Ultracold neutrons in the low curvature limit: Remarks on the post-Newtonian effects

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Ultracold neutrons are great experimental tools to explore the gravitational interaction in the regime of quantized states. From a theoretical perspective, starting from a Dirac equation in curved spacetime, we applied a perturbative scheme to systematically derive the nonrelativistic Schrödinger equation that governs the evolution of the neutron's wave function in the Earth's gravitational field. At the lowest order, this procedure reproduces a Schrödinger system affected by a linear Newtonian potential, but corrections due to both curvature and relativistic effects are present. Here, we argue that one should be very careful when going one step further in the perturbative expansion. Proceeding methodically with the help of the Foldy-Wouthuysen transformation and a formal post-Newtonian $1/c^2$ expansion, we derive the nonrelativistic Hamiltonian for a generic static spacetime. By employing Fermi coordinates within this framework, we calculate the next-to-leading-order corrections to the neutron's energy spectrum. Finally, we evaluate them for typical experimental configurations, such as that of qBounce, and note that, while the current precision for observations of ultracold neutrons may not yet enable to probe them, they could still be relevant in the future or in alternative circumstances.

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I. INTRODUCTION

In the last decades, there has been a surging interest in a wide variety of small-size and tabletop experiments exploring the fundamental properties of the gravitational interaction: starting from optical [1] and atom interferometry [2–4], also with the inclusion of optical lattices [5,6], getting to more exotic ideas, like Bose-Einstein condensate [7], geonium atoms [8,9], and eventual gravitational wave detectors [10].

A very interesting possibility is offered by ultracold neutrons (UCN) [11], particles with such low energies and velocities that their wavelength become larger than typical atomic interspacing, and can therefore be stored much more easily, since they get totally reflected by many materials. Recently, UCNs have also been employed to investigate the quantum nature of gravitational interaction. Remarkably, experiments such as qBounce [12,13] and GRANIT [14,15] have successfully observed gravitationally induced quantum states [16–18]. As a result, experiment involving UCN are becoming a standard option to probe fundamental physics [19–21], in particular extensions of general relativity (GR), like beyond-Riemannian models [22,23], Torsion contributions [24], emergent gravity proposals [25–27], and much more [28].

Within this framework, GR or Standard Model extensions are usually described starting from a generalized Dirac equation, depending on the theory under analysis, embedded in the curved spacetime sourced by the Earth. From there, one can obtain the corresponding nonrelativistic Hamiltonian by taking the low-curvature and low-velocity limit. Usually, the final result of this procedure can be split into the GR contribution and additional terms which parametrize the extension under consideration.

Contrary to the prevailing trend, our paper uniquely concentrates on precisely the GR contribution. At leading order, one expects to recover the Schrödinger equation describing a particle in the Earth's Newtonian potential, which is the theoretical picture considered in the interpretation of gravitational experiments with UCNs [29]. However, when going deeper into the perturbative expansion, new terms get to influence the quantum dynamics and since experiments like qBounce have already reached the stunning sensibility of 10^{-17} eV, one should start asking oneself to which extent will the trivial Newtonian picture hold on. Then, while most of noninertial and rotational effects are, in principle, already taken into account implicitly by the effective local acceleration value for the qBounce setup, one of the pragmatic purposes of this work is exactly to quantify the next-to-leading-order (NLO) corrections to UCNs energy spectrum in the Earth's gravitational field.

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To complete this task we have to go through several technical steps which, despite the amount of work already present in the literature, remain nontrivial. For example, the aforementioned corrections get typically sorted out by the powers of the inverse fermion mass $1/m$. Nevertheless, if not carefully considered, this choice can result in inconsistencies when dealing with the gravitational interaction (whose source is the mass itself) [30], while still being perfectly fine in the electromagnetic sector. Also for this reason, we decided to adopt for our work the post-Newtonian approach, in which perturbations are categorized by their inverse c^2 power. More details on that are spread throughout the paper.

Therefore, the outline of this work is as follows: After a small summary illustrating our conventions, in Sec. II, we start from the Dirac equation in curved spacetime to evaluate the respective Hamiltonian for a static spacetime and, then, we take the low-curvature limit and introduce the post-Newtonian expansion. In Sec. III we exploit the Foldy-Wouthuysen (FW) transformation to perform the nonrelativistic limit, while in Sec. IV, with the help of Fermi coordinates, we take the perspective of an accelerated laboratory frame on the surface of the Earth. Finally, in Sec. V we derive the next-to-leading-order corrections to the neutron's spectrum and determine their magnitude for current experiments, like qBounce. To avoid cluttering in the main text with too many calculations, we include some details on the more lengthy ones in the Appendix of this work.

A. Notation and conventions

Before we get into the calculations, let us set up the notational conventions used along the paper; hereon, we will use Greek and Latin characters to label respectively spacetime and tangent space indices. As usual, the tetrad field $e^a{}_\mu$ will be used to "translate" spacetime indices into tangent space indices and vice versa. Let us also note that to avoid confusion with the tetrads and their inverse, which are the only objects that intrinsically mix the two types of indices, we will place the (upper or lower) tangent space index as the first one appearing from left to right. Finally, we will refer to time components with "t" when dealing with spacetime indices and with "0" in the tangent space, while spacetime spatial components $\{1,2,3\}$ will be labeled by lower case letters $\{i,j,k,\ldots\}$ and tangent space spatial components by capital letters $\{I,J,K,\ldots\}$ in the set $\{1,2,3\}$ as well.

Since our final aim is to pursue a nonrelativistic expansion within the approach of the post-Newtonian approximation [31], we will not work in natural units to still be able to keep track of powers of c.

For the Dirac matrices γ^a in flat spacetime we choose the standard representation,

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \qquad \gamma^I = \begin{pmatrix} 0 & \sigma^I \\ -\sigma^I & 0 \end{pmatrix}, \tag{1}$$

with $\mathbf{1}$ representing a two by two identity matrix and $\boldsymbol{\sigma}$ the usual Pauli's matrices. From those we can define their curved spacetime version

$$\underline{\gamma}^\mu \equiv e_a{}^\mu \gamma^a, \tag{2}$$

which satisfy the consistent curved spacetime Clifford algebra,

$$\{\underline{\gamma}^\mu, \underline{\gamma}^\nu\} = 2g^{\mu\nu}. \tag{3}$$

We further introduce σ^{ab} and Σ matrices

$$\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b], \qquad \Sigma^I = \begin{pmatrix} \sigma^I & 0 \\ 0 & \sigma^I \end{pmatrix}. \tag{4}$$

Finally, for the Minkowski metric we pick the mostly minus convention $\eta_{\mu\nu} = \mathrm{diag}\{1,-1,-1,-1\}$.

II. DIRAC HAMILTONIAN IN CURVED SPACETIME

Let us start our analysis from the Dirac equation in curved spacetime [32,33],

$$(i\hbar \underline{\gamma}^\mu D_\mu - mc)\psi = 0, \tag{5}$$

with $D_\mu = \partial_\mu + \Gamma_\mu$ representing the spinor covariant derivative and the spinor connection Γ_μ [34]. The latter is expressed through the inverse tetrads $e_a{}^\mu$ as

$$\Gamma_\mu = \frac{1}{2}\sigma^{ab} e_a{}^\nu \nabla_\mu e_{b\nu} = \frac{1}{2}\sigma^{ab} g_{\nu\rho} e_a{}^\nu \nabla_\mu e_b{}^\rho, \tag{6}$$

where

$$\nabla_\mu e_b{}^\rho = \partial_\mu e_b{}^\rho + \left\{{}^{\rho}_{\mu\alpha}\right\} e_b{}^\alpha \tag{7}$$

is the usual GR covariant derivative constructed from the Christoffel symbols $\left\{{}^{\rho}_{\mu\alpha}\right\}$,

$$\left\{{}^{\rho}_{\mu\alpha}\right\} = \frac{1}{2} g^{\rho\beta}(\partial_\mu g_{\alpha\beta} + \partial_\alpha g_{\mu\beta} - \partial_\beta g_{\mu\alpha}). \tag{8}$$

Multiplying Eq. (5) by $(g^{tt})^{-1}\underline{\gamma}^t$, we can manipulate it to obtain a time-evolution equation,

$$\mathscr{H}_D \psi = i\hbar \frac{\partial \psi}{\partial t}, \tag{9}$$

where we used $x^t = ct$, and $\mathscr{H}_D$ is then the Dirac-Hamiltonian for a generic spacetime

$$\mathscr{H}_D = mc^2 (g^{tt})^{-1}\underline{\gamma}^t - i\hbar c \Gamma_t - i\hbar c (g^{tt})^{-1} \underline{\gamma}^t \underline{\gamma}^i D_i, \tag{10}$$

where repeated spatial indices are summed. Since the inclusion of rotational effects goes beyond the scope of this paper, and keeping in mind a practical approximation to the qBounce experimental setup, in the following we will limit ourselves to static spacetimes, effectively neglecting the effects of Earth's rotation. This choice coincides, for example, with the one in [20,23] where the authors assume the laboratory frame to be inertial. Then, the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ for a generic static spacetime is

$$ds^2 = V^2 (cdt)^2 + g_{ij} dx^i dx^j, \tag{11}$$

where V and g_{ij} are functions of the spatial coordinates. Remembering that the tetrads must satisfy the condition $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, we conveniently choose their expressions such that they do not mix time and spatial coordinates

$$e^0{}_i = e^I{}_t = 0, \qquad e_I{}^t = e_0{}^i = 0, \tag{12}$$

and that

$$e^0{}_t = (e_0{}^t)^{-1} = V. \tag{13}$$

In this way, the Dirac matrices with the indices referring to the curved spacetime coordinates read

$$\begin{aligned} \underline{\gamma}^t &= e_a{}^t \gamma^a = \frac{1}{V} \gamma^0, \\ \underline{\gamma}^i &= e_a{}^i \gamma^a = e_J{}^i \gamma^J. \end{aligned} \tag{14}$$

With the above considerations, we can rewrite Eq. (10) for the particular case of a static spacetime (11)

$$\mathscr{H}_D = mc^2 V \gamma^0 - i\hbar c \Gamma_t - i\hbar c V \gamma^0 \gamma^J e_J{}^i (\partial_i + \Gamma_i),$$

which, using the explicit expression (A3) of Γ_t combined with the one for Γ_i (see Appendix A), becomes

$$\mathscr{H}_D = mc^2 V \beta - i\hbar c \alpha^J e_J{}^i \left(V \partial_i + \frac{1}{2} \partial_i V + V \Gamma_i \right). \tag{15}$$

Here, we define the matrices $\beta \equiv \gamma^0$ and $\alpha^i \equiv \gamma^0 \gamma^i$. Note that, in curved space, the Hamiltonian (15) is Hermitian only with respect to the right scalar product measure [35] including $J \equiv \sqrt{-\det(g_{ij})}$,

$$d^3x J = d^3x \sqrt{-\det(g_{ij})}. \tag{16}$$

Equivalently, we can also implement Hermiticity through the following redefinitions for the spinor and Hamiltonian operator [36–38],

$$\begin{aligned} \tilde{\psi} &= J^{\frac{1}{2}} \psi, \\ \tilde{\mathscr{H}}_D &= J^{\frac{1}{2}} \mathscr{H}_D J^{-\frac{1}{2}}. \end{aligned} \tag{17}$$

In this way, the Hamiltonian becomes

$$\begin{aligned} \tilde{\mathscr{H}}_D = {} & mc^2 V \beta \\ & - i\hbar c \alpha^J e_J{}^i \left(V \partial_i + \frac{1}{2} \partial_i V - \frac{1}{2} V J^{-1} \partial_i J + V \Gamma_i \right), \end{aligned} \tag{18}$$

which is now Hermitian respect to the flat measure. From here on we will drop the "tilde" notation for the sake of simplicity.

