_{12}^2}{N_{12}N_{23}} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{119}$$

In this case, there are 2 degrees of freedom of the vector, ${}^t(\Omega_{12}, \Omega_{23}, \Omega_{31})$, which satisfies Eq. (118). Indeed, that vector can be written as

$$\begin{pmatrix} \Omega_{12} \\ \Omega_{23} \\ \Omega_{31} \end{pmatrix} = \tau_N \begin{pmatrix} N_{12} \\ N_{23} \\ N_{31} \end{pmatrix} + \tau_{N^{-1}} \begin{pmatrix} N_{23}N_{31} - N_{12}N_{33} \\ N_{31}N_{12} - N_{23}N_{11} \\ N_{12}N_{23} - N_{31}N_{22} \end{pmatrix}, \tag{120}$$

and then, from Eq. (117), we can obtain that

$$\begin{pmatrix} \Delta\Omega_{12} \\ \Delta\Omega_{23} \\ \Delta\Omega_{31} \end{pmatrix} = \tau_N \begin{pmatrix} \Delta N_{12} \\ \Delta N_{23} \\ \Delta N_{31} \end{pmatrix} + \tau_{N^{-1}} \begin{pmatrix} N_{31}^2 - N_{12}^2 - \Delta N_{12}N_{33} \\ N_{12}^2 - N_{31}^2 - \Delta N_{23}N_{11} \\ N_{23}^2 - N_{12}^2 - \Delta N_{31}N_{22} \end{pmatrix}. \tag{121}$$

Thus, the moduli can be written by

$$\begin{aligned} \Omega &= \tau \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \tau_N/p_N \begin{pmatrix} N_{11} & N_{12} & N_{31} \\ N_{12} & N_{22} & N_{23} \\ N_{31} & N_{23} & N_{33} \end{pmatrix} \\ &\quad + \tau_{N^{-1}}/p_{N^{-1}} \begin{pmatrix} N_{22}N_{33} - N_{23}^2 & N_{23}N_{31} - N_{12}N_{33} & N_{12}N_{23} - N_{31}N_{22} \\ N_{23}N_{31} - N_{12}N_{33} & N_{33}N_{11} - N_{31}^2 & N_{31}N_{12} - N_{23}N_{11} \\ N_{12}N_{23} - N_{31}N_{22} & N_{31}N_{12} - N_{23}N_{11} & N_{11}N_{22} - N_{12}^2 \end{pmatrix} \\ &= \tau \sum B_{ii} + \tau_N \sum N_{ij}/p_N B_{ij} + \tau_{N^{-1}} \sum \tilde{N}_{ij}/p_{N^{-1}} B_{ij} \\ &\equiv \tau B_{I_3} + \tau_N B_{N_3} + \tau_{N^{-1}} B_{N_3^{-1}}, \end{aligned} \tag{122}$$

where p_N and $p_{N^{-1}}$ denote the greatest common divisor of all elements of the N matrix and the adjugate matrix $\tilde{N}$ [defined by $N^{-1} = (\det N)^{-1}\tilde{N}$], respectively. Note that the class (3-3) is the specific case of the class (3-4-a). Then, we can consider S, T_{I_3}, T_{N_3} with $B = B_{N_3}$, and $T_{N_3^{-1}}$ with $B = B_{N_3^{-1}}$, which generate the $N_3^{(3-4-b)}(H)$ transformation. The stabilizer is $H = \{\pm I_6\} = \mathbb{Z}_2^t$, which acts on $z = {}^t(z^1, z^2, z^3)$ as Eq. (109). Hence, the $T^6/\mathbb{Z}_2^t$ twisted orbifold also has the same modular symmetry.

Class (3-4-b) In this class, we consider the case that

$$N = \begin{pmatrix} n_{11} & n_{12} & n_{31} \\ n_{12} & n_{22} & n_{23} \\ n_{31} & n_{23} & n_{33} \end{pmatrix}, \begin{cases} n_{11} = n + \frac{n_{31}n_{12}}{n_{23}} \in \mathbb{Z} \\ n_{22} = n + \frac{n_{12}n_{23}}{n_{31}} \in \mathbb{Z}, \\ n_{33} = n + \frac{n_{23}n_{31}}{n_{12}} \in \mathbb{Z} \end{cases} \tag{123}$$

which satisfies

$$\begin{pmatrix} \frac{\Delta N_{12}}{N_{12}} - \frac{N_{31}^2 - N_{23}^2}{N_{23}N_{31}} \\ \frac{\Delta N_{23}}{N_{23}} - \frac{N_{12}^2 - N_{31}^2}{N_{31}N_{12}} \\ \frac{\Delta N_{31}}{N_{31}} - \frac{N_{23}^2 - N_{12}^2}{N_{12}N_{23}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{124}$$

Equation (118) is satisfied for $\forall\,\Omega_{12}$, $\forall\,\Omega_{23}$, and $\forall\,\Omega_{31}$. By combining Eq. (117), the moduli in this case are given by

$$\begin{aligned}\Omega &= \tau\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \tau'_{12}\begin{pmatrix}(N_{31}^2-N_{23}^2)/p_{12}^2 & N_{23}N_{31}/p_{12}^2 & 0\\ N_{23}N_{31}/p_{12}^2 & 0 & 0\\ 0 & 0 & N_{31}^2/p_{12}^2\end{pmatrix}\\ &\quad + \tau'_{23}\begin{pmatrix}N_{12}^2/p_{23}^2 & 0 & 0\\ 0 & (N_{12}^2-N_{31}^2)/p_{23}^2 & N_{31}N_{12}/p_{23}^2\\ 0 & N_{31}N_{12}/p_{23}^2 & 0\end{pmatrix}\\ &\quad + \tau'_{31}\begin{pmatrix}0 & 0 & N_{12}N_{23}/p_{31}^2\\ 0 & N_{23}^2/p_{31}^2 & 0\\ N_{12}N_{23}/p_{31}^2 & 0 & (N_{23}^2-N_{12}^2)/p_{31}^2\end{pmatrix}\end{aligned} \tag{125}$$

$$\begin{aligned}&= \tau\sum_{i=1,2,3}B_{ii} + \tau'_{12}(N_{23}N_{31}/p_{12}^2B_{12} + N_{31}^2/p_{12}^2B_{33} + (N_{31}^2-N_{23}^2)/p_{12}^2B_{11})\\ &\quad + \tau'_{23}(N_{31}N_{12}/p_{23}^2B_{23} + N_{12}^2/p_{23}^2B_{11} + (N_{12}^2-N_{31}^2)/p_{23}^2B_{22})\\ &\quad + \tau'_{31}(N_{12}N_{23}/p_{31}^2B_{13} + N_{23}^2/p_{31}^2B_{22} + (N_{23}^2-N_{12}^2)/p_{31}^2B_{33})\\ &\equiv \tau B_{I_3} + \tau'_{12}B'_{12} + \tau'_{23}B_{23} + \tau'_{31}B'_{13},\end{aligned} \tag{126}$$

where $p_{12}=\gcd(N_{23},N_{31})$, $p_{23}=\gcd(N_{31},N_{12})$, and $p_{31}=\gcd(N_{12},N_{23})$. Then, we can consider S, T_{I_3}, T'_{12} with $B=B'_{12}$, T'_{23} with $B=B'_{23}$, and T'_{31} with $B=B'_{31}$, which generate the $N_3^{(3-4-b)}(H)$ transformation. In particular, when $N_{12}=N_{23}=N_{31}$, among T'_{12}, T'_{23}, T'_{31}, and S, the following relations:

$$\begin{aligned}(ST'_{12})^3 &= A_{B_{12}+B_{33}}, & (ST'_{12})^6 &= I_6,\\ (ST'_{23})^3 &= A_{B_{23}+B_{11}}, & (ST'_{23})^6 &= I_6,\\ (ST'_{31})^3 &= A_{B_{31}+B_{22}}, & (ST'_{31})^6 &= I_6,\end{aligned} \tag{127}$$

are satisfied. The stabilizer is generally $H=\{\pm I_6\}=\mathbb{Z}_2^t$, which acts on $z={}^t(z^1,z^2,z^3)$ as Eq. (109). Hence, the $T^6/\mathbb{Z}_2^t$ twisted orbifold also has the same modular symmetry. In particular, when $\tau'_{12}=\tau'_{23}=\tau'_{31}$ is also satisfied, the stabilizer becomes $H=\{\pm I_6, \pm A_{B_{12}+B_{33}}, \pm A_{B_{23}+B_{11}}, \pm A_{B_{31}+B_{22}}, \pm A_{P_{231}}, \pm A_{P_{312}}\}=\mathbb{Z}_2^t\times S_3$, which acts on $z={}^t(z^1,z^2,z^3)$ as

$$\pm A_{P_{231}}: \begin{pmatrix}z^1\\z^2\\z^3\end{pmatrix} \to \begin{pmatrix}\pm z^2\\ \pm z^3\\ \pm z^1\end{pmatrix}, \tag{128}$$

$$\pm A_{P_{312}}: \begin{pmatrix}z^1\\z^2\\z^3\end{pmatrix} \to \begin{pmatrix}\pm z^3\\ \pm z^1\\ \pm z^2\end{pmatrix}, \tag{129}$$

$$\pm A_{B_{31}+B_{22}}: \begin{pmatrix}z^1\\z^2\\z^3\end{pmatrix} \to \begin{pmatrix}\pm z^3\\ \pm z^2\\ \pm z^1\end{pmatrix}, \tag{130}$$

in addition to Eqs. (109), (110), and (116). We note that

$$\begin{aligned}A_{B_{12}+B_{33}}A_{B_{31}+B_{22}} &= A_{B_{31}+B_{22}}A_{B_{23}+B_{11}}\\ &= A_{B_{23}+B_{11}}A_{B_{12}+B_{33}} = A_{P_{231}},\\ A_{B_{31}+B_{22}}A_{B_{12}+B_{33}} &= A_{B_{23}+B_{11}}A_{B_{31}+B_{22}}\\ &= A_{B_{12}+B_{33}}A_{B_{23}+B_{11}} = A_{P_{312}}.\end{aligned} \tag{131}$$

Hence, the $T^6/(\mathbb{Z}_2^t\times S_3)$ orbifold also has the same modular symmetry.

In the above cases, not only the magnetic flux F but the gauge potential A and the covariant derivative D as well as the Dirac operator $i\not{D}$ are invariant under the corresponding modular transformation.

B. Modular symmetry of wave functions on magnetized T^{2g} and orbifold models

Now, let us see the modular symmetry of wave functions on magnetized T^{2g} in Eq. (51). First, for the boundary conditions of wave functions on magnetized T^{2g} in Eqs. (48) and (49) to be consistent with the modular transformation, particularly for S and T transformations, the following conditions:

$$S\,\text{transformation} \Rightarrow \beta_k^S = \alpha_k^S = 0, \quad \text{or} \quad 1/2, \tag{132}$$

$$T \text{ transformation} \Rightarrow (NB)_{ii} + (2{}^t\alpha^S B)_i = \sum_k (N_{ik} + 2\alpha_k^S)(B)_{ki} \in 2\mathbb{Z}, \quad (133)$$

are required,[5] in addition to Eqs. (62)–(65), where B denotes a g-dimensional symmetric matrix generated by some combinations of B_{ab}. In particular, to consider the full $\Gamma_g = Sp(2g, \mathbb{Z})$ transformation for wave functions on magnetized T^{2g}, it is required that the N matrix is in the class (g-1-a) with $n \in 2\mathbb{Z}$ and the SS phases are $\alpha_k^S = \beta_k^S = 0$. Hereafter, we consider vanishing SS phases, and then we omit the SS phase indices. In this case, to satisfy Eq. (133), the N matrix in each class must be $\forall N_{ij} \in 2\mathbb{Z}$. In the following, we often denote the N matrix as

$$N = sN', \quad (134)$$

where s is the greatest common divisor of all components of the N matrix, $\forall N_{ij}$.

