Symmetry charges on reduced phase space and asymptotic flatness

Muxin Han[1,2,*] Zichang Huang[3] and Hongwei Tan[1,†]

[1]*Department of Physics, Florida Atlantic University,*
777 Glades Road, Boca Raton, Florida 33431-0991, USA
[2]*Institut für Quantengravitation, Universität Erlangen-Nürnberg,*
Staudtstr. 7/B2, 91058 Erlangen, Germany
[3]*College of Science, University of Shanghai for Science and Technology,*
Shanghai 200093, People's Republic of China

This paper studies the reduced phase space formulation (relational formalism) of gravity coupling to the Brown-Kuchař dust for asymptotic flat spacetimes. A set of boundary conditions for the asymptotic flatness are formulated for Dirac observables on the reduced phase space. The physical Hamiltonian generates the time translation of the dust clock. We compute the boundary term of the physical Hamiltonian, which is identical to the Arnowitt-Deser-Misner mass. We construct a set of the symmetry charges on the reduced phase space, which are conserved by the physical Hamiltonian evolution. The symmetry charges generate transformations preserving the asymptotically flat boundary condition. Under the reduced-phase-space Poisson bracket, the symmetry charges form an infinite dimensional Lie algebra $\mathcal{A}_G$ after adding a central charge. A suitable quotient of $\mathcal{A}_G$ is analogous to the Bondi-Metzner-Sachs algebra at spatial infinity by Henneaux and Troessaert.

I. INTRODUCTION

The canonical formulation of gravity is a theory of constraints [1,2]. The Hamiltonian of four-dimensional gravity consists of four constraints: one Hamiltonian constraint and three components of diffeomorphism constraint. All hypersurface deformations generated by these constraints are gauge transformations. Constructing gauge invariant quantities is important for understanding the physics of gravitational field at both classical and quantum levels [3–5].

An interesting approach of constructing gauge invariant observables is the relational formalism [6–25]. The idea is to couple gravity with certain matter fields, e.g., pressureless dust fields or scalar fields. In this work, we consider the Brown-Kuchař (BK) dust as the matter and refer to this formalism as BK formalism [18,26]. In this formalism, BK dust fields play the role of rulers and clocks that provide a physical reference frame at every spacetime point, and they relate to the gauge fixing scheme that allows us to construct a reduced phase space of the gravity-dust system. The gauge invariant Dirac observables defined on the reduced phase space are canonical fields evaluated in the physical reference frame made by the dust fields. Importantly, when formulating the

dynamics on the reduced phase space in terms of Dirac observables, the gravity-dust system in the BK formalism becomes free of any constraint, since the Hamiltonian and diffeomorphism constraints are resolved on the reduced phase space. As a result, the formalism provides a physical Hamiltonian $\mathbf{H}_{\text{phys}} := \int d^3\sigma H_{\text{phys}}$, which generates the physical time evolution of the Dirac observables. $\vec{\sigma}$ and τ are the spatial and time coordinates in the physical reference frame. There are infinitely many conserved charges $H_{\text{phys}}(\vec{\sigma})$ and $C_j(\vec{\sigma})$ descended from the hypersurface deformation. $H_{\text{phys}}(\vec{\sigma})$ and $C_j(\vec{\sigma})$ relates to the time translation and spatial diffeomorphism in the space of $(\tau, \vec{\sigma})$.

The purpose of this work is to apply the relational formalism to spacetimes with boundary at infinity. In particular we focus on the situation of asymptotically flat spacetimes. We propose a set of asymptotic boundary conditions for the asymptotically flatness in the BK formalism. These boundary conditions, discussed in Sec. III, are formulated for Dirac observables on the reduced phase space, and thus they implement the asymptotically flatness in the gauge-invariant manner. In addition, the compatibility with the asymptotically flatness requires the BK dust density to have a falloff behavior of $O(r^{-4} \log r)$ as the radius $r \to \infty$.

As a technical aspect, the BK dust turns out to either be incompatible to the usual falloff behavior and the parity condition of the canonical fields in the literature, or it rules

---
[*]hanm@fau.edu
[†]Corresponding author: htan2018@fau.edu

out many interesting boundary charges (e.g., in [27–29]). As the resolution, some additional $r^{-n} \log(r)$-terms are added to the falloff behavior, and the parity condition is relaxed and replaced by other suitable boundary conditions. In this case, a counterterm has to be added to the symplectic form to remove the divergence. These treatments are partially inspired by [30,31].

A key development is to construct the boundary-preserving symmetry charge on the reduced phase space:

$$G(\xi, \vec{\xi}) = \int \mathrm{d}^3\sigma(\xi H_{\text{phys}} + \xi^j C_j) + \mathcal{B}(\xi, \vec{\xi}). \quad (1.1)$$

The smearing fields $\xi, \vec{\xi}$ satisfy certain boundary conditions relating to a subset of the traditional Poincaré transformations and the supertranslations. Unlike the usual situations in general relativity, the bulk terms of the symmetry charge do not vanish on shell, and they generate physical symmetries in the bulk of the dust space (the space of $\vec{\sigma}$). The symmetry charge contains the boundary term $\mathcal{B}(\xi, \vec{\xi})$, which we call the boundary charge. $\mathcal{B}(\xi, \vec{\xi})$ is determined by the compatibility to the asymptotically flat boundary condition and making the variation $\delta G(\xi, \vec{\xi})$ well defined on the reduced phase space. The boundary charge $\mathcal{B}(\xi, \vec{\xi})$ is computed explicitly in Sec. IV.

We show that the boundary-preserving symmetry charge $G(\xi, \vec{\xi})$ enjoys the following important properties [the first and second properties are what we mean by $G(\xi, \vec{\xi})$ being symmetry and boundary-preserving]:

(1) It commutes with the total physical Hamiltonian $\mathbf{H}_{\text{phys}} + \mathbf{H}_{\text{bdy}}$, where $\mathbf{H}_{\text{bdy}}$ is the boundary term relating to the Arnowitt-Deser-Misner (ADM) mass.

(2) The Hamiltonian flow generated by $G(\xi, \vec{\xi})$ preserves the boundary conditions.

(3) With the Poisson bracket on the reduced phase space, the set of charges $G(\xi, \vec{\xi})$ for all $\xi, \vec{\xi}$ form an infinite-dimensional Lie algebra up to a central charge $\mathcal{C}$, namely

$$\{G(\xi_1, \vec{\xi}_1), G(\xi_2, \vec{\xi}_2)\} = G(\hat{\xi}, \hat{\vec{\xi}}) + \mathcal{C}(\hat{\xi}, \hat{\vec{\xi}}), \quad (1.2)$$

where $\hat{\xi} = \xi_1^i D_i \xi_2 - \xi_2^i D_i \xi_1$ and $\hat{\xi}^j = [\vec{\xi}_1, \vec{\xi}_2]^j$. In other words, the boundary-preserving symmetry charge G and the central charge $\mathcal{C}$ form a closed Lie algebra. The derivation of the algebra is given in Sec. V.

We denote by $\mathcal{A}_G$ the algebra of the boundary-preserving symmetry charges $G(\xi, \vec{\xi})$, and we would like to compare $\mathcal{A}_G$ to the original Bondi-Metzner-Sachs (BMS) algebra, $\mathcal{A}_{\text{BMS}}$ [32–35]. The original BMS charges are defined as boundary charges that generate symmetry transformations on the null infinity of an asymptotically flat spacetime.

These symmetry transformations include the traditional Poincaré transformations and the supertranslations. The supertranslation charges form an infinitesimal Abelian subalgebra of $\mathcal{A}_{\text{BMS}}$. In contrast, the charges $G(\xi, \vec{\xi})$ defined in this paper incorporate both bulk terms and boundary terms and are Dirac observables. The bulk terms in the charges do not correspond to any constraint and thus nonvanishing. Indeed, $\mathcal{A}_G$ contains an ideal $\tilde{\mathcal{A}}_G$ of the symmetries in the bulk of the dust space that are not extended to the asymptotic boundary. The quotient Lie algebra $\hat{\mathcal{A}}_G = \mathcal{A}_G / \tilde{\mathcal{A}}_G$ is analog to $\mathcal{A}_{\text{BMS}}$. There is one issue to emphasize: We release the parity conditions of the canonical variables as mentioned previously, which may result in the divergency at the null infinity and prevent one from constructing the BMS charges there [36–38]. Therefore, in this paper we only compare the algebraic structures between these two algebras, setting aside the associated physical interpretations.

As mentioned earlier, the supertranslations themselves form an Abelian subalgebra within the original BMS algebra. Correspondingly, the supertranslations in $\hat{\mathcal{A}}_G$ along the physical time and radial directions form an Abelian subalgebra when the central charge vanishes. If we impose appropriate parity conditions on the parameters of the supertranslations, then the central extension vanishes, and we obtain a subalgebra of $\hat{\mathcal{A}}_G$, which is analogous to the original BMS subalgebra of supertranslations. In our work, we do not generally require the parameters of the supertranslations to satisfy parity conditions, and we introduce two additional degrees of freedom contributed by the angular components of the supertranslations in the spatial direction. Therefore, $\hat{\mathcal{A}}_G$ represents a generalized algebraic structure of $\mathcal{A}_{\text{BMS}}$ as far as the supertranslations are concerned.

The asymptotic spatial rotations in $\hat{\mathcal{A}}_G$ form an SO(3) algebra thus is analogous to the corresponding subalgebra in BMS. The commutator between the supertranslations in the time direction and in the spatial directions involves the central charge of the algebra. In this work, we only consider the isometric spatial rotations generated by the Killing vectors on S^2. It would be interesting to generalize the analysis to the conformal transformations on S^2. But we leave this aspect to future research.

The symmetry algebra $\hat{\mathcal{A}}_G$ does not take account of the asymptotic boost transformation, since it is difficult to make the boost transformation preserving the boundary condition. The detailed discussion is given in Sec. VI.

$\hat{\mathcal{A}}_G$ closely relates to the BMS group at spatial infinity proposed in [28]. In particular, certain restriction of $\xi, \vec{\xi}$ selects a subalgebra of $\hat{\mathcal{A}}_G$ that recovers the algebraic structures of the BMS algebra at spatial infinity in [28] with vanishing boost generator, as shown in Sec. V B.

The relational formalism enables the gravity-dust system to be formulated on the reduced phase space and free of constraints, and the dynamics are expressed similarly to common Hamiltonian dynamical systems. Intriguingly, the symmetry of the gravity-dust system can be formulated similarly to symmetries in typical Hamiltonian dynamical systems. This is illustrated in this study for asymptotically flat spacetimes: All symmetry charges $G(\xi, \vec{\xi})$ are obtained as phase space functions that Poisson commute with the physical Hamiltonian. The symmetry charge, when expressed in terms of Dirac observables, includes both the bulk and boundary contributions. We are concentrating on asymptotically flat spacetimes at this stage because our aim is to analyze the similarity between the algebra of the symmetry charges obtained in this case and the familiar result on the BMS algebra [32–35]. However, the same symmetry analysis can be carried out for other boundary conditions, or even spacetimes without a boundary.

The relational formalism is an approach for resolving the problem of time in classical and quantum gravity. On the reduced phase space, the physical Hamiltonian generates the physical time translation, whose quantization makes sense the unitarity of quantum gravity [17,39,40]. In contrast, in the usual formulation of gravity, the time translation and unitarity are meaningful only on the boundary of the spacetime, and this is the idea behind the holographic duality. It is worth exploring the relation between the physical time evolution formulated on the reduce phase space and the time evolution of the holographic boundary dynamics. This work may present an initial effort towards understanding this relation, by comparing $\mathcal{A}_G$ to the BMS algebra on the boundary.

This paper is organized as follows: Sec. II reviews the relational formalism. Section III introduces the asymptotic boundary conditions and defines the finite symplectic structure with a counterterm. Section IV constructs the boundary-preserving symmetry charges. Section V compute the Poisson bracket between a pair of the boundary-preserving symmetry charges. Section VI demonstrates that the generator of the boost term does not satisfy the definition of the boundary-preserving generator. Section VII summarizes the results and discusses a few future perspectives.

II. A REVIEW OF THE BROWN-KUCHAŘ FORMALISM

A. Lagrangian formalism

The BK formalism is a realization of relational formalism with the BK dust field consisting of four scalars $T, S^{j=1,2,3}$. The total action in the BK formalism is

$$S = S_{\text{EH}} + S_{\text{dust}}. \qquad (2.1)$$

S_{EH} is the Einstein-Hilbert action.[1] The BK dust action S_{dust} depends on the dust fields T, S^j and Lagrangian multipliers ρ, W_j [26]:

$$S_{\text{dust}} = -\frac{1}{2} \int_M \mathrm{d}^4 x \sqrt{|\det(g)|} \rho[g^{\mu\nu} U_\mu U_\nu + 1], \qquad (2.2)$$

with

$$U_\mu = -\partial_\mu T + W_j \partial_\mu S^j. \qquad (2.3)$$

The equation of motion $\frac{\delta S}{\delta \rho} = 0$ yields

$$g^{\mu\nu} U_\mu U_\nu = -1. \qquad (2.4)$$

Thus, U^μ is the 4-velocity of the dust. Another equation of motion $\frac{\delta S}{\delta g_{\mu\nu}} = 0$ relates the Einstein tensor to the energy-stress tensor of the dust field:

$$T_{\mu\nu}^{\text{dust}} = \rho U_\mu U_\nu, \qquad (2.5)$$

which indicates that the BK dust is a pressure-less perfect fluid. ρ is interpreted as the dust density, and

$$\rho = \begin{cases} > 0 & \text{physical dust} \\ < 0 & \text{phantom dust} \end{cases}. \qquad (2.6)$$

Note that the case of phantom dust can still fulfill the energy condition when coupling to additional matter fields [18]. Our following discussion applies to both cases of physical and phantom dusts.

Other equations of motion $\frac{\delta S}{\delta T} = 0$, $\frac{\delta S}{\delta S^j} = 0$, and $\frac{\delta S}{\delta W_j} = 0$ provide

$$U^\nu \nabla_\nu U_\mu = 0, \qquad U^\nu \nabla_\nu T = 1, \qquad U^\nu \nabla_\nu S_j = 0. \quad (2.7)$$

The first two equations in (2.7) indicate that the integral line of U^μ is a timelike geodesic and T is its line-length parameter. The last one indicates that S_j are constants along the integral line of U^μ. The dust field naturally introduces a reference frame to the spacetime. On this spacetime, the dust fields T and $\vec{S}$ can be viewed as providing a coordinate system where T and $\vec{S}$ are the time and space coordinates, respectively.

B. Hamiltonian analysis

Following the ADM formalism, a spacetime is decomposed into $\Sigma \times \mathbb{R}$, where Σ is the three-dimensional spatial slice. We denote the 3-metric on Σ by q_{ab} and its conjugate momentum by p^{ab}.[2] The Hamiltonian analysis provides four first-class constraints

---

[1]In our convention, the constant $\kappa = 16\pi G$ is set to be 1.
[2]The latin letters a, b are the indices of Σ.

$$c^{\mathrm{tot}} = c^{\mathrm{geo}} + c^{\mathrm{dust}}, \tag{2.8}$$

$$c_a^{\mathrm{tot}} = c_a^{\mathrm{geo}} + c_a^{\mathrm{dust}}, \tag{2.9}$$

with

$$c^{\mathrm{geo}} = \frac{1}{\sqrt{\det(q)}} \left[q_{ac} q_{bd} - \frac{1}{2} q_{ab} q_{cd} \right] p^{ab} p^{cd} \\ - \sqrt{\det(q)} {}^{(3)}R_0, \tag{2.10}$$

$$c^{\mathrm{dust}} = \frac{1}{2} \left[\frac{P^2/\rho}{\sqrt{\det(q)}} + \sqrt{\det(q)} \rho (q^{ab} U_a U_b + 1) \right], \tag{2.11}$$

$$c_a^{\mathrm{geo}} = -2 q_{ac} D_b p^{bc}, \tag{2.12}$$

$$c_a^{\mathrm{dust}} = P[T_{,a} - W_j S_{,a}^j], \tag{2.13}$$

where P and P_j are conjugate momenta of T and S^j, respectively, and ${}^{(3)}R_0$ is the Ricci scalar of q_{ab}. There are eight second-class constraints. Four of them are

$$W_j = -P_j/P, \tag{2.14}$$

$$\rho^2 = \frac{P^2}{\det(q)} (q^{ab} U_a U_b + 1)^{-1}. \tag{2.15}$$

Equation (2.15) gives

$$\rho = \epsilon \frac{P}{\sqrt{\det(q)}} (q^{ab} U_a U_b + 1)^{-1/2}, \qquad \epsilon = \pm 1. \tag{2.16}$$

The future-pointing timelike U^μ fixes $\epsilon = 1$ [18]. By (2.8), $c^{\mathrm{tot}} = 0$ yields

$$c^{\mathrm{geo}} = -\sqrt{\det(q)} \rho (q^{ab} U_a U_b + 1). \tag{2.17}$$

With (2.3), (2.13), and $c_a^{\mathrm{tot}} = 0$, we get

$$U_a = \frac{c_a^{\mathrm{geo}}}{P}. \tag{2.18}$$

These formulas are useful in analyzing the asymptotic behaviors of the dust density ρ, which we will see in the next section. Other second-class constraints are less important for our discussion, so we refer the reader to [18] for further details.