A. Low-curvature limit and post-Newtonian expansion

At this point, we are ready to take the low-curvature or weak-gravity limit. The way we perform it is by realizing a formal $1/c$ expansion of the geometrical objects around flat spacetime quantities [8], in the same fashion as post-Newtonian (PN) expansions

$$\begin{aligned} v &\equiv V - 1 \sim \mathcal{O}(c^{-2}), \\ h_{ij} &\equiv g_{ij} - \eta_{ij} \sim \mathcal{O}(c^{-2}), \\ \varepsilon_J{}^i &\equiv e_J{}^i - \delta_J{}^i \sim \mathcal{O}(c^{-2}), \\ \tilde{\varepsilon}^I{}_j &\equiv e^I{}_j - \delta^I{}_j \sim \mathcal{O}(c^{-2}), \end{aligned} \tag{19}$$

with the perturbative objects defined above containing all the corrections starting from the smallest $1/c^2$ order up to the largest one allowed by the context of the expansion. That also implies the following form of the Jacobian in (16)

$$J = \sqrt{-\det(\eta_{ij} + h_{ij})} \simeq 1 - \frac{1}{2} h + \mathcal{O}(c^{-4}), \qquad h \equiv \sum_i h_{ii}. \tag{20}$$

Note that the perturbation $\tilde{\varepsilon}$ and ε in (19) are respectively related to the tetrad and inverse tetrad.

Replacing definitions (19) and (20) into the Hamiltonian (18) and keeping everything up to order $1/c^2$, we have

$$\begin{aligned} \mathscr{H}_D \simeq {} & mc^2 \beta + mc^2 \beta v - i\hbar c \alpha^J \left((1+v) \delta_J{}^i + \varepsilon_J{}^i \right) \partial_i \\ & - i\hbar c \alpha^J \delta_J{}^i \left(\frac{1}{2} \partial_i v + \Gamma_i + \frac{1}{4} \partial_i h \right), \end{aligned} \tag{21}$$

where each perturbation term v, h or ε is intended to be expanded up to highest possible order while keeping the Hamiltonian at order $1/c^2$.

In practice, this approach is equivalent to the one used for PN calculations [31,39], which have demonstrated to be very powerful when dealing with gravitational systems. By virtue of this analogy, we borrow some of the PN vocabulary to classify corrections in a convenient way; in particular, terms of order $1/c^N$ in our scheme, will correspond to $\frac{N}{2}$ PN corrections. Please see Table I for a

TABLE I. Correspondence table between our formal expansion and the post-Newtonian one (also look at Fig. 1 in [63]).

$1/c$ order	PN equivalent
1	0 PN
c^{-1}	0.5 PN
c^{-2}	1 PN

correspondence between our formal expansion and the standard PN notation.

In this sense, the Hamiltonian (21) must be interpreted as the 1 PN-version of the complete expression (18). Thus, apart from the 0 PN Newtonian contribution, it will also include 0.5 PN and 1 PN corrections that are the focus of this work. However, it is important to highlight that, even if here "hybrid" 0.5 PN contributions are present due to the structure of the Dirac Hamiltonian, in the rest of the paper we will see that those are going to become additional 1 PN perturbations after the nonrelativistic limit is taken.

From Eq. (21) we can also note that, at this stage of the calculation, the distinction between capital and lowercase spatial indices becomes irrelevant. In fact, the information on the perturbations around the flat spacetime (up to the relevant order) is already encoded in the v and ε objects.

Therefore, with a little abuse of notation, from now on we will drop this distinction to avoid unnecessary complexities in the reading,

$$\alpha^I \to \alpha^i, \quad \delta_J{}^i \to \delta_j{}^i, \quad \varepsilon_J{}^i \to \varepsilon_j{}^i, \quad \tilde{\varepsilon}^I{}_j \to \tilde{\varepsilon}^i{}_j. \tag{22}$$

This allows us to write the Hamiltonian (21) in the following convenient way:

$$\mathscr{H}_D \simeq mc^2\beta + mc^2\beta v - i\hbar c\alpha^i\left((1+v)\partial_i + \varepsilon_i{}^j\partial_j + \frac{1}{2}\partial_i v + \Gamma_i + \frac{1}{4}\partial_i h\right). \tag{23}$$

B. Simplified expression for Γ_i

Starting from Eq. (A5) for Γ_i, we now want to expand it and obtain its 1.5 PN expression, which is relevant for Hamiltonian (23). Making use of the new notation (22), we have the result,

$$\begin{aligned}\Gamma_i &\simeq \frac{1}{2}\sigma^{kl}(\eta_{mn}\delta_k{}^m\partial_i\varepsilon_l{}^n + \delta_k{}^m\delta_l{}^n\partial_n h_{im})\\ &\simeq \frac{1}{2}\sigma^{kl}(\partial_i\varepsilon_k{}^l - \partial_k h_{il}),\end{aligned} \tag{24}$$

where repeated spatial indices are summed up independently of their upper or lower position. Nevertheless, there is another simplification that can be done in a few steps. First of all, let us observe that, expanding up the tetrad conditions $e_a{}^\mu e^a{}_\nu = \delta^\mu{}_\nu$ and $g_{\mu\nu} = e^a{}_\nu e^a{}_\nu \eta_{ab}$, we obtain, respectively, the equations

$$\begin{aligned}\tilde{\varepsilon}^j{}_i &= -\varepsilon_i{}^j,\\ h_{ij} &= -\tilde{\varepsilon}^i{}_j - \tilde{\varepsilon}^j{}_i = \varepsilon_i{}^j + \varepsilon_j{}^i.\end{aligned} \tag{25}$$

Thus, it is straightforward to see that we can always make the following choice for the tetrads,

$$\varepsilon_i{}^j = -\tilde{\varepsilon}^j{}_i = \frac{1}{2}h_{ij}, \tag{26}$$

so that the conditions (25) are fullfilled up to the relevant $1/c^3$ order. This way, the matrices representing the tetrad tensors perturbations will be symmetric,

$$\varepsilon_i{}^j = \varepsilon_j{}^i, \quad \tilde{\varepsilon}^j{}_i = \tilde{\varepsilon}^i{}_j. \tag{27}$$

On the other hand, the above relations imply that contractions of the type $\sigma^{kl}\varepsilon_k{}^l$ vanish, because of the (anti) symmetries of the involved objects. Therefore, we can finally simplify the Γ_i to the expression,

$$\Gamma_i \simeq -\frac{1}{2}\sigma^{kl}\partial_k h_{il}. \tag{28}$$

In the rest of the paper, all the eventual $\tilde{\varepsilon}$ will be expressed in terms of the inverse tetrad perturbation ε, by virtue of their interconnection (25).

III. NONRELATIVISTIC LIMIT

We shall now proceed with the low-velocity or nonrelativistic limit $|\boldsymbol{p}| \ll mc$. In order to do that, we will apply the FW transformation [40], a well-known procedure to decouple Dirac spinors into its positive and negative energy components. This is usually valid up to some order in m^{-1}, which would otherwise get mixed by the α matrices. However, since we are working in the PN hierarchy of perturbative corrections, we will keep using $1/c$ as our formal expansion parameter. That also allows us to avoid the inconsistencies raised in [30] when using the standard FW transformation in the gravitational context.

A. Foldy-Wouthuysen transformation

The first step in the FW approach is to divide the Hamiltonian (23) into the "even" (nonmixing) operator $\mathcal{E}$ and the "odd" (mixing) operator Θ [41],

$$\mathscr{H}_D = \beta mc^2 + \mathcal{E} + \Theta, \tag{29}$$

with

$$\begin{cases}\Theta = -i\hbar c\alpha^i\left((1+v)\partial_i + \varepsilon_i{}^j\partial_j + \frac{1}{4}\partial_i(2v+h) + \Gamma_i\right),\\ \mathcal{E} = mc^2\beta v.\end{cases} \tag{30}$$

The even and odd operators satisfy the following (anti) commutation relation,

$$[\mathcal{E},\beta]=0,\{\alpha^i,\beta\}=0\to\{\Theta,\beta\}=0, \tag{31}$$

and we can easily see that, due to their expressions and the relations (19), we have at lowest order

$$\mathcal{E}\sim\mathcal{O}(1),\qquad \Theta\sim\mathcal{O}(c). \tag{32}$$

In our formalism, the typical unitary FW transformation $U=e^{iS}$, with S Hermitian, is defined as

$$S=-i\frac{\beta}{mc^2}\Theta\sim\mathcal{O}(c^{-1}). \tag{33}$$

Then, the following expansion is valid up to 1 PN order:

$$\begin{aligned}\mathscr{H}_{FW}&=e^{iS}\mathscr{H}_De^{-iS}\\&=\mathscr{H}_D+i[S,\mathscr{H}_D]\\&\quad+\frac{i^2}{2!}[S,[S,\mathscr{H}_D]]+\frac{i^3}{3!}[S,[S,[S,\mathscr{H}_D]]]\\&\quad+\frac{i^4}{4!}[S,[S,[S,[S,\mathscr{H}_D]]]]+\mathcal{O}(c^{-3}).\end{aligned} \tag{34}$$

Starting from these settings, after three consecutive FW transformations, we end up with the Hamiltonian [42],

$$\begin{aligned}\mathscr{H}_{FW}&=\beta mc^2+\mathcal{E}+\frac{\beta\Theta^2}{2mc^2}\\&\quad+\frac{\beta}{8m^2c^4}[[\Theta,\mathcal{E}],\Theta]-\frac{\beta}{8m^3c^6}\Theta^4+\mathcal{O}(c^{-3}).\end{aligned} \tag{35}$$

in which the matter and antimatter sectors have been decoupled up to order $1/c^3$. Thus, each term in (35) must be calculated up to the relevant order for our approximation. For example, to keep only terms at most 1 PN, we should compute $[[\Theta,\mathcal{E}],\Theta]$ at least to order c^2, and so on. More details on the FW transformation are included in Appendix B.

At this point, it is straightforward to obtain the nonrelativistic Hamiltonian H_{NR} describing the fermion dynamics by simply selecting the positive energy solutions of $\mathscr{H}_{FW}$, and neglecting its constant mass term that would only produce an overall shift to the energy spectrum. After working out every single commutator in (35) and defining the shifted spatial metric correction $\tilde{h}_{ij}$,

$$\tilde{h}_{ij}\equiv h_{ij}+v\delta_{ij}, \tag{36}$$

the final result can be expressed in a surprisingly compact and practical form

$$\begin{aligned}H_{NR}&=mc^2v-\frac{\hbar^4\partial_i^4}{8m^3c^2}\\&\quad-\frac{\hbar^2}{2m}\bigg(\partial_i^2+\tilde{h}_{ij}\partial_i\partial_j+\partial_i\tilde{h}_{ij}\partial_j+\frac{1}{4}\partial_i\partial_j\tilde{h}_{ij}\\&\quad+\frac{i}{2}\epsilon^{ijk}\sigma^k\partial_i\tilde{h}_{jl}\partial_l+\frac{i}{4}\epsilon^{ijk}\sigma^k\partial_i\partial_l\tilde{h}_{jl}\bigg),\end{aligned} \tag{37}$$

with ϵ^{ijk} being the Levi-Civita symbol. For the consistency of the expression, v must be calculated here up to order $1/c^4$ while $\tilde{h}_{ij}$ up to $1/c^2$. This is one of the main results of this paper due to its compactness and general validity for static weak gravity scenarios. We summarize again the considerations used to achieve it:

(i) $g_{0i}=0$ (11);
(ii) $v,h_{ij},\varepsilon_j{}^i,\tilde{\varepsilon}^i{}_j\sim\mathcal{O}(c^{-2})$ (19);
(iii) $\varepsilon_i{}^j=\varepsilon_j{}^i$ (27).