When the above conditions, including conditions in individual classes discussed in the previous subsection, are satisfied, wave functions on magnetized T^{2g} in Eq. (51) transform under the modular transformation [34,87] as

$$\tilde{\gamma}: \psi_{T^{2g}}^{J,N}(z, \Omega) \to \psi_{T^{2g}}^{J,N}(\tilde{\gamma}(z, \Omega)) = \tilde{J}_{1/2}(\tilde{\gamma}, \Omega)\rho_{T^{2g}}(\tilde{\gamma})_{JK}\psi_{T^{2g}}^{K,N}(z, \Omega), \quad (135)$$

$$\tilde{S} = [S, (-1)^g]: \tilde{J}_{1/2}(\tilde{S}, \Omega) = (-1)^g(\det(-\Omega))^{1/2},$$
$$\rho_{T^{2g}}(\tilde{S})_{JK} = \frac{(-e^{\pi i/4})^g}{\sqrt{\det N}} e^{2\pi i {}^t J N^{-1} K}, \quad (136)$$

$$\tilde{T} = [T, 1]: \tilde{J}_{1/2}(\tilde{T}, \Omega) = 1, \quad \rho_{T^{2g}}(\tilde{T})_{JK} = e^{\pi i {}^t J B N^{-1} J}\delta_{J,K}. \quad (137)$$

This means that the wave functions behave as the Siegel modular forms of weight $1/2$. Then, as mentioned in Sec. III, the corresponding 4D chiral fields, $\varphi^J(x)$, also transform as

$$\tilde{\gamma}: \varphi^J(x) \to \tilde{J}_{-1/2}(\tilde{\gamma}, \Omega)\bar{\rho}_{T^{2g}}(\tilde{\gamma})_{JK}\varphi^K(x). \quad (138)$$

This means that the 4D chiral fields transform nontrivially under the $\bar{\rho}_{T^{2g}}(\tilde{\gamma})$ modular flavor transformation with modular weight $-1/2$,[6] which can also be found from Eq. (55). In the following, we study under what modular flavor group the wave functions transform nontrivially. Here, we note that the following relation [34,87] should, in particular, be satisfied,

$$\psi_{T^{2g}}^{J,N}(\tilde{A}_X^{\pm}(z, \Omega)) = \tilde{J}_{1/2}(\tilde{A}_X^{\pm}, \Omega)\rho_{T^{2g}}(\tilde{A}_X^{\pm})_{JK}\psi_{T^{2g}}^{K,N}(z, \Omega) = \psi_{T^{2g}}^{XJ,N}(z, \Omega),$$
$$\tilde{A}_X^{\pm} \equiv [A_X, \pm 1]: \tilde{J}_{1/2}(\tilde{A}_X^{\pm}, \Omega) = \pm(\det X)^{1/2},$$
$$\rho_{T^{2g}}(\tilde{A}_X^{\pm})_{JK} = \pm(\det X)^{-1/2}\delta_{XJ,K}. \quad (139)$$

Indeed, as for the $S^2 = A_{-I_g} = -I_{2g}$ transformation, we can find that

$$\rho_{T^{2g}}(\tilde{R})_{JK} = \rho_{T^{2g}}(\tilde{S})^2_{JK} = e^{\pi i g/2}\delta_{-J,K},$$
$$\rho_{T^{2g}}(\tilde{R})^2_{JK} = \rho_{T^{2g}}(\tilde{S})^4_{JK} = (-1)^g\delta_{J,K},$$
$$\rho_{T^{2g}}(\tilde{R})^4_{JK} = \rho_{T^{2g}}(\tilde{S})^8_{JK} = \delta_{J,K}. \quad (140)$$

As for the $(ST)^n = A_{\Omega_0^n}$ transformation, it depends on the detailed structure of the B matrix in the T transformation, as in Appendix A. For example, when $B^2 = I_g$ is satisfied, we can find that

$$[\rho_{T^{2g}}(\tilde{S})\rho_{T^{2g}}(\tilde{T})]^3_{JK} = e^{-\pi i n_-^B/2}\delta_{BJ,K},$$
$$[\rho_{T^{2g}}(\tilde{S})\rho_{T^{2g}}(\tilde{T})]^6_{JK} = (-1)^{n_-^B}\delta_{J,K},$$
$$[\rho_{T^{2g}}(\tilde{S})\rho_{T^{2g}}(\tilde{T})]^{12}_{JK} = \delta_{J,K}, \quad (141)$$

where n_-^B denotes the number of negative eigenvalues of the B matrix and we apply the g-dimensional Landsberg-Schaar relation,

$$\frac{1}{\sqrt{\det N}}\sum_{K \in \Lambda_N} e^{\pi i {}^t K N^{-1} B K} = \frac{e^{\pi i (g - 2n_-^B)/4}}{\sqrt{|\det B|}}\sum_{K \in \Lambda_B} e^{-\pi i {}^t K N B^{-1} K}, \quad (142)$$

for the above calculations. (See Appendix B for the Landsberg-Schaar relation.) By considering Eqs. (140) and (141), we can also find that

$$[\rho_{T^{2g}}(\tilde{S})\rho_{T^{2g}}(\tilde{T})]^6_{JK} = (-1)^{g'-1}e^{3\pi i (g-g')/2}\delta_{(-I_g+2B_{ab}^2)J,K},$$
$$[\rho_{T^{2g}}(\tilde{S})\rho_{T^{2g}}(\tilde{T})]^{12}_{JK} = (-1)^{g-g'}\delta_{J,K},$$
$$[\rho_{T^{2g}}(\tilde{S})\rho_{T^{2g}}(\tilde{T})]^{24}_{JK} = \delta_{J,K}. \quad (143)$$

Similarly, by utilizing the relation in Eq. (142), we can calculate the other relations among S and T, e.g.,

$$[\rho_{T^4}(\tilde{S})\rho_{T^4}(\tilde{T})]^5_{JK} = -\delta_{J,K} \qquad (T = T_{aa}T_{12}(a = 1, 2)),$$
$$[\rho_{T^4}(\tilde{S})\rho_{T^4}(\tilde{T})]^{10}_{JK} = \delta_{J,K}, \quad (144)$$

$$[\rho_{T^4}(\tilde{S})\rho_{T^4}(\tilde{T})]^4_{JK} = -\delta_{\Omega_0^4 J,K}$$
$$(T = T_{ab}T_{bc}((a, b, c) = (1, 2, 3), (2, 3, 1), (3, 1, 2))),$$
$$[\rho_{T^4}(\tilde{S})\rho_{T^4}(\tilde{T})]^8_{JK} = \delta_{J,K}. \quad (145)$$

[5] When $B = B_{I_g}$, it is consistent with the analysis in Ref. [60].
[6] That is, the sign of the modular weight of 4D fields is flipped from one in extra dimensions.

In addition, we obtain the following relations among T transformations:

$$\rho_{T^{2g}}(\tilde{T}_X)^{h_X}_{JK} = \delta_{J,K}, h_X = \begin{cases} s\det N' & ((B_X\tilde{N}')_{ii} \in 2\mathbb{Z} \text{ for } \forall i) \\ 2s\det N' & (\text{otherwise}) \end{cases},$$
$$[\rho_{T^{2g}}(\tilde{T}_X)\rho_{T^{2g}}(\tilde{T}_Y)]_{JK} = [\rho_{T^{2g}}(\tilde{T}_Y)\rho_{T^{2g}}(\tilde{T}_X)]_{JK}, \qquad (146)$$

where T_X and T_Y denote T transformations with $B = B_X$ and $B = B_Y$, respectively. Thus, the wave functions on magnetized T^{2g} in Eq. (51) behave as the Siegel modular forms of weight $1/2$ and $\tilde{N}_g(H,h)$, and then they transform nontrivially under the $\tilde{N}_{g,h}(H) = \tilde{N}_g(H)/\tilde{N}_g(H,h)$ modular flavor transformation, where h denotes the least common multiple of all orders of h_X. In particular, we denote $\tilde{N}_g^{(g-1-a)}(H) \equiv \tilde{\Gamma}_g$, $\tilde{N}_g^{(g-1-a)}(H,h) \equiv \tilde{\Gamma}_g(h)$, and then $\tilde{N}_{g,h}^{(g-1-a)} \equiv \tilde{\Gamma}_{g,h}$, with $h = 2n$.[7]

Furthermore, when we consider the T^{2g} orbifold constructed by identifying a stabilizer $A_X \in \mathbb{Z}_2$ transformation (e.g., discussed in Ref. [33][8]), the wave functions on the magnetized $T^{2g}/\mathbb{Z}_2$, $\psi^{J,N}_{T^{2g}/\mathbb{Z}_2^m}(z,\Omega)$, which satisfy the boundary condition for the $A_X \in \mathbb{Z}_2$ transformation,

$$\psi^{J,N}_{T^{2g}/\mathbb{Z}_2^m}(Xz,\Omega) = (-1)^m \psi^{J,N}_{T^{2g}/\mathbb{Z}_2^m}(z,\Omega), \qquad (147)$$

as well as the boundary conditions in Eqs. (48) and (49), can be expanded by wave functions on the magnetized T^{2g} as

$$\psi^{J,N}_{T^{2g}/\mathbb{Z}_2^m}(z,\Omega) = \mathcal{N}_{T^{2g}/\mathbb{Z}_2}(\psi^{J,N}_{T^{2g}}(z,\Omega) + (-1)^m \psi^{{}^tXJ,N}_{T^{2g}}(z,\Omega)), \qquad (148)$$

and then they transform under the A_X transformation as

$$\begin{aligned}\psi^{J,N}_{T^{2g}}(\tilde{A}_X^{\pm}(z,\Omega)) &= \tilde{J}_{1/2}(\tilde{A}_X^{\pm},\Omega)\rho_{T^{2g}/\mathbb{Z}_2^m}(\tilde{A}_X^{\pm})_{JK}\psi^{K,N}_{T^{2g}/\mathbb{Z}_2^m}(z,\Omega) \\ &= (-1)^m \psi^{J,N}_{T^{2g}/\mathbb{Z}_2^m}(z,\Omega), \\ \tilde{A}_X^{\pm} &\equiv [A_X, \pm 1] : \tilde{J}_{1/2}(\tilde{A}_X^{\pm},\Omega) \\ &= \pm(\det X)^{1/2}, \rho_{T^{2g}/\mathbb{Z}_2^m}(\tilde{A}_X^{\pm})_{JK} \\ &= \pm(-1)^m(\det X)^{-1/2}\delta_{J,K}, \end{aligned} \qquad (149)$$

where m and $\mathcal{N}_{T^{2g}/\mathbb{Z}_2}$ denote the eigenvalue of the $A_X \in \mathbb{Z}_2$ transformation and the normalization factor such that Eq. (55) is also satisfied, respectively. Similarly, we can find that the behavior of wave functions on the magnetized $T^{2g}/\mathbb{Z}_2$ under the modular transformation is the same[9] as that of wave functions on the magnetized T^{2g}. The difference is just the basis of the representation. In particular, in the orbifold eigenbasis, the representation can be block diagonalized by orbifold eigenvalues. In other words, once the orbifold eigenvalue is fixed, we can obtain a smaller representation in the orbifold case than in the T^{2g} case, in general. Hence, we basically consider the orbifold case, hereafter.