In this section, we do not introduce the boundary condition and boundary terms to the constraints or Dirac observables. We will introduce the asymptotically flat boundary condition directly at the level of reduced phase space, as to be discussed in Sec. III.

C. Dirac observables and the bulk symmetry charges

BK formalism defines the gauge invariant quantities—Dirac observables through the deparametrization process [18]. First, the first-class constraints (2.8) and (2.9) are solved by

$$P = \mathrm{sgn}(P)h, \qquad P_j = -h_j, \tag{2.19}$$

with

$$h = \sqrt{(c^{\mathrm{geo}})^2 - q^{ab} c_a^{\mathrm{geo}} c_b^{\mathrm{geo}}}, \qquad h_j = S_j^a(-h T_{,a} + c_a^{\mathrm{geo}}), \tag{2.20}$$

and S_j^a is the inverse matrix of $\partial_a S^j$, and h does not depend on (T, P).

Second, the smeared constraint is

$$K_\beta \equiv \int_\Sigma \mathrm{d}^3 x [\beta(\vec{x}) c^{\mathrm{tot}}(\vec{x}) + \beta^j(\vec{x}) c_j^{\mathrm{tot}}(\vec{x})]. \tag{2.21}$$

The Dirac observable from a generic phase space function f is constructed by the exponential map generated by K_β

$$O_f[\tau, \sigma] \equiv \left[\sum_{n=0}^\infty \frac{1}{n!} \{f, K_\beta\}_{(n)} \right]_{\substack{\beta \to \tau - T \\ \beta^j \to \sigma^j - S^j}}. \tag{2.22}$$

Here $\{f, K_\beta\}_{(n)}$ means doing n times of the Poisson bracket between f and K_β. For example: $\{f, K_\beta\}_{(0)} = f$, and $\{f, K_\beta\}_{(2)} = \{f, \{f, K_\beta\}\}$. $\tau, \sigma^j \in \mathbb{R}$ are coordinates in the reference frame defined by the dust fields. One can prove that $O_f[\tau, \sigma]$ is gauge invariant on the constraint surface, namely, it is a (weakly) Dirac observable [2]. O_f is a field defined on the dust space $\mathcal{S}$.[3] In this way, we construct the Dirac observables of q_{ab} and p^{ab} and denote them by g_{ij} and π^{ij}, respectively.[4] The symplectic structure on the reduced phase space is then given by

$$\Omega := \int_\mathcal{S} \mathrm{d}^3 \sigma \delta g_{ij} \wedge \delta \pi^{ij}. \tag{2.23}$$

We have ignored the possible modification of Ω. In the case that $\mathcal{S}$ has a boundary, we will come back to the symplectic structure in the next section.

c^{geo} and c_a^{geo} can be promoted to the Dirac observables C and C_j by substituting (q_{ab}, p^{ab}) with (g_{ij}, π^{ij}) in their expressions:

---

[3]Here the $\mathcal{S}$ is the spacelike hypersurface whose coordinates are given by σ^j. We have a dust space $\mathcal{S}[\tau]$ at each instance of the physical time τ.

[4]The indices of the Dirac observables are the greek letters since the observables are defined on the dust space.

$$C = \frac{1}{\sqrt{\det(g)}} \left[g_{ik} g_{jl} - \frac{1}{2} g_{ij} g_{kl} \right] \pi^{ij} \pi^{kl} - \sqrt{\det(g)} {}^{(3)}R,$$

$$(2.24)$$

$$C_j = -2 g_{jk} D_i \pi^{ik}, \qquad (2.25)$$

with ${}^{(3)}R$ is the Ricci scalar of g_{ij}. The Dirac observable of h is given by

$$H_{\mathrm{phys}} = \sqrt{C^2 - g^{ij} C_i C_j}. \qquad (2.26)$$

Ignoring again the situation that $\mathcal{S}$ has a boundary, the integration of H_{phys} over $\mathcal{S}$ gives the physical Hamiltonian

$$\mathbf{H}_{\mathrm{phys}} := \int_{\mathcal{S}} \mathrm{d}^3 \sigma H_{\mathrm{phys}}. \qquad (2.27)$$

Given a generic Dirac observable F on the reduced phase space, its physical time evolution with respect to τ is generated by $\mathbf{H}_{\mathrm{phys}}$

$$\frac{\mathrm{d}F}{\mathrm{d}\tau} := \{F, \mathbf{H}_{\mathrm{phys}}\}. \qquad (2.28)$$

The variation of the H_{phys} is given by

$$\delta \mathbf{H}_{\mathrm{phys}} = \int_{\mathcal{S}} \mathrm{d}^3 \sigma \left(N \delta C + N^i \delta C_i + \frac{1}{2} H_{\mathrm{phys}} N^i N^j \delta g_{ij} \right).$$

$$(2.29)$$

In analogy to the lapse function and the shift vector field in general relativity, N and N^i are called the dynamical lapse function and the dynamical shift vector field, respectively. The expressions of N^i and N are

$$N^i = -\frac{g^{ij} C_j}{H_{\mathrm{phys}}}, \qquad (2.30)$$

$$N = C/H_{\mathrm{phys}} = \begin{cases} -\sqrt{1 + g^{jk} N_j N_k} & \text{physical dust} \\ \sqrt{1 + g^{jk} N_j N_k} & \text{phantom dust} \end{cases}.$$

$$(2.31)$$

A difference between the general relativity and the BK formalism is that both N and N^j depend on the canonical variables. When ignoring the boundary of $\mathcal{S}$, we have infinitely many conserved charges $H_{\mathrm{phys}}(\vec{\sigma})$ and $C_j(\vec{\sigma})$

$$\{C_j(\vec{\sigma}), \mathbf{H}_{\mathrm{phys}}\} = 0, \qquad \{H_{\mathrm{phys}}(\vec{\sigma}), \mathbf{H}_{\mathrm{phys}}\} = 0. \qquad (2.32)$$

The Hamiltonian density $H_{\mathrm{phys}}(\vec{\sigma})$ and the dust-space diffeomorphism $C_j(\vec{\sigma})$ generate symmetries that are analogs of the hypersurface deformations in canonical general relativity, although here they are physical symmetries on the reduced phase space rather than gauge symmetries [26]. The modification of these symmetries charges in presence of boundary will be discussed in Sec. IV.

III. ASYMPTOTIC BOUNDARY CONDITIONS

Starting from this section, we analyze the situation that the dust space $\mathcal{S}$ has an asymptotic boundary at infinity. We consider the asymptotically flat boundary condition, and accordingly, we modify the symmetry charges H_{phys} and C_j by adding the corresponding boundary charges. Section III A introduces the boundary conditions and the symplectic structure. Subsection III B shows that the boundary charge of the physical Hamiltonian corresponds to the ADM mass.

A. The phase space and the symplectic structure

In any asymptotically flat spacetime, there exits an asymptotic Cartesian coordinate X^α, such that the 4-metric has the following asymptotic behavior at $r_0 \to \infty$:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \frac{f_{\alpha\beta}(X^0, \vec{X}/r_0)}{r_0} + o(r_0^{-1}), \qquad (3.1)$$

where $r_0 = \sqrt{X^i X_i}$, and $f_{\alpha\beta}(X^0, \vec{X}/r_0)$ is a smooth tensor field on S^2 at X^0.[5] To study the asymptotically flat spacetime in the reduced phase space, we propose that the material reference frame given by the dust is asymptotically Cartesian. Namely, we identify $\tau = X^0$, $\vec{\sigma} = \vec{X}$, and we define $r = \sqrt{\sigma^i \sigma_i}$. Then Eq. (3.1) leads to the following asymptotic behaviors of q_{ij} and π^{ij} on the dust space $\mathcal{S}$:

$$g_{ij} = \delta_{ij} + \frac{\bar{h}_{ij}}{r} + o(r^{-1}), \qquad \pi^{ij} = \frac{\bar{\pi}^{ij}}{r^2} + o(r^{-1}). \qquad (3.2)$$

A naively approach might be to adapt the standard procedure [27–29] in our context, by expanding

$$g_{ij} = \delta_{ij} + \frac{\bar{h}_{ij}}{r} + \frac{h_{ij}^{(2)}}{r^2} + o(r^{-2}), \qquad (3.3)$$

$$\pi^{ij} = \frac{\bar{\pi}^{ij}}{r^2} + \frac{\pi^{(2)ij}}{r^3} + \frac{\pi^{(3)ij}}{r^4} + o(r^{-4}), \qquad (3.4)$$

and imposing parity conditions to $\bar{h}_{ij}$ and $\bar{\pi}^{ij}$.[6] The reason of the parity condition is to make the following symplectic form finite as $R \to \infty$

---

[5]In our notations, $o(r^a)$ means decaying faster than r^a while $O(r^a)$ means decaying as fast as r^a for some constant a.

[6]For a function $f(\vec{\sigma})$ on S^2, $f(\vec{x})$ has odd parity if $f(-\vec{\sigma}) = -f(\vec{\sigma})$; $f(\vec{\sigma})$ has even parity if $f(-\vec{\sigma}) = f(\vec{\sigma})$.

$$\Omega_0 := \int_{\mathcal{S}_R} \mathrm{d}^3\sigma \delta g_{ij} \wedge \delta\pi^{ij}, \tag{3.5}$$

where $\mathcal{S}_R$ is the dust space with radial cutoff R. Ω_0 diverges logarithmically as $R \to \infty$ without the parity conditions.

However, as to be clarified in a moment, the above naive approach fails here because it is incompatible with neither physical dust nor phantom dust (see the remark at the end of this subsection), or it rules out many interesting boundary charges. We make two modifications: (1) We consider more general boundary condition by relaxing the parity conditions, similar to the approach in [30,31]. Then a counterterm has to be added to Ω_0 to cancel the divergent terms in (3.5), as we will see in a moment. (2) We consider the following more general falloff conditions including logarithmic terms

$$g_{ij} = \delta_{ij} + \frac{\bar{h}_{ij}}{r} + \frac{\log r}{r^2}h_{ij}^{(\log)} + \frac{h_{ij}^{(2)}}{r^2} + o(r^{-2}), \tag{3.6}$$

$$\pi^{ij} = \frac{\bar{\pi}^{ij}}{r^2} + \frac{\log r}{r^3}\pi^{(\log)ij} + \frac{\pi^{(2)ij}}{r^3} + \frac{\log r}{r^4}\pi^{(ll)ij}$$
$$+ \frac{\pi^{(3)ij}}{r^4} + o(r^{-4}). \tag{3.7}$$

Here, $\bar{h}_{ij}$, $\bar{\pi}^{ij}$, $h_{ij}^{(\log)}$, $\pi^{(\log)ij}$, $h_{ij}^{(2)}$, $\pi^{(2)ij}$, $\pi^{(ll)ij}$, and $\pi^{(3)ij}$ are called the boundary fields, since they are functions on S^2, which is the boundary of $\mathcal{S}$. In order to analyze the higher order asymptotic behavior of C_j, we expand π^{ij} to $O(r^{-4})$. The motivation of adding log r-terms in (3.6) and (3.7) is to include more general solutions of $C_j = 0$ (see [31] and Appendix B of [29]). We next transform (3.6) and (3.7) to spherical coordinates $\{r, \sigma^A\}(A = 1, 2)$. Here σ^A are the angular coordinates, and they are the functions of the traditional spherical coordinates θ, φ. They have odd parity $\sigma^A(\pi - \theta, \varphi + \pi) = -\sigma^A(\pi, \varphi)$.

$$g_{rr} = 1 + \frac{1}{r}\bar{h}_{rr} + \frac{\log r}{r^2}h_{rr}^{(\log)} + \frac{1}{r^2}h_{rr}^{(2)} + o(r^{-2}), \tag{3.8}$$

$$g_{rA} = \bar{h}_{rA} + \frac{\log r}{r}h_{rA}^{(\log)} + \frac{1}{r}h_{rA}^{(2)} + o(r^{-1}), \tag{3.9}$$

$$g_{AB} = r^2\bar{\gamma}_{AB} + r\bar{h}_{AB} + \log(r)h_{AB}^{(\log)} + h_{AB}^{(2)} + o(r^0), \tag{3.10}$$

$$\pi^{rr} = \bar{\pi}^{rr} + \frac{\log r}{r}\pi^{(\log)rr} + \frac{1}{r}\pi^{(2)rr} + \frac{\log r}{r^2}\pi^{(ll)rr}$$
$$+ \frac{\pi^{(3)rr}}{r^2} + o(r^{-2}), \tag{3.11}$$

$$\pi^{rA} = \frac{1}{r}\bar{\pi}^{rA} + \frac{\log r}{r^2}\pi^{(\log)rA} + \frac{1}{r^2}\pi^{(2)rA} + \frac{\log r}{r^3}\pi^{(ll)rA}$$
$$+ \frac{\pi^{(3)rA}}{r^3} + o(r^{-3}), \tag{3.12}$$

$$\pi^{AB} = \frac{1}{r^2}\bar{\pi}^{AB} + \frac{\log r}{r^3}\pi^{(\log)AB} + \frac{1}{r^3}\pi^{(2)AB} + \frac{\log r}{r^4}\pi^{(ll)AB}$$
$$+ \frac{\pi^{(3)AB}}{r^4} + o(r^{-4}). \tag{3.13}$$

Here A and B are indices of the angular coordinates on S^2, and $\bar{\gamma}_{AB}$ is the metric on a unit sphere. We introduce some short-hand notations for the following discussion

$$\bar{\lambda} = \frac{1}{2}\bar{h}_{rr}, \qquad \tilde{\bar{k}}_{AB} = \frac{1}{2}\bar{h}_{AB} + \bar{\lambda}\bar{\gamma}_{AB}, \qquad \bar{\lambda}_A = \bar{h}_{rA},$$
$$\bar{p} = 2(\bar{\pi}^{rr} - \bar{\pi}_A^A), \qquad \pi_{(k)}^{AB} = 2\bar{\pi}^{AB}. \tag{3.14}$$

We need to impose some additional boundary conditions to g_{ij} and π^{ij}. First, there is a parity condition that is imposed to $\pi^{(\log)rA}$ and is compatible to the dust:

$$\pi^{(\log)rA}(-\vec{\sigma}) = \pi^{(\log)rA}(\vec{\sigma}). \tag{3.15}$$

This parity condition results in that the ADM angular momentum becomes finite (it will be shown in Sec. IV). Equations (3.8)–(3.13) yield

$$C_r = \frac{C_r^{(1)}}{r} + \frac{\log(r)}{r^2}C_r^{(\log)} + \frac{C_r^{(2)}}{r^2} + \frac{\log r}{r^3}C_r^{(ll)} + O(r^{-3}), \tag{3.16}$$

$$C_A = C_A^{(0)} + \frac{\log(r)}{r}C_A^{(\log)} + \frac{C_A^{(1)}}{r} + \frac{\log r}{r^2}C_A^{(ll)} + O(r^{-2}), \tag{3.17}$$

$$C = \frac{C^{(1)}}{r} + \frac{\log(r)}{r^2}C^{(\log)} + O(r^{-2}), \tag{3.18}$$

where

$$C_r^{(1)} = 2(\bar{\pi}_A^A - \partial_A\bar{\pi}^{rA}), \tag{3.19}$$

$$C_r^{(\log)} = 2(\pi^{(\log)rr} - \partial_A\pi^{(\log)rA} + \pi^{(\log)A}_A), \tag{3.20}$$

$$C_A^{(0)} = -2(\bar{\pi}_A^r + \bar{D}_B\bar{\pi}_A^B), \tag{3.21}$$

$$C_A^{(\log)} = -2\bar{D}_B\pi^{(\log)B}_A, \tag{3.22}$$

$$C^{(1)} = -2\sqrt{\bar{\gamma}}(\bar{D}_A\bar{D}_B\tilde{\bar{k}}^{AB} - \bar{D}_A\bar{D}^A\tilde{\bar{k}} + \bar{D}_A\bar{\lambda}^A). \tag{3.23}$$

The explicit expression of $C^{(\log)}$ is complicate and unnecessary for the following discussions. We will provide further comments on this term in a moment.