B. Examples and comparison

In this section, we would like to consider a few applications for Eq. (37) and some comparisons with the literature.

1. Diagonal spacetime metrics

Let us start by considering the special case of a diagonal static metric with the form

$$g_{\mu\nu}=\mathrm{diag}\{V^2,-W^2,-W^2,-W^2\}. \tag{38}$$

When assuming the weak gravity limit in (19), we define the additional perturbative quantity

$$w\equiv W-1\sim\mathcal{O}(c^{-2}). \tag{39}$$

Therefore, it will be sufficient to replace $h_{ij}=-2w\delta_{ij}$ in Eq. (37) to obtain the relevant Hamiltonian expression for this case, that becomes

$$\begin{aligned}H_{NR}&=mc^2v-\frac{\hbar^4\partial_i^4}{8m^3c^2}\\&\quad-\frac{\hbar^2}{2m}\bigg((1+v-2w)\partial_i^2+\partial_i(v-2w)\partial_i\\&\quad+\frac{1}{4}\partial_i^2(v-2w)+\frac{i}{2}\epsilon^{ijk}\sigma^k\partial_i(v-2w)\partial_j\bigg),\end{aligned} \tag{40}$$

which corresponds to the result in [43].

2. Schwarzschild metric

A straightforward application of the previous formulae is for the Schwarzschild spacetime. In fact, considering the low-curvature limit in isotropic coordinates, we obtain the following spacetime element:

$$ds^2 = \left(1 + \frac{2\Phi_S}{c^2} + \frac{2\Phi_S^2}{c^4}\right)(cdt)^2 - \left(1 - \frac{2\Phi_S}{c^2}\right)\boldsymbol{dx}^2, \quad (41)$$

with $\Phi_S \equiv -\frac{GM}{r}$ the gravitational potential external to the spherical mass source M, and $r = \sqrt{x^2 + y^2 + z^2}$ the coordinate distance from its center. In the temporal and spatial components of the metric, we have kept terms up to the relevant order for our case. This expression leads us to

$$v = \frac{\Phi_S}{c^2} + \frac{\Phi_S^2}{2c^4}, \qquad w = -\frac{\Phi_S}{c^2}, \quad (42)$$

which replaced into (40) gives back the nonrelativistic Hamiltonian for the Schwarzschild metric,

$$H_S = m\Phi_S\left(1 + \frac{\Phi_S}{2c^2}\right) - \frac{\hbar^2}{2m}\left(1 + \frac{3\Phi_S}{c^2}\right)\partial_i^2 - \frac{\hbar^4\partial_i^4}{8m^3c^2} + \frac{3\hbar^2}{8mc^2}\boldsymbol{\partial}\cdot\boldsymbol{g} + \frac{3\hbar^2}{2mc^2}\boldsymbol{g}\cdot\boldsymbol{\partial} + \frac{3i\hbar^2}{4mc^2}\boldsymbol{\sigma}\cdot(\boldsymbol{g}\times\boldsymbol{\partial}), \quad (43)$$

where we defined the "Newtonian" gravitational acceleration vector $\boldsymbol{g}$ as

$$g^i \equiv -\partial_i\Phi_S. \quad (44)$$

Expression (43) matches the results in [38,41,44], and also [45] if we neglect the Darwin term $\propto \boldsymbol{p}\cdot\boldsymbol{g} \sim \partial^2\Phi_S$, which outside the source of the gravitational field does not matter anyway.

3. Eddington-Robertson metric

Finally, we want to discuss here a particular case that will also come in handy later in the paper; the Eddington-Robertson (ER) parametrized post-Newtonian metric [46,47],

$$ds^2 = \left(1 + \frac{2\Phi_N}{c^2} + \frac{2\beta\Phi_N^2}{c^4}\right)(cdt)^2 - \left(1 - \frac{2\gamma\Phi_N}{c^2}\right)\boldsymbol{dx}^2, \quad (45)$$

where Φ_N is the usual Newtonian potential for an extended classical body with density $\rho(\boldsymbol{x})$

$$\Phi_N(\boldsymbol{x}) = G\int_{\text{Source}} d^3x' \frac{\rho(\boldsymbol{x}')}{\|\boldsymbol{x} - \boldsymbol{x}'\|}, \quad (46)$$

while the parameters β and γ account for possible deviations from GR (in which $\beta = \gamma = 1$), and should not be mistaken with the above-presented Dirac matrices. We re-label these parameters respectively as b (for β) and d (for γ) to avoid confusion. The GR limit of (45) is the solution of Einstein equation in a $1/c$ expansion for a static source. The ER metric is the simplest example of a metric in the general parameterized post-Newtonian (PPN) formalism, which provides a general framework for testing metric theories of gravity in the weak-field regime. For a more exhaustive discussion on this topic, see [48].

Considering a spherically symmetric source and limiting ourselves to its exterior, Φ_N reduces to the Schwarzschild expression $\Phi_S = -\frac{GM}{r}$, yielding what we call the ER-Schwarzschild (ERS) metric

$$ds^2 = \left(1 + \frac{2\Phi_S}{c^2} + \frac{2b\Phi_S^2}{c^4}\right)(cdt)^2 - \left(1 - \frac{2d\Phi_S}{c^2}\right)\boldsymbol{dx}^2, \quad (47)$$

which is trivially equivalent to (41) when $b = d = 1$ as expected. Following the same procedure as before, the ERS Hamiltonian reads,

$$H_{\text{ERS}} = m\Phi_S\left(1 + (2b-1)\frac{\Phi_S}{2c^2}\right) - \frac{\hbar^2}{2m}\left(1 + (1+2d)\frac{\Phi_S}{c^2}\right)\partial_i^2 - \frac{\hbar^4\partial_i^4}{8m^3c^2} + (1+2d)\frac{\hbar^2}{8mc^2}\boldsymbol{\partial}\cdot\boldsymbol{g} + (1+2d)\frac{\hbar^2}{2mc^2}\boldsymbol{g}\cdot\boldsymbol{\partial} + (1+2d)\frac{i\hbar^2}{4mc^2}\boldsymbol{\sigma}\cdot(\boldsymbol{g}\times\boldsymbol{\partial}), \quad (48)$$

from which we recover again (43) if $b = d = 1$.

IV. PROPER LABORATORY FRAME

Our final goal is to apply our derivation to experiments and observations made in small laboratories on Earth's surface. For this purpose, (almost) global coordinates $\{x^\mu\}$ like the Schwarschild ones are clearly not the most suitable option, since they would lead to difficulties with the definitions of time intervals and physical distances.

The most natural possibility, instead, is to work in the proper reference frame of the laboratory, which can be done exploiting Fermi coordinates (FC) [49–51] extended to the case of accelerated motion [52] thanks to Fermi-Walker transport [53]. In fact, the laboratory does not follow a geodesic motion, since it is accelerated upwards by the normal force exerted by the Earth's surface itself. Therefore, this should be the natural framework when one is interested into local observations. This approach also has the advantage to make coordinate time and lengths coincides with their corresponding physical quantities, avoiding any possible confusion and need for rescalings.

A. Fermi coordinates

The main philosophy of FC is to approximate a small enough region of spacetime around the worldline $\xi^\mu(x)$ of an observer. This task is achieved by considering the observer's proper time τ and constructing an Euclidean grid $\{X^i\}$ comoving with the observer. For further details on the geometric construction of FC see [54,55] and references therein. In the following, objects evaluated on the observer's worldline (i.e., $X^i = 0$) are denoted when possible by a bar over them: $O|_\xi = \bar{O}$. Furthermore, since here we are particularly interested in static observers, we are free to align the Z-direction with the local acceleration $\boldsymbol{a}$ along all the path of the worldline.

The consistency of this treatment is governed by a new small parameter to be introduced in the picture, $\|X\|/\mathcal{R} \ll 1$, with $\|X\|$ representing the typical length scale of the experiment and $\mathcal{R}$ symbolically defined by [56,57]

$$\mathcal{R} = \min\left(\|R_{\mu\nu\rho\sigma}\|^{-\frac{1}{2}}, \frac{\|R_{\mu\nu\rho\sigma}\|}{\|R_{\mu\nu\rho\sigma;\alpha}\|}, \frac{c^2}{\|\boldsymbol{a}\|}\right), \qquad (49)$$

where we used the semicolon in the Riemann tensor $R_{\mu\nu\rho\sigma;\alpha}$ to indicate its covariant derivative.

B. Metric in Fermi coordinates

When mapped to our notational conventions, the general form of the Fermi metric in the proper reference frame experiencing an acceleration $\boldsymbol{a}$ is [57]

$$\begin{aligned} g^F_{\tau\tau} &= \left(1 + \frac{\boldsymbol{a}\cdot\boldsymbol{X}}{c^2}\right)^2 - \bar{R}^F_{tltm}X^lX^m + \mathcal{O}(\|X\|^3), \\ g^F_{\tau i} &= -\frac{2}{3}\bar{R}^F_{tlim}X^lX^m + \mathcal{O}(\|X\|^3) \\ g^F_{ij} &= -\delta_{ij} - \frac{1}{3}\bar{R}^F_{ijlm}X^lX^m + \mathcal{O}(\|X\|^3), \end{aligned} \qquad (50)$$

where $\bar{R}^F_{\mu\nu\alpha\beta}$ is the Riemann tensor in FC evaluated on the observer's worldline. Its Fermi expression, due to gauge covariance, can be evaluated starting from the Riemann tensor in some prior coordinates,

$$\begin{aligned} &\bar{R}^F_{\mu\nu\alpha\beta} = \bar{R}_{\rho\sigma\kappa\gamma}\bar{\Lambda}_\mu{}^\rho\bar{\Lambda}_\nu{}^\sigma\bar{\Lambda}_\alpha{}^\kappa\bar{\Lambda}_\beta{}^\gamma, \\ \text{with}\quad &R_{\mu\nu\alpha\beta} = g_{\mu\rho}R^\rho{}_{\nu\alpha\beta}, \end{aligned} \qquad (51)$$

where $\bar{\Lambda}$ represents the coordinate transformation matrix evaluated on the worldline and, by construction [51], it coincides with the tetrad matrices.

Clearly, in our weak-gravity framework, the Riemann tensor $R_{\mu\nu\alpha\beta}$ should be treated as an $\mathcal{O}(c^{-2})$ object, since its leading contributions are at least linear in h_{ij} [58]. Note also that, working in a static context, we will have $\bar{R}_{tlim} = 0$ implying $g^F_{\tau i} = 0$, as expected. There exists in the literature a more general metric than (50) in which the effects of laboratory's rotation are also taken into account [59]. However, in this work we shall limit ourselves to (50) for consistency.