Now, let us see concrete modular flavor symmetry of lower-dimensional (two-, three-, and four-dimensional) wave functions on magnetized T^{2g} orbifolds. In particular, we mainly discuss $g = 2$ cases. For $g = 1$ cases, we have studied $g = 1$ cases in Refs. [59,60]. On the other hand, while a lot of $g = 3$ cases can be studied in ways similar to $g = 2$ cases, the generation numbers become larger in general, and then the modular flavor groups become larger and the analysis becomes more complicated. Hereafter, we often use the following notation:

$$|J_1,\ldots,J_g\rangle \equiv \psi^{J,N}_{T^{2g}}(z,\Omega) \quad \text{with} \quad J = {}^t(J_1,\ldots,J_g). \qquad (150)$$

At first, let us see the class (g-1-a) with $n \in 2\mathbb{Z}$. Wave functions on magnetized T^{2g} as well as $T^{2g}/\mathbb{Z}_2^t$ transform nontrivially under the $\tilde{\Gamma}_{g,2n}$ modular flavor transformation. For example, $n = 2$ is the minimal example. For the $g = 1$ case, in Ref. [60],[10] it was found that two-dimensional $\mathbb{Z}_2^t$ twisted even ($m^t = 0$) modes on magnetized $T^2/\mathbb{Z}_2^t$ transform nontrivially under $\tilde{\Gamma}_{1,4} \simeq T' \rtimes Z_4 \simeq \tilde{S}_4$ [92]. Let us see the $g = 2$ case, i.e.,

$$N = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I_2,$$
$$\Omega = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} = \sum_{i,j=1,2,3} \tau_{ij}B_{ij}. \qquad (151)$$

In this case, the following four-dimensional $\mathbb{Z}_2^t$ twisted even ($m^t = 0$) modes[11] on magnetized $T^2/\mathbb{Z}_2^t$:

$$|J_1,J_2\rangle = \begin{pmatrix} |0,0\rangle \\ |1,0\rangle \\ |0,1\rangle \\ |1,1\rangle \end{pmatrix}, \qquad (152)$$

transform under S and T_{ab} transformations as

[7] For $g = 1$, it has been studied in Ref. [59].

[8] See also Refs. [34,35].

[9] As discussed in Ref. [70], if $\mathbb{Z}_2$ eigenvalues around orbifold singular points convert into localized fluxes on the orbifold singular points, which can be potentially required even for the total magnetic flux on the fundamental domain of the T^{2g} orbifold to be integer quantized, the modular weight can be shifted.

[10] See also Ref. [91].

[11] The four-dimensional modes can be modular forms at $z = 0$.

$$\rho_{T^4/\mathbb{Z}_2^0}(\tilde{S}) = \frac{i}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \qquad \rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{12}) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix},$$
$$\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{11}) = \begin{pmatrix} 1 & & & \\ & i & & \\ & & 1 & \\ & & & i \end{pmatrix}, \qquad \rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{22}) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & i & \\ & & & i \end{pmatrix}, \tag{153}$$

which satisfy the following relations:

$$\begin{aligned}
&\rho_{T^4/\mathbb{Z}_2^0}(\tilde{S})^2 = -I, \\
&[\rho_{T^4/\mathbb{Z}_2^0}(\tilde{S})\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{12})]^6 = -I, \\
&[\rho_{T^4/\mathbb{Z}_2^0}(\tilde{S})\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{11})]^6 = [\rho_{T^4/\mathbb{Z}_2^0}(\tilde{S})\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{22})]^6 = -iI, \\
&\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{12})^2 = I, \qquad \rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{11})^4 = \rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{22})^4 = I, \\
&\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{ab})\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{cd}) = \rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{cd})\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{ab}) \qquad (a,b,c,d \in \{1,2\}).
\end{aligned} \tag{154}$$

Then, they transform nontrivially under the $\tilde{\Gamma}_{2,4}$ modular flavor transformation.[12] We note that we can similarly consider the $g=3$ case; eight-dimensional $\mathbb{Z}_2^t$ twisted even $(m^t=0)$ modes on magnetized $T^6/\mathbb{Z}_2$ with $N=2I_3$ transform nontrivially under the $\tilde{\Gamma}_{3,4}$ modular flavor transformation. Moreover, by combining the classes (1-1-a) and (2-1-a), we can similarly consider the classes (2-1-b) and (3-1-b). In the above example, if τ_{12} is restricted to $\tau_{12}=0$, it corresponds to the specific case of the class (2-1-b). Hence, the four-dimensional $\mathbb{Z}_2^{t_k}$ twisted even $(m^{t_k}=0, k=1,2)$ modes on magnetized $T^2/\mathbb{Z}_2^{t_1}\times T^2/\mathbb{Z}_2^{t_2}$ in Eq. (152) transform nontrivially under the $\tilde{\Gamma}_{1,4}\times\tilde{\Gamma}_{1,4} = \tilde{S}_4\times\tilde{S}_4$ modular flavor transformation. Furthermore, if there is a constraint between τ_{11} and τ_{22} such that $\tau_{11}=\tau_{22}=\tau$ in addition to $\tau_{12}=0$, i.e., $\Omega=\tau I$, the three-dimensional $\mathbb{Z}_2^t$ twisted even $(m^t=0)$ and $\mathbb{Z}_2^p$ permutation even $(m^p=0)$ modes[13] on magnetized $T^2/(\mathbb{Z}_2^t\times\mathbb{Z}_2^m)$,

$$|J_1, J_2\rangle = \begin{pmatrix} |0,0\rangle \\ \frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle) \\ |1,1\rangle \end{pmatrix}, \tag{155}$$

transform nontrivially under the $\Gamma'_{1,4}\simeq S'_4\simeq\Delta'(24)\simeq[(Z_2\times Z'_2)\rtimes Z_3]\rtimes Z_4$ modular flavor transformation with weight 1 [60,66], where we can write the generators of Z_2, Z'_2, Z_3, and Z_4 as

$$\begin{aligned}
&a = T^2ST^2ST^2, \qquad a' = ST^2S^{-1}T^{-2}, \\
&b = TS^3T^2, \qquad c = ST^2ST^3,
\end{aligned} \tag{156}$$

respectively, and we can indeed check the relations among them;

$$\begin{aligned}
&a^2 = a'^2 = b^3 = c^4 = 1 \qquad (c=-1), \\
&aa' = a'a, \qquad bab^{-1} = a^{-1}a'^{-1}, \qquad ba'b^{-1} = a, \\
&cac^{-1} = a'^{-1}, \qquad ca'c^{-1} = a^{-1}, \qquad cbc^{-1} = b^{-1}.
\end{aligned} \tag{157}$$

On the other hand, if the constraints are $\tau_{11}=\tau_{22}=\tau$ and $\tau_{12}\neq 0$, it corresponds to the specific case of the class (2-2-a). Hence, the three-dimensional modes on magnetized $T^2/(\mathbb{Z}_2^t\times\mathbb{Z}_2^m)$ in Eq. (155) transform under the S, $T_{I_2}=T_{11}T_{22}$ and T_{12} transformations as

$$\begin{aligned}
&\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{0_p})}(\tilde{S}) = \frac{i}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \\
&\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{0_p})}(\tilde{T}_{I_2}) = \begin{pmatrix} 1 & & \\ & i & \\ & & -1 \end{pmatrix}, \\
&\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{0_p})}(\tilde{T}_{12}) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix},
\end{aligned} \tag{158}$$

which satisfy the following relations:

[12]The order is too large to specify the concrete modular flavor group.

[13]The three-dimensional modes can be modular forms at $z=0$.

$$
\begin{aligned}
&\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{0_p})}(\tilde{S})^2=-I,\\
&[\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{0_p})}(\tilde{S})\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{0_p})}(\tilde{T}_{I_2})]^3=I,\\
&[\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{0_p})}(\tilde{S})\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{0_p})}(\tilde{T}_{12})]^3=-iI,\\
&\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{0_p})}(\tilde{T}_{12})^2=I,\qquad \rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{0_p})}(\tilde{T}_{I_2})^4=I,\\
&\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{0_p})}(\tilde{T}_{I_2})\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{0_p})}(\tilde{T}_{12})=\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{0_p})}(\tilde{T}_{12})\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{0_p})}(\tilde{T}_{I_2}).
\end{aligned}\tag{159}
$$

Then, they transform under the $\tilde{N}_{2,4}^{(2-2-a)}(H)$ modular flavor transformation. Actually, when we define

$$
\begin{aligned}
&\tilde{a}=T_{I_2}T_{12}ST_{12}T_{I_2}S^{-1}T_{I_2}T_{12}(ST_{12})^3,\qquad \tilde{a}'=ST_{I_2}T_{12}S^{-1}T_{12}^{-1}T_{I_2}^{-1},\\
&b=T_{I_2}S^3T_{I_2}^2,\qquad c'=ST_{I_2}^2ST_{I_2}^3(ST_{12})^3,\\
&d=(ST_{12})^3,
\end{aligned}\tag{160}
$$

we can find that they satisfy the following relations:

$$
\begin{aligned}
&\tilde{a}^4=\tilde{a}'^4=b^3=c'^2=d^4=1,\\
&\tilde{a}\tilde{a}'=\tilde{a}'\tilde{a},\qquad b\tilde{a}b^{-1}=\tilde{a}^{-1}\tilde{a}'^{-1},\qquad b\tilde{a}'b^{-1}=\tilde{a},\qquad c'\tilde{a}c'^{-1}=\tilde{a}'^{-1},\qquad c'\tilde{a}'c'^{-1}=\tilde{a}^{-1},\qquad c'bc'^{-1}=b^{-1},\\
&dx=xd(x=\tilde{a},\tilde{a}',b,c');
\end{aligned}\tag{161}
$$

that is, the three-dimensional modes in Eq. (155) transform nontrivially under the $\Delta(96)\times Z_4$ modular flavor transformation. Notice that we can find the following relation between $\tilde{a}^{(\prime)}$ in Eq. (160) and $a^{(\prime)}$ in Eq. (156):

$$
(\tilde{a}^{(\prime)})^2=a^{(\prime)}.\tag{162}
$$

We can check it in Appendix C. Therefore, we can obtain two patterns of breaking chains, i.e.,

$$
\begin{array}{ccccc}
 & \overset{\tau_{12}=0}{\swarrow} & \widetilde{\Gamma}_{2,4} & \overset{\tau_{11}=\tau_{22}}{\searrow} & \\
\widetilde{S}_4\times\widetilde{S}_4 & & & & \Delta(96)\times Z_4\ . \\
 & \underset{\tau_{11}=\tau_{22}}{\searrow} & S_4' & \underset{\tau_{12}=0}{\swarrow} &
\end{array}\tag{163}
$$

As another example of the class (2-2-a), let us consider the case that

$$
\begin{aligned}
&N=\begin{pmatrix}4&-2\\-2&4\end{pmatrix}=2\begin{pmatrix}2&-1\\-1&2\end{pmatrix}=2N',\qquad \det N=4\det N'=12,\qquad \tilde{N}=2\tilde{N}=2\begin{pmatrix}2&1\\1&2\end{pmatrix},\\
&\Omega=\begin{pmatrix}\tau&\tau_{12}\\\tau_{12}&\tau\end{pmatrix}=\tau B_{I_2}+\tau_{12}B_{12}.
\end{aligned}\tag{164}
$$

In this case, the following three-dimensional $\mathbb{Z}_2^t$ twisted odd $(m^t=1)$ and $\mathbb{Z}_2^p$ permutation odd $(m^p=1)$ modes:

$$
|J_1,J_2\rangle=\begin{pmatrix}\frac{1}{\sqrt{2}}(|-1,3\rangle-|3,-1\rangle)\\ \frac{1}{2}(|1,0\rangle-|0,1\rangle+|2,1\rangle-|1,2\rangle)\\ \frac{1}{\sqrt{2}}(|2,0\rangle-|0,2\rangle)\end{pmatrix},\tag{165}
$$

transform under the S, T_{I_2}, and T_{12} transformations as

$$\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{S}) = -\frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix},$$
$$\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2}) = \begin{pmatrix} e^{\pi i/3} & & \\ & e^{\pi i/3} & \\ & & e^{4\pi i/3} \end{pmatrix} = e^{\pi i/3}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix},$$
$$\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12}) = \begin{pmatrix} e^{-\pi i/3} & & \\ & e^{\pi i/6} & \\ & & e^{2\pi i/3} \end{pmatrix} = e^{-\pi i/3}\begin{pmatrix} 1 & & \\ & i & \\ & & -1 \end{pmatrix}, \tag{166}$$

which satisfy the following relations:

$$\begin{aligned}
&\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{S})^2 = I, \\
&[\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{S})\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2})]^3 = I, \\
&[\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{S})\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12})]^3 = iI, \\
&\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2})^2 = \rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12})^4 = e^{2\pi i/3}I, \\
&\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2})\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12}) = \rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12})\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2}).
\end{aligned} \tag{167}$$