Generically, the physical Hamiltonian density H_{phy} decays as $O(r^{-1})$ by (3.16)–(3.18) and (2.26). Then the bulk physical Hamiltonian diverges logarithmically,

$$\mathbf{H}_{\text{phys}} = \lim_{R \to \infty} \int_{\mathcal{S}_R} \mathrm{d}^3 \mathcal{V} H_{\text{phys}} = \lim_{R \to \infty} |C^{(1)}| \log(R) + \text{finite}. \tag{3.24}$$

This integral is written in the spherical coordinates. The coordinate volume is

$$\mathrm{d}^3 \mathcal{V} = \mathrm{d}r \mathrm{d}\sigma^1 \mathrm{d}\sigma^2. \tag{3.25}$$

To remove the logarithmic divergence, we introduce the boundary condition

$$C^{(1)} = 0. \tag{3.26}$$

The leading order contribution to H_{phys} is $C^{(\log)}$. The parity condition (3.15) has no impact on $C^{(\log)}$ based on power counting. It is convenient to analyze this in asymptotic Cartesian coordinates, where the contribution of $C^{(\log)}$ is in $O(\frac{\log r}{r^4})$. On the other hand, with (2.24) and (3.7), one finds that the leading order of the contribution from $\pi^{(\log)ij}$ in C is at $O(\frac{\log r}{r^5})$. As the result, regardless what parity condition assigned to $\pi^{(\log)ij}$, it does not influence $C^{(\log)}$.

Given a functional f of the fields g_{ij}, π^{ij}, the boundary term of δf is a differential if

$$\delta f = A_{ij} \delta \pi^{ij} + B^{ij} \delta g_{ij} - \delta \mathcal{B}_f|_{r \to \infty}, \tag{3.27}$$

for $\mathcal{B}_f|_{r \to \infty}$ depending on only boundary fields. In this case, the variation of $f + \mathcal{B}_f$ is well defined, i.e., $f + \mathcal{B}_f$ is a differentiable functional. If f relates to a symmetry, then we call $f + \mathcal{B}_f$ the symmetry charge and $\mathcal{B}_f$ the boundary charge. If we only impose (3.6) and (3.7), then the boundary term of $\delta \mathbf{H}_{\text{phys}}$ is not a differential thus cannot be canceled by adding boundary charges. A resolution is to impose some additional boundary conditions:

$$C_A^{(0)} = C_A^{(\log)} = C_A^{(1)} = C_r^{(1)} = C_r^{(\log)} = C_r^{(2)} = 0,$$
$$C^{(\log)} \neq 0. \tag{3.28}$$

In vacuum gravity, the boundary conditions for C and C_j are adapted on shell due to C and C_j are constraints. All the solutions of the equations of motion satisfy $C = 0$ and $C_j = 0$. In our case of gravity coupled to BK dust, $C \neq 0$ and $C_j \neq 0$ in general, since the boundary conditions (3.26)

and (3.28) only impose C and C_j to be 0 up to certain orders in the asymptotic expansion. Equations (3.26) and (3.28) are viewed as additional restrictions, which restrict the solution space of the equations of motion. In Sec. III B, we will show that these boundary conditions ensure the non-differential boundary terms in $\delta \mathbf{H}_{\text{phys}}$ decaying fast enough and vanishing in the spatial infinity.

Inspired by [30,31], the symplectic form Ω becomes finite at $R \to \infty$ after adding a counterterm:

$$\Omega = \lim_{R \to \infty} \int_{\mathcal{S}_R} \mathrm{d}^3 \sigma \delta g_{ij} \wedge \delta \pi^{ij} - \log(R) \oint_{S^2} \mathrm{d}^2 S \delta \bar{h}_{ij} \wedge \delta \bar{\pi}^{ij}, \tag{3.29}$$

with $\mathrm{d}^2 S = \mathrm{d}\sigma^1 \mathrm{d}\sigma^2$, and the δ indicates the variation of fields satisfying the falloff conditions (3.6) and (3.7) and boundary conditions (3.15), (3.26), and (3.28). The counter term of (3.29) may relate to the corner term in the formalism of [41]. Note that Ω does not depend on the variations of the boundary fields $\bar{\pi}^{ij}, \bar{h}_{ij}, \pi^{(\log)ij}, h_{ij}^{(\log)}$, etc., in the expansions (3.6) and (3.7). This can be seen by inserting the expansions of g_{ij}, π^{ij} in Ω. Indeed, in Cartesian coordinate, the boundary contribution of (3.29) is $\frac{\log(R)}{R} \oint_{S^2} \mathrm{d}^2 S(\delta \bar{\pi}^{ij} \wedge \delta \bar{h}_{ij}^{(\log)} + \delta \pi^{(\log)ij} \wedge \delta \bar{h}_{ij}) + O(R^{-1})$, and it vanishes as $R \to \infty$.

We define the reduced phase space $\mathcal{P}$ to be the space of fields g_{ij}, π^{ij} satisfying the falloff conditions (3.6) and (3.7) and boundary conditions (3.15), (3.26), and (3.28). $\mathcal{P}$ is equipped with the symplectic form Ω. For any differentiable function f on $\mathcal{P}$, its variation on $\mathcal{P}$ is given by

$$\delta f = \int_{\mathcal{S}} \mathrm{d}^3 \sigma \left[\frac{\delta f}{\delta g_{ij}(\sigma)} \delta g_{ij}(\sigma) + \frac{\delta f}{\delta \pi^{ij}(\sigma)} \delta \pi^{ij}(\sigma) \right], \tag{3.30}$$

where the variations δg_{ij} and $\delta \pi^{ij}$ preserve the falloff conditions. If two differentiable functions f, f' have the corresponding Hamiltonian vector fields, their Poisson bracket is given by

$$\{f, f'\} := \int_{\mathcal{S}} \mathrm{d}^3 \sigma \left[\frac{\delta f}{\delta g_{ij}(\sigma)} \frac{\delta f'}{\delta \pi^{ij}(\sigma)} - \frac{\delta f}{\delta \pi^{ij}(\sigma)} \frac{\delta f'}{\delta g_{ij}(\sigma)} \right]. \tag{3.31}$$

In the above two formulas and formulas in the following discussion, we adopt the following notation:

$$\int_{\mathcal{S}} \cdots = \lim_{R \to \infty} \int_{\mathcal{S}_R} \cdots. \tag{3.32}$$

There are two remarks about the boundary conditions:
(1) By (3.16)–(3.18) and (3.28), we have $C = O(\frac{\log r}{r^4})$ and $\frac{C_j}{C} = O(r^{-1})$ in the Cartesian coordinates. Then (2.26), (2.19), and (2.18) give

$U_j = \frac{C_j}{P} = O(r^{-1})$. Recall (2.17), we have $C = -\sqrt{\det(g)}\rho(g^{ij}U_iU_j + 1)$ in the dust space, and $g^{ij}U_iU_j = O(r^{-2})$. Thus, $\rho = O(\frac{\log r}{r^4})$. Reference [42] analyzes the spacetime coupling to a perfect fluid with certain boundary conditions. The asymptotic flatness conditions would be violated if the energy density ρ decays as $O(r^{-2})$. In our case, the asymptotic flatness conditions, however, are preserved due to the faster falloff behavior of ρ.

(2) As mentioned above, naively adding parity conditions to the expansions (3.3) and (3.4) does not work in the BK formalism. In the cases without introducing logarithmic terms to the falloff conditions (3.2), we need to impose $C^{(2)} \neq 0$ as the boundary condition. The explicit expression of $C^{(2)}$ is in Appendix B of [28], where $C^{(2)}$ contains the contributions of the canonical variables with odd parity. It turns out that the sign of $C^{(2)}$ can flip under an antipodal map. Since $C \propto \sqrt{g}\rho$, ρ can be smaller than 0 in some regions and be greater than 0 in other regions, whereas the BK formalism requires $\mathrm{sgn}(\rho)$ to be constant. Introducing logarithmic terms to the falloff conditions with appropriate parity conditions of $h_{ij}^{(\log)}$ removes this problem. In Cartesian coordinates, the leading order of C is $\frac{\log(r)}{r^4}C^{(\log)}$ with $C^{(1)} = 0$. The terms contribute to $\frac{\log(r)}{r^4}C^{(\log)}$ are the second derivative of $\frac{\log(r)}{r^2}h_{ij}^{(\log)}$. Henceforth, introducing even parity to $h_{ij}^{(\log)}$ ensures $C^{(\log)}$ has even parity. Its sign does not flip under the antipodal map. Nevertheless, relaxing the parity conditions give us more general results. We will concentrate on the relaxed parity conditions cases in our discussions throughout this paper.

B. The boundary term of the physical Hamiltonian

In this subsection, we show that the boundary term that arises from the variation $\delta\mathbf{H}_{\mathrm{phys}}$ is a differential under our boundary conditions. Thus, we can add a boundary Hamiltonian $\mathbf{H}_{\mathrm{bdy}}$ whose variation cancels the boundary term. $\mathbf{H}_{\mathrm{bdy}}$ turns out to be the ADM mass.

Recall (2.29), $\int_{\mathcal{S}} d^3\mathcal{V}g^{ij}N_i\delta C_j$ and $\int_{\mathcal{S}} d^3\mathcal{V}N\delta C$ may give boundary terms after integration by parts. First,

$$\int_{\mathcal{S}} d^3\mathcal{V}g^{ij}N_i\delta C_j = \int_{\mathcal{S}} d^3\mathcal{V}(\delta\pi^{ij}\mathcal{L}_{\vec{N}}g_{ij} - \delta g_{ij}\mathcal{L}_{\vec{N}}\pi^{ij})$$
$$- \lim_{R\to\infty} \oint_{S^2} d^2S(2N^j\delta(\pi^{rk}g_{jk}) - N^r\pi^{ij}\delta g_{ij}).$$
$$(3.33)$$

The boundary conditions turns out to make the boundary terms in (3.33) vanish at $R \to \infty$. Indeed, recall (2.30), we have

$$N_i = -\frac{C_i}{H_{\mathrm{phys}}}. \qquad (3.34)$$

By (3.8)–(3.13), we find that the boundary term in (3.33) vanishes at $R \to \infty$ once C_j decays faster than H_{phys} in Cartesian coordinates. Equation (3.2) indicates C_j decays as fast as H_{phys} without additional conditions and their leading terms are $O(r^{-3})$. The approach in Ref. [18] is to add an ϵ regulator to the leading order of C_j such that C_j decays as $C_j = O(r^{-3-\epsilon})$. Here ϵ is a positive constant. Then

$$N_j = O(r^{-\epsilon}). \qquad (3.35)$$

With (3.35), the boundary terms of (3.33) vanish. In this paper, we take a different approach by imposing (3.26)–(3.28). First, in spherical coordinates (3.34) reads

$$N_r = -\frac{C_r}{H_{\mathrm{phys}}}, \qquad N_A = -\frac{C_A}{H_{\mathrm{phys}}}. \qquad (3.36)$$

By (2.30), (3.16)–(3.17), (3.26), and (3.28), we find the following asymptotic behaviors of N_r, N_A:

$$N_r = O(r^{-1}), \qquad N_A = O(1). \qquad (3.37)$$

Or for N^r and N^A,

$$N^r = O(r^{-1}), \qquad N^A = O(r^{-2}). \qquad (3.38)$$

Then we analyze the boundary terms in (3.33) by using (3.8)–(3.13), and we obtain

$$\oint_{S^2} d^2SN^j\delta(\pi^{rk}g_{jk}) = \oint_{S^2} d^2S[N^r\delta(\pi^{rr}g_{rr} + \pi^{rD}g_{rD})$$
$$+ N^C\delta(\pi^{rr}g_{Cr} + \pi^{rD}g_{CD})],$$
$$= O(r^{-1}), \qquad (3.39)$$

and

$$\oint_{S^2} d^2SN^r\pi^{jk}\delta g_{jk} = \oint_{S^2} d^2SN^r(\pi^{rr}\delta g_{rr}$$
$$+ 2\pi^{rA}\delta g_{rA} + \pi^{AB}\delta g_{AB}),$$
$$= O(r^{-2}). \qquad (3.40)$$

When taking the limit $R \to \infty$, both terms vanish:

$$\lim_{R\to\infty} \oint_{S^2} \mathrm{d}^2 S (2N^j \delta(\pi^{rk} g_{jk}) - N^r \pi^{ij} \delta g_{ij}) = 0. \tag{3.41}$$

On the other hand,

$$\int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} N\delta C = \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} (\mathcal{A}_{ij}\delta\pi^{ij} - \mathcal{B}^{ij}\delta g_{ij}) + \mathcal{K}_N. \tag{3.42}$$

The explicit expressions of $\mathcal{A}_{ij}$ and $\mathcal{B}^{ij}$ can be found in [18]:

$$\mathcal{A}_{ij} = 2Ng^{-\frac{1}{2}}\left(\pi_{ij} - \frac{1}{2}g_{ij}\pi\right),$$

$$\mathcal{B}^{ij} = -Ng^{\frac{1}{2}}\left(R^{ij} - \frac{1}{2}g^{ij}R\right) + \frac{1}{2}Ng^{-\frac{1}{2}}g^{ij}\left(\pi_{mn}\pi^{mn} - \frac{1}{2}\pi^2\right) - 2Ng^{-\frac{1}{2}}\left(\pi^{im}\pi^j_m - \frac{1}{2}\pi^{ij}\pi\right)$$
$$+ g^{\frac{1}{2}}(D^i D^j N - g^{ij} D^m D_m N). \tag{3.43}$$

Here N is the dynamical lapse function, rather than the Lagrange multiplier in the original ADM formalism. The boundary term $\mathcal{K}_N$ reads (see Appendix A for more details):

$$\mathcal{K}_N = \lim_{R\to\infty} \oint_{S^2} \mathrm{d}^2 S \sqrt{\gamma}\left(-\frac{1}{\lambda}\gamma^{AB}(D_r N)\delta\gamma_{AB} + \frac{\lambda^A}{\lambda}\gamma^{BC}(D_A N)\delta\gamma_{BC}\right) - 2\lim_{R\to\infty}\oint_{S^2}\mathrm{d}^2 S (N\delta K + N\delta(\gamma_{AC})\gamma^{AB}K^C_B)$$

$$+ \lim_{R\to\infty}\oint_{S^2}\mathrm{d}^2 S\sqrt{\gamma}\frac{N}{\lambda}\delta\left[\left(-\frac{\lambda^A}{\lambda}(\partial_A\lambda + K_{AC}\lambda^C) + D_A\lambda^A\right) - D_A\delta\lambda^A\right] - \lim_{R\to\infty}\oint_{S^2}\mathrm{d}^2 S\sqrt{\gamma}N\frac{\lambda^A}{\lambda}\left(\frac{1}{2}\gamma^{BC}D_A\delta\gamma_{BC} - \delta\left(\frac{\lambda^B}{\lambda}K_{BA}\right)\right)$$

$$+ \lim_{R\to\infty}\oint_{S^2}\mathrm{d}^2 S\sqrt{\gamma}N\frac{\lambda^A}{\lambda}\delta\left(\frac{1}{\lambda}(\partial_A\lambda + K_{AB}\lambda^B)\right) + \lim_{R\to\infty}\oint_{S^2}\mathrm{d}^2 S\lambda\sqrt{\gamma}\left(-N\frac{\lambda^A\lambda^B}{\lambda^5}\delta K_{AB} + NK_{AB}\frac{\lambda^A\lambda^B}{\lambda^4}\frac{\delta\lambda}{\lambda^2}\right)$$

$$+ \lim_{R\to\infty}\oint_{S^2}\mathrm{d}^2 S\sqrt{\gamma}\left(-ND^A\left(\frac{1}{\lambda}\right)\delta\lambda_A - \frac{\lambda^A}{\lambda}\gamma^{BC}(D_C N)\delta\gamma_{AB}\right). \tag{3.44}$$

Recall (2.31), we find

$$N = \pm\sqrt{1 + g^{ij}N_i N_j}. \tag{3.45}$$

Here the "+" sign is for the phantom dust and the "−" sign is for the physical dust. In Cartesian coordinates, the asymptotic behaviors of N_j (3.6) read

$$N_j = O(r^{-1}). \tag{3.46}$$

It turns out that[7]

$$N = \pm 1 + O(r^{-2}). \tag{3.47}$$

Here γ_{AB} is the 2-metric on the S^2 in the "2 + 1" decomposition [28], which satisfies

---

$$\gamma_{AB} = g_{AB},$$

$$\gamma^{AB} = g^{AB} - \frac{\lambda^A\lambda^B}{\lambda^2}. \tag{3.48}$$

D_A is the covariant derivative compatible with γ_{AB}, and the angular indexes A and B are lowered and raised by γ_{AB} and its inverse γ^{AB}. By (3.8)–(3.13) and (3.47), the boundary term $\mathcal{K}_N$ to be a differential

$$\mathcal{K}_N = -\delta\mathbf{H}_{\text{bdy}},$$

$$= \mp\oint_{S^2}\mathrm{d}^2 S 2\sqrt{\bar\gamma}\delta(2\bar\lambda + \bar D_A\bar\lambda^A),$$

$$= \mp 4\oint_{S^2}\mathrm{d}^2 S\sqrt{\bar\gamma}\delta\bar\lambda. \tag{3.49}$$

The last step is because the integration region is closed. Here $\bar D_A$ is the covariant derivative compatible with $\bar\gamma_{AB}$, and the angular indices A and B are lowered and raised by $\bar\gamma_{AB}$ and its inverse $\bar\gamma^{AB}$. We can add the boundary term $\mathbf{H}_{\text{bdy}}$ to the physical Hamiltonian to make its variation well defined. Where

$$\mathbf{H}_{\text{bdy}} = \pm 4 \oint_{S^2} \mathrm{d}^2 S \sqrt{\bar{\gamma}} \bar{\lambda}, \tag{3.50}$$

and it is interpreted as the ADM mass. Here the "+" sign is for the phantom dust, while the "−" sign is for the physical dust. These signs are consistent with the convention that the physical time evolution corresponding to the phantom dust is future toward, whereas the physical time evolution corresponding to the physical dust is past toward, see [18].