Therefore, the relevant quantities to consider for the calculation of (37) are

$$\begin{aligned} v^F &= \frac{\boldsymbol{a}\cdot\boldsymbol{X}}{c^2} - \frac{1}{2}\bar{R}^F_{\tau l\tau m}X^lX^m, \\ h^F_{ij} &= -\frac{1}{3}\bar{R}^F_{iljm}X^lX^m. \end{aligned} \qquad (52)$$

Here $\bar{R}^F_{\tau l\tau m}$ has to be calculated up to order $1/c^4$, starting from Eq. (51)

$$\bar{R}^F_{\tau l\tau m}X^lX^m = (1+2\bar{v})\bar{R}_{tltm}X^lX^m + 2\bar{\varepsilon}_l{}^k\bar{R}_{tktm}X^lX^m, \qquad (53)$$

while $\bar{R}^F_{iljm}$, at order $1/c^2$, just coincides with its expression in the prior coordinates system

$$\bar{R}^F_{iljm} \simeq \bar{R}_{iljm}. \qquad (54)$$

C. Fermi Hamiltonian

Replacing (52) in the Hamiltonian (37), we can split it into the sum of two contributions

$$H_{NR} = H_N + H_{\mathrm{NLO}}, \qquad (55)$$

with the Newtonian Hamiltonian H_N defined as

$$H_N = m\boldsymbol{a}\cdot\boldsymbol{X} - \frac{\hbar^2}{2m}\partial_i^2 = maZ - \frac{\hbar^2}{2m}\partial_i^2, \qquad (56)$$

while the NLO part, containing all 1PN and $O(\|X\|^2)$ corrections, is

$$\begin{aligned} H_{\mathrm{NLO}} = &-\frac{mc^2}{2}\bar{R}^F_{tltm}X^lX^m - \frac{\hbar^4\partial_i^4}{8m^3c^2} \\ &-\frac{\hbar^2}{2m}\left(\tilde{h}^F_{ij}\partial_i\partial_j + \partial_i\tilde{h}^F_{ij}\partial_j + \frac{1}{4}\partial_i\partial_j\tilde{h}^F_{ij}\right. \\ &\left.+\frac{i}{2}\epsilon^{ijk}\sigma^k\partial_i\tilde{h}^F_{jl}\partial_l + \frac{i}{4}\epsilon^{ijk}\sigma^k\partial_i\partial_l\tilde{h}^F_{jl}\right). \end{aligned} \qquad (57)$$

To make this picture coherent with our formulation, we assume that $\boldsymbol{a}$ does not depend on time, or that its dependence is sufficiently weak to be negligible over the relevant time scale involved in the physical process we want to study.

As clearly shown above (52), to proceed with the calculations we also have to make some assumptions about spacetime geometry and its "prior" coordinates. To avoid additional complications at this stage, in the following we will model the Earth as a sphere of radius R, which naturally leads us to the choice of the ERS metric (47). This decision also allows us to look after eventual GR departures.

Note that, as far as static observers on the Earth's surface are concerned, the spatial components of their quadrivelocity U^α are zero. Thus, their spatial position will be constant and simply set to

$$\xi^i(x) = (0, 0, R), \tag{58}$$

due to our reference frame choices. The full expression for the NLO Hamiltonian in the ERS spacetime together with other details are included in Appendix C.

D. Theoretical local acceleration value

In the next sections, we will consider the local acceleration just as a parameter that is determined experimentally. Nevertheless, for completeness, we would like to include here some remarks on the theoretical values predicted for the experienced acceleration. Working with a static observer, we know its spacetime acceleration $a^\mu = U^\alpha \nabla_\alpha U^\mu$ along ξ^μ, when translated in the mostly minus convention, must be given by [60,61]

$$a_\mu = -c^2 \nabla_\mu \ln V|_\xi = -\frac{c^2}{1+\bar{v}} \partial_\mu v|_\xi. \tag{59}$$

In our perturbative scheme, considering the ERS metric (47) with $v = \frac{\Phi_S}{c^2} + \frac{2b-1}{2}\frac{\Phi_S^2}{c^4}$ and keeping everything up to 1PN order, we have

$$\begin{aligned} a_\mu &= -\frac{c^2}{1+\frac{\bar{\Phi}_S}{c^2}+\frac{2b-1}{2}\frac{\bar{\Phi}_S^2}{c^4}} \partial_\mu \left(\frac{\Phi_S}{c^2} + \frac{2b-1}{2}\frac{\Phi_S^2}{c^4}\right)\Big|_\xi \\ &\simeq -\left(1 + 2(b-1)\frac{\bar{\Phi}_S}{c^2}\right)\partial_\mu \Phi_S|_\xi, \end{aligned} \tag{60}$$

from which we readily see that $a_t = 0$. Therefore, raising the acceleration index with the metric, we get

$$a^t = \bar{g}^{t\mu} a_\mu = 0, \tag{61}$$

$$\begin{aligned} a^i &= \bar{g}^{i\mu} a_\mu = -\left(1 + \frac{2d\bar{\Phi}_S}{c^2}\right) a_i \\ &= \left(1 + 2(b+d-1)\frac{\bar{\Phi}_S}{c^2}\right)\partial_i \Phi_S|_\xi. \end{aligned} \tag{62}$$

To calculate the corresponding value on the Fermi frame we just exploit the coordinate transformation Λ

$$\begin{aligned} (a^F)^\tau &= \bar{\Lambda}^\tau{}_\nu a^\nu = \bar{\Lambda}^\tau{}_t a^t = 0, \\ (a^F)^i &= \bar{\Lambda}^i{}_\nu a^\nu = a^i + \bar{\varepsilon}^i{}_j a^j = \left(1 - \frac{d\bar{\Phi}_S}{c^2}\right) a^i. \end{aligned} \tag{63}$$

Finally, remembering the definition (44) of the gravitational acceleration vector, the effective acceleration experienced by the Fermi observer will simply be

$$(a^F)^i = -\left(1 + (2b+d-2)\frac{\bar{\Phi}_S}{c^2}\right)\bar{g}^i, \tag{64}$$

which, as expected, coincides with the Newtonian result at leading order and is in agreement with the equivalence principle by construction. Clearly, in real-life experiments, there are other effects influencing the effective acceleration value, like Earth's rotation itself, which however here is not explicitly accounted for.

Note that the above result for the acceleration can also be obtained starting directly from the ERS Hamiltonian (48), and performing the Fermi transformation Λ *a posteriori*; in fact, after the transformation, it is sufficient to identify the "effective acceleration" as the coupling to the term linear in Z and proportional to m. Embracing this philosophy, we also have to remember that coordinate time in this case does not coincide with the proper one, leading to a rescaling also of the Hamiltonian itself $H^{(\tau)} = \frac{1}{1+\bar{v}} H^{(t)}$. The equivalence of this approach points towards the fact that the order in which one performs the FW transformation and the change to Fermi coordinates, at the end of the day, does not affect relevant quantities, as one could also expect. However, we will not include more details on that since it goes beyond the scope of this paper.

E. Remarks on higher in $\|X\|$-orders

An analysis close to ours is carried out in [57], where the authors argue that strictly speaking, if one would also like to formally expand the Hamiltonian in $\|X\|$, then the derivatives should be considered as order $\sim 1/\|X\|$ implying the need to take into account orders higher than $\|X\|^2$ in (50). However, here we adopt another view for the Fermi expansion; by momentarily adopting dimensionless quantities, let us imagine to divide the Hamiltonian (57) by mc^2. Defining the momentum operator

$$p^i \equiv -i\hbar \partial_i, \tag{65}$$

we can then think for all terms enclosed by round brackets in (57) to be of order $\sim (\frac{p}{mc})^2 (1 + \tilde{h}^F)$, where we remember that $\tilde{h}^F_{ij}$ is already dimensionless. Thus, it is clear that all terms $\mathcal{O}(\|X\|^3)$ we could consider in the Fermi metric expansion would give smaller and smaller corrections. Thus, holding onto to the principle of "least action", here meaning smallest modification with largest consequences, we stick with perturbations at most $\sim \|X\|^2$ in $g^F_{\mu\nu}$.

V. APPLICATION TO UCN AND QBOUNCE

In this section, we would like to quantitatively study the effects of the NLO corrections within an experimental setup analogous to the one of qBounce [12,29,62]. Values for kinematic parameters and constants in this configuration are summarized in Table II.

TABLE II. Some constants and parameters values used to fit the qBounce experiment [12]. The local acceleration value here, is determined through a falling corner cube classical experiment.

qBounce parameters	Values
Neutron mass m	1.675×10^{-27} kg
Earth mass M	5.9726×10^{24} kg
Newton constant G	6.6743×10^{-11} $\frac{\mathrm{N\cdot m^2}}{\mathrm{kg^2}}$
Local acceleration a	9.8049 m/s^2
Longitudinal velocity $v_\perp$	∼4–10 m/s
Vertical velocity v^Z	∼10 cm/s

A. Decoupling the XY-dynamics

At leading order, neutrons are just affected by a Z-dependent potential given in (56). This implies that their dynamics can be factorized into a longitudinal XY-component and a transverse Z-component, which is reflected in their wave function as

$$\Psi(\boldsymbol{X}) = \phi(X,Y)\varphi(Z). \tag{66}$$

The free XY-motion in these experiments can be well-described by semiclassical laws, considering the UCN's longitudinal states as normalized wave packets centered on their classical trajectory [21,64]. For our purposes, it can be therefore modeled as

$$\phi(\boldsymbol{X}_\perp,\tau) \equiv \frac{1}{\sqrt{\pi}\sigma} e^{\frac{i}{\hbar}\boldsymbol{k}_\perp\cdot\boldsymbol{X}_\perp - \frac{(\boldsymbol{X}_\perp-\boldsymbol{X}^{cl}_\perp)^2}{2\sigma^2}}, \tag{67}$$

with $\boldsymbol{X}_\perp = (X,Y)$ and

$$\begin{cases} \boldsymbol{X}^{cl}_\perp(\tau) = (X_{cl}(\tau), Y_{cl}(\tau)) \\ \boldsymbol{k}_\perp = (k^x, k^y), \end{cases} \tag{68}$$

respectively indicating the classical horizontal coordinates and momenta of the UCNs.

Later on, we will be mainly interested in the energy spectrum. Then, for us it is sufficient to consider Z-energy eigenstates and thus set the XY-initial state in the origin of the laboratory frame,

$$\phi(\boldsymbol{X}_\perp) \equiv \phi(\boldsymbol{X}_\perp,\tau=0) = \frac{1}{\sqrt{\pi}\sigma} e^{\frac{i}{\hbar}\boldsymbol{k}_\perp\cdot\boldsymbol{X}_\perp - \frac{\boldsymbol{X}^2_\perp}{2\sigma^2}}. \tag{69}$$

In the following we want to study the details of the fully quantum transverse dynamics, by integrating out the horizontal degrees of freedom. Doing this, we can derive an effective one-dimensional Hamiltonian guiding the vertical Z-evolution

$$H^{(Z)} = \int d^2X_\perp \phi^*(\boldsymbol{X}_\perp) H \phi(\boldsymbol{X}_\perp). \tag{70}$$

Note that the spatial spreading $\Delta X = \Delta Y = \sigma/\sqrt{2}$ is not directly known for the UCNs in the qBounce experiment. Nevertheless, we know for sure it has to be bigger than nuclear spacing $\sim10^{-10}$ m and smaller than the characteristic size of the experiment. A useful educated guess is the UCN's de Broglie wavelength

$$\sigma \sim \frac{h}{m\|\boldsymbol{v}\|} \sim 10^{-8}\ \mathrm{m}, \tag{71}$$

which is approximately the same result we would get from the Heisenberg principle for the minimal value of the position uncertainty, being the velocity dispersion Δv for qBounce UCN about ~1 m/s. In fact, from the state (69) we find

$$m\Delta v = \frac{\hbar}{\sqrt{2}\sigma} \to \sigma = \frac{\hbar}{\sqrt{2}m\Delta v} = 4.45\times10^{-8}\ \mathrm{m}. \tag{72}$$

B. Newtonian Schrödinger problem

At the lowest order (c^0), the theoretical description of the UCNs in the qBounce setting is given by the Hamiltonian H_N in (56), where spin does not play any role. Using Eqs. (70) and (56), we can calculate the leading-order effective Hamiltonian,

$$H^{(Z)}_N = \frac{\hbar^2}{2m\sigma^2} + \frac{\boldsymbol{k}^2_\perp}{2m} + \frac{\hbar^2}{2m}\partial^2_Z + maZ. \tag{73}$$

Its spectrum $\mathcal{E}^{(0)}$ is given by solving the correspondent secular equation,

$$\left(\frac{\hbar^2}{2m}\partial^2_Z + maZ\right)\varphi(Z) = E\varphi(Z), \tag{74}$$

with $E \equiv \mathcal{E}^{(0)} - \frac{\boldsymbol{k}^2_\perp}{2m} - \frac{\hbar^2}{2m\sigma^2}$. The presence of the qBounce bottom mirror is simulated by setting boundary conditions at $Z=0$. Fortunately, the solution of the above equation is well-known and is given by Airy functions [65],

$$\varphi_n(Z) = C_n \mathrm{Ai}\left(\frac{Z-Z_n}{Z_0}\right) \tag{75}$$

with

$$C_n = \frac{Z_0^{-\frac{1}{2}}}{\mathrm{Ai}'\left(-\frac{Z_n}{Z_0}\right)}, \quad Z_0 = \left(\frac{\hbar^2}{2m^2a}\right)^{\frac{1}{3}}, \quad Z_n = \frac{E_n}{ma}, \tag{76}$$

where the $\mathrm{Ai}'(\zeta)$ represents the derivative of $\mathrm{Ai}(\zeta)$ with respect to its argument $\zeta \equiv \frac{Z-Z_n}{Z_0}$. The E_n values are determined by the quantization condition obtained by setting the wave function at $Z=0$ to zero,

$$\mathrm{Ai}\left(-\frac{E_n}{maZ_0}\right) = 0. \tag{77}$$

Thus, zeroth-order spectrum for UCN is

$$\mathcal{E}_n^{(0)} = \frac{\hbar^2}{2m\sigma^2} + \frac{1}{2}mv_\perp^2 + E_n, \quad v_\perp \equiv \frac{\|k_\perp\|}{m}. \tag{78}$$

Altenatively, this 0 PN problem can also be tackled by a Green's function approach, as presented for the Lloyd interferometry setup discussed in [66].