Then, they transform nontrivially under the $\tilde{N}_{2,12}^{(2-2-a)}(H)$ modular flavor transformation. We note that wave functions in the class (2-2-a) generally transform nontrivially under the $\tilde{N}_{2,2s\det N'}^{(2-2-a)}(H)$ modular flavor transformation. Actually, similar to Eq. (158), we can find that the three-dimensional modes in Eq. (165) transform nontrivially under the $\Delta(96)\times Z_4\times Z_3$ modular flavor transformation, where the generators of $\Delta(96)\times Z_4$ are in Eq. (160) replacing S, T_{I_2}, and T_{12} with the following s, t, and t_{12},:

$$s=S(ST_{12})^{-3},\quad t=T_{12}T_{I_2}^{-2}(ST_{12})^{-6},\quad t_{12}=T_{I_2}T_{12}^4(ST_{12})^6, \tag{168}$$

respectively, while the Z_3 generator is $e=T_{I_2}^2=T_{12}^4$, satisfying

$$e^3=1,\qquad ex=xe\ (x=\tilde{a},\tilde{a}',b,c',d). \tag{169}$$

Here, if τ_{12} is restricted to $\tau_{12}=0$, i.e., $\Omega=\tau I$, the three-dimensional modes in Eq. (165) transform nontrivially under the $S_3\times Z_3\simeq(Z_3'\rtimes Z_2)\times Z_3$ modular flavor transformation with weight 1, where the generators Z_3', Z_2, and Z_3 are written as

$$p=ST_{I_2},\qquad q=T_{I_2}^3,\qquad e=T_{I_2}^2, \tag{170}$$

satisfying

$$p^3=q^2=e^3=1,\quad q^{-1}pq=p^2,\quad ex=xe\ (x=p,q). \tag{171}$$

Therefore, we can obtain the following breaking pattern:

$$\Delta(96)\times Z_4\times Z_3 \xrightarrow{\tau_{12}=0} S_3\times Z_3. \tag{172}$$

So far, we have seen the modular flavor symmetry of three-dimensional $\mathbb{Z}_2^t$ twisted odd $(m^t=1)$ and $\mathbb{Z}_2^p$ permutation odd $(m^p=1)$ modes in Eq. (165). On the other hand, the following two-dimensional $\mathbb{Z}_2$ twisted even $(m^t=0)$ and $\mathbb{Z}_2$ permutation odd $(m^p=1)$ modes,

$$|J_1,J_2\rangle = \begin{pmatrix} \frac{1}{2}(|1,0\rangle-|0,1\rangle-|2,1\rangle+|1,2\rangle) \\ \frac{1}{\sqrt{2}}(|2,-1\rangle-|-1,2\rangle) \end{pmatrix}, \tag{173}$$

transform under S, T_{I_2}, and T_{12} transformations as

$$\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{S}) = -\frac{i}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix},$$
$$\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2}) = \begin{pmatrix} e^{\pi i/3} & \\ & -1 \end{pmatrix},$$
$$\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12}) = \begin{pmatrix} e^{\pi i/6} & \\ & -i \end{pmatrix}, \tag{174}$$

which satisfy the following relations:

$$\begin{aligned}
&\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{S})^2=-I,\\
&[\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{S})\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2})]^3=I,\\
&[\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{0_p})}(\tilde{S})\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12})]^3=iI,\\
&\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2})=\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12})^2,\\
&\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12})^3=iI,\\
&\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2})^3=\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12})^6=-I,\\
&\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2})\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12})\\
&\quad=\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12})\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2}). \qquad (175)
\end{aligned}$$

Then, they transform nontrivially under the $\tilde{N}_{2,12}^{(2-2-a)}(H)$ modular flavor transformation. Actually, when we define

$$s=S^{-1},\qquad t=S^2T_{I_2},\qquad c=ST_{I_2}T_{12},\qquad (176)$$

we can find that they satisfy the following relations:

$$\begin{aligned}
&s^2=-1,\qquad s^4=t^3=(st)^3=c^2=1,\\
&c^{-1}xc=x'\ (x,x'\in T'). \qquad (177)
\end{aligned}$$

Thus, the two-dimensional modes in Eq. (173) transform nontrivially under the $T'\rtimes Z_2$ modular flavor transformation. Here, if τ_{12} is restricted to $\tau_{12}=0$, i.e., $\Omega=\tau I$, the two-dimensional modes in Eq. (173) transform nontrivially under the T' modular flavor transformation with weight 1, where the generators are s and t in Eq. (D15). Therefore, we can obtain the following breaking pattern:

$$T'\rtimes Z_2\xrightarrow{\tau_{12}=0}T'. \qquad (178)$$

Similarly, let us see the following example of the class (2-2-b) with the $\Delta N_{12}=N_{12}$ case:

$$\begin{aligned}
&N=\begin{pmatrix}2&-2\\-2&4\end{pmatrix}=2\begin{pmatrix}1&-1\\-1&2\end{pmatrix}=2N',\\
&\det N=4\det N'=4,\qquad \tilde{N}=2\tilde{N}=2\begin{pmatrix}1&-1\\-1&2\end{pmatrix},\\
&\Omega=\begin{pmatrix}\tau+\tau_N&\tau_N\\\tau_N&\tau\end{pmatrix}=\tau B_{I_2}+\tau_N B_{N_2}. \qquad (179)
\end{aligned}$$

In this case, the following four-dimensional $\mathbb{Z}_2^t$ twisted even $(m^t=0)$ modes[14]:

$$|J_1,J_2\rangle=\begin{pmatrix}|0,0\rangle\\|0,1\rangle\\|1,-1\rangle\\|-1,2\rangle\end{pmatrix}, \qquad (180)$$

transform under the S, T_{I_2}, and T_{N_2} transformations as

$$\begin{aligned}
&\rho_{T^4/\mathbb{Z}_2^0}(\tilde{S})=\frac{i}{2}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix},\\
&\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{I_2})=\begin{pmatrix}1&&&\\&i&&\\&&i&\\&&&-1\end{pmatrix},\\
&\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{N_2})=\begin{pmatrix}1&&&\\&i&&\\&&1&\\&&&-i\end{pmatrix}, \qquad (181)
\end{aligned}$$

which satisfy the following relations:

$$\begin{aligned}
&\rho_{T^4/\mathbb{Z}_2^0}(\tilde{S})^2=-I,\\
&[\rho_{T^4/\mathbb{Z}_2^0}(\tilde{S})\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{I_2})]^3=I,\\
&[\rho_{T^4/\mathbb{Z}_2^0}(\tilde{S})\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{N_2})]^5=-I,\\
&\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{I_2})^4=I,\qquad \rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{N_2})^4=I,\\
&\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{I_2})\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{N_2})=\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{N_2})\rho_{T^4/\mathbb{Z}_2^0}(\tilde{T}_{I_2}). \qquad (182)
\end{aligned}$$

Then, they transform nontrivially under the $\tilde{N}_{2,4}^{(2-2-b)}(H)$ modular flavor transformation. We note that wave functions in the class (2-2-b) generally transform nontrivially under the $\tilde{N}_{2,2s\det N'}^{(2-2-b)}(H)$ modular flavor transformation. Here, the above S, $T_{I_2}=T_{11}T_{22}$, and $T_{N_2}=T_{11}T_{12}$ transformations in Eq. (181) correspond to the S, $T_{11}T_{22}$, and $T_{11}T_{12}$ transformations in Eq. (153), respectively. Actually, we can find that the four-dimensional modes in Eq. (180) transform nontrivially under the $[(Z_4\times Z_2\times Z_2)\rtimes(Z_2\times Z_2)]\rtimes A_5$ modular flavor transformation. If τ_N is restricted to $\tau_N=0$, i.e., $\Omega=\tau I$, the four-dimensional modes in Eq. (180) transform nontrivially under the $\tilde{\Gamma}'_{1,4}\simeq S'_4$ modular flavor transformation with weight 1. Therefore, we can obtain the following breaking pattern:

$$[(Z_4\times Z_2\times Z_2)\rtimes(Z_2\times Z_2)]\rtimes A_5\xrightarrow{\tau_N=0}S'_4. \qquad (183)$$

Now, we can similarly consider the case of the class (3-2-a). As a specific case, let us see the following example of the class (3-2-b) with the $N_{11}=N_{22}=n=4$ and $N_{33}=n+N_{12}=4-2=2$ case, i.e.,

[14]The four-dimensional modes can be modular forms at $z=0$.

$$
\begin{aligned}
N &= \begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2N', \\
\Omega &= \begin{pmatrix} \tau & \tau_{12} & \tau' \\ \tau_{12} & \tau & \tau' \\ \tau' & \tau' & \tau_{33} \end{pmatrix} = \tau B_{I_2} + \tau_{12}B_{12} + \tau_{33}B_{33} + \tau' B'_{+}.
\end{aligned} \tag{184}
$$

In this case, the following four-dimensional $\mathbb{Z}_2^t$ twisted even $(m^t = 0)$ and $\mathbb{Z}_2^p$ permutation odd $(m^p = 1)$ modes:

$$
\begin{aligned}
&\rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{S}) = \frac{e^{3\pi i/4}}{\sqrt{6}}\begin{pmatrix} 1 & \sqrt{2} & 1 & \sqrt{1} \\ \sqrt{2} & -1 & \sqrt{2} & -1 \\ 1 & \sqrt{2} & -1 & -\sqrt{2} \\ \sqrt{2} & -1 & -\sqrt{2} & 1 \end{pmatrix}, \\
&\rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2}) = \begin{pmatrix} e^{\pi i/3} & & & \\ & -1 & & \\ & & e^{\pi i/3} & \\ & & & -1 \end{pmatrix}, \qquad \rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12}) = \begin{pmatrix} e^{\pi i/6} & & & \\ & -i & & \\ & & e^{\pi i/6} & \\ & & & -i \end{pmatrix}, \\
&\rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{33}) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & i & \\ & & & i \end{pmatrix}, \qquad \rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}'_{+}) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix},
\end{aligned} \tag{185}
$$

satisfy the following relations:

$$
\begin{aligned}
&\rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{S})^2 = -iI, \\
&[\rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{S})\rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2})]^6 = -iI, \qquad [\rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{S})\rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{33})]^6 = -I, \\
&[\rho_{T^6/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{0_p})}(\tilde{S})\rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12})]^6 = iI, \qquad [\rho_{T^6/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{0_p})}(\tilde{S})\rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}'_{+})]^4 = I, \\
&\rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2}) = \rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12})^2, \qquad \rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}'_{+}) = \rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{33})^2, \\
&\rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12})^3 = iI, \\
&\rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2})^3 = \rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{12})^6 = -I, \qquad \rho_{T^6/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}'_{+})^2 = \rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{33})^4 = I, \\
&\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_X)\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_Y) = \rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_Y)\rho_{T^4/(\mathbb{Z}_2^{0_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_X) \qquad (X, Y = I_2, 12, 33, +).
\end{aligned} \tag{186}
$$

Then, they transform nontrivially under the $\tilde{N}_{3,12}^{(3-2-b)}(H)$ modular flavor transformation. Actually, we can similarly find that the four-dimensional modes transform nontrivially under the $(T' \rtimes A_4) \rtimes Z_4$ modular flavor transformation. If τ'_{+} is restricted to $\tau'_{+} = 0$, they transform nontrivially under the $(T' \rtimes Z_2) \times (T' \rtimes Z_4)$ modular flavor transformation, which is nothing but the direct product of $T^4/(\mathbb{Z}_2^{0_t} \times \mathbb{Z}_2^{1_p})$ with the N matrix in Eq. (164) and $T^2/\mathbb{Z}_2^{0_t}$ with $N = 2$. Therefore, we can obtain the following breaking pattern:

$$
(T' \rtimes A_4) \rtimes Z_4 \xrightarrow{\tau'_{+}=0} (T' \rtimes Z_2) \times (T' \rtimes Z_4). \tag{187}
$$