$\mathbf{H}_{\text{bdy}}$ is conserved under the physical time evolution, i.e., $\frac{\mathrm{d}\mathbf{H}_{\text{bdy}}}{\mathrm{d}\tau} = 0$. Note that $\sqrt{\bar{\gamma}}$ is fiducial in (3.50), we only need to compute the physical time derivative of $\bar{\lambda}$. We first compute the physical time derivative of g_{rr}. Since asymptotically flat cases necessitate the consideration of boundary terms, the physical time derivative formula (2.28) needs to be modified as

$$\frac{\mathrm{d}g_{rr}}{\mathrm{d}\tau} = \{g_{rr}, \mathbf{H}_{\text{phys}} + \mathbf{H}_{\text{bdy}}\}, \tag{3.51}$$

and the result is

$$\frac{\mathrm{d}g_{rr}}{\mathrm{d}\tau} = N g^{-\frac{1}{2}} \left(\pi_{rr} - \frac{1}{2} g_{rr} \pi \right) + \mathcal{L}_{\vec{N}} g_{rr} = O(r^{-2}) \tag{3.52}$$

with the falloff conditions and boundary conditions introduced before. On the other hand, $\bar{\lambda}$ is proportional to the coefficient of the $O(r^{-1})$ term in g_{rr}, and its physical time derivative is induced by expanding (3.52) to the boundary. Therefore,

$$\frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}\tau} = 0, \tag{3.53}$$

and thus

$$\frac{\mathrm{d}\mathbf{H}_{\text{bdy}}}{\mathrm{d}\tau} = 0. \tag{3.54}$$

The similar boundary Hamiltonian is obtained by the approach with the ϵ regulator in [18]. However, as an advantage of our approach, the boundary conditions (3.26)–(3.28) result in more general symmetry charges, which turn out to relate to supertranslations on the boundary. We will show this in the next section.

IV. BOUNDARY-PRESERVING SYMMETRY CHARGES AND BOUNDARY CHARGES

In this section, we firstly define the boundary-preserving symmetry charges. Then these symmetry charges are explicitly constructed as phase space functions. The expressions of the symmetry charges contain both bulk and boundary terms. The boundary terms are constructed such that the variations of the functions are well defined on the phase space.

Definition 1. G is a boundary-preserving symmetry charge if

  (i) G commutes with the physical Hamiltonian:

$$\{G, \mathbf{H}_{\text{phys}} + \mathbf{H}_{\text{bdy}}\} = 0. \tag{4.1}$$

  (ii) The Hamiltonian flow of G preserves the falloff and boundary conditions in Sec. III A.

We first introduce the vector fields used to construct the boundary-preserving charges, which are denoted as $\xi^\mu := (\xi, \xi^j)$. As $r \to \infty$, ξ^μ falls off as

$$\xi = f + O(r^{-1}), \qquad \xi^r = W + O(r^{-1}),$$
$$\xi^A = Y^A + \frac{1}{r} I^A + O(r^{-2}). \tag{4.2}$$

For convenience, we require that the expansions of $(\xi, \vec{\xi})$ only contain integer order of r^{-1}. f, W, and I^A are functions and vector field on S^2, Y^A is the Killing vector field on the S^2. Their geometric meaning will be discussed shortly. The boundary-preserving symmetry $G(\xi, \vec{\xi})$ is constructed as

$$G(\xi, \vec{\xi}) := J(\xi, \vec{\xi}) + \mathcal{B}(\xi, \vec{\xi}), \tag{4.3}$$

where $J(\xi, \vec{\xi})$ is the bulk term

$$J(\xi, \vec{\xi}) = \mathcal{T}(\xi) + \mathcal{P}(\vec{\xi}), \tag{4.4}$$

with

$$\mathcal{T}(\xi) = \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} \xi H_{\text{phys}},$$
$$\mathcal{P}(\vec{\xi}) = \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} \xi^i C_i. \tag{4.5}$$

$\mathcal{B}(\xi, \vec{\xi})$ is the boundary term, whose variation should cancel the boundary term from $\delta J(\xi, \vec{\xi})$, where

$$\delta J(\xi, \vec{\xi}) = \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} \left[\tilde{\xi} \delta C + (\xi^r + \tilde{\xi}^r) \delta C_r + (\xi^A + \tilde{\xi}^A) \delta C_A \right.$$
$$\left. + \frac{\xi}{2} H_{\text{phys}} N^i N^j \delta g_{ij} \right],$$
$$= \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} [\mathcal{G}_{ij} \delta \pi^{ij} - \mathcal{F}^{ij} \delta g_{ij}] + \mathcal{K}(\xi, \vec{\xi}). \tag{4.6}$$

In (4.6), $\tilde{\xi} = \xi N$, $\tilde{\xi}^i = \xi N^i$, and $\mathcal{K}(\xi, \vec{\xi})$ is the boundary term. The bulk terms are

$$\mathcal{G}_{ij} = 2\tilde{\xi}g^{-\frac{1}{2}}\left(\pi_{ij} - \frac{1}{2}g_{ij}\pi\right) + \mathcal{L}_{\vec{\xi}}g_{ij} + \mathcal{L}_{\vec{\tilde{\xi}}}g_{ij},$$

$$\mathcal{F}^{ij} = -\tilde{\xi}g^{\frac{1}{2}}\left(R^{ij} - \frac{1}{2}g^{ij}R\right) + \frac{1}{2}\tilde{\xi}g^{-\frac{1}{2}}g^{ij}\left(\pi_{mn}\pi^{mn} - \frac{1}{2}\pi^2\right)$$

$$- 2\tilde{\xi}g^{-\frac{1}{2}}\left(\pi^{im}\pi_m^j - \frac{1}{2}\pi^{ij}\pi\right) + g^{\frac{1}{2}}(D^iD^j\tilde{\xi} - g^{ij}D^mD_m\tilde{\xi})$$

$$+ \mathcal{L}_{\vec{\xi}}\pi^{ij} + \mathcal{L}_{\vec{\tilde{\xi}}}\pi^{ij} + \frac{\xi}{2}H_{\text{phys}}N^iN^j. \tag{4.7}$$

$$G(f) := \int_{\mathcal{S}} \mathrm{d}^3\mathcal{V}fH_{\text{phys}} + \mathcal{B}(f),$$

$$G(W) := \int_{\mathcal{S}} \mathrm{d}^3\mathcal{V}WC_r + \mathcal{B}(W),$$

$$G(\vec{Y}) := \int_{\mathcal{S}} \mathrm{d}^3\mathcal{V}Y^AC_A + \mathcal{B}(\vec{Y}),$$

$$G(\vec{I}) := \int_{\mathcal{S}} \mathrm{d}^3\mathcal{V}\frac{I^A}{r}C_A + \mathcal{B}(\vec{I}). \tag{4.8}$$

Unlike the typical general relativity, the $\tilde{\xi}$ and the $\tilde{\xi}^j$ in (4.7) are dynamical. The terms $\mathcal{L}_{\vec{\tilde{\xi}}}g_{ij}$ and $\mathcal{L}_{\vec{\tilde{\xi}}}\pi^{ij}$ are contributed by $\xi\delta H_{\text{phys}}$. The term $\frac{\xi}{2}N^iN^j\delta g_{ij}$ is the correction due to the dust field.

We discuss the geometric meaning of the parameters in (4.2). The boundary-preserving symmetry charges generate the diffeomorphisms in the bulk and the asymptotic symmetry transformations at the boundary. The asymptotic symmetry transformations at the boundary preserve the falloff conditions and the boundary conditions. The charges are classified into the following four types:

$\mathcal{B}(f), \mathcal{B}(W), \mathcal{B}(\vec{Y}), \mathcal{B}(\vec{I})$ are the boundary charges. Physically, these four types of charges generate the following four classes of the asymptotic symmetry transformations at the boundary:

(1) $G(\vec{Y})$: spatial rotations.
(2) $G(f)$: supertranslations in the physical time direction.
(3) $G(W)$: The radius component of the supertranslations in the spatial direction.
(4) $G(\vec{I})$: The angular components of the supertranslations in the spatial direction.

When $f = 1$, $G(f)$ reduces to the physical Hamiltonian with the ADM mass as the boundary charge, and it generates the physical time translation. As is shown in [28], when

$$W = W_T = T_A\sin\sigma_\theta\cos\sigma_\varphi + T_Y\sin\sigma_\theta\sin\sigma_\varphi + T_Z\cos\sigma_\theta,$$

$$I^A = I^A_T = \bar{D}^AW_T, \tag{4.9}$$

where T_X, T_Y, T_Z are constants and other components of $(\xi, \vec{\xi})$ are 0, $G(\xi, \vec{\xi})$ generates spatial translation since

$$W_T\frac{\partial}{\partial r} + \frac{I^A_T}{r}\frac{\partial}{\partial\sigma^A} = T_X\frac{\partial}{\partial\sigma_X} + T_Y\frac{\partial}{\partial\sigma_Y} + T_Z\frac{\partial}{\partial\sigma_Z}. \tag{4.10}$$

In order to obtain the boundary charge $\mathcal{B}(\xi, \vec{\xi})$, we compute $\mathcal{K}(\xi, \vec{\xi})$ (see Appendix A for more details)

$$\mathcal{K}(\xi, \vec{\xi}) = \lim_{R\to\infty}\oint_{S^2}\mathrm{d}^2S(-2u^i\delta\pi^r_i + u^r\pi^{ij}\delta g_{ij}) + \lim_{R\to\infty}\oint_{S^2}\mathrm{d}^2S\sqrt{\gamma}\left(-\frac{1}{\lambda}\gamma^{AB}(D_r\tilde{\xi})\delta\gamma_{AB} + \frac{\lambda^A}{\lambda}\gamma^{BC}(D_A\tilde{\xi})\delta\gamma_{BC}\right)$$

$$- \lim_{R\to\infty}\oint_{S^2}\mathrm{d}^2S\sqrt{\gamma}(2\tilde{\xi}\delta K + \tilde{\xi}\delta(\gamma_{AC})\gamma^{AB}K^C_B) + \lim_{R\to\infty}\oint_{S^2}\mathrm{d}^2S\sqrt{\gamma}\frac{\tilde{\xi}}{\lambda}\left[\delta\left(-\frac{\lambda^A}{\lambda}(\partial_A\lambda + K_{AC}\lambda^C) + D_A\lambda^A\right) - D_A\delta\lambda^A\right]$$

$$- \lim_{R\to\infty}\oint_{S^2}\mathrm{d}^2S\sqrt{\gamma}\tilde{\xi}\frac{\lambda^A}{\lambda}\left(\frac{1}{2}\gamma^{BC}D_A\delta\gamma_{BC} - \delta\left(\frac{\lambda^B}{\lambda}K_{BA}\right)\right) + \lim_{R\to\infty}\oint_{S^2}\mathrm{d}^2S\sqrt{\gamma}\tilde{\xi}\frac{\lambda^A}{\lambda}\delta\left(\frac{1}{\lambda}(\partial_A\lambda + K_{AB}\lambda^B)\right)$$

$$+ \lim_{R\to\infty}\oint_{S^2}\mathrm{d}^2S\lambda\sqrt{\gamma}\left(-\tilde{\xi}\frac{\lambda^A\lambda^B}{\lambda^5}\delta K_{AB} + \tilde{\xi}K_{AB}\frac{\lambda^A\lambda^B}{\lambda^4}\frac{\delta\lambda}{\lambda^2}\right) + \lim_{R\to\infty}\oint_{S^2}\mathrm{d}^2S\sqrt{\gamma}\left(-\tilde{\xi}D^A\left(\frac{1}{\lambda}\right)\delta\lambda_A - \frac{\lambda^A}{\lambda}\gamma^{BC}(D_C\tilde{\xi})\delta\gamma_{AB}\right), \tag{4.11}$$

with $u^i = \xi^i + \tilde{\xi}^i$. From (3.8)–(3.13) and (3.38),

$$\mathcal{K}(\xi, \vec{\xi}) = \lim_{R \to \infty} \oint_{S^2} d^2 S(-2RY^A \delta \bar{\pi}^r_A - 2Y^A \log(R) \delta \pi^{(\log)r}_A$$
$$- 2Y^A \delta(\bar{\pi}^{rr} \bar{\lambda}_A + \pi^{r(2)}_A + \bar{\pi}^{rB} \bar{h}_{AB})$$
$$- 2I^A \delta \bar{\pi}^r_A - 2W \delta \bar{\pi}^{rr} \mp 2\sqrt{\bar{\gamma}} f \delta(2\bar{\lambda} + \bar{D}_A \bar{\lambda}^A)). \tag{4.12}$$

Similar to the discussion in Sec. III B, we want to show $\mathcal{K}$ as a differential of certain boundary charge $\mathcal{B}$. There are two terms in $\mathcal{K}$ that would be divergent without the suitable boundary condition (see Appendix B of [29] for similar results): As $R \to \infty$, the term $2R \oint_{S^2} d^2 SY^A \delta \bar{\pi}^r_A$ is generically divergent linearly but vanishes by imposing the boundary condition $C^{(0)}_A = 0$ and the Killing equation of Y^A. Note that Y^A has odd parity and $\pi^{(\log)r}_A = \bar{\gamma}_{AB} \pi^{(\log)rB}$, the term $-2 \oint_{S^2} d^2 SY^A \log(R) \delta \pi^{(\log)r}_A$ is generically divergent logarithmically but vanishes by (3.15). Finally,

$$\mathcal{K}(\xi, \vec{\xi}) = \oint_{S^2} d^2 S(-2Y^A \delta(\bar{\pi}^{rr} \bar{\lambda}_A + \pi^{r(2)}_A + \bar{\pi}^{rB} \bar{h}_{AB}) - 2I^A \delta \bar{\pi}^r_A$$
$$- 2W \delta \bar{\pi}^{rr} \mp 2\sqrt{\bar{\gamma}} f \delta(2\bar{\lambda} + \bar{D}_A \bar{\lambda}^A)). \tag{4.13}$$