C. Next-to-leading-order corrections

We are now ready to study the energy corrections due to NLO contributions; the first step is to integrate out the XY dynamics also from the NLO Hamiltonian contribution in (C3) as

$$H_{\mathrm{NLO}}^{(Z)} = \int d^2X_\perp \phi^*(\boldsymbol{X}_\perp) H_{\mathrm{NLO}} \phi(\boldsymbol{X}_\perp), \tag{79}$$

whose full expression is provided in (C4). At this point we are left with a perturbation to the Newtonian Hamiltonian (73). Since NLO terms introduce operators involving Pauli matrices, for each value of the quantum number n we now have a two-dimensional eigenspace, spanned by the degenerate eigenvectors of the unperturbed problem,

$$\langle Z|\varphi_n, \uparrow\rangle = C_n \mathrm{Ai}\left(\frac{Z-Z_n}{Z_0}\right)\begin{pmatrix}1\\0\end{pmatrix},$$
$$\langle Z|\varphi_n, \downarrow\rangle = C_n \mathrm{Ai}\left(\frac{Z-Z_n}{Z_0}\right)\begin{pmatrix}0\\1\end{pmatrix}. \tag{80}$$

To apply standard quantum perturbation theory, we must first calculate the matrix elements $W_{\alpha\beta}^n$ of the perturbation within the degenerate unperturbed subspaces,

$$W_{\alpha\beta}^n \equiv \langle \varphi_n, \alpha | H_{\mathrm{NLO}}^{(Z)} | \varphi_n, \beta\rangle = \langle \varphi_n | (H_{\mathrm{NLO}}^{(Z)})_{\alpha\beta} | \varphi_n\rangle, \tag{81}$$

where $(H_{\mathrm{NLO}}^{(Z)})_{\alpha\beta}$ are the matrix components of the NLO Hamiltonian in the two-dimensional spin subspace with $\alpha, \beta = \uparrow, \downarrow$. To calculate the first-order corrections $\mathcal{E}_n^{(1)}$ we therefore have to diagonalize the $W_{\alpha\beta}^n$ matrices to find their eigenvalues as solutions of the equation,

$$\det(W^n - \mathcal{E}_n^{(1)}\mathbf{1}) = 0, \tag{82}$$

with $\mathbf{1}$ the two-by-two identity matrix. Expanding the determinant, we arrive at the secular equation,

$$(W_{\uparrow\uparrow}^n - \mathcal{E}_n^{(1)})^2 - |W_{\uparrow\downarrow}^n|^2 = 0, \tag{83}$$

where we used the fact that, in our case, we have

$$W_{\uparrow\uparrow}^n = W_{\downarrow\downarrow}^n, \quad W_{\uparrow\downarrow}^n = (W_{\downarrow\uparrow}^n)^*. \tag{84}$$

Thus, the next-to-leading order corrections to the spectrum will be given by

$$\mathcal{E}_{n,\pm}^{(1)} = W_{\uparrow\uparrow}^n \pm W_{\uparrow\downarrow}^n. \tag{85}$$

To compute the $W_{\alpha\beta}^n$, we now have to calculate the averages of all the Z-dependent quantities appearing in Eq. (C4), which involve integrals of Airy functions multiplied by powers of Z and ∂_Z. Sur·pris·ing·ly, this can be done analytically, by using the method reported in [65]. The calculation technique involves shifting the argument of the second Ai by a small quantity λ and then taking the limit $\lambda \to 0$ after performing the integrals. We include details about this procedure in Appendix D.

Assuming the following general notation for the averages over the Airy eigenstates:

$$\langle O\rangle_n = \langle \psi_n^{(0)} | O | \psi_n^{(0)}\rangle, \tag{86}$$

at the end, we explicitly obtain the following mean values,

$$\begin{cases} \langle \partial_Z\rangle_n = 0, \quad \langle \partial_Z^2\rangle_n = -\frac{1}{3}\frac{Z_n}{Z_0^3}, \quad \langle Z\rangle_n = \frac{2}{3}Z_n, \\ \langle Z^2\rangle_n = \frac{8}{15}Z_n^2, \quad \langle Z\partial_Z^2\rangle_n = -\frac{2}{15}\frac{Z_n^2}{Z_0^3}, \quad \langle \partial_Z^4\rangle_n = \frac{1}{5}\frac{Z_n^2}{Z_0^6}, \\ \langle Z\partial_Z\rangle_n = -\frac{1}{2}, \quad \langle Z^2\partial_Z^2\rangle_n = \frac{3}{7} - \frac{8}{105}\frac{Z_n^3}{Z_0^3}. \end{cases} \tag{87}$$

Combining formulas (85), (87), (C4) and neglecting constant shift terms, we obtain

$$\begin{aligned} \mathcal{E}_{n,\pm}^{(1)} = & -\frac{8mGMZ_n^2}{15R^3} - (5-6b-5d)\frac{4mG^2M^2Z_n^2}{15c^2R^4} \\ & + \frac{maZ_n}{6c^2}\left(v_\perp^2 + \frac{\hbar^2}{m^2\sigma^2} + (3-2d)\frac{GM\sigma^2}{3R^3}\right) \\ & + \frac{ma^2Z_n^2}{30c^2} - (d+3)\frac{4mGMZ_n^2}{45c^2R^3}\left(v_\perp^2 + \frac{\hbar^2}{m^2\sigma^2}\right) \\ & - \frac{8maGMZ_n^3}{105c^2R^3} \pm \frac{\hbar v_\perp}{4c^2}\left((d+2)\frac{2GMZ_n}{3R^3} - a\right). \end{aligned} \tag{88}$$

Note that we grouped v^F-corrections (52) in the first line, while mixed $\tilde{h}_{ij}^F$ corrections are present in the remaining lines.

D. Remarks on NLO contributions in literature

As previously mentioned in the Introduction and in Sec. III for the FW transformation, NLO corrections to the Hamiltonian for fermions in weak gravitational fields are often packed in a $1/m$ expansion. This choice can lead to inconsistent result, particularly when coupled with the additional caveat of keeping terms at most linear in Φ_S; consider the case of the following first-order Hamiltonian (in natural units $c = \hbar = 1$) [22,23],

$$H^{(\mathrm{lit})}_{\mathrm{NLO}} = \frac{3}{4m}(\boldsymbol{\sigma}\times\boldsymbol{p})\cdot\boldsymbol{g} - \frac{3}{4m}(\boldsymbol{p}^2\boldsymbol{g}\cdot\boldsymbol{z} + \boldsymbol{g}\cdot\boldsymbol{z}\boldsymbol{p}^2). \tag{89}$$

Already from the averages in (87), we can observe that there are pieces missing from (89) at this level of approximation. In fact, since

$$\frac{\langle p_z^2\rangle_n}{m^2} = a\langle z\rangle_n \simeq \Phi_S - \bar{\Phi}_S, \tag{90}$$

we can identify the following terms to generate corrections of the same order of magnitude,

$$\Phi_S\frac{p_z^2}{m^2} \sim \Phi_S^2 \sim \frac{p_z^4}{m^4}. \tag{91}$$

This fact directly highlights two problems:

(i) Working in the framework of a linear weak-gravity expansion, leads to the partial exclusion of z^2-corrections coming from $1/c^4$ contributions to v or equivalently Φ_S^2 contributions in g_{00}. This is especially true if the additional assumption $\boldsymbol{g} = \mathrm{const}$ is taken.

(ii) The misleading use of the $\frac{1}{m}$-expansion parameter can also lead to erroneously neglect the first special-relativistic corrections $\propto \partial_z^4$.

Both these exclusions lead to neglecting terms whose corrections are of the same order of the one given by (89), and should therefore be included.

Those considerations strengthen our choice of a $1/c$–expansion and the successive use of the Fermi coordinates, two prescriptions which also other authors started to adopt in recent years [57].

E. Estimation for qBounce

Let us finally give some estimations for the effects in the qBounce context. Combining the results (78) and (88), with a little algebra and the exclusion of n-independent terms, we get

$$\begin{aligned}\mathcal{E}_{n,\pm} = &\left[1+\frac{1}{6c^2}\left(v_\perp^2+\frac{\hbar^2}{m^2\sigma^2}+(3-2d)\frac{GM\sigma^2}{3R^3}\right)\right]E_n \\ &-\frac{1}{5m}\left[\frac{8GM}{3a^2R^3}-\frac{1}{6c^2}-(5-6b-5d)\frac{4G^2M^2}{3a^2c^2R^4}\right. \\ &\left.+(d+3)\frac{4GM}{9c^2a^2R^3}\left(v_\perp^2+\frac{\hbar^2}{m^2\sigma^2}\right)\right]E_n^2 \\ &+\frac{8GME_n^3}{105ma^2c^2R^3}\pm\frac{\hbar v_\perp}{4c^2}\left((d+2)\frac{2GME_n}{3maR^3}-a\right).\end{aligned} \tag{92}$$

Using the values in Table II and using the conservative estimate $\sigma \sim 10^{-8}$ m, we easily see that by far the largest perturbation comes from the first term in the squared parenthesis proportional to E_n^2, which would be order $10^{-12}\times E_n$. Actually, that term is analogous to what we would get expanding the $1/r$ potential around the Earth's surface to quadratic order and, in principle, it should not be considered as a part of the PN corrections but of the higher-order $\|X\|$ corrections. In our scheme, in fact, it is generated by the first term in Eq. (52) when considering $1/c^2$ contributions of the Riemann tensor in Eq. (51).

The consequence of these considerations are twofold:

(i) First of all, since the terms involving the PPN parameters b and d are the smallest ones, it is unlikely that useful upper bounds can be put through these type of experiments.

(ii) Secondly, the NLO corrections will not be observable in this class of experiments in the near future.

F. Local–a tension

Recently, an interesting discrepancy among the local acceleration value measured by a classical experiment $a_{\mathrm{cl}} = 9.8049 m/s^2$ and by the qBounce experiment $a_{\mathrm{qB}} = 9.8120 m/s^2$ has been reported in [67]. This inconsistency was observed by deducing the effective value of the acceleration by studying the transition among the energy level $n = 1$ to $n = 6$. The experimentally derived value for the transition frequency $\nu_{1\to6}$ is

$$\nu^{\mathrm{obs}}_{1\to6} = 972.81\ \mathrm{Hz}, \tag{93}$$

while the predicted value from (78) corresponds to 972.35 Hz. Since the statistical significance of this result already reached several sigmas, assuming there is no systematical flaw in the experimental derivation, one should start to search for causes of this shift. The cautious way to go here, before thinking to some new physics hint, is to take into consideration NLO effects and see whether they can account for the discrepancy.