Finally, let us see the specific cases of the classes (3-3) and (3-4). First, let us see the specific case of the class (3-3) [and also the class (3-4-a)] with $N_{11} = N_{22} = N_{33}$ and $N_{12} = N_{13}$, i.e.,

$$N = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 4 \end{pmatrix} = 2\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} = 2N',$$

$$\det N = 8\det N' = 32, \quad \tilde{N} = 4\tilde{N} = 4\begin{pmatrix} 4 & -2 & -2 \\ -2 & 4 & 1 \\ -2 & 1 & 4 \end{pmatrix},$$

$$\Omega = \begin{pmatrix} \tau + \tau_{N^{-1}} & \tau_N & \tau_N \\ \tau_N & \tau & \tau_{N^{-1}} \\ \tau_N & \tau_{N^{-1}} & \tau \end{pmatrix} = \tau B_{I_3} + \tau_N B_{N_3} + \tau_{N^{-1}} B_{N_3^{-1}}. \tag{188}$$

In this case, the generation number of zero modes on magnetized $T^2/(\mathbb{Z}_2^t \times \mathbb{Z}_2^p)$ with $(m^t, m^p) = (0,1), (1,0), (1,1)$ is 6 while that of zero modes on magnetized $T^2/(\mathbb{Z}_2^t \times \mathbb{Z}_2^p)$ with $(m^t, m^p) = (0,0)$ is 14. Hence, the modular flavor groups, generated by those dimensional representations of S, T_{I_3}, T_{N_3}, and $T_{N_3^{-1}}$ transformations, become larger and complicated, and then we were not able to specify the modular flavor groups. Next, let us see the specific case of the class (3-4-b) with $N_{11} = N_{22} = N_{33}$ and $N_{12} = N_{23} = N_{31}$, i.e.,

$$N = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} = 2\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = 2N',$$

$$\det N = 8\det N' = 32, \quad \tilde{N} = 4\tilde{N} = 4\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix},$$

$$\Omega = \begin{pmatrix} \tau + \tau'_{23} & \tau'_{12} & \tau'_{31} \\ \tau'_{12} & \tau + \tau'_{31} & \tau'_{23} \\ \tau'_{31} & \tau'_{23} & \tau + \tau'_{12} \end{pmatrix} = \tau B_{I_3} + \tau'_{12} B'_{12} + \tau'_{23} B'_{23} + \tau'_{31} B'_{31}. \tag{189}$$

In this case, the generation number of $\mathbb{Z}_2^t$ twisted even $(m^t = 0)$ modes on magnetized $T^2/\mathbb{Z}_2^t$ is 20 while that of $\mathbb{Z}_2^t$ twisted odd $(m^t = 1)$ modes on magnetized $T^2/\mathbb{Z}_2^t$ is 12. Hence, the modular flavor groups, generated by those dimensional representations of S, T_{I_3}, T'_{12}, T'_{23}, and T'_{31} transformations, become larger and complicated, and then we were not able to specify the modular flavor groups.

V. CONCLUSION

We have studied the modular symmetry in magnetized T^{2g} and orbifold models. There is $\Gamma_g = Sp(2g, \mathbb{Z})$ modular symmetry on T^{2g} and its orbifold by the stabilizer H. When a magnetic flux is introduced on T^{2g} as well as its orbifold, the modular symmetry is reduced from Γ_g to a certain normalizer $N_g(H)$. We have classified the remaining modular symmetry by magnetic flux matrix types in Sec. IV A. Furthermore, we have studied modular symmetry for wave functions on the magnetized T^{2g} and certain orbifolds in Sec. IV B. We have found that wave functions on magnetized T^{2g} as well as its orbifolds behave as the Siegel modular forms of weight $1/2$ and $\tilde{N}_g(H, h)$, which is the metaplectic congruence subgroup of the double covering group of $N_g(H)$, $\tilde{N}_g(H)$. Then, they transform nontrivially under the quotient group, $\tilde{N}_{g,h} = \tilde{N}_g(H)/\tilde{N}_g(H, h)$, where the level h is related to the determinant of the magnetic flux matrix. Accordingly, the corresponding 4D chiral fields also transform nontrivially under the $\tilde{N}_{g,h}$ modular flavor transformation with modular weight $-1/2$. We have also studied concrete modular flavor symmetries of wave functions on the magnetized T^{2g} orbifold. The study in this paper is extended from the studies in Refs. [59,60,66] and one specific application of the study in Ref. [77].

Our results are important to study four-dimensional effective field theory derived by torus and orbifold compactifications with a magnetic flux background, in particular, the realization of quark and lepton masses. We would investigate the realistic model building in the magnetized T^{2g} orbifold models elsewhere to understand quark and lepton masses as well as their mixing angles from their modular flavor symmetries.

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APPENDIX A: $(ST)^n$ TRANSFORMATION

In this appendix, we discuss the algebraic relations between the S transformation and the general T transformation generated by the T_{ab} transformation. Generally, $(ST)^n$ can be written as

$$(ST)^n = \begin{pmatrix} -b_{n-1} & b_n \\ -b_n & b_{n+1} \end{pmatrix}, \quad b_{n+1} = -b_n B - b_{n-1}, \tag{A1}$$

with

$$ST = \begin{pmatrix} -b_0 & b_1 \\ -b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 0 & I_g \\ -I_g & -B \end{pmatrix}, \tag{A2}$$

where B denotes a general $g \times g$ symmetric matrix. Here, we assume Ω_0 ($\det \Omega_0 \neq 0$) such that it satisfies

$$\Omega_0 + B = -\Omega_0^{-1}, \tag{A3}$$

$$\Leftrightarrow -(\Omega_0 + B)^{-1} = \Omega_0; \tag{A4}$$

that is, Ω_0 is nothing but the fixed point of the ST transformation in moduli space. Then, the recursion formula in Eq. (A1) can be rewritten by

$$(b_{n+1}\Omega_0^n) - (b_n\Omega_0^{n-1})\Omega_0^2 = I_g. \tag{A5}$$

Furthermore, by introducing

$$\xi \equiv (I_g - \Omega_0^2)^{-1} \qquad (\det(I_g \pm \Omega_0) \neq 0), \tag{A6}$$

the recursion formula can be solved as

$$b_n = (I_g - \Omega_0^2)^{-1}(I_g - \Omega_0^{2n})\Omega_0^{-(n-1)} = \sum_{k=0}^{n-1} \Omega_0^{2k+1-n}, \tag{A7}$$

where we use the following relation:

$$(I_g - \Omega_0^2)^{-1}(I_g - \Omega_0^{2n}) = \sum_{k=0}^{n-1} \Omega_0^{2k}, \tag{A8}$$

for the last equality. Now, we can rewrite Eq. (A1) as

$$\begin{aligned}(ST)^n &= \begin{pmatrix} -b_n\Omega_0 + \Omega_0^n & b_n \\ -b_n & b_n\Omega_0^{-1} + \Omega_0^n \end{pmatrix}, \\ b_n &= (I_g - \Omega_0^2)^{-1}(I_g - \Omega_0^{2n})\Omega_0^{-(n-1)}. \end{aligned} \tag{A9}$$

In particular, when it is satisfied that $\Omega_0^{2n} = I_g \Leftrightarrow b_n = 0$, $(ST)^n$ can be written as

$$\begin{aligned}(ST)^n &= \begin{pmatrix} \Omega_0^n & 0 \\ 0 & \Omega_0^n \end{pmatrix} = A_{\Omega_0^n}, \\ (ST)^{2n} &= I_{2g}. \end{aligned} \tag{A10}$$

Then, in the case that $\Omega_0^{2n+1} = I_g$, we can find that $(ST)^{2n+1} = I_{2g}$.

Let us see the meaning of the result in detail. When we consider $\Omega = \Omega_0$, (z, Ω_0) transform under the ST transformation as

$$ST\colon (z, \Omega_0) \to (\Omega_0 z, \Omega_0). \tag{A11}$$

In addition, if $\Omega_0^n = I_g \Leftrightarrow (ST)^n = I_{2g}$, we can find that

$$(ST)^n = I_{2g}\colon\ (z, \Omega_0) \to (\Omega_0^n z, \Omega_0) = (z, \Omega_0). \tag{A12}$$

This means the $\mathbb{Z}_n$ transformation.

From now on, we discuss how to determine the fixed point Ω_0 and the order n such that $(ST)^n = I_{2g}$. First, the fixed point Ω_0 with $\Omega_0^n = I_g$ is written by the diagonalized matrix and a real orthogonal matrix O as

$$\begin{aligned}\Omega_0 &= O^{-1}\mathrm{diag}(e^{2\pi i k_i/n})O, \\ \mathrm{Re}\Omega_0 + i\mathrm{Im}\Omega_0 &= O^{-1}\mathrm{diag}(\cos(2\pi k_i/n))O \\ &\quad + iO^{-1}\mathrm{diag}(\sin(2\pi k_i/n))O. \end{aligned} \tag{A13}$$

In addition, from Eq. (A3), we can obtain that

$$\mathrm{Re}\Omega_0 = O^{-1}\mathrm{diag}(\cos(2\pi k_i/n))O = -\frac{1}{2}B. \tag{A14}$$

Thus, the B matrix determines $\mathrm{Re}\Omega_0$, the eigenvalues, and the O matrix. In addition, from the result, we can also find $\mathrm{Im}\Omega_0$ and the order n. Here, we note that $B_{ij} = 0, \pm 1$ by considering that the fixed point Ω_0 is on the fundamental region, $|(2\mathrm{Re}\Omega_0)_{ij}| \leq 1$. In the following, we show the above analysis concretely through examples.

First, let us consider the $g = 1$ case:

(i) When we consider $B = 1$, that is ST transformation, we obtain $\mathrm{Re}\Omega_0 = -1/2 = \cos(2\pi/3)$, and then

$$\begin{aligned}\Omega_0 &= e^{2\pi i/3}, \\ \Omega_0^3 &= I_1 \Leftrightarrow (ST)^3 = I_2. \end{aligned} \tag{A15}$$

(ii) When we consider $B = 0$, that is S transformation, we obtain $\mathrm{Re}\Omega_0 = 0 = \cos(2\pi/4)$, and then

$$\begin{aligned}\Omega_0 &= e^{2\pi i/4}, \\ \Omega_0^2 &= -I_1 \Leftrightarrow S^2 = -I_2, \\ \Omega_0^4 &= I_1 \Leftrightarrow S^4 = I_2. \end{aligned} \tag{A16}$$

(iii) When we consider $B = -1$, that is ST^{-1} transformation, we obtain $\mathrm{Re}\Omega_0 = 1/2 = \cos(2\pi/6)$, and then

$$\begin{aligned}\Omega_0 &= e^{2\pi i/6}, \\ \Omega_0^6 &= -I_1 \Leftrightarrow (ST^{-1})^6 = -I_2, \\ \Omega_0^6 &= I_1 \Leftrightarrow (ST^{-1})^6 = I_2. \end{aligned} \tag{A17}$$

Next, let us consider the $g = 2$ case:

(i) When we consider $B = \left(\begin{smallmatrix} b_{11} & 0 \\ 0 & b_{22} \end{smallmatrix}\right)$, that is $ST_{11}^{b_{11}}T_{22}^{b_{22}}$ transformations, with $b_{ii} = 0, \pm 1$, we obtain

$$\begin{aligned}\Omega_0 &= \begin{pmatrix} e^{2\pi i/n_1} & 0 \\ 0 & e^{2\pi i/n_2} \end{pmatrix}, \\ n_i &= \begin{cases} 3 & (b_{ii} = 1) \\ 4 & (b_{ii} = 0) \\ 6 & (b_{ii} = -1) \end{cases}, \\ \Omega_0^{\mathrm{lcm}(n_1,n_2)} &= I_2 \Leftrightarrow (ST_{11}^{b_{11}}T_{22}^{b_{22}})^{\mathrm{lcm}(n_1,n_2)} = I_4. \end{aligned} \tag{A18}$$

These are nothing but $T^2 \times T^2$ cases.