In order to fulfill $\mathcal{K}(\xi, \vec{\xi}) = -\delta \mathcal{B}(\xi, \vec{\xi})$,

$$\mathcal{B}(\xi, \vec{\xi}) = \oint_{S^2} d^2 SY^A (4(\tilde{\bar{k}}_{AB} - \bar{\lambda} \bar{\gamma}_{AB}) \bar{\pi}^{rB}$$
$$+ 2\bar{\gamma}_{AB} \pi^{(2)rB} + \bar{\lambda}_A (\bar{p} + 2\bar{\pi}^B_B))$$
$$+ \oint_{S^2} d^2 S(2I^A \bar{\gamma}_{AB} \bar{\pi}^{rB} + W(\bar{p} + 2\bar{\pi}^A_A)$$
$$\pm 2\sqrt{\bar{\gamma}} f(2\bar{\lambda} + \bar{D}_A \bar{\lambda}^A))$$
$$= \mathcal{B}(\xi) + \mathcal{B}(\vec{\xi}), \tag{4.14}$$

where

$$\mathcal{B}(\xi) = \mathcal{B}(f),$$
$$\mathcal{B}(\vec{\xi}) = \mathcal{B}(\vec{Y}) + \mathcal{B}(\vec{I}) + \mathcal{B}(W). \tag{4.15}$$

Here $\mathcal{B}(f)$, $\mathcal{B}(\vec{Y})$, $\mathcal{B}(\vec{I})$, and $\mathcal{B}(W)$ are the four types of boundary charges:

$$\mathcal{B}(\vec{Y}) = \oint_{S^2} d^2 SY^A (4(\tilde{\bar{k}}_{AB} - \bar{\lambda} \bar{\gamma}_{AB}) \bar{\pi}^{rB} + 2\bar{\gamma}_{AB} \pi^{(2)rB}$$
$$+ \bar{\lambda}_A (\bar{p} + 2\bar{\pi}^B_B)),$$
$$\mathcal{B}(\vec{I}) = \oint_{S^2} d^2 S 2I^A \bar{\gamma}_{AB} \bar{\pi}^{rB},$$
$$\mathcal{B}(W) = \oint_{S^2} d^2 SW(\bar{p} + 2\bar{\pi}^A_A),$$
$$\mathcal{B}(f) = \pm \oint_{S^2} d^2 S 2\sqrt{\bar{\gamma}} f(2\bar{\lambda} + \bar{D}_A \bar{\lambda}^A). \tag{4.16}$$

The boundary charges (4.14) are finite. By (3.16)–(3.18), (3.26), (3.28), and (4.2), $J(\xi, \vec{\xi})$ are also finite. Thus, the boundary-preserving charges $G(\xi, \vec{\xi}) = J(\xi, \vec{\xi}) + \mathcal{B}(\xi, \vec{\xi})$ are finite on the phase space. Additionally, by adding the boundary terms $\mathcal{B}(\xi, \vec{\xi})$, the variation of $G(\xi, \vec{\xi})$ is well defined. Furthermore, similar to the discussions at the end of Sec. III B, one can show the boundary charges in (4.14) are conserved, and the detailed discussion is in Appendix B.

The symplectic form defined by Eq. (3.29) is preserved by $G(\xi, \vec{\xi})$. First, Eq. (3.29) only contains bulk terms, as the boundary term of the integration $\lim_{R \to \infty} \int_{S_R} d^3 \sigma \delta g_{ij} \wedge \delta \pi^{ij}$ is canceled by the counterterm, as previously discussed. Second, the variation of $G(\xi, \vec{\xi})$ is well defined, implying the existence of corresponding Hamiltonian vector field. Consequently, the transformation on the phase space generated by $G(\xi, \vec{\xi})$ is a canonical one, which preserves the symplectic structure (see Appendix C).

Before ending this section, we would like to compare our results with the results in [28]. First, our work is conceptually different from [28]. The result in [28] bases on the traditional formulation of general relativity, where the bulk is a totally constrained system, thus the symmetry charges vanish in the bulk and only have nonvanished terms at the boundary. The result in our case, however, is based on the reduced phase space formulation. The gravity-dust system on the reduced phase space is not a constraint system any longer and has a physical Hamiltonian. Therefore, our symmetry charges have both nonvanishing bulk and boundary terms. The transformations generated by these charges are symmetries but not gauge symmetries. Second, although some boundary terms look similar to the ones in [28], there are several additional boundary terms in our results, including $2 \oint_{S^2} d^2 SW \bar{\pi}^A_A$ and $4 \oint_{S^2} d^2 SY^A \tilde{\bar{k}}_{AB} \bar{\pi}^{rB}$. These terms vanish when the parity condition in [28] is imposed. Finally, [28] considers the symmetry charge corresponding to the boundary boost, which requires $\bar{\lambda}_A = 0$. However, we do not take into account the boundary boost in $G(\xi, \vec{\xi})$, so we do not need to introduce $\bar{\lambda}_A = 0$ here.

V. ALGEBRA OF THE BOUNDARY-PRESERVING SYMMETRY CHARGES

In this section, we obtain the algebra of the boundary-preserving symmetry charges by computing their Poisson brackets. Section V A presents some detailed computations of the Poisson bracket between a pair of $G(\xi, \vec{\xi})$ for $\xi, \vec{\xi}$ satisfying the boundary condition (4.2). In Section V B, we show that $G(\xi, \vec{\xi})$ form a closed Poisson algebra up to a central charge, and we also discuss the relation between this algebra and BMS algebra. Section V C checks that the boundary conditions (3.28) are preserved by the transformations generated by $G(\xi, \vec{\xi})$.

A. Computing Poisson brackets between the boundary-preserving symmetry charges

We first derive the following useful relations. In these derivations, the vector field $(\xi, \vec{\xi})$ satisfies (4.2). The boundary charge $\mathcal{B}(\vec{\xi})$ is given by the second equation of (4.15) and $\tilde{\mathcal{B}}(\xi) = \pm \mathcal{B}(\xi) = \oint_{S^2} d^2 S 2\sqrt{\bar{\gamma}} f(2\bar{\lambda} + \bar{D}_A \bar{\lambda}^A)$, where $\mathcal{B}(\xi)$ is given by the first equation of (4.15). First,

$$\left\{ \int_S d^3\mathcal{V} \xi_1^i C_i + \mathcal{B}(\vec{\xi}_1), \int_S d^3\mathcal{V} \xi_2^i C_i + \mathcal{B}(\vec{\xi}_2) \right\} = \int_S d^3\mathcal{V}[-\mathcal{L}_{\vec{\xi}_1}\pi^{ij}\mathcal{L}_{\vec{\xi}_2}g_{ij} + \mathcal{L}_{\vec{\xi}_1}g_{ij}\mathcal{L}_{\vec{\xi}_2}\pi^{ij}],$$

$$= \int_S d^3\mathcal{V}[\vec{\xi}_1, \vec{\xi}_2]^j C_j + 2 \lim_{R\to\infty} \int_{S^2} d^2 S[\vec{\xi}_1, \vec{\xi}_2]^j \pi_j^r. \tag{5.1}$$

Second,

$$\left\{ \int_S d^3\mathcal{V} \xi_1 C + \tilde{\mathcal{B}}(\xi_1), \int_S d^3\mathcal{V} \xi_2 C + \tilde{\mathcal{B}}(\xi_2) \right\} = \int_S d^3\mathcal{V} \left[\xi_1 g^{\frac{1}{2}}\left(R^{ij} - \frac{1}{2}g^{ij}R \right) - \frac{1}{2}\xi_1 g^{ij}g^{-\frac{1}{2}}\left(\pi_{mn}\pi^{mn} - \frac{1}{2}\pi^2 \right) \right.$$

$$\left. + 2\xi_1 g^{-\frac{1}{2}}\left(\pi^{im}\pi_m^j - \frac{1}{2}\pi^{ij}\pi \right) - g^{\frac{1}{2}}(D^iD^j\xi_1 - g^{ij}D_mD^m\xi_1) \right] \xi_2 g^{-\frac{1}{2}}\left(\pi_{ij} - \frac{1}{2}g_{ij}\pi \right) - (1 \leftrightarrow 2),$$

$$= \int_S d^3\mathcal{V}(\xi_1 D^j\xi_2 - \xi_2 D^j\xi_1)C_j + 2 \lim_{R\to\infty} \oint_{S^2} d^2 S(\xi_1 D^j\xi_2 - \xi_2 D^j\xi_1)\pi_j^r. \tag{5.2}$$

Third,

$$\left\{ \int_S d^3\mathcal{V} \xi_1^i C_i + \mathcal{B}(\vec{\xi}_1), \int_S d^3\mathcal{V} \xi_2 C + \tilde{\mathcal{B}}(\xi_2) \right\} = \int_S d^3\mathcal{V} \left\{ -2\mathcal{L}_{\vec{\xi}_1}\pi^{ij}\xi_2 g^{-\frac{1}{2}}\left(\pi_{ij} - \frac{1}{2}g_{ij}\pi \right) \right.$$

$$- \mathcal{L}_{\vec{\xi}_1}g_{ij}\left[\xi_2 g^{\frac{1}{2}}(R^{ij} - g^{ij}R) - \frac{1}{2}\xi_2 g^{ij}C \right.$$

$$\left. + 2\xi_2 g^{-\frac{1}{2}}\left(\pi^{im}\pi_m^j - \frac{1}{2}\pi^{ij}\pi \right) - g^{\frac{1}{2}}(D^iD^j\xi_2 - g^{ij}D_mD^m\xi_2) \right] \right\},$$

$$= \int_S d^3\mathcal{V} \xi_1^i \partial_i(\xi_2)C - \delta_{\vec{\xi}_1}\tilde{\mathcal{B}}(\xi_2) - \lim_{R\to\infty} \oint_{S^2} d^2 S \xi_1^r \xi_2 C. \tag{5.3}$$

Before continuing, we introduce the notation $\delta_{\vec{\xi}}$. For a vector field $\vec{\xi}$ in (4.2), $\delta_{\vec{\xi}}g_{ij}$ and $\delta_{\vec{\xi}}\pi^{ij}$ are given by

$$\delta_{\vec{\xi}}g_{ij} := \left\{ g_{ij}, \int_S d^3\mathcal{V} \xi_1^i C_i + \mathcal{B}(\vec{\xi}_1) \right\} = \mathcal{L}_{\vec{\xi}}g_{ij} \tag{5.4}$$

and

$$\delta_{\vec{\xi}}\pi^{ij} := \left\{ \pi^{ij}, \int_S d^3\mathcal{V} \xi_1^i C_i + \mathcal{B}(\vec{\xi}_1) \right\} = \mathcal{L}_{\vec{\xi}}\pi^{ij}, \tag{5.5}$$

respectively. We only consider $\delta_{\vec{\xi}}$ acting on phase space linear functions $f[g_{ij}, \pi^{ij}]$,

$$\delta_{\vec{\xi}}f[g_{ij}, \pi^{ij}] := f[\delta_{\vec{\xi}}g_{ij}, \delta_{\vec{\xi}}\pi^{ij}]. \tag{5.6}$$

The boundary-preserving symmetry charges $G(\xi, \vec{\xi})$ can split into the temporal part $G(\xi)$ and the spatial part $G(\vec{\xi})$:

$$G(\xi, \vec{\xi}) = G(\xi) + G(\vec{\xi}), \tag{5.7}$$

where

$$G(\xi) = \int_S d^3\mathcal{V} \xi H_{\text{phys}} + \mathcal{B}(\xi),$$

$$G(\vec{\xi}) = \int_S d^3\mathcal{V} \xi^i C_i + \mathcal{B}(\vec{\xi}). \tag{5.8}$$

The Poisson bracket is bilinear,

$$
\begin{aligned}
\{G(\xi_1, \vec{\xi}_1), G(\xi_2, \vec{\xi}_2)\} &= \{G(\xi_1) + G(\vec{\xi}_1), G(\xi_2) + G(\vec{\xi}_2)\} \\
&= \{G(\vec{\xi}_1), G(\vec{\xi}_2)\} + \{G(\vec{\xi}_1), G(\xi_2)\} + \{G(\xi_1), G(\vec{\xi}_2)\} + \{G(\xi_1), G(\xi_2)\}.
\end{aligned}
\tag{5.9}
$$

By (5.1), the first term of (5.9) becomes

$$
\{G(\vec{\xi}_1), G(\vec{\xi}_2)\} = \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} [\vec{\xi}_1, \vec{\xi}_2]^j C_j + 2 \lim_{R \to \infty} \oint_{S^2} \mathrm{d}^2 S [\vec{\xi}_1, \vec{\xi}_2]^j \pi_j^r.
\tag{5.10}
$$

By (2.29), (5.1), and (5.2),

$$
\begin{aligned}
\{G(\vec{\xi}_1), G(\xi_2)\} &= \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} \xi_1^i \partial_i(\tilde{\xi}_2) C - \frac{1}{2} \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} \xi_2 H_{\text{phys}} N^j N^k \mathcal{L}_{\vec{\xi}_1} g_{jk} + \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} [\vec{\xi}_1, \vec{\xi}_2]^j C_j \\
&\quad - \delta_{\vec{\xi}_1} \tilde{\mathcal{B}}(\tilde{\xi}_2) - \lim_{R \to \infty} \oint_{S^2} \mathrm{d}^2 S \xi_1^r \tilde{\xi}_2 C + 2 \lim_{R \to \infty} \oint_{S^2} \mathrm{d}^2 S [\vec{\xi}_1, \vec{\xi}_2]^j \pi_j^r, \\
&= \lim_{R \to \infty} \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} \xi_1^i D_i(\xi_2) H_{\text{phys}} - \delta_{\vec{\xi}_1} \tilde{\mathcal{B}}(\tilde{\xi}_2) - \lim_{R \to \infty} \oint_{S^2} \mathrm{d}^2 S \xi_1^r \tilde{\xi}_2 C + 2 \lim_{R \to \infty} \oint_{S^2} \mathrm{d}^2 S [\vec{\xi}_1, \vec{\xi}_2]^j \pi_j^r.
\end{aligned}
\tag{5.11}
$$

Similarly,

$$
\begin{aligned}
\{G(\xi_1), G(\vec{\xi}_2)\} &= -\{G(\vec{\xi}_2), G(\xi_1)\}, \\
&= -\int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} \xi_2^i D_i(\xi_1) H_{\text{phys}} + \delta_{\vec{\xi}_2} \tilde{\mathcal{B}}(\tilde{\xi}_1) + \lim_{R \to \infty} \oint_{S^2} \mathrm{d}^2 S \xi_2^r \tilde{\xi}_1 C - 2 \lim_{R \to \infty} \oint_{S^2} \mathrm{d}^2 S [\vec{\xi}_2, \vec{\xi}_1]^j \pi_j^r.
\end{aligned}
\tag{5.12}
$$

By (5.1)–(5.3),

$$
\begin{aligned}
\{G(\xi_1), G(\xi_2)\} &= \mathcal{B} + 2 \lim_{R \to \infty} \oint_{S^2} \mathrm{d}^2 S (\tilde{\xi}_1 D^j \tilde{\xi}_2 - \tilde{\xi}_2 D^j \tilde{\xi}_1) \pi_j^r - \delta_{\vec{\xi}_1} \tilde{\mathcal{B}}(\tilde{\xi}_2) \\
&\quad - \lim_{R \to \infty} \oint_{S^2} \mathrm{d}^2 S \tilde{\xi}_1^r \tilde{\xi}_2 C + \delta_{\vec{\xi}_2} \tilde{\mathcal{B}}(\tilde{\xi}_1) + \lim_{R \to \infty} \oint_{S^2} \mathrm{d}^2 S \tilde{\xi}_2^r \tilde{\xi}_1 C + 2 \lim_{R \to \infty} \oint_{S^2} \mathrm{d}^2 S [\vec{\xi}_1, \vec{\xi}_2]^j \pi_j^r,
\end{aligned}
\tag{5.13}
$$

where the bulk term $\mathcal{B}$ is

$$
\begin{aligned}
\mathcal{B} &= \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} (\tilde{\xi}_1 D^j \tilde{\xi}_2 - \tilde{\xi}_2 D^j \tilde{\xi}_1) C_j + \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} \xi_1^j D_j(\tilde{\xi}_2) C - \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} \xi_2^j D_j(\tilde{\xi}_1) C + \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} [\vec{\xi}_1, \vec{\xi}_2]^j C_j \\
&\quad + \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} H_{\text{phy}} N^i N^j \xi_1 \tilde{\xi}_2 g^{-\frac{1}{2}} \left(\pi_{ij} - \frac{1}{2} \pi g_{ij} \right) - \frac{1}{2} \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} H_{\text{phy}} N^i N^j \xi_2 \tilde{\xi}_1 g^{-\frac{1}{2}} \left(\pi_{ij} - \frac{1}{2} \pi g_{ij} \right) \\
&\quad + \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} H_{\text{phy}} N^i N^j \xi_1 D_i(\tilde{\xi}_2)_j - \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} H_{\text{phy}} N^i N^j \xi_2 D_i(\tilde{\xi}_1)_j.
\end{aligned}
\tag{5.14}
$$

This term turns out to vanish, in agree with the result of [39]. The detailed computation is provided in Appendix E.