Therefore, we calculate $\nu_{1\to6}$ with our corrected spectrum (92), while fixing $b = d = 1$ and letting σ as the only "free" parameter (being the only quantity whose value could in principle lay in a range spanning several orders of magnitude). This way we find

$$\nu^{\mathrm{NLO}}_{1\to6} = \left(972.35 + 9.24\{\sigma\}^2\times10^{-22} + \frac{7.15}{\{\sigma\}^2}\times10^{-30}\right)\ \mathrm{Hz}, \tag{94}$$

where with $\{\sigma\}$ we are indicating just the numerical value of σ when expressed in meters. Thus, we see that to obtain the experimental value (93) we should have $\sigma = 3.94\times10^{-15}$ m (curiously close to the physical diameter of neutrons) or $\sigma = 2.23\times10^{10}$ m. Such values are, however, not likely since they are way out the allowed region defined by the qBounce setting. This is just another confirmation of the fact that for NLO corrections to be relevant we have to push parameters to values which lie outside their realistic ranges for current experiments. Thus, the local-a tension still awaits for an explanation.

VI. CONCLUSIONS

In this work, we have calculated the nonrelativistic and low-curvature corrections to the Schrödinger equation for a ultracold neutron in a static spacetime. We have done that starting from the Dirac equation on the curved spacetime generated by the Earth's gravitional field. The whole process involves many different technicalities, like the FW transformation and the proper reference frame choice, which makes it highly nontrivial, despite the amount of literature on the subject. In fact, terms that could seem negligible at first glance end up being of the same order of the other perturbative corrections to the neutron energy spectrum, when doing things consistently. In this sense, we have seen that a post-Newtonian approach can help to avoid these difficulties. In the near future, we plan to extend our analysis for spacetimes with $g_{0i} \neq 0$, therefore including rotational effects into the picture, which are known to also contribute to NLO corrections [68,69].

Finally, we analyzed our results from an experimental perspective and found that, with the current level of precision, post-Newtonian corrections will not play a role in near-future observations for experiments like qBounce or GRANIT, unless drastic changes to the setups. That also implies that UCN experiments may not be useful in determining deviations from GR predictions for the PPN parameters. Nevertheless, the positive side of the story is that any new tension that may appear in those measurements, like the local acceleration one mentioned in the text, could be regarded as a sign of new physics, after carefully excluding any alternative origin for such systematic errors.

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APPENDIX A: EXPRESSIONS FOR Γ_μ

Here we calculate the expression of the spin-connection Γ_μ. Let us start with the time component,

$$\Gamma_t = \frac{1}{2}\sigma^{ab} g_{\nu\rho} e_a{}^\nu \nabla_t e_b{}^\rho = \frac{1}{2}\sigma^{ab} g_{\nu\rho} e_a{}^\nu e_b{}^\gamma \left\{ {\rho \atop t\gamma} \right\}, \tag{A1}$$

where we already used the fact that working with static metrics, nothing can depend on time. Expanding the expression for the Christoffel symbol and remembering that $g_{ti} = e_0{}^i = 0$, we have

$$\begin{aligned}\Gamma_t &= \frac{1}{4}\sigma^{ab} g_{\nu\rho} g^{\rho\alpha} e_a{}^\nu e_b{}^\gamma (\partial_\gamma g_{\alpha t} - \partial_\alpha g_{\gamma t}) \\ &= \frac{1}{4}\sigma^{ab} e_a{}^\nu e_b{}^\gamma (\partial_\gamma g_{\nu t} - \partial_\nu g_{\gamma t}) \\ &= \frac{1}{2}\sigma^{ab} e_a{}^t e_b{}^j \partial_j g_{tt} = \frac{1}{2}\sigma^{0I} e_0{}^t e_I{}^j \partial_j g_{tt}.\end{aligned} \tag{A2}$$

Using the expressions in (11) and (13), we finally obtain

$$\Gamma_t = \frac{1}{2V}\sigma^{0I} e_I{}^j \partial_j V^2 = \sigma^{0I} e_I{}^j \partial_j V. \tag{A3}$$

The spatial components of Γ_i can be calculated in an analogous fashion, by taking into consideration that

$$\left\{ {t \atop i\,j} \right\} = \left\{ {j \atop i\,t} \right\} = 0. \tag{A4}$$

At the end of the day, we get

$$\Gamma_i = \frac{1}{2}\sigma^{KL}(g_{mn} e_K{}^m \partial_i e_L{}^n + e_K{}^m e_L{}^n \partial_n g_{im}). \tag{A5}$$

APPENDIX B: FOLDY-WOUTHUYSEN TRANSFORMATION

The Foldy-Wouthuysen transformation is a technical tool to construct a connection between Dirac theories and their Schrödinger equivalent in the nonrelativistic limit $|\boldsymbol{p}| \ll mc$. In the following, we include details on some calculations that lead to results used in the main text. Remembering that $S = -i\frac{\beta\Theta}{2mc^2} \sim \mathcal{O}(c^{-1})$, we can easily find the structures of the commutators in Eq. (34) up to the needed order of $1/c^2$

$$\begin{aligned}[S, \mathscr{H}_D] &= i\Theta - i\frac{\beta}{2mc^2}[\Theta, \mathcal{E}] - i\frac{\beta\Theta^2}{mc^2}, \\ [S, [S, \mathscr{H}_D]] &= \frac{\beta\Theta^2}{mc^2} - \frac{1}{4m^2c^4}[[\Theta, \mathcal{E}], \Theta] + \frac{\Theta^3}{m^2c^4}, \\ [S, [S, [S, \mathscr{H}_D]]] &\simeq i\frac{\Theta^3}{2m^2c^4} - i\frac{\beta\Theta^4}{2m^3c^6}, \\ [S, [S, [S, [S, \mathscr{H}_D]]]] &\simeq \frac{\beta\Theta^4}{6m^3c^6}.\end{aligned} \tag{B1}$$

Putting all of these expressions together we obtain,

$$\mathscr{H}^{I}_{FW} \simeq \beta mc^2 + \mathcal{E} + \frac{1}{2mc^2}\beta[\Theta,\mathcal{E}] + \frac{1}{2mc^2}\beta\Theta^2 + \frac{1}{8m^2c^4}[[\Theta,\mathcal{E}],\Theta] - \frac{1}{3m^2c^4}\Theta^3 - \frac{1}{8m^3c^6}\Theta^4. \quad \text{(B2)}$$

Repeating the same procedure two more times, we can completely get rid off the odd terms up to order c^{-2},

$$\mathscr{H}^{III}_{FW} = \beta mc^2 + \mathcal{E} + \frac{\beta\Theta^2}{2mc^2} + \frac{\beta}{8m^2c^4}[[\Theta,\mathcal{E}],\Theta] - \frac{\beta}{8m^3c^6}\Theta^4 + \mathcal{O}(c^{-3}). \quad \text{(B3)}$$

We now just have to calculate each one of the structures appearing in the Hamiltonian above up to the relevant perturbative order. This task can be completed quite straightforwardly taking into account the expressions for even and odd operators in (30) and the commutation rules for Dirac matrices. In fact, when calculating objects like Θ^2 one should be very careful since, for example, σ^{KL} does not simply commute with the α-matrices

$$\begin{aligned}\alpha^i\sigma^{kl}\alpha^j &= -\frac{1}{4}\gamma^i[\gamma^k,\gamma^l]\gamma^j \\ &= -\gamma^i\gamma^j\sigma^{kl} - \gamma^i\gamma^l\delta^{jk} + \gamma^i\gamma^k\delta^{jl} \\ &= \alpha^i\alpha^j\sigma^{kl} + \alpha^i\alpha^l\delta^{jk} - \alpha^i\alpha^k\delta^{jl}, \end{aligned} \quad \text{(B4)}$$

where we had to take into account that

$$\gamma^i\gamma^j = 2\eta^{ij} - \gamma^j\gamma^i = -2\delta^{ij} - \gamma^j\gamma^i. \quad \text{(B5)}$$

We remember that at this point of the calculations we are already adopting the new convention (22) to simplify the reading.

1. Nonrelativistic Hamiltonian

After performing all the above cited calculations and adding up the pieces, we end up with

$$\begin{aligned}\mathscr{H}^{III}_{FW} = \beta mc^2 + \beta mc^2 v - \frac{\hbar^4\partial_i^4}{8m^3c^2} - \frac{\beta\hbar^2}{2m}\Big\{&(1+v)\partial_i^2 + 2\varepsilon_i{}^j\partial_i\partial_j + 2\Gamma_i\partial_i + \partial_i\Gamma_i \\ &+ \frac{1}{2}\partial_i(2v\delta_{il} + \varepsilon_l{}^i + \varepsilon_i{}^l + h_{il})\partial_l \\ &+ \frac{1}{4}\partial_i^2 v + \frac{1}{4}\partial_i^2 h + i\epsilon^{ijk}\Sigma^k\partial_i\Gamma_j \\ &+ \frac{i}{2}\epsilon^{ijk}\Sigma^k\partial_i(v\delta_{jl} + \varepsilon_j{}^l + \varepsilon_l{}^j - h_{jl})\partial_l\Big\}. \end{aligned} \quad \text{(B6)}$$

Exploiting the properties in (25) we obtain

$$\begin{aligned}\mathscr{H}^{III}_{FW} = \beta mc^2 + \beta mc^2 v - \frac{\hbar^4\partial_i^4}{8m^3c^2} - \frac{\beta\hbar^2}{2m}\Big\{&(1+v)\partial_i^2 + h_{ij}\partial_i\partial_j + 2\Gamma_i\partial_i + \partial_i\Gamma_i \\ &+ \partial_i(v\delta_{il} + h_{il})\partial_l + \frac{i}{2}\epsilon^{ijk}\Sigma^k\partial_i v\partial_j \\ &+ \frac{1}{4}\partial_i^2(v+h) + i\epsilon^{ijk}\Sigma^k\partial_i\Gamma_j\Big\}, \end{aligned} \quad \text{(B7)}$$

which is already a quite compact form. To simplify it even more, we make use of the choice (27), which in our expansion scheme is always a possible one; in fact, in this case the Γ_j reduce to (28) and therefore,

$$\Gamma_j = \frac{i}{4}\epsilon^{jkl}\Sigma^j\partial_k h_{il}, \quad \text{(B8)}$$

which when replaced into (B7), defining $\tilde{h}_{ij} \equiv h_{ij} + v\delta_{ij}$, directly lead to the final formula (57) in the main text.