(ii) When we consider $B = \left(\begin{smallmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{smallmatrix}\right)$, that is $ST_{11}T_{22}T_{12}^{\pm 1}$ transformation, and $B = \left(\begin{smallmatrix} -1 & \pm 1 \\ \pm 1 & -1 \end{smallmatrix}\right)$, that is $ST_{11}^{-1}T_{22}^{-1}T_{12}^{\pm 1}$ transformation, we obtain

$$\Omega_0 = O^{-1}\begin{pmatrix} e^{2\pi i/4} & 0 \\ 0 & \pm 1 \end{pmatrix} O. \tag{A19}$$

From it, however, there is no solution since $\det(I - \Omega_0^2) = 0$.

(iii) When we consider $B = \left(\begin{smallmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{smallmatrix}\right)$, that is $ST_{12}^{\pm 1}$ transformations, we obtain

$$\begin{aligned}
\Omega_0 &= O^{-1}\begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{2\pi i/6} \end{pmatrix} O \\
&= \begin{pmatrix} \frac{\sqrt{3}}{2} i & \mp \frac{1}{2} \\ \mp \frac{1}{2} & \frac{\sqrt{3}}{2} i \end{pmatrix}, \\
\Omega_0^3 &= \pm B_{12} \Leftrightarrow (ST_{12}^{\pm 1})^3 = \pm A_{B_{12}}, \\
\Omega_0^6 &= I_2 \Leftrightarrow (ST_{12}^{\pm 1})^6 = I_4.
\end{aligned} \tag{A20}$$

(iv) When we consider $B = \left(\begin{smallmatrix} 1 & \pm 1 \\ \pm 1 & -1 \end{smallmatrix}\right)$, that is $ST_{11}T_{22}^{-1}T_{12}^{\pm 1}$ transformation, we obtain

$$\begin{aligned}
\Omega_0 &= O^{-1}\begin{pmatrix} e^{2\pi i/8} & 0 \\ 0 & e^{6\pi i/8} \end{pmatrix} O \\
&= \begin{pmatrix} -\frac{1}{2} + \frac{i}{\sqrt{2}} & \mp \frac{1}{2} \\ \mp \frac{1}{2} & \frac{1}{2} + \frac{i}{\sqrt{2}} \end{pmatrix}, \\
\Omega_0^4 &= -I_2 \Leftrightarrow (ST_{11}T_{22}^{-1}T_{12}^{\pm 1})^4 = -I_4, \\
\Omega_0^8 &= I_2 \Leftrightarrow (ST_{11}T_{22}^{-1}T_{12}^{\pm 1})^8 = I_4.
\end{aligned} \tag{A21}$$

Similarly, we can find that $(ST_{11}^{-1}T_{22}T_{12}^{\pm 1})^5 = I_4$.

(v) When we consider $B = \left(\begin{smallmatrix} 1 & \pm 1 \\ \pm 1 & 0 \end{smallmatrix}\right)$, that is $ST_{11}T_{12}^{\pm 1}$ transformation, we obtain

$$\begin{aligned}
\Omega_0 &= O^{-1}\begin{pmatrix} e^{2\pi i/5} & 0 \\ 0 & e^{4\pi i/5} \end{pmatrix} O \\
&= \begin{pmatrix} -\frac{1}{2} + \frac{i}{2}\left(\sqrt{\frac{5+\sqrt{5}}{10}} + \sqrt{\frac{5-\sqrt{5}}{10}}\right) & \mp\left(\frac{1}{2} + \frac{i}{2}\left(\sqrt{\frac{5+\sqrt{5}}{10}} - \sqrt{\frac{5-\sqrt{5}}{10}}\right)\right) \\ \mp\left(\frac{1}{2} + \frac{i}{2}\left(\sqrt{\frac{5+\sqrt{5}}{10}} - \sqrt{\frac{5-\sqrt{5}}{10}}\right)\right) & \frac{i}{2}\left(\sqrt{\frac{5+2\sqrt{5}}{5}} + \sqrt{\frac{5-2\sqrt{5}}{5}}\right) \end{pmatrix}, \\
\Omega_0^5 &= I_2 \Leftrightarrow (ST_{11}T_{12}^{\pm 1})^5 = I_4.
\end{aligned} \tag{A22}$$

Similarly, we can find that $(ST_{22}T_{12}^{\pm 1})^5 = I_4$.

(vi) When we consider $B = \left(\begin{smallmatrix} -1 & \pm 1 \\ \pm 1 & 0 \end{smallmatrix}\right)$, that is $ST_{11}^{-1}T_{12}^{\pm 1}$ transformation, we obtain

$$\begin{aligned}
\Omega_0 &= O^{-1}\begin{pmatrix} e^{2\pi i/10} & 0 \\ 0 & e^{6\pi i/10} \end{pmatrix} O \\
&= \begin{pmatrix} \frac{1}{2} + \frac{i}{2}\left(\sqrt{\frac{5+\sqrt{5}}{10}} + \sqrt{\frac{5-\sqrt{5}}{10}}\right) & \mp\left(\frac{1}{2} - \frac{i}{2}\left(\sqrt{\frac{5+\sqrt{5}}{10}} - \sqrt{\frac{5-\sqrt{5}}{10}}\right)\right) \\ \mp\left(\frac{1}{2} - \frac{i}{2}\left(\sqrt{\frac{5+\sqrt{5}}{10}} - \sqrt{\frac{5-\sqrt{5}}{10}}\right)\right) & \frac{i}{2}\left(\sqrt{\frac{5+2\sqrt{5}}{5}} + \sqrt{\frac{5-2\sqrt{5}}{5}}\right) \end{pmatrix}, \\
\Omega_0^5 &= -I_2 \Leftrightarrow (ST_{11}^{-1}T_{12}^{\pm 1})^5 = -I_4, \\
\Omega_0^{10} &= I_2 \Leftrightarrow (ST_{11}^{-1}T_{12}^{\pm 1})^{10} = I_4.
\end{aligned} \tag{A23}$$

Similarly, we can find that $(ST_{22}^{-1}T_{12}^{\pm 1})^{10} = I_4$.

Finally, let us consider the $g = 3$ case:

(i) When we consider $B = \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{12} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}$, that is $ST_{11}^{b_{11}}T_{22}^{b_{22}}T_{33}^{b_{33}}T_{12}^{b_{12}}$ transformation, $B = \begin{pmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & 0 \\ b_{13} & 0 & b_{33} \end{pmatrix}$, that is $ST_{11}^{b_{11}}T_{22}^{b_{22}}T_{33}^{b_{33}}T_{13}^{b_{13}}$ transformation, and $B = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & b_{23} \\ 0 & b_{23} & b_{33} \end{pmatrix}$, that is $ST_{11}^{b_{11}}T_{22}^{b_{22}}T_{33}^{b_{33}}T_{23}^{b_{23}}$ transformation, with $b_{ij} = 0, \pm 1$, they are nothing but $T^4 \times T^2$ cases.

(ii) When we consider $B=\begin{pmatrix} 0 & 0 & \pm_1 1 \\ 0 & 0 & \pm_2 1 \\ \pm_1 1 & \pm_2 1 & 0 \end{pmatrix}$, that is $ST_{13}^{\pm_1 1}T_{23}^{\pm_2 1}$ transformation, we obtain

$$\begin{aligned}
\Omega_0 &= O^{-1}\begin{pmatrix} e^{6\pi i/8} & 0 & 0 \\ 0 & e^{2\pi i/8} & 0 \\ 0 & 0 & e^{4\pi i/8} \end{pmatrix} O \\
&= \begin{pmatrix} \left(\frac{1}{2}+\frac{1}{2\sqrt{2}}\right)i & (\pm_1 1)(\pm_2 1)\left(-\frac{1}{2}+\frac{1}{2\sqrt{2}}\right)i & \mp_1 \frac{1}{2} \\ (\pm_1 1)(\pm_2 1)\left(-\frac{1}{2}+\frac{1}{2\sqrt{2}}\right)i & \left(\frac{1}{2}+\frac{1}{2\sqrt{2}}\right)i & \mp_2 \frac{1}{2} \\ \mp_1 \frac{1}{2} & \mp_2 \frac{1}{2} & \frac{i}{\sqrt{2}} \end{pmatrix}, \\
\Omega_0^4 &= -((\pm_1 1)(\pm_2 1)B_{12}+B_{33}) \Leftrightarrow (ST_{13}^{\pm_1 1}T_{23}^{\pm_2 1})^4 = -A_{(\pm_1 1)(\pm_2 1)B_{12}+B_{33}}, \\
\Omega_0^8 &= I_3 \Leftrightarrow (ST_{13}^{\pm_1 1}T_{23}^{\pm_2 1})^8 = I_6.
\end{aligned} \tag{A24}$$

Similarly, we can find that $(ST_{12}^{\pm_1 1}T_{23}^{\pm_2 1})^8 = (ST_{12}^{\pm_1 1}T_{13}^{\pm_2 1})^8 = I_6$.

APPENDIX B: THE LANDSBERG-SCHAAR RELATION

We derive the g-dimensional Landsberg-Schaar relation with N and B matrices, satisfying

$$\begin{aligned}
N_{ij} &= N_{ji}, \\
B_{ij} &= B_{ji}, \\
(NB)_{ij} &= (NB)_{ji}, \\
(NB)_{ii} &\in 2\mathbb{Z},
\end{aligned} \tag{B1}$$

where $i,j\in\{1,2,\ldots,g\}$. In the process of the derivation of the Landsberg-Schaar relation, we use the Poisson resummation formula, and to make it converge, we introduce an infinitesimal positive number ϵ ($0<\epsilon\ll 1$).

First, we introduce the following function $f(A)$ to describe the Poisson resummation formula with $g\times g$ symmetric matrix A:

$$f(A)=\sum_{K\in\mathbb{Z}^g} e^{-\pi {}^tKAK}. \tag{B2}$$

We find the Poisson resummation formula

$$f(A)=\frac{1}{\sqrt{\det A}}f(A^{-1}). \tag{B3}$$

Next, we define A^{-1} as

$$A^{-1}=iB^{-1}N+\epsilon I_g. \tag{B4}$$

We will take the limit $\epsilon\to+0$ later to obtain the Landsberg-Schaar relation. Hereafter, we ignore higher-order terms of ϵ because the final result we obtain is unaffected. Then, A can be written as

$$A=-iBN^{-1}+\epsilon B^2N^{-2}. \tag{B5}$$

Now, $f(A)$ can be written as

$$\begin{aligned}
f(A) &= \sum_{K\in\mathbb{Z}^g} e^{-\pi {}^tK(-iBN^{-1}+\epsilon B^2N^{-2})K} \\
&= \sum_{K\in\mathbb{Z}^g} e^{-\pi\epsilon {}^t(BN^{-1}K)(BN^{-1}K)} e^{\pi i {}^tKBN^{-1}K} \\
&= \sum_{L\in\Lambda_N}\sum_{J\in\mathbb{Z}^g} e^{-\pi\epsilon({}^tJB+{}^tLBN^{-1})(BJ+BN^{-1}L)} \\
&\quad\times e^{\pi i({}^tJN+{}^tL)BN^{-1}(NJ+L)} \\
&= \sum_{L\in\Lambda_N} e^{\pi i {}^tLBN^{-1}L}\sum_{J\in\mathbb{Z}^g} e^{-\pi\epsilon {}^t(J+N^{-1}L)B^2(J+N^{-1}L)},
\end{aligned} \tag{B6}$$

where we write $K=NJ+L$ ($J\in\mathbb{Z}^g, L\in\Lambda_N$) in the third equality and we use $(NB)_{ii}\in 2\mathbb{Z}$ in the fourth equality. In the limit $\epsilon\to+0$, we can obtain

$$\lim_{\epsilon\to+0} f(A) = \lim_{\epsilon\to+0}\frac{1}{|\det B|}\frac{1}{\epsilon^{g/2}}\sum_{L\in\Lambda_N} e^{\pi i {}^tLBN^{-1}L}, \tag{B7}$$

where we use the formula of the Gaussian integral with multiple variables. On the other hand, $f(A^{-1})/\sqrt{\det A}$ can be written as