In summary, with generic $\xi, \vec{\xi}$, the general form of the Poisson bracket between a pair of $G(\xi, \vec{\xi})$ reads

$$
\begin{aligned}
\{G(\xi_1, \vec{\xi}_1), G(\xi_2, \vec{\xi}_2)\} = &\int_{\mathcal{S}} d^3\mathcal{V}[\vec{\xi}_1, \vec{\xi}_2]^j C_j + \int_{\mathcal{S}} d^3\mathcal{V}[\xi_1^i D_i(\xi_2) - \xi_2^i D_i(\xi_1)] H_{\text{phys}} \\
&- \delta_{\tilde{\xi}_1} \tilde{\mathcal{B}}(\tilde{\xi}_2) - \lim_{R\to\infty} \oint_{S^2} d^2 S \xi_1^r \tilde{\xi}_2 C + 2\lim_{R\to\infty} \oint_{S^2} d^2 S[\vec{\xi}_1, \vec{\tilde{\xi}}_2]^j \pi_j^r + \delta_{\tilde{\xi}_2} \tilde{\mathcal{B}}(\tilde{\xi}_1) \\
&+ \lim_{R\to\infty} \oint_{S^2} d^2 S \xi_2^r \tilde{\xi}_1 C - 2\lim_{R\to\infty} \oint_{S^2} d^2 S[\vec{\xi}_2, \vec{\tilde{\xi}}_1]^j \pi_j^r + 2\lim_{R\to\infty} \oint_{S^2} d^2 S(\tilde{\xi}_1 D^j \tilde{\xi}_2 - \tilde{\xi}_2 D^j \tilde{\xi}_1) \pi_j^r \\
&- \delta_{\vec{\tilde{\xi}}_1} \tilde{\mathcal{B}}(\tilde{\xi}_2) - \lim_{R\to\infty} \oint_{S^2} d^2 S \tilde{\xi}_1^r \tilde{\xi}_2 C + \delta_{\vec{\tilde{\xi}}_2} \tilde{\mathcal{B}}(\tilde{\xi}_1) + \lim_{R\to\infty} \oint_{S^2} d^2 S \tilde{\xi}_2^r \tilde{\xi}_1 C \\
&+ 2\lim_{R\to\infty} \oint_{S^2} d^2 S[\vec{\tilde{\xi}}_1, \vec{\tilde{\xi}}_2]^j \pi_j^r + 2\lim_{R\to\infty} \oint_{S^2} d^2 S[\vec{\xi}_1, \vec{\tilde{\xi}}_2]^j \pi_j^r,
\end{aligned}
\tag{5.15}
$$

where $\tilde{\xi} = \xi N$, $\tilde{\xi}^i = \xi N^i$.

B. The Poisson algebra of symmetry charges

We take into account the boundary condition (4.2) for $\xi, \vec{\xi}$. The bulk terms of (5.15) become

$$
\int_{\mathcal{S}} d^3\mathcal{V}(\hat{\xi}^j C_j + \hat{\xi} H_{\text{phys}}),
\tag{5.16}
$$

where $\hat{\xi}, \hat{\xi}^j$ are

$$
\hat{\xi} = \xi_1^i D_i \xi_2 - \xi_2^i D_i \xi_1, \qquad \hat{\xi}^j = [\vec{\xi}_1, \vec{\xi}_2]^j.
\tag{5.17}
$$

The smearing fields $\xi_{1,2}, \vec{\xi}_{1,2}$ satisfying (4.2) imply that $\hat{\xi}, \hat{\xi}^j$ also satisfy (4.2), see Appendix D. The boundary data of $\hat{\xi}, \hat{\xi}^j$ is denoted by $\widehat{Y}, \widehat{I}, \widehat{f}, \widehat{W}$, and they have the following relations with the boundary data of $\xi_{1,2}, \vec{\xi}_{1,2}$, denoted by $\widehat{Y}_{1,2}, \widehat{I}_{1,2}, \widehat{f}_{1,2}, \widehat{W}_{1,2}$

$$
\begin{aligned}
\widehat{Y}^A &= Y_1^B \bar{D}_B Y_2^A - (1 \leftrightarrow 2), \\
\hat{f} &= Y_1^A \partial_A f_2 - (1 \leftrightarrow 2), \\
\widehat{W} &= Y_1^A \partial_A W_2 - (1 \leftrightarrow 2), \\
\hat{I}^A &= Y_1^B \bar{D}_B I_2^A + I_1^B \bar{D}_B Y_2^A - (1 \leftrightarrow 2).
\end{aligned}
\tag{5.18}
$$

The boundary terms of (5.15) include four parts:

$$
\begin{aligned}
\mathcal{B}_{\xi_1, \xi_2} = &-\delta_{\vec{\tilde{\xi}}_1} \tilde{\mathcal{B}}(\tilde{\xi}_2) - \lim_{R\to\infty} \oint_{S^2} d^2 S \tilde{\xi}_1^r \tilde{\xi}_2 C + \delta_{\vec{\tilde{\xi}}_2} \tilde{\mathcal{B}}(\tilde{\xi}_1) + \lim_{R\to\infty} \oint_{S^2} d^2 S \tilde{\xi}_2^r \tilde{\xi}_1 C \\
&+ 2\lim_{R\to\infty} \oint_{S^2} d^2 S[\vec{\tilde{\xi}}_1, \vec{\tilde{\xi}}_2]^j \pi_j^r + 2\lim_{R\to\infty} \oint_{S^2} d^2 S(\tilde{\xi}_1 D^j \tilde{\xi}_2 - \tilde{\xi}_2 D^j \tilde{\xi}_1) \pi_j^r,
\end{aligned}
\tag{5.19}
$$

$$
\mathcal{B}_{\vec{\xi}_1, \vec{\xi}_2} = 2\lim_{R\to\infty} \oint_{S^2} d^2 S[\vec{\xi}_1, \vec{\xi}_2]^j \pi_j^r,
\tag{5.20}
$$

$$
\mathcal{B}_{\vec{\xi}_1, \xi_2} = -\delta_{\vec{\tilde{\xi}}_1} \tilde{\mathcal{B}}(\tilde{\xi}_2) - \lim_{R\to\infty} \oint_{S^2} d^2 S \xi_1^r \tilde{\xi}_2 C + 2\lim_{R\to\infty} \oint_{S^2} d^2 S[\vec{\xi}_1, \vec{\tilde{\xi}}_2]^j \pi_j^r,
\tag{5.21}
$$

$$
\mathcal{B}_{\xi_1, \vec{\xi}_2} = \delta_{\vec{\tilde{\xi}}_2} \tilde{\mathcal{B}}(\tilde{\xi}_1) + \lim_{R\to\infty} \oint_{S^2} d^2 S \xi_2^r \tilde{\xi}_1 C - 2\lim_{R\to\infty} \oint_{S^2} d^2 S[\vec{\xi}_2, \vec{\tilde{\xi}}_1]^j \pi_j^r.
\tag{5.22}
$$

They come from $\{G(\xi_1), G(\xi_2)\}$, $\{G(\vec{\xi}_1), G(\vec{\xi}_2)\}$, $\{G(\vec{\xi}_1), G(\xi_2)\}$, and $\{G(\xi_1), G(\vec{\xi}_2)\}$, respectively, and the boundary term of $\{G(\xi_1, \vec{\xi}_1), G(\xi_2, \vec{\xi}_2)\}$ is $\mathcal{B}_{\xi_1, \xi_2} + \mathcal{B}_{\vec{\xi}_1, \vec{\xi}_2} + \mathcal{B}_{\vec{\xi}_1, \xi_2} + \mathcal{B}_{\xi_1, \vec{\xi}_2}$. Next, we evaluate them individually. By (3.8)–(3.13), (3.38), and (4.2),

$$
\begin{aligned}
\mathcal{B}_{\xi_1, \xi_2} &= -\delta_{\tilde{\xi}_1} \tilde{\mathcal{B}}(\tilde{\xi}_2) - \lim_{R \to \infty} \oint_{S^2} d^2 S \tilde{\xi}_1^r \tilde{\xi}_2 C + \delta_{\tilde{\xi}_2} \tilde{\mathcal{B}}(\tilde{\xi}_1) + \lim_{R \to \infty} \oint_{S^2} d^2 S \tilde{\xi}_2^r \tilde{\xi}_1 C \\
&\quad + 2 \lim_{R \to \infty} \oint_{S^2} d^2 S [\vec{\tilde{\xi}}_1, \vec{\tilde{\xi}}_2]^j \pi_j^r + 2 \lim_{R \to \infty} \oint_{S^2} d^2 S (\tilde{\xi}_1 D^j \tilde{\xi}_2 - \tilde{\xi}_2 D^j \tilde{\xi}_1) \pi_j^r, \\
&= -\delta_{\tilde{\xi}_1} \mathcal{B}(\tilde{\xi}_2) + \delta_{\tilde{\xi}_2} \tilde{\mathcal{B}}(\tilde{\xi}_1) - \lim_{R \to \infty} \oint_{S^2} d^2 S \xi_1 \xi_2 N^r N C + \lim_{R \to \infty} \oint_{S^2} d^2 S \xi_1 \xi_2 N^r N C \\
&\quad + 2 \lim_{R \to \infty} \oint_{S^2} d^2 S (\xi_1 N^j N^k \partial_k(\xi_2) - \xi_2 N^j N^k \partial_k(\xi_1)) \pi_j^r + 2 \lim_{R \to \infty} \oint_{S^2} d^2 S (\xi_1 D^j \xi_2 - \xi_2 D^j \xi_1) N^2 \pi_j^r, \\
&= -\delta_{\tilde{\xi}_1} \tilde{\mathcal{B}}(\tilde{\xi}_2) + \delta_{\tilde{\xi}_2} \tilde{\mathcal{B}}(\tilde{\xi}_1).
\end{aligned}
\tag{5.23}
$$

Recall (4.16), we have

$$
\delta_{\tilde{\xi}_1} \tilde{\mathcal{B}}(\tilde{\xi}_2) = \mp \oint_{S^2} d^2 S \sqrt{\bar{\gamma}} f_2 (4 \delta_{\tilde{\xi}_1} \bar{\lambda} + 2 \bar{D}_A(\delta_{\tilde{\xi}_1} \bar{\lambda}^A)).
\tag{5.24}
$$

Here $\delta_{\tilde{\xi}_1} \bar{\lambda}$ is determined by the coefficient of $O(r^{-1})$ of $\delta_{\tilde{\xi}_1} g_{rr}$ and $\delta_{\tilde{\xi}_1} \bar{\lambda}_A$ is determined by the coefficient of $O(1)$ of $\delta_{\tilde{\xi}_1} g_{rA}$. With (3.8)–(3.10) and (3.38), we have

$$
\delta_{\tilde{\xi}_1} g_{rr} = O(r^{-2}), \qquad \delta_{\tilde{\xi}_1} g_{rA} = O(r^{-1}).
\tag{5.25}
$$

Henceforth,

$$
\delta_{\tilde{\xi}_1} \bar{\lambda} = 0, \qquad \delta_{\tilde{\xi}_1} \bar{\lambda}_A = 0.
\tag{5.26}
$$

Consequently, we have

$$
\delta_{\tilde{\xi}_1} \tilde{\mathcal{B}}(\tilde{\xi}_2) = \delta_{\tilde{\xi}_2} \tilde{\mathcal{B}}(\tilde{\xi}_1) = 0.
\tag{5.27}
$$

Therefore, we obtain $\mathcal{B}_{\xi_1, \xi_2} = 0$, so the symmetry charges $G(\xi)$ form an Abelian subalgebra. By (3.8)–(3.13), (3.28), (4.2), (4.16), and (5.10),

$$
\begin{aligned}
\mathcal{B}_{\vec{\xi}_1, \vec{\xi}_2} &= 2 \lim_{R \to \infty} \oint_{S^2} d^2 S [\vec{\xi}_1, \vec{\xi}_2]^j \pi_j^r, \\
&= 2 \lim_{R \to \infty} \oint_{S^2} d^2 S [\vec{\xi}_1, \vec{\xi}_2]^r \pi_r^r + 2 \lim_{R \to \infty} \oint_{S^2} d^2 S [\vec{\xi}_1, \vec{\xi}_2]^A \pi_A^r, \\
&= 2 \lim_{R \to \infty} \oint_{S^2} d^2 S \hat{\xi}^r \pi_r^r + 2 \lim_{R \to \infty} \oint_{S^2} d^2 S \hat{\xi}^A \pi_A^r, \\
&= \oint_{S^2} d^2 S \hat{Y}^A (4(\bar{\tilde{k}}_{AB} - \bar{\lambda} \bar{\gamma}_{AB}) \bar{\pi}^{rB} + 2 \bar{\gamma}_{AB} \pi^{(2)rB} + \bar{\lambda}_A (\bar{p} + 2 \bar{\pi}_B^B)) + \oint_{S^2} d^2 S 2 \hat{I}^A \bar{\gamma}_{AB} \bar{\pi}^{rB} + \oint_{S^2} d^2 S \hat{W}(\bar{p} + 2 \bar{\pi}_A^A) \\
&\quad + 2 \lim_{R \to \infty} \oint_{S^2} d^2 S R \hat{Y}^A \bar{\pi}_A^r + 2 \lim_{R \to \infty} \oint_{S^2} d^2 S \log(R) \hat{Y}^A \pi_A^{(\log)r}, \\
&= \mathcal{B}(\hat{W}) + \mathcal{B}(\hat{\tilde{Y}}) + \mathcal{B}(\hat{\tilde{I}}) + 2 \lim_{R \to \infty} \oint_{S^2} d^2 S R \hat{Y}^A \bar{\pi}_A^r + 2 \lim_{R \to \infty} \oint_{S^2} d^2 S \log(R) \hat{Y}^A \pi_A^{(\log)r}.
\end{aligned}
\tag{5.28}
$$

The commutator of Killing vector fields $\hat{Y}^A = [\vec{Y}_1, \vec{Y}_2]^A$ is still a Killing vector field on S^2 and it has odd parity. Similar to the discussions in Sec. IV, the terms $\lim_{R\to\infty} \oint_{S^2} d^2 SR \hat{Y}^A \bar{\pi}_A^r$ and $\lim_{R\to\infty} \oint_{S^2} d^2 S \log(R) \hat{Y}^A \pi_A^{(\log)r}$ are actually vanishing rather than divergent. As the result, we obtain

$$\mathcal{B}_{\vec{\xi}_1, \vec{\xi}_2} = \mathcal{B}(\hat{W}) + \mathcal{B}(\hat{\vec{Y}}) + \mathcal{B}(\hat{\vec{I}}). \tag{5.29}$$

By (5.11), we obtain

$$\begin{aligned}
\mathcal{B}_{\vec{\xi}_1, \xi_2} &= -\delta_{\vec{\xi}} \tilde{\mathcal{B}}(\tilde{\xi}_2) - \lim_{R\to\infty} \oint_{S^2} d^2 S \xi_1^r \tilde{\xi}_2 C \\
&\quad + 2 \lim_{R\to\infty} \oint_{S^2} d^2 S [\vec{\xi}_1, \vec{\tilde{\xi}}_2]^j \pi_j^r, \\
&= \mp \oint_{S^2} d^2 S \sqrt{\bar{\gamma}} f_2 (4\delta_{\vec{\xi}_1} \bar{\lambda} + 2\bar{D}_A (\delta_{\vec{\xi}_1} \bar{\lambda}^A)) \\
&\quad - \lim_{R\to\infty} \oint_{S^2} d^2 S \xi_1^r N \xi_2 C + 2 \lim_{R\to\infty} \oint_{S^2} d^2 S [\vec{\xi}_1, \xi_2 \vec{N}]^j \pi_j^r.
\end{aligned} \tag{5.30}$$