APPENDIX C: HAMILTONIAN FOR ERS SPACETIME IN FERMI COORDINATES

Here, we will include the expression for the Fermi Hamiltonian (55) taking the ERS spacetime as our prior spacetime structure. The relevant quantities to use are

$$v^F = \frac{aZ}{c^2} + \frac{GM(\boldsymbol{X}_\perp^2 - 2Z^2)}{2c^2R^3} + \frac{G^2M^2}{2c^4R^4}\left[(5d+6b-5)Z^2 - (2b+3d-2)\boldsymbol{X}_\perp^2\right], \quad \text{(C1)}$$

$$h^F_{ij} = \frac{dGM}{3c^2R^3}\begin{pmatrix} 2Y^2 - Z^2 & -2XY & XZ \\ -2XY & 2X^2 - Z^2 & YZ \\ XZ & YZ & -\boldsymbol{X}_\perp^2 \end{pmatrix}, \quad \text{(C2)}$$

which were calculated by evaluating the Riemann tensor components on the observer's worldline, setting (58). Combining these expressions with Eq. (57) we obtain the NLO Hamiltonian correction

$$H_{\rm NLO} = \frac{GMm}{2R^3}(\boldsymbol{X}_\perp^2 - 2Z^2) + \frac{mG^2M^2}{2c^2R^4}((2-2b-3d)\boldsymbol{X}_\perp^2 - (5-6b-5d)Z^2)$$
$$- \frac{\hbar^2 a}{2mc^2}\partial_Z + \frac{d-3}{6}\frac{\hbar^2 GM}{mc^2R^3}(\boldsymbol{X}_\perp\cdot\boldsymbol{\partial}_\perp - 2Z\partial_Z) + \frac{i\hbar^2}{4mc^2}\left(a-(d+2)\frac{GM}{R^3}Z\right)(\sigma^X\partial_Y - \sigma^Y\partial_X)$$
$$+ \frac{i\hbar^2 GM}{4mc^2R^3}\left((d-1)(\sigma^X Y - \sigma^Y X)\partial_Z + (2d+1)\sigma^Z(Y\partial_X - X\partial_Y)\right)$$
$$- \frac{\hbar^2 a}{2mc^2}Z\partial_i^2 + (2d-3)\frac{\hbar^2 GM}{12mc^2R^3}X_\perp^2\partial_Z^2 - (4d+3)\frac{\hbar^2 GM}{12mc^2R^3}(X^2\partial_Y^2 + Y^2\partial_X^2) - \frac{\hbar^2 GM}{4mc^2R^3}(X^2\partial_X^2 + Y^2\partial_Y^2)$$
$$+ \frac{\hbar^2 GM}{2mc^2R^3}Z^2\partial_Z^2 + (d+3)\frac{\hbar^2 GM}{6mc^2R^3}Z^2\partial_\perp^2 - \frac{d}{3}\frac{\hbar^2 GM}{mc^2R^3}(XZ\partial_X\partial_Z + YZ\partial_Y\partial_Z - 2XY\partial_X\partial_Y) - \frac{\hbar^4\partial_i^4}{8m^3c^2}. \quad \text{(C3)}$$

which is sorted in order of increasing derivatives. Note that the first term in the above expression is just the second order contribution to the expansion of the classical Newtonian $1/r$ potential.

At this point, integrating out the XY dynamics as in (79), we get the final perturbation form to calculate the spectrum's corrections,

$$H_{\rm NLO}^{(Z)} = \left((2d-3)\frac{GM\sigma^2}{6R^3} + \frac{\boldsymbol{k}_\perp^2}{2m^2} + \frac{\hbar^2}{2m^2\sigma^2}\right)\frac{\hbar^2\partial_Z^2}{2mc^2}$$
$$+\left(\boldsymbol{k}_\perp^2 + \frac{\hbar^2}{\sigma^2}\right)\frac{aZ}{2mc^2} - \left(1 + \frac{(d+3)\hbar^2}{6m^2c^2\sigma^2}\left(\boldsymbol{k}_\perp^2 + \frac{\hbar^2}{\sigma^2}\right)\right.$$
$$\left. -(5-6b-5d)\frac{GM}{2c^2R}\right)\frac{GMm}{R^3}Z^2 - \frac{\hbar^4\partial_Z^4}{8m^3c^2}$$
$$- \frac{\hbar^2 a}{2mc^2}(1+Z\partial_Z)\partial_Z + \frac{\hbar^2 GM}{mc^2R^3}\left(Z\partial_Z + \frac{Z^2\partial_Z^2}{2}\right)$$
$$+ \frac{\hbar}{4mc^2}\left((d+2)\frac{GMZ}{R^3} - a\right)(\sigma^X k^Y - \sigma^Y k^X), \quad \text{(C4)}$$

from which, being interested in transition energies, we already removed the Z-independent terms, since their effect would get cancelled in the differences between energy levels.

To not include here even more large formulas, we avoid to write the expressions for the single matrix components $W_{\alpha\beta}^n$ defined in (81). Their calculation is, in fact, trivial starting from (C4) and exploiting the relations (87).

APPENDIX D: INTEGRALS OF AIRY FUNCTIONS

The problem we want to discuss in this appendix is the one related with calculating the analytic form of integrals $I[O] \equiv \langle O\rangle_n$ of the type

$$I[O(Z)] = \int_0^\infty dZ\, \mathrm{Ai}\left(\frac{Z-Z_n}{Z_0}\right) O(Z)\mathrm{Ai}\left(\frac{Z-Z_n}{Z_0}\right), \quad \text{(D1)}$$

where $O(Z)$ represents here a generic operator depending on Z and acting on the second Airy function. Note that we will always consider $Z=0$ as a starting point for the integration since we are assuming no "floor-leakage", which would be equivalent to the addition of a Heaviside function $\theta(Z)$ in the wave functions.

We follow the strategy outlined in [65]. First of all we make the change of variable $\zeta \equiv \frac{Z-Z_n}{Z_0}$ for integral (D1),

$$I[O] = Z_0 \int_{-\frac{Z_n}{Z_0}}^\infty d\zeta\, \mathrm{Ai}(\zeta) O(Z_n + Z_0\zeta)\mathrm{Ai}(\zeta). \quad \text{(D2)}$$

At this point, the strategy is to introduce an infinitesimal shift λ in the argument of the second Airy function, that depending on the form of O will allow to easily realize the integral, so that at the end we can take the $\lambda \to 0$ again,

$$I_\lambda[Z\partial_Z^2] = Z_0 \int_{-\frac{Z_n}{Z_0}}^\infty d\zeta \mathrm{Ai}(\zeta) O(Z_n + Z_0\zeta)\mathrm{Ai}(\zeta-\lambda). \quad \text{(D3)}$$

As an example, let us consider the case of $O = Z\partial_Z^2$. Observing that $\partial_Z = \frac{1}{Z_0}\partial_\zeta$ we have

$$I_\lambda[Z\partial_Z^2] = \frac{1}{Z_0}\int_{-\frac{Z_n}{Z_0}}^\infty d\zeta \mathrm{Ai}(\zeta)(Z_n + Z_0\zeta)\partial_\zeta^2 \mathrm{Ai}(\zeta-\lambda). \quad \text{(D4)}$$

Since $\partial_{\zeta-\lambda} = -\partial_\lambda = \partial_\zeta$, we have

$$I_\lambda[Z\partial_Z^2] = \frac{1}{Z_0}\partial_\lambda^2 \int_{-\frac{Z_n}{Z_0}}^\infty d\zeta \mathrm{Ai}(\zeta)(Z_n + Z_0\zeta)\mathrm{Ai}(\zeta-\lambda), \quad \text{(D5)}$$

which reduces to

$$I_\lambda[Z\partial_Z^2] = \frac{Z_n}{Z_0}\partial_\lambda^2 \int_{-\frac{Z_n}{Z_0}}^\infty d\zeta \mathrm{Ai}(\zeta)\mathrm{Ai}(\zeta-\lambda)$$
$$+ \partial_\lambda^2 \int_{-\frac{Z_n}{Z_0}}^\infty d\zeta \mathrm{Ai}(\zeta)\zeta\mathrm{Ai}(\zeta-\lambda)$$
$$= \partial_\lambda^2\left(\left\{\frac{Z_n}{Z_0}\right\}_\lambda + \{\zeta\}_\lambda\right), \quad \text{(D6)}$$

where we introduce the general notation

$$\{P(\zeta)\}_\lambda = \int_{-\frac{Z_n}{Z_0}}^{\infty} d\zeta \mathrm{Ai}(\zeta) P(\zeta) \mathrm{Ai}(\zeta - \lambda). \tag{D7}$$

For these shifted ζ–integrals we can use formulas (A31) and (A37) from [65], which remembering that $\mathrm{Ai}(-\frac{Z_n}{Z_0}) = \mathrm{Ai}(\infty) = \mathrm{Ai}'(\infty) = 0$, lead to

$$\begin{aligned}
\{1\}_\lambda &= \frac{1}{\lambda}\mathrm{Ai}'\left(-\frac{Z_n}{Z_0}\right)\mathrm{Ai}\left(-\frac{Z_n}{Z_0} - \lambda\right), \\
\{\zeta\}_\lambda &= -\frac{2 + \lambda^2(-\frac{Z_n}{Z_0})}{\lambda^3}\mathrm{Ai}'\left(-\frac{Z_n}{Z_0}\right)\mathrm{Ai}\left(-\frac{Z_n}{Z_0} - \lambda\right) \\
&\quad - \frac{2}{\lambda^2}\mathrm{Ai}'\left(-\frac{Z_n}{Z_0}\right)\mathrm{Ai}'\left(-\frac{Z_n}{Z_0} - \lambda\right).
\end{aligned} \tag{D8}$$

After replacing the above expressions back into (D6), the following steps are to take the λ derivatives and carefully expand all the shifted Ai functions for small values of λ, as shown in (A8) of [65], where the authors also take in consideration that Airy's functions satisfy,

$$\partial_\zeta^2 \mathrm{Ai}(\zeta) = \zeta \mathrm{Ai}(\zeta). \tag{D9}$$

At the end of the day, if one does things correctly, it should end up with an expression for which it is easy to take the $\lambda \to 0$ limit, obtaining

$$\begin{aligned}
\partial_\lambda^2\{1\}_\lambda|_{\lambda\to 0} &= \frac{1}{3}\left(-\frac{Z_n}{Z_0}\right)\left(\mathrm{Ai}'\left(-\frac{Z_n}{Z_0}\right)\right)^2, \\
\partial_\lambda^2\{\zeta\}_\lambda|_{\lambda\to 0} &= \frac{1}{5}\left(-\frac{Z_n}{Z_0}\right)^2\left(\mathrm{Ai}'\left(-\frac{Z_n}{Z_0}\right)\right)^2.
\end{aligned} \tag{D10}$$

Putting all together, we finally find that,

$$\begin{aligned}
I_{\lambda\to 0}(Z\partial_Z^2) &= \frac{1}{5}\left(\frac{Z_n}{Z_0}\right)^2\left(\mathrm{Ai}'\left(-\frac{Z_n}{Z_0}\right)\right)^2 \\
&\quad - \frac{1}{3}\left(\frac{Z_n}{Z_0}\right)^2\left(\mathrm{Ai}'\left(-\frac{Z_n}{Z_0}\right)\right)^2 \\
&= -\frac{2}{15}\left(\frac{Z_n}{Z_0}\right)^2\left(\mathrm{Ai}'\left(-\frac{Z_n}{Z_0}\right)\right)^2,
\end{aligned} \tag{D11}$$

from which we can derive directly

$$\langle Z\partial_Z^2\rangle_n = C_n^2 I_{\lambda\to 0}[Z\partial_Z^2] = -\frac{2}{15}\frac{Z_n^2}{Z_0^3}. \tag{D12}$$

All the other mean values $\langle\rangle_n$ in the main text can be obtained with an analogous procedure.

[1] A. Brodutch, A. Gilchrist, T. Guff, A. R. Smith, and D. R. Terno, Post-newtonian gravitational effects in optical interferometry, Phys. Rev. D **91**, 064041 (2015).

[2] H. Müller, S.-w. Chiow, S. Herrmann, S. Chu, and K.-Y. Chung, Atom-interferometry tests of the isotropy of post-Newtonian gravity, Phys. Rev. Lett. **100**, 031101 (2008).

[3] C. Ufrecht, F. Di Pumpo, A. Friedrich, A. Roura, C. Schubert, D. Schlippert, E. M. Rasel, W. P. Schleich, and E. Giese, Atom-interferometric test of the universality of gravitational redshift and free fall, Phys. Rev. Res. **2**, 043240 (2020).