$$
\begin{aligned}
&\frac{1}{\sqrt{\det A}} f(A^{-1}) \\
&= \frac{1}{\sqrt{\det A}} \sum_{K\in\mathbb{Z}^g} e^{-\pi {}^tK(iB^{-1}N+\epsilon I_g)K} \\
&= \frac{1}{\sqrt{\det A}} \sum_{K\in\mathbb{Z}^g} e^{-\pi\epsilon {}^tKK} e^{-\pi i {}^tKB^{-1}NK} \\
&= \frac{1}{\sqrt{\det A}} \sum_{L\in\Lambda_B} \sum_{J\in\mathbb{Z}^g} e^{-\pi\epsilon {}^t(BJ+L)(BJ+L)} e^{-\pi i {}^t(BJ+L)B^{-1}N(BJ+L)} \\
&= \frac{1}{\sqrt{\det A}} \sum_{L\in\Lambda_B} e^{-\pi i {}^tLB^{-1}NL} \sum_{J\in\mathbb{Z}^g} e^{-\pi\epsilon {}^t(J+B^{-1}L)B^2(J+B^{-1}L)}, \qquad \text{(B8)}
\end{aligned}
$$

where we write $K = BJ + L$ ($J\in\mathbb{Z}^g$, $L\in\Lambda_B$) in the third equality and we use $(NB)_{ii}\in 2\mathbb{Z}$ in the fourth equality. In the limit $\epsilon\to+0$, we can obtain

$$
\begin{aligned}
\lim_{\epsilon\to+0} \frac{1}{\sqrt{\det A}} f(A^{-1}) &= \lim_{\epsilon\to+0} \sqrt{|\det N|} \frac{e^{\pi i(g+2(n_-^N-n_-^B))/4}}{\sqrt{|\det B|}} \\
&\times \frac{1}{|\det B|} \frac{1}{\epsilon^{g/2}} \sum_{L\in\Lambda_B} e^{-\pi i {}^tLB^{-1}NL}, \qquad \text{(B9)}
\end{aligned}
$$

where we use the formula of the Gaussian integral with multiple variables and we take $(\pm i)^{x/2} = e^{\pm\pi i x/4}$ with the number of negative eigenvalues of the B- (N-)matrix, n_-^B (n_-^N).

Thus, we can obtain the g-dimensional Landsberg-Schaar relation:

$$
\frac{e^{-\pi i n_-^N/2}}{\sqrt{|\det N|}} \sum_{K\in\Lambda_N} e^{\pi i {}^tKN^{-1}BK} = \frac{e^{\pi i g/4} e^{-\pi i n_-^B/2}}{\sqrt{|\det B|}} \sum_{K\in\Lambda_B} e^{-\pi i {}^tKNB^{-1}K}. \qquad \text{(B10)}
$$

In Eq. (142), we assume that all eigenvalues of the N matrix are positive, i.e., $n_-^N = 0$.

APPENDIX C: THE GENERATORS OF $\Delta(96)\times Z_4$

In this appendix, we prove the generators in Eq. (160) satisfy the algebraic relations of $\Delta(96)\times Z_4$ in Eq. (161), where S, T_{I_2}, and T_{12} satisfy the following relations:

$$
\begin{aligned}
&T_{12}^2 = 1, \\
&(ST_{12})^3 = -i1, \\
&(ST_{12})^6 = S^2 = -1, \\
&(ST_{12})^{12} = S^4 = (ST_{I_2})^3 = 1, \\
&T_{I_2}^4 = 1. \qquad \text{(C1)}
\end{aligned}
$$

In addition, from Eq. (70) in Ref. [60], the above Eq. (158) also satisfies

$$
\begin{aligned}
1 &= (S^{-1}T_{I_2}^{-1}ST_{I_2})^3 \\
&= (T_{I_2}ST_{I_2}SST_{I_2})^3 \\
&= (T_{I_2}S^3T_{I_2}^2)^3 \\
&= (T_{I_2}^2S^3T_{I_2})^3 \\
&= (S^3T_{I_2}^3). \qquad \text{(C2)}
\end{aligned}
$$

We note that, in Ref. [60], we have already proved that the generators in Eq. (156) satisfy the algebraic relation of $S_4' \simeq \Delta'(24)$ in Eq. (157). First, we can easily check that $d = (ST_{12})^3$ satisfies

$$
d^4 = 1, \qquad \text{(C3)}
$$

$$
dx = xd(x = \tilde{a}, \tilde{a}', b, c'). \qquad \text{(C4)}
$$

Next, $\tilde{a}'$ can be rewritten as

$$
\tilde{a}' = ST_{I_2}T_{12}S^{-1}T_{12}^{-1}T_{I_2}^{-1} \qquad \text{(C5)}
$$

$$
\begin{aligned}
&= ST_{I_2}ST_{12}S^{-1}T_{I_2}^{-1}(ST_{12})^3 \\
&= T_{I_2}^{-1}S^{-1}T_{I_2}^{-1}T_{12}T_{I_2}ST_{I_2}S(ST_{12})^3 \\
&= T_{I_2}^{-1}ST_{12}ST_{I_2}S^{-1}(ST_{12})^3 \\
&= T_{I_2}^{-1}T_{12}^{-1}ST_{12}T_{I_2}S^{-1} \\
&= T_{12}^{-1}T_{I_2}^{-1}ST_{I_2}T_{12}S^{-1}, \qquad \text{(C6)}
\end{aligned}
$$

and then we can prove that

$$
\begin{aligned}
\tilde{a}'^2 &= ST_{I_2}T_{12}S^{-1}T_{12}^{-1}T_{I_2}^{-1}ST_{I_2}T_{12}S^{-1}T_{12}^{-1}T_{I_2}^{-1} \\
&= ST_{I_2}T_{12}S^{-1}ST_{I_2}T_{12}S^{-1}T_{12}^{-2}T_{I_2}^{-2} \\
&= ST_{I_2}^2S^{-1}T_{I_2}^{-2} \\
&= a'. \qquad \text{(C7)}
\end{aligned}
$$

Similarly, $\tilde{a}$ can be rewritten as

$$
\begin{aligned}
\tilde{a} &= T_{I_2}T_{12}ST_{12}T_{I_2}S^{-1}T_{I_2}T_{12}(ST_{12})^3 \\
&= T_{I_2}^2ST_{12}T_{I_2}S^{-1}(ST_{12})^3 \\
&= ST_{12}T_{I_2}S^{-1}T_{I_2}^2(ST_{12})^3, \qquad \text{(C8)}
\end{aligned}
$$

and then we can prove that

$$
\begin{aligned}
\tilde{a}^2 &= T_{I_2}^2ST_{12}T_{I_2}S^{-1}(ST_{12})^3ST_{12}T_{I_2}S^{-1}T_{I_2}^2(ST_{12})^3 \\
&= T_{I_2}^2ST_{I_2}^2ST_{I_2}^2 \\
&= a. \qquad \text{(C9)}
\end{aligned}
$$

They are the proof of Eq. (162). Hence, by considering Eq. (157), we can obtain that

$$
\tilde{a}^4 = 1, \qquad \text{(C10)}
$$

$$\tilde{a}'^4 = 1. \tag{C11}$$

We can also check that

$$\tilde{a}\,\widetilde{a'} = \tilde{a}'\tilde{a} = ST_{I_2}^2 S^{-1} T_{I_2} T_{12} (ST_{12})^3. \tag{C12}$$

Similarly, since b in Eq. (160) and b in Eq. (156) are the same, we can obtain that

$$b^3 = 1. \tag{C13}$$

On the other hand, by considering that c in Eq. (156) satisfies that

$$c^2 = S^2, \tag{C14}$$

we can obtain that

$$c'^2 = 1. \tag{C15}$$

We also obtain

$$c' b c'^{-1} = c b c^{-1} = b^{-1}. \tag{C16}$$

The other relations can be checked that

$$\begin{aligned} b\tilde{a}b^{-1} &= T_{I_2} S^3 T_{I_2}^2 S T_{12} T_{I_2} S^{-1} T_{I_2}^2 T_{I_2}^{-2} S^{-3} T_{I_2}^{-1} (ST_{12})^3 \\ &= S^{-1} T_{I_2}^{-1} S T_{I_2} S T_{12} (ST_{12})^3 \\ &= S^{-1} T_{I_2}^{-2} S^{-1} T_{I_2}^{-1} T_{12}^{-1} (ST_{12})^3 \\ &= (ST_{12})^{-3} T_{12}^{-1} T_{I_2}^{-1} S T_{I_2}^{-2} S^{-1} \\ &= \tilde{a}^{-1} \tilde{a}'^{-1}, \end{aligned} \tag{C17}$$

$$\begin{aligned} b\tilde{a}'b^{-1} &= T_{I_2} S^3 T_{I_2}^2 T_{I_2}^{-1} T_{12}^{-1} S T_{12} T_{I_2} S^{-1} T_{I_2}^{-2} S^{-3} T_{I_2}^{-1} \\ &= T_{I_2} S T_{I_2} T_{12} S T_{12} T_{I_2}^2 T_{I_2}^{-1} S^{-1} T_{I_2}^{-1} T_{I_2}^{-1} S^{-1} T_{I_2}^{-1} \\ &= S^{-1} T_{I_2}^{-1} S^{-1} S^{-1} T_{12}^{-1} S^{-1} T_{I_2}^2 S T_{I_2} S S T_{I_2} S S^{-2} (ST_{12})^{-3} \\ &= S^{-1} T_{I_2}^{-1} T_{12}^{-1} S^{-1} T_{I_2}^2 S T_{I_2}^2 S (ST_{12})^3 \\ &= S^{-1} T_{12} T_{I_2}^{-1} S^{-1} S T_{I_2}^2 S T_{I_2}^2 (ST_{12})^3 \\ &= S T_{12} T_{I_2} S^{-1} T_{I_2}^2 (ST_{12})^3 \\ &= \tilde{a}, \end{aligned} \tag{C18}$$

$$\begin{aligned} & c'\tilde{a}c'^{-1} \\ &= ST_{I_2}^2 S T_{I_2}^3 S T_{I_2} T_{12} S^{-1} T_{I_2}^2 T_{I_2}^{-3} S^{-1} T_{I_2}^{-2} S^{-1} (ST_{12})^3 \\ &= ST_{I_2}^2 S^3 T_{I_2} T_{I_2}^2 S^3 T_{I_2} T_{12} S^{-1} T_{I_2}^{-1} S^{-1} T_{I_2}^{-1} T_{I_2}^{-1} S^{-1} (ST_{12})^3 \\ &= ST_{I_2}^{-1} S^{-3} T_{I_2}^{-2} T_{12} T_{I_2} S S T_{I_2} S T_{I_2} (ST_{12})^3 \\ &= S^{-1} T_{I_2}^{-1} S T_{12} S T_{I_2} (ST_{12})^3 \\ &= S^{-1} T_{I_2}^{-1} T_{12}^{-1} S T_{12} T_{I_2} \\ &= T_{I_2} T_{12} S T_{12}^{-1} T_{I_2}^{-1} S^{-1} \\ &= \tilde{a}'^{-1}, \end{aligned} \tag{C19}$$

$$\begin{aligned} c'\tilde{a}'c'^{-1} &= ST_{I_2}^2 S T_{I_2}^3 T_{I_2}^{-1} T_{12}^{-1} S T_{I_2} T_{12} S^{-1} T_{I_2}^{-3} S^{-1} T_{I_2}^{-2} S^{-1} \\ &= T_{I_2}^{-1} T_{12}^{-1} S T_{I_2}^2 S^3 T_{I_2} T_{I_2}^2 S^3 T_{I_2} T_{12} T_{I_2} T_{I_2}^{-1} S^{-3} T_{I_2}^{-2} T_{I_2}^{-1} S^{-3} T_{I_2}^{-2} S^{-1} \\ &= T_{I_2}^{-1} T_{12}^{-1} S T_{I_2}^{-1} S^{-3} T_{I_2}^{-2} T_{12} T_{I_2} T_{I_2}^2 S^3 T_{I_2} S^{-1} \\ &= T_{12}^{-1} T_{I_2}^{-1} S^{-1} T_{I_2}^{-1} S^{-1} T_{I_2}^{-1} T_{12} T_{I_2}^2 S^3 T_{I_2} S^{-1} \\ &= T_{12} S T_{12} T_{I_2}^2 S^3 T_{I_2} S^{-1} \\ &= S T_{12} S T_{I_2}^2 S T_{I_2} S^{-1} (ST_{12})^{-3} \\ &= S T_{12}^{-1} T_{I_2}^{-1} T_{I_2} S T_{I_2} T_{I_2} S T_{I_2} S^{-1} (ST_{12})^{-3} \\ &= S T_{12}^{-1} T_{I_2}^{-1} S^{-1} T_{I_2}^{-2} \\ &= \tilde{a}^{-1}, \end{aligned} \tag{C20}$$

where we also used the following relation proved in Ref. [60]:

$$ST_{I_2}^{2p} S^{-1} T_{I_2}^{2q} = (ST_{I_2}^2 S^{-1})^p T_{I_2}^{2q} = T_{I_2}^{2q} S T_{I_2}^{2p} S^{-1}, \quad p, q \in \mathbb{Z}. \tag{C21}$$

Therefore, the generators in Eq. (160) satisfy the algebraic relations of $\Delta(96) \times Z_4$ in Eq. (161).