By (3.8)–(3.13), (3.18), (3.26), and (3.38), we find the last two terms vanish. Therefore,

$$\mathcal{B}_{\vec{\xi}_1, \xi_2} = \mp \oint_{S^2} d^2 S \sqrt{\bar{\gamma}} f_2 (4\delta_{\vec{\xi}_1} \bar{\lambda} + 2\bar{D}_A (\delta_{\vec{\xi}_1} \bar{\lambda}^A)). \tag{5.31}$$

Here

$$\delta_{\vec{\xi}_1} \bar{\lambda} = Y_1^A \bar{D}_A \bar{\lambda}, \tag{5.32}$$

$$\delta_{\vec{\xi}_1} \bar{\lambda}_A = \mathcal{L}_{\vec{Y}_1} \bar{\lambda}_A + \bar{D}_A W_1 - \bar{\gamma}_{AB} I^B{}_1. \tag{5.33}$$

In Appendix F, we prove

$$\bar{D}_A (\mathcal{L}_{\vec{Y}_1} \bar{\lambda}_B) = \mathcal{L}_{\vec{Y}_1} (\bar{D}_A \bar{\lambda}_B). \tag{5.34}$$

By this result, (5.31) becomes

$$\begin{aligned}
\mathcal{B}_{\vec{\xi}_1, \xi_2} &= \pm \oint_{S^2} d^2 S (2\sqrt{\bar{\gamma}} Y_1^B \bar{D}_B (f_2)(2\bar{\lambda} + \bar{D}_A \bar{\lambda}^A) \\
&\quad - 2\sqrt{\bar{\gamma}} f_2 (\bar{D}^A \bar{D}_A W_1 - \bar{D}_A I_1^A)).
\end{aligned} \tag{5.35}$$

Here we have used the Killing equations of Y^A. Similarly, we obtain

$$\begin{aligned}
\mathcal{B}_{\xi_1, \vec{\xi}_2} &= \mp \oint_{S^2} d^2 S (2\sqrt{\bar{\gamma}} Y_2^B \bar{D}_B (f_1)(2\bar{\lambda} + \bar{D}_A \bar{\lambda}^A) \\
&\quad - 2\sqrt{\bar{\gamma}} f_1 (\bar{D}^A \bar{D}_A W_2 - \bar{D}_A I_2^A)).
\end{aligned} \tag{5.36}$$

Then we have

$$\begin{aligned}
\mathcal{B}_{\vec{\xi}_1, \xi_2} + \mathcal{B}_{\xi_1, \vec{\xi}_2} &= \pm \oint_{S^2} d^2 S (2\sqrt{\bar{\gamma}} Y_1^B \bar{D}_B (f_2)(2\bar{\lambda} + \bar{D}_A \bar{\lambda}^A) - 2\sqrt{\bar{\gamma}} f_2 (\bar{D}^A \bar{D}_A W_1 - \bar{D}_A I_1^A)) \\
&\quad \mp \oint_{S^2} d^2 S (2\sqrt{\bar{\gamma}} Y_2^B \bar{D}_B (f_1)(2\bar{\lambda} + \bar{D}_A \bar{\lambda}^A) - 2\sqrt{\bar{\gamma}} f_1 (\bar{D}^A \bar{D}_A W_2 - \bar{D}_A I_2^A)), \\
&= \mathcal{B}(\hat{f}) + \mathcal{C}(\hat{\xi}, \hat{\vec{\xi}}),
\end{aligned} \tag{5.37}$$

where

$$\mathcal{B}(\hat{f}) = \pm \oint_{S^2} d^2 S 2\sqrt{\bar{\gamma}} (Y_1^B \bar{D}_B f_2 - Y_2^B \bar{D}_B f_1)(2\bar{\lambda} + \bar{D}_A \bar{\lambda}^A), \tag{5.38}$$

and

$$\begin{aligned}
\mathcal{C}(\hat{\xi}, \hat{\vec{\xi}}) &= \mp 2 \oint_{S^2} d^2 S (\sqrt{\bar{\gamma}} f_2 (\bar{D}^A \bar{D}_A W_1 - \bar{D}_A I_1^A) \\
&\quad - \sqrt{\bar{\gamma}} f_1 (\bar{D}^A \bar{D}_A W_2 - \bar{D}_A I_2^A)).
\end{aligned} \tag{5.39}$$

Therefore, we have the following Poisson algebra of $G(\xi, \vec{\xi})$

$$\{G(\xi_1, \vec{\xi}_2), G(\xi_2, \vec{\xi}_2)\} = G(\hat{\xi}, \hat{\vec{\xi}}) + \mathcal{C}(\hat{\xi}, \hat{\vec{\xi}}). \tag{5.40}$$

Although the Poisson algebra of $G(\xi, \vec{\xi})$ is not immediately close, the quantity $\mathcal{C}(\hat{\xi}, \hat{\vec{\xi}})$ is a phase-space independent constant, and furthermore it is a central charge (see Appendix G for detailed discussion). As a result, we find the symmetry charges $G(\xi, \vec{\xi})$ together with the central charge $\mathcal{C}(\hat{\xi}, \hat{\vec{\xi}})$ form an infinite-dimensional Lie algebra under the Poisson bracket. This algebra is denoted by $\mathcal{A}_G$.

In particular, if the $G(\xi_2, \vec{\xi}_2)$ in (5.40) is restricted to the physical Hamiltonian with boundary term, i.e., $\xi_2 = 1$, $\vec{\xi}_2 = 0$, the result of (5.40) vanishes. Indeed, it is obvious that the bulk terms of (5.40) vanish. For the boundary terms, (5.37) reads

$$\mathcal{B}_{\bar{\xi}_1,1} + \mathcal{B}_{\bar{\xi}_1,0} = \mp 2 \oint_{S^2} \mathrm{d}^2 S \sqrt{\bar{\gamma}} (\bar{D}^A \bar{D}_A W_1 - \bar{D}_A I_1^A) = 0.$$

$$(5.41)$$

The last step is because it is integrated on a closed surface. Therefore, the boundary-preserving symmetry charges commute with the physical Hamiltonian, which satisfies the first requirement of Definition 4.1.

The central charge $\mathcal{C}(\hat{\xi}, \hat{\bar{\xi}})$ vanishes when we require the boundary data of $\xi, \vec{\xi}$ to satisfy certain conditions. For example,

(1) $I^A = \bar{D}^A W$ for the boundary data of $\vec{\xi}$.
(2) The parity condition: we assign the odd parity to W and the even parity to f and I^A.[8]

We next compare $\mathcal{A}_G$ to the original BMS algebra, $\mathcal{A}_{\mathrm{BMS}}$ [32–35]. Before discussing the similarities, it is important to note the distinctions between the two algebras. The original BMS charges are defined as boundary charges at the null infinity of an asymptotically flat spacetime. In contrast, the charges in $\mathcal{A}_G$ have both bulk and boundary terms, while being Dirac observables on the reduced phase space. Furthermore, the boundary in our study is defined at spatial infinity. In addition, the fact of no imposing suitable parity conditions on the leading order fields at spatial infinity yields logarithmic divergences at null infinity, preventing in this way to find BMS algebra there [36–38]. Therefore, here we compare the algebraic structures of these two kinds of algebra, setting aside the associated physical interpretations.

The boundary-preserving generators defined in this work are significantly larger than the original BMS charges. In the bulk, they generate arbitrary physical transformations. Consequently, comparing $\mathcal{A}_G$ with the traditional BMS algebra directly is meaningless. Nevertheless, we can construct a quotient Lie algebra of $\mathcal{A}_G$, which is denoted as $\hat{\mathcal{A}}_G$. $\hat{\mathcal{A}}_G$ is determined by boundary data of ξ^μ, and we will compare it with the original BMS algebra. We consider the vector fields $v^\mu = (v, \vec{v})$, which satisfy $f = W = 0$, $Y^A = I^A = 0$ in (4.2). The symmetry charges corresponding to v^μ constitute a subalgebra of $\mathcal{A}_G$. We denote these charges by $\tilde{G}(v, \vec{v})$ and their algebra by $\tilde{\mathcal{A}}_G$. $\tilde{G}(v, \vec{v})$ is given by

$$\tilde{G}(v, \vec{v}) := \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} \, v H_{\mathrm{phys}} + \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} \, v^j C_j. \qquad (5.42)$$

Note that (5.42) contains no boundary term. We next demonstrate that $\tilde{\mathcal{A}}_G$ is an ideal of $\mathcal{A}_G$. First, it is obvious that $\tilde{\mathcal{A}}_G$ is a subalgebra of $\mathcal{A}_G$. The algebraic structure of

---

[8]$\bar{D}_A \bar{D}^A$ preserves the parity of W: Note that $\bar{D}_A \bar{D}^A W = \partial_A \partial^A W + {}^{\bar{\gamma}}\Gamma^A_{AB} \partial^B W$. The first term preserves the parity of W obviously. For the second term, note that $\bar{\gamma}_{AB}$ has even parity, and ${}^{\bar{\gamma}}\Gamma^A_{AB}$ is first derivative of $\bar{\gamma}_{AB}$, thus ${}^{\bar{\gamma}}\Gamma^A_{AB}$ has odd parity. Therefore, the term ${}^{\bar{\gamma}}\Gamma^A_{AB} \partial^B W$ also preserves the parity conditions of W.

$\mathcal{A}_G$ naturally induces an algebraic structure of $\tilde{\mathcal{A}}_G$, which is given by the Poisson bracket of the generators. $\forall \ \tilde{G}(v, \vec{v}) \in \tilde{\mathcal{A}}_G$, $G(\xi, \vec{\xi}) \in \mathcal{A}_G$, we have

$$\{\tilde{G}(v, \vec{v}), G(\xi, \vec{\xi})\} = \tilde{G}(\hat{v}, \hat{\vec{v}}), \qquad (5.43)$$

with

$$\hat{v} = v^j \partial_j \xi - \xi^j \partial_j v, \qquad (5.44)$$

and

$$\hat{v}^j = [\vec{v}, \vec{\xi}]^j. \qquad (5.45)$$

Therefore, $\tilde{G}(\hat{v}, \hat{\vec{v}}) \in \tilde{\mathcal{A}}_G$. Consequently, $\tilde{\mathcal{A}}_G$ is an ideal of $\mathcal{A}_G$. Next, we define a equivalent relation between the elements of $\mathcal{A}_G$:

Definition 5.1. Given $G(\xi_1, \vec{\xi}_1)$, $G(\xi_2, \vec{\xi}_2) \in \mathcal{A}_G$. $G(\xi_1, \vec{\xi}_1)$ is equivalent to $G(\xi_2, \vec{\xi}_2)$ if

$$G(\xi_1, \vec{\xi}_1) - G(\xi_2, \vec{\xi}_2) \in \tilde{\mathcal{A}}_G. \qquad (5.46)$$

With the equivalent relation given above, we define a quotient Lie algebra of $\mathcal{A}_G$

$$\hat{\mathcal{A}}_G := \mathcal{A}_G / \tilde{\mathcal{A}}_G. \qquad (5.47)$$

The algebraic structure of $\hat{\mathcal{A}}_G$ is naturally induced by the algebraic structure of $\mathcal{A}_G$. First, we define a projection $\Pi : \mathcal{A}_G \to \hat{\mathcal{A}}_G$ as follows:

Definition 5.2. $\forall \ G(\xi, \vec{\xi}) \in \mathcal{A}_G$, $\tilde{G}(v, \vec{v}) \in \tilde{\mathcal{A}}_G$, the projection Π is a linear map $\Pi : \mathcal{A}_G \to \hat{\mathcal{A}}_G$, s.t.,

$$\Pi[G(\xi, \vec{\xi})] = \Pi[G(\xi_1, \vec{\xi}_1) + \tilde{G}(v, \vec{v})] = \hat{G}(\xi, \vec{\xi}), \qquad (5.48)$$

with $\hat{G}(\xi, \vec{\xi}) \in \hat{\mathcal{A}}_G$.

The value of $\hat{G}(\xi, \vec{\xi})$ only depends on the boundary data of ξ^μ. The algebraic structure of $\mathcal{A}_G$ naturally induces a algebraic structure of $\hat{\mathcal{A}}_G$ by Π:

(1) Addition: Given $G(\xi_1, \vec{\xi}_1), G(\xi_2, \vec{\xi}_2) \in \mathcal{A}_G$. Then we have

$$\hat{G}(\xi_1, \vec{\xi}_1) = \Pi[G(\xi_1, \vec{\xi}_1)],$$
$$\hat{G}(\xi_2, \vec{\xi}_2) = \Pi[G(\xi_2, \vec{\xi}_2)]. \qquad (5.49)$$

Then the addition of $\hat{\mathcal{A}}_G$ is given by

$$\hat{G}(\xi_1, \vec{\xi}_1) + \hat{G}(\xi_2, \vec{\xi}_2) := \Pi[G(\xi_1, \vec{\xi}_1) + G(\xi_2, \vec{\xi}_2)]. \qquad (5.50)$$

The self-consistency of this definition is easy to check: Given $\tilde{G}(v_1, \vec{v}_1), \tilde{G}(v_2, \vec{v}_2) \in \tilde{\mathcal{A}}_G$, then

$$\hat{G}(\xi_1, \vec{\xi}_1) = \Pi[G(\xi_1, \vec{\xi}_1) + \tilde{G}(v_1, \vec{v}_1)],$$
$$\hat{G}(\xi_2, \vec{\xi}_2) = \Pi[G(\xi_2, \vec{\xi}_2) + \tilde{G}(v_2, \vec{v}_2)]. \quad (5.51)$$

As the result, we have

$$\hat{G}(\xi_1, \vec{\xi}_1) + \hat{G}(\xi_2, \vec{\xi}_2) = \Pi[G(\xi_1, \vec{\xi}_1) + \tilde{G}(v_1, \vec{v}_1)$$
$$+ G(\xi_2, \vec{\xi}_2) + \tilde{G}(v_2, \vec{v}_2)],$$
$$= \Pi[G(\xi_1, \vec{\xi}_1) + G(\xi_2, \vec{\xi}_2)].$$
$$(5.52)$$

(2) Scalar multiplication: Given $G(\xi, \vec{\xi}) \in \mathcal{A}_G$, with $\hat{G}(\xi, \vec{\xi}) = \Pi[G(\xi, \vec{\xi})]$. $\forall \alpha \in R$, the scalar Multiplication is given by

$$\alpha \hat{G}(\xi, \vec{\xi}) = \Pi[\alpha G(\xi, \vec{\xi})]. \quad (5.53)$$

(3) Multiplication: The multiplication is essential in the definition of an algebra. In our case, we denote it as $\{\cdot, \cdot\}_B$, which is a bilinear map $\{\cdot, \cdot\}_B : \hat{\mathcal{A}}_G \times \hat{\mathcal{A}}_G \to \hat{\mathcal{A}}_G$, and it is defined as following: Given $G(\xi_1, \vec{\xi}_1), G(\xi_2, \vec{\xi}_2) \in \mathcal{A}_G$. Then we have

$$\hat{G}(\xi_1, \vec{\xi}_1) = \Pi[G(\xi_1, \vec{\xi}_1)],$$
$$\hat{G}(\xi_2, \vec{\xi}_2) = \Pi[G(\xi_2, \vec{\xi}_2)]. \quad (5.54)$$

The $\{\cdot, \cdot\}_B$ is defined as

$$\{\hat{G}(\xi_1, \vec{\xi}_1), \hat{G}(\xi_2, \vec{\xi}_2)\}_B := \Pi[\{G(\xi_1, \vec{\xi}_1), G(\xi_2, \vec{\xi}_2)\}]. \quad (5.55)$$

Next, we check the self-consistency of the definition of $\{\cdot, \cdot\}_B$. Given $\tilde{G}(v_1, \vec{v}_1), \tilde{G}(v_2, \vec{v}_2) \in \tilde{\mathcal{A}}_G$, then

$$\hat{G}(\xi_1, \vec{\xi}_1) = \Pi[G(\xi_1, \vec{\xi}_1) + \tilde{G}(v_1, \vec{v}_1)],$$
$$\hat{G}(\xi_2, \vec{\xi}_2) = \Pi[G(\xi_2, \vec{\xi}_2) + \tilde{G}(v_2, \vec{v}_2)]. \quad (5.56)$$