[4] G. M. Tino, Testing gravity with cold atom interferometry: Results and prospects, Quantum Sci. Technol. **6**, 024014 (2021).

[5] O. Boada, A. Celi, J. I. Latorre, and M. Lewenstein, Dirac equation for cold atoms in artificial curved spacetimes, New J. Phys. **13**, 035002 (2011).

[6] C. D. Panda, M. J. Tao, M. Ceja, and H. Müller, Measuring gravity by holding atoms, arXiv:2310.01344.

[7] G. A. Sekh, Bouncing dynamics of Bose-Einstein condensates under the effects of gravity, Phys. Lett. A **381**, 852 (2017).

[8] S. Ulbricht, R. Müller, and A. Surzhykov, Gravitational effects on geonium and free electron g_s-factor measurements in a penning trap, Phys. Rev. D **100**, 064029 (2019).

[9] A. Ito, Inertial and gravitational effects on a geonium atom, Classical Quantum Gravity **38**, 195015 (2021).

[10] G. Tino and F. Vetrano, Is it possible to detect gravitational waves with atom interferometers?, Classical Quantum Gravity **24**, 2167 (2007).

[11] V. K. Ignatovich, *The Physics of Ultracold Neutrons* (Oxford University Press, 1986).

[12] H. Abele, G. Cronenberg, P. Geltenbort, T. Jenke, T. Lins, and H. Saul, qbounce, the quantum bouncing ball experiment, Phys. Procedia **17**, 4 (2011).

[13] T. Jenke, D. Stadler, H. Abele, and P. Geltenbort, Q-bounce—experiments with quantum bouncing ultracold neutrons, Nucl. Instrum. Methods Phys. Res., Sect. A **611**, 318 (2009).

[14] S. Baessler, M. Beau, M. Kreuz, V. N. Kurlov, V. V. Nesvizhevsky, G. Pignol, K. V. Protasov, F. Vezzu, and A. Y. Voronin, The granit spectrometer, C.R. Phys. **12**, 707 (2011).

[15] B. Clément, S. Baeßler, V. V. Nesvizhevsky, E. Perry, G. Pignol, J. A. Pioquinto, K. V. Protasov, D. Rebreyend, D. Roulier, L. Shen *et al.*, Manipulation of gravitational quantum states of a bouncing neutron with the granit spectrometer, arXiv:2205.11130.

[16] V. V. Nesvizhevsky, Near-surface quantum states of neutrons in the gravitational and centrifugal potentials, Phys. Usp. **53**, 645 (2010).

[17] V. V. Nesvizhevsky, H. Börner, A. Gagarski, A. Petoukhov, G. Petrov, H. Abele, S. Baeßler, G. Divkovic, F. Rueß, T. Stöferle *et al.*, Measurement of quantum states of neutrons in the earth's gravitational field, Phys. Rev. D **67**, 102002 (2003).

[18] G. Cronenberg, H. Filter, M. Thalhammer, T. Jenke, H. Abele, and P. Geltenbort, A gravity of earth measurement with a qbounce experiment, *Proc. Sci.*, EPS-HEP2015 (2015) 408 [arXiv:1512.09134].

[19] S. Sponar, R. I. Sedmik, M. Pitschmann, H. Abele, and Y. Hasegawa, Tests of fundamental quantum mechanics and dark interactions with low-energy neutrons, Nat. Rev. Phys. **3**, 309 (2021).

[20] A. Ivanov, M. Wellenzohn, and H. Abele, Probing of violation of lorentz invariance by ultracold neutrons in the standard model extension, Phys. Lett. B **797**, 134819 (2019).

[21] C. Escobar, A. Martín-Ruiz, A. Escobar-Ruiz, and R. Linares, Testing the scalar sector of the standard-model extension with neutron gravity experiments, Eur. Phys. J. Plus **137**, 1186 (2022).

[22] V. A. Kostelecký and Z. Li, Searches for beyond-Riemann gravity, Phys. Rev. D **104**, 044054 (2021).

[23] A. Ivanov, M. Wellenzohn, and H. Abele, Quantum gravitational states of ultracold neutrons as a tool for probing of beyond-Riemann gravity, Phys. Lett. B **822**, 136640 (2021).

[24] A. Ivanov and M. Wellenzohn, Nonrelativistic approximation of the Dirac equation for slow fermions coupled to the chameleon and torsion fields in the gravitational field of the earth, Phys. Rev. D **92**, 065006 (2015).

[25] A. J. Schimmoller, G. McCaul, H. Abele, and D. I. Bondar, Decoherence-free entropic gravity: Model and experimental tests, Phys. Rev. Res. **3**, 033065 (2021).

[26] E. J. Sung, A. G. Campos, H. Abele, and D. I. Bondar, Decoherence-free entropic gravity for a Dirac fermion, Phys. Rev. D **108**, 104036 (2023).

[27] A. Kobakhidze, Gravity is not an entropic force, Phys. Rev. D **83**, 021502 (2011).

[28] H. Fischer, C. Käding, H. Lemmel, S. Sponar, and M. Pitschmann, Search for dark energy with neutron interferometry, Prog. Theor. Exp. Phys. **2024**, 023E02 (2024).

[29] M. Suda, M. Faber, J. Bosina, T. Jenke, C. Käding, J. Micko, M. Pitschmann, and H. Abele, Spectra of neutron wave functions in Earth's gravitational field, Z. Naturforsch. A **77**, 875 (2022).

[30] Y. N. Obukhov, On gravitational interaction of fermions, Fortschr. Phys. **50**, 711 (2002).

[31] S. Chandrasekhar, The post-Newtonian equations of hydrodynamics in general relativity, Astrophys. J. **142**, 1488 (1965).

[32] C. De Oliveira and J. Tiomno, Representations of Dirac equation in general relativity, Il Nuovo Cimento (1955–1965) **24**, 672 (1962).

[33] M. Pollock, On the Dirac equation in curved space-time, Acta Phys. Pol. B **41**, 1827 (2010).

[34] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, Inc., New York, 1972).

[35] M. Arminjon, Post-Newtonian equation for the energy levels of a Dirac particle in a static metric, Phys. Rev. D **74**, 065017 (2006).

[36] L. Parker, One-electron atom as a probe of spacetime curvature, Phys. Rev. D **22**, 1922 (1980).

[37] X. Huang and L. Parker, Hermiticity of the Dirac Hamiltonian in curved spacetime, Phys. Rev. D **79**, 024020 (2009).

[38] A. N. Ivanov and M. Pitschmann, Nonrelativistic approximation of the Dirac equation for slow fermions in static metric spacetimes, Phys. Rev. D **90**, 045040 (2014).

[39] R. A. Nelson, Post-Newtonian approximation for an accelerated, rotating frame of reference, Gen. Relativ. Gravit. **22**, 431 (1990).

[40] L. L. Foldy and S. A. Wouthuysen, On the Dirac theory of spin $1/2$ particles and its non-relativistic limit, Phys. Rev. **78**, 29 (1950).

[41] U. D. Jentschura and J. Noble, Foldy-Wouthuysen transformation, scalar potentials and gravity, J. Phys. A **47**, 045402 (2014).

[42] F. W. Hehl and W.-T. Ni, Inertial effects of a Dirac particle, Phys. Rev. D **42**, 2045 (1990).

[43] A. J. Silenko and O. V. Teryaev, Semiclassical limit for Dirac particles interacting with a gravitational field, Phys. Rev. D **71**, 064016 (2005).

[44] U. D. Jentschura and J. Noble, Nonrelativistic limit of the Dirac-Schwarzschild Hamiltonian: Gravitational Zitterbewegung and gravitational spin-orbit coupling, Phys. Rev. A **88**, 022121 (2013).

[45] E. Fischbach, B. S. Freeman, and W.-K. Cheng, General-relativistic effects in hydrogenic systems, Phys. Rev. D **23**, 2157 (1981).

[46] D. Giulini, A. Großardt, and P. K. Schwartz, Coupling quantum matter and gravity, in *Modified and Quantum Gravity: From Theory to Experimental Searches on All Scales* (Springer, New York, 2023), pp. 491–550.

[47] C. Lämmerzahl, A Hamilton operator for quantum optics in gravitational fields, Phys. Lett. A **203**, 12 (1995).

[48] E. Poisson and C. M. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic* (Cambridge University Press, Cambridge, England, 2014).

[49] L. Ó Raifeartaigh, Fermi coordinates, in *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences* (JSTOR, 1957), Vol. 59, pp. 15–24.

[50] F. Manasse and C. W. Misner, Fermi normal coordinates and some basic concepts in differential geometry, J. Math. Phys. (N.Y.) **4**, 735 (1963).

[51] J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing Co., Amsterdam, 1960).

[52] W.-Q. Li and W.-T. Ni, Coupled inertial and gravitational effects in the proper reference frame of an accelerated, rotating observer, J. Math. Phys. (N.Y.) **20**, 1473 (1979).

[53] P. M. Alsing, G. J. Stephenson Jr, and P. Kilian, Spin-induced non-geodesic motion, gyroscopic precession, Wigner rotation and EPR correlations of massive spin $1/2$ particles in a gravitational field, arXiv:0902.1396.

[54] J. Maluf and F. Faria, On the construction of Fermi-Walker transported frames, Ann. Phys. (Berlin) **520**, 326 (2008).

[55] D. Klein and P. Collas, General transformation formulas for Fermi-Walker coordinates, Classical Quantum Gravity **25**, 145019 (2008).

[56] J. Audretsch and K.-P. Marzlin, Ramsey fringes in atomic interferometry: Measurability of the influence of space-time curvature, Phys. Rev. A **50**, 2080 (1994).

[57] A. Alibabaei, P. K. Schwartz, and D. Giulini, Geometric post-Newtonian description of massive spin-half particles in curved spacetime, Classical Quantum Gravity **40**, 235014 (2023).

[58] K.-P. Marzlin, Fermi coordinates for weak gravitational fields, Phys. Rev. D **50**, 888 (1994).

[59] K. S. Thorne, J. A. Wheeler, and C. W. Misner, *Gravitation* (Freeman San Francisco, CA, 2000).

[60] S. M. Carroll, *Spacetime and Geometry* (Cambridge University Press, Cambridge, England, 2019).

[61] F. Dahia and P. F. da Silva, Static observers in curved spaces and non-inertial frames in Minkowski spacetime, Gen. Relativ. Gravit. **43**, 269 (2011).

[62] R. I. P. Sedmik *et al.*, Proof of principle for ramsey-type gravity resonance spectroscopy with qBounce, EPJ Web Conf. **219**, 05004 (2019).

[63] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, Black hole binary dynamics from the double copy and effective theory, J. High Energy Phys. 10 (2019) 206.

[64] A. Martín-Ruiz and C. Escobar, Testing Lorentz and *CPT* invariance with ultracold neutrons, Phys. Rev. D **97**, 095039 (2018).

[65] M. Pitschmann and H. Abele, Schrödinger equation for a non-relativistic particle in a gravitational field confined by two vibrating mirrors, arXiv:1912.12236.

[66] C. Käding, M. Pitschmann, and H. Lemmel, Green's function analysis of the neutron lloyd interferometer, Z. Naturforsch. **78**, 651 (2023).

[67] J. Micko, J. Bosina, S. Cranganore, T. Jenke, M. Pitschmann, S. Roccia, R. Sedmik, and H. Abele, qbounce: Systematic shifts of transition frequencies of gravitational states of ultra-cold neutrons using ramsey gravity resonance spectroscopy, arXiv:2301.08583.

[68] Y. N. Obukhov, A. J. Silenko, and O. V. Teryaev, General treatment of quantum and classical spinning particles in external fields, Phys. Rev. D **96**, 105005 (2017).

[69] M. A. Oancea and A. Kumar, Semiclassical analysis of Dirac fields on curved spacetime, Phys. Rev. D **107**, 044029 (2023).