APPENDIX D: MODULAR FLAVOR SYMMETRY WITH $\Omega = \tau I_g$ CASE

In this appendix, let us see the modular flavor symmetry of wave functions on magnetized T^{2g} and its orbifolds, in particular, for the $\Omega = \tau I_g$ case. First, in the $\Omega = \tau I_g$ case, we can consider the $\Gamma_1 = SL(2, \mathbb{Z})$ modular transformation for any N matrices. In addition, in order for the modular transformation to be consistent with the boundary conditions of wave functions on magnetized T^{2g} as well as its orbifolds, it is required that the diagonal elements of the N matrix must be even ($N_{ii} \in 2\mathbb{Z}$) in the case of the vanishing SS phases. In this case, the wave functions on magnetized T^{2g} transform under the modular transformation as

$$\begin{aligned} \tilde{\Gamma}_1 \ni \tilde{\gamma}\colon \psi^{J,N}_{T^{2g}}(z,\Omega) &\to \psi^{J,N}_{T^{2g}}(\tilde{\gamma}(z,\Omega)) \\ &= \tilde{J}_{g/2}(\tilde{\gamma},\Omega)\rho_{T^{2g}}(\tilde{\gamma})_{JK}\psi^{K,N}_{T^{2g}}(z,\Omega), \end{aligned} \tag{D1}$$

$$\begin{aligned} \tilde{S} = [S,(-1)^g]\colon \tilde{J}_{g/2}(\tilde{S},\Omega) = (-1)^g(-\tau)^{g/2}, \\ \rho_{T^{2g}}(\tilde{S})_{JK} = \frac{(-e^{\pi i/4})^g}{\sqrt{\det N}} e^{2\pi i {}^t J N^{-1} K}, \end{aligned} \tag{D2}$$

$$\begin{aligned} \tilde{T}_{I_g} = [T_{I_g},1]\colon \tilde{J}_{g/2}(\tilde{T}_{I_g},\Omega) = 1, \\ \rho_{T^{2g}}(\tilde{T}_{I_g})_{JK} = e^{\pi i {}^t J N^{-1} J}\delta_{J,K}, \end{aligned} \tag{D3}$$

where $\rho_{T^{2g}}$ satisfy the algebraic relations in Eq. (140), the top of Eq. (141) with $B = I_g$ and $n^B_- = 0$, and Eq. (146). Thus, the wave functions behave as the modular forms of weight $g/2$ and $\tilde{\Gamma}_1(h)$, and then they transform under the $\tilde{\Gamma}_{1,h} = \tilde{\Gamma}_1/\tilde{\Gamma}_1(h)$ modular flavor transformation nontrivially. Note that, for $g = 2$, it corresponds to the $\Gamma'_{1,h} = \Gamma_1/\Gamma_1(h)$ modular flavor transformation. From now on, we show concrete modular flavor symmetries of three-dimensional wave functions. $g = 1$ cases have been studied in Refs. [59,60]. Once we study $g = 2$ cases, we can similarly study the $g = 3$ case, and the result is similar to $g = 1$ cases by replacing the modular weight from 1/2 to 3/2. Hence, we show examples of $g = 2$ cases, in particular. Moreover, the cases of the class (2-1) have been studied in Refs. [60,66]. Then, let us see the case of the class (2-2).

First, let us see the case of the class (2-2-b). Reference [33] shows that only $\mathbb{Z}_2^t$ twisted odd ($m^t = 1$) modes with $\det N = 7$ are three-dimensional modes on magnetized $T^4/\mathbb{Z}_2^t$ with vanishing SS phases and $N_{ii} \in 2\mathbb{Z}$. In this case, we can find that $s = 1$ and $\tilde{N}'_{ii} = \tilde{N}_{ii} \in 2\mathbb{Z}$. Hence, the order h is determined as $h = \det N = 7$. In addition, by considering that the wave functions are $\mathbb{Z}_2^t$ twisted odd modes, we obtain the following algebraic relations:

$$\begin{aligned} \rho_{T^4/\mathbb{Z}_2^1}(\tilde{S})^2 &= I, \\ [\rho_{T^4/\mathbb{Z}_2^1}(\tilde{S})\rho_{T^4/\mathbb{Z}_2^1}(\tilde{T}_{I_2})]^3 &= I, \\ \rho_{T^4/\mathbb{Z}_2^1}(\tilde{T}_{I_2})^7 &= I. \end{aligned} \tag{D4}$$

Thus, the three-dimensional modes transform nontrivially under the $\Gamma_{1,7} = PSL(2,\mathbb{Z}_7)$ modular flavor transformation. Indeed, let us see the following example:

$$N = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} = N', \quad \det N^{(t)} = 7, \quad \tilde{N}^{(t)} = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}. \tag{D5}$$

The three-dimensional $\mathbb{Z}_2^t$ twisted odd ($m^t = 1$) modes,

$$|J_1,J_2\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}}(|1,1\rangle - |2,4\rangle) \\ \frac{1}{\sqrt{2}}(|1,2\rangle - |2,3\rangle) \\ \frac{1}{\sqrt{2}}(|1,3\rangle - |2,2\rangle) \end{pmatrix}, \tag{D6}$$

transform under the S and T_{I_2} transformations as

$$\begin{aligned} \rho_{T^4/\mathbb{Z}_2^1}(\tilde{S}) &= \frac{2}{\sqrt{7}}\begin{pmatrix} \sin\left(\frac{6\pi}{7}\right) & \sin\left(\frac{4\pi}{7}\right) & \sin\left(\frac{2\pi}{7}\right) \\ \sin\left(\frac{4\pi}{7}\right) & -\sin\left(\frac{2\pi}{7}\right) & \sin\left(\frac{6\pi}{7}\right) \\ \sin\left(\frac{2\pi}{7}\right) & \sin\left(\frac{6\pi}{7}\right) & -\sin\left(\frac{4\pi}{7}\right) \end{pmatrix}, \\ \rho_{T^4/\mathbb{Z}_2^1}(\tilde{T}_{I_2}) &= \begin{pmatrix} e^{4\pi i/7} & & \\ & e^{8\pi i/7} & \\ & & e^{2\pi i/7} \end{pmatrix}, \end{aligned} \tag{D7}$$

which satisfy the algebraic relations in Eq. (D14) and also

$$[\rho_{T^4/\mathbb{Z}_2^1}(\tilde{S})^{-1}\rho_{T^4/\mathbb{Z}_2^1}(\tilde{T}_{I_2})^{-1}\rho_{T^4/\mathbb{Z}_2^1}(\tilde{S})\rho_{T^4/\mathbb{Z}_2^1}(\tilde{T}_{I_2})]^4 = I. \tag{D8}$$

Thus, they transform nontrivially under the $\Gamma_{1,7} = PSL(2,\mathbb{Z}_7)$ modular flavor transformation.

Next, let us see the case of the class (2-2-a). There are two types of examples, besides one in Eqs. (164)–(172), such that there appear three-dimensional modes on magnetized $T^4/(\mathbb{Z}_2^t \times \mathbb{Z}_2^p)$ with vanishing SS phases and $N_{11} = N_{22} = n \in 2\mathbb{Z}$. The first one is that the N matrix is given by

$$N = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} = N', \quad \det N^{(t)} = 7, \quad \tilde{N}^{(t)} = \begin{pmatrix} 4 & -3 \\ -3 & 4 \end{pmatrix}. \tag{D9}$$

The three-dimensional $\mathbb{Z}_2^t$ twisted odd ($m^t = 1$) and $\mathbb{Z}_2^p$ permutation even ($m^p = 0$) modes,[15]

$$|J_1,J_2\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}}(|3,3\rangle - |4,4\rangle) \\ \frac{1}{\sqrt{2}}(||2,2\rangle - |5,5\rangle) \\ \frac{1}{\sqrt{2}}(|1,1\rangle - |6,6\rangle) \end{pmatrix}, \tag{D10}$$

[15]There are no $\mathbb{Z}_2^t$ twisted odd and $\mathbb{Z}_2^p$ permutation odd modes. In other words, all $\mathbb{Z}_2^t$ twisted odd modes are $\mathbb{Z}_2^p$ permutation even.

also transform under S and T_{I_2} transformations as Eqs. (D7) and (D8), and then they also transform nontrivially under the $\Gamma_{1,7} = PSL(2,\mathbb{Z}_7)$ modular flavor transformation. On the other hand, the second one is that the N matrix is given by

$$N = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} = N', \quad \det N^{(t)} = 15, \quad \tilde{N}^{(t)} = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}. \tag{D11}$$

The three-dimensional $\mathbb{Z}_2^t$ twisted odd ($m^t = 1$) and $\mathbb{Z}_2^p$ permutation odd ($m^p = 1$) modes,

$$|J_1, J_2\rangle = \begin{pmatrix} \frac{1}{2}(|2,1\rangle - |1,2\rangle - |3,4\rangle + |4,3\rangle) \\ \frac{1}{2}(|3,1\rangle - |1,3\rangle - |2,4\rangle + |4,2\rangle) \\ \frac{1}{\sqrt{2}}(|3,2\rangle - |2,3\rangle) \end{pmatrix}, \tag{D12}$$

transform under S and T_{I_2} transformations as

$$\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{S}) = -\frac{1}{\sqrt{5}} \begin{pmatrix} 2\sin\left(\frac{3\pi}{10}\right) & 2\sin\left(\frac{\pi}{10}\right) & -\sqrt{2} \\ 2\sin\left(\frac{\pi}{10}\right) & 2\sin\left(\frac{3\pi}{10}\right) & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} & -1 \end{pmatrix},$$

$$\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2}) = \begin{pmatrix} e^{16\pi i/15} & & \\ & e^{4\pi i/15} & \\ & & e^{10\pi i/15} \end{pmatrix}, \tag{D13}$$

which satisfy the following algebraic relations:

$$\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{S})^2 = I,$$
$$[\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{S})\rho_{T^4/\mathbb{Z}_2^1}(\tilde{T}_{I_2})]^3 = I,$$
$$\rho_{T^4/(\mathbb{Z}_2^{1_t}\times\mathbb{Z}_2^{1_p})}(\tilde{T}_{I_2})^5 = e^{4\pi i/3} I. \tag{D14}$$

Then, they transform nontrivially under the $\Gamma_{1,15}$ modular flavor transformation. Actually, when we define

$$s = S, \qquad t = T_{I_2}^6, \qquad c = T_{I_2}^5, \tag{D15}$$

we can find that they satisfy the following relations:

$$s^2 = 1, \qquad t^5 = (st)^3 = c^3 = 1, \qquad cx = xc \ (x = s, t). \tag{D16}$$

Thus, they transform nontrivially under the $A_5 \times Z_3$ modular flavor transformation.

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