Note that $\tilde{\mathcal{A}}_G$ is an ideal of $\mathcal{A}_G$, we have

$$\{G(\xi_1, \vec{\xi}_1) + \tilde{G}(v_1, \vec{v}_1), G(\xi_2, \vec{\xi}_2) + \tilde{G}(v_2, \vec{v}_2)\}$$
$$= \{G(\xi_1, \vec{\xi}_1), G(\xi_2, \vec{\xi}_2)\} + \{G(\xi_1, \vec{\xi}_1), \tilde{G}(v_2, \vec{v}_2)\}$$
$$+ \{\tilde{G}(v_1, \vec{v}_1), G(\xi_2, \vec{\xi}_2)\} + \{\tilde{G}(v_1, \vec{v}_1), \tilde{G}(v_2, \vec{v}_2)\},$$
$$= \{G(\xi_1, \vec{\xi}_1), G(\xi_2, \vec{\xi}_2)\} + \tilde{G}(v, \vec{v}), \quad (5.57)$$

with

$$\tilde{G}(v, \vec{v}) = \{G(\xi_1, \vec{\xi}_1), \tilde{G}(v_2, \vec{v}_2)\}$$
$$+ \{\tilde{G}(v_1, \vec{v}_1), G(\xi_2, \vec{\xi}_2)\}$$
$$+ \{\tilde{G}(v_1, \vec{v}_1), \tilde{G}(v_2, \vec{v}_2)\}, \quad (5.58)$$

and $\tilde{G}(v, \vec{v}) \in \tilde{\mathcal{A}}_G$. Consequently, we find

$$\{\hat{G}(\xi_1, \vec{\xi}_1), \hat{G}(\xi_2, \vec{\xi}_2)\}_B = \Pi[\{G(\xi_1, \vec{\xi}_1)$$
$$+ \tilde{G}(v_1, \vec{v}_1), G(\xi_2, \vec{\xi}_2)$$
$$+ \tilde{G}(v_2, \vec{v}_2)\}],$$
$$= \Pi[\{G(\xi_1, \vec{\xi}_1), G(\xi_2, \vec{\xi}_2)\}]. \quad (5.59)$$

In the original BMS algebra, the supertranslations form an Abelian subalgebra, which is denoted as $\mathcal{A}_{\text{BMS}}^{(ab)}$ in this paper. Similarly, in our work, as indicated by (5.40), $G(f)$ and $G(W)$ also form an Abelian subalgebra when the central term $\mp 2 \oint_{S^2} d^2S (\sqrt{\bar{\gamma}} f_2 \bar{D}^A \bar{D}_A W_1 - \sqrt{\bar{\gamma}} f_1 \bar{D}^A \bar{D}_A W_2)$ vanishes. This central term arises from the contribution of W in (5.39), and it vanishes if we assign even parity to f and odd parity to W. Therefore, we have an Abelian subalgebra of $\hat{\mathcal{A}}_G$, which is denoted as $\hat{\mathcal{A}}_G^{(ab)}$. Next, we show that $\hat{\mathcal{A}}_G^{(ab)}$ is analog to $\mathcal{A}_{\text{BMS}}^{(ab)}$. Indeed, Ref. [28] proves that in vacuum gravity, the algebra of the supertranslations at the spatial infinity is isomorphic to $\mathcal{A}_{\text{BMS}}^{(ab)}$ (which is at the null infinity). We provide a brief review of the proof below. We first rewrite the asymptotically flat metric (3.1) using a generalization of the Beig-Schmidt ansatz [31,43,44]

$$g_{\mu\nu} dx^\mu dx^\nu = \left(1 + \frac{2\sigma}{\eta} + \frac{\sigma^2}{\eta^2} + o(\eta^{-2})\right) d\eta^2 + o(\eta^{-1}) d\eta dx^b$$
$$+ (\eta^2 h_{ab}^{(0)} + \eta(k_{ab} - 2\sigma h_{ab}^{(0)}) + \log \eta i_{ab}$$
$$+ h_{ab}^{(2)} + o(\eta^0)) dx^a dx^b. \quad (5.60)$$

Here

$$\eta^2 = -t^2 + r^2. \quad (5.61)$$

The coordinates x^a are

$$x^a = (s, x^A) \ (A = 1, 2), \quad (5.62)$$

where $X^A (A = 1, 2)$ are the angular coordinates, and

$$s = \pm t/r. \quad (5.63)$$

σ, k_{ab}, i_{ab}, and $h_{ab}^{(n)}$ depend on x^a only. The spatial infinity is given by the limitation $\eta \to \infty$, and the boundary metric $h_{ab}^{(0)}$ is the metric of a unit hyperboloid, which reads

$$h_{ab}^{(0)} dx^a dx^b = \frac{-1}{(1-s^2)^2} ds^2 + \frac{1}{1-s^2} \gamma_{AB} dx^A dx^B. \quad (5.64)$$

The covariant derivative compatible with $h_{ab}^{(0)}$ is denoted as $\mathcal{D}_a$.

Next, we aim to establish a connection between the null infinity $\mathcal{I}^{\pm}$ and the spatial infinity i^0 of the asymptotically flat spacetime. In the geometrical definition of the asymptotically flat spacetime $(M, g_{\mu\nu})$, one requires that there exists a conformal spacetime $(\tilde{M}, \tilde{g}_{\mu\nu})$, such that

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}. \quad (5.65)$$

Where Ω is a scalar field satisfying certain conditions [45]. To construct Ω explicitly, we follow the scheme described in [44]. Assuming some smoothness conditions for the metric around the spatial infinity (for more details, refer to [46–48]), we can introduce a new coordinate system in a neighborhood of i^0, which is (ρ, s, x^A). ρ is a new coordinate we introduce, which is given by

$$\eta = \frac{1}{\rho\sqrt{1-s^2}}. \quad (5.66)$$

Note that the spatial infinity is given by $\eta \to \infty$. Therefore, it is also given by $\rho \to 0$. Then Ω reads

$$\Omega = \tilde{\Omega}\left(1 - \left(\frac{s}{\tilde{\omega}}\right)^2\right), \quad (5.67)$$

where $\tilde{\Omega}$ and $\tilde{\omega}$ are some scalar functions satisfying

$$\lim_{\rho \to 0} \rho^{-1}\tilde{\Omega} = 1, \qquad \lim_{\rho \to 0} \tilde{\omega} = 1. \quad (5.68)$$

The null infinity $\mathcal{I}^{\pm}$ locates at

$$s = \pm\tilde{\omega}. \quad (5.69)$$

At the neighborhood of i^o, the null infinity $\mathcal{I}^{\pm}$ locates at

$$s = \pm 1. \quad (5.70)$$

Then we can connect spatial infinity and the null infinity by taking the following limitation:

$$\rho \to 0, \qquad s \to \pm 1. \quad (5.71)$$

The parameter that characterizes the supertranslation is govern by the following equation

$$\mathcal{D}_a \mathcal{D}^a \omega = 0, \quad (5.72)$$

where ω is a function of (s, x^A). Reference [44] states that the solutions with odd parity[9] of (5.72) correspond to the supertanslations, which are isomorphic to the usual BMS supertanslations. These solutions are

$$\omega = (1-s^2)^{-1/2}\hat{\omega}. \quad (5.73)$$

Here $\hat{\omega}$ can be expanded by the spherical harmonics

$$\hat{\omega} = \sum_{lm} \omega_{l,m}\Psi_l(s)Y_{l,m}(x^A), \quad \Psi_l = \frac{1}{2}(1-s^2)^2\partial_s^2 Q_l. \quad (5.74)$$

Here $Q_l(s)$ are Legendre functions of the second kind and can be written in terms of Legendre Polynomials $P_l(s)$ as

$$Q_l(s) = P_l(s)\frac{1}{2}\log\left(\frac{1+s}{1-s}\right) + \tilde{Q}_l(s), \quad (5.75)$$

where $\tilde{Q}_l(s)$ are polynomials. Furthermore, we have the following limitation:

$$\lim_{s \to 1} \Psi_l = 1, \quad (5.76)$$

which can be found in [44]. We then introduce

$$\mathcal{T} = \lim_{s \to 1}(\sqrt{1-s^2}\omega). \quad (5.77)$$

Here $\mathcal{T}$ can be viewed as the parameter of usual BMS supertranslations, and it is expanded as

$$\mathcal{T} = \sum_{lm} \omega_{l,m}Y_{l,m}(x^A). \quad (5.78)$$

In Ref. [28], f is assigned with even parity and W is assigned with odd parity. Then the subalgebra formed by f and W is isomorphic to the subalgebra formed by $\mathcal{T}$. To see this, we need to show the coefficients ω_{ml} in (5.78) can be determined by f and W completely, and vice versa. Note that $\hat{\omega}$ is odd, we can set its initial conditions at $s = 0$:

$$\omega|_{s=0} = \hat{\omega}|_{s=0} = W(x^A), \quad \partial_s\omega|_{s=0} = \partial_s\hat{\omega}|_{s=0} = f(x^A). \quad (5.79)$$

Note that W is odd and f is even, they are expanded by the spherical harmonics as

---

[9]Here odd parity means $\omega(s, x^A) = -\omega(-s, -x^A)$.

$$W = \sum_k \sum_{m=-2k-1}^{2k+1} W_{2k+1,m} Y_{2k+1,m},$$

$$f = \sum_k \sum_{m=-2k}^{2k} f_{2k,m} Y_{2k,m}. \tag{5.80}$$

As the result, the coefficients ω_{ml} are determined by

$$\omega_{2k+1,m} \Psi_{2k+1}|_{s=0} = W_{2k+1,m}, \qquad \omega_{2k,m} \partial_s \Psi_{2k}|_{s=0} = f_{2k,m}. \tag{5.81}$$

Therefore, the algebra of the spuertranslations obtained in [28] and $\mathcal{A}_{\mathrm{BMS}}^{(ab)}$ are isomorphic.

In our case, the BMS charge might be ill defined at the null infinity due to the relaxation of the parity conditions of the canonical variables, as previously discussed. Nevertheless, it is still interesting to analyze the similarity between $\hat{\mathcal{A}}_{\mathrm{G}}^{(ab)}$ and the original BMS supertranslations. Here we emphasize that we only compare the algebraic structure of these two algebras, setting aside the associated physical interpretations. Note that $\hat{\mathcal{A}}_{\mathrm{G}}^{(ab)}$ is formally equivalent to the algebra of the spuertranslations obtained in [28]. Consequently, $\hat{\mathcal{A}}_{\mathrm{G}}^{(ab)}$ can be considered as an analogy of $\mathcal{A}_{\mathrm{BMS}}^{(ab)}$ at the algebraic level. In general, f and W are not required to have the parity conditions. Moreover, we have additionally the supertranslations contributed by I^A. Therefore, the subalgebra in $\hat{\mathcal{A}}_{\mathrm{G}}$ of supertranslations represents a generalized algebraic structure of the original BMS subalgebra of supertranslations.

The asymptotic spatial rotations in $\hat{\mathcal{A}}_{\mathrm{G}}$ form an SO(3) algebra thus is analogous to the corresponding subalgebra in BMS. The commutator between the supertranslations and spatial rotations also involves the central charge of the algebra.[10]

It is also interesting to compare $\hat{\mathcal{A}}_{\mathrm{G}}$ to the results in [28], which constructs the BMS group at the spatial infinity. The analysis there relies on the usual falloff (3.3) and (3.4) and the parity conditions on $\bar{h}_{ij}$ and $\bar{\pi}_{ij}$. The resulting BMS algebra in [28] does not have the central charge due to the additional boundary condition $\bar{\lambda}_A = 0$. Preserving $\bar{\lambda}_A = 0$ needs to require $I^A = \frac{2b}{\sqrt{\bar{\gamma}}} \bar{\pi}^{rA} + \bar{D}^A W$, where b is the boost parameter. $\hat{\mathcal{A}}_{\mathrm{G}}$ does not take into account the boost, and $b = 0$ precisely reduces this requirement to $I^A = \bar{D}^A W$ for vanishing $\mathcal{C}(\xi, \vec{\xi})$ mentioned above. Another way to make $\mathcal{C}(\xi, \vec{\xi})$ vanishing is to assign even parity to f and odd parity to W. If we impose these parity conditions to f and W and require $I^A = \bar{D}^A W$, then $\hat{\mathcal{A}}_{\mathrm{G}}$ recovers the BMS algebra at

---

[10]As a comparison, the central charges obtained in [35] are field dependent, whereas the central charge here is not.

spatial infinity given by [28] with vanishing b (vanishing boost generators).

C. Preservation of the falloff conditions and the boundary conditions

This subsection shows that the boundary-preserving symmetry transformations preserve the falloff conditions (3.8)–(3.13) and the boundary conditions (3.28). This means that the transformations generated by the boundary-preserving charges are restricted in the phase space defined in Sec. III.

As stated in Sec. IV, $G(\xi, \vec{\xi})$ is finite and its variation is well defined in the phase space. Therefore, the Hamiltonian flow of $G(\xi, \vec{\xi})$ is well defined and we can compute the transformations of the canonical variables with Poisson bracket:

$$\delta_G g_{ij} = \{g_{ij}, G(\xi, \vec{\xi})\},$$
$$= 2\tilde{\xi} g^{-\frac{1}{2}} \left(\pi_{ij} - \frac{1}{2} g_{ij} \pi \right) + 2D_{(i}\tilde{\xi}_{j)} + 2D_{(i}\xi_{j)}, \tag{5.82}$$

$$\delta_G \pi^{ij} = \{\pi^{ij}, G(\xi, \vec{\xi})\},$$
$$= -\tilde{\xi} g^{\frac{1}{2}} \left(g^{ik} g^{jl(3)} R_{lk} - \frac{1}{2} g^{ij(3)} R \right)$$
$$+ \frac{1}{2} \tilde{\xi} g^{-\frac{1}{2}} g^{ij} \left(\pi_{mn} \pi^{mn} - \frac{1}{2} \pi^2 \right)$$
$$- 2\tilde{\xi} g^{-\frac{1}{2}} \left(\pi^{im} \pi_m^j - \frac{1}{2} \pi^{ij} \pi \right) + g^{\frac{1}{2}} (D^i D^j \tilde{\xi} - g^{ij} D_m D^m \tilde{\xi})$$
$$+ \mathcal{L}_{\vec{\tilde{\xi}}} \pi^{ij} + \frac{\xi}{2} H_{\mathrm{phys}} N^i N^j + \mathcal{L}_{\vec{\xi}} \pi^{ij}. \tag{5.83}$$

Here $((\xi, \vec{\xi})$ is given by (4.2) and $\tilde{\xi}^j = N\xi^j$. $^{(3)}R_{lk}$ is the Ricci tensor of g_{ij} and $^{(3)}R$ is the corresponding Ricci scalar. With (3.6), we have

$$^{(3)}R_{lk} = \frac{\bar{R}_{lk}}{r^3} + \frac{\log(r)}{r^4} R_{lk}^{(\log)} + O(r^{-4}) \tag{5.84}$$

in the asymptotically Cartesian coordinates. Then we find

$$\delta_G g_{ij} = \frac{\bar{\mathcal{H}}_{ij}}{r} + \frac{\log(r)}{r^2} \mathcal{H}_{ij}^{(\log)} + O(r^{-2}), \tag{5.85}$$

$$\delta_G \pi^{ij} = \frac{\bar{\mathcal{P}}^{ij}}{r^2} + \frac{\log(r)}{r^3} \mathcal{P}^{(\log)ij} + \frac{\mathcal{P}^{(2)ij}}{r^3}$$
$$+ \frac{\log(r)}{r^4} \mathcal{P}^{(ll)ij} + O(r^{-4}). \tag{5.86}$$

We have used $\bar{\mathcal{H}}_{ij}, \bar{\mathcal{P}}^{ij}, \mathcal{H}_{ij}^{(\log)}, \mathcal{P}^{(\log)ij}, \mathcal{P}^{(2)ij}, \mathcal{P}^{(ll)ij}$ to label the expansions coefficients, whose explicit expression may

not be useful for the discussion. Equations (5.85)–(5.86) satisfy the falloff conditions (3.6)–(3.7). Additionally, (5.86) gives

$$\delta_G \pi^{rA} = \frac{\bar{\mathcal{P}}^{rA}}{r} + \frac{\log(r)}{r^2} \mathcal{P}^{(\log)rA} + \frac{\mathcal{P}^{(2)rA}}{r^2}$$
$$+ \frac{\log(r)}{r^3} \mathcal{P}^{(ll)rA} + O(r^{-3}). \qquad (5.87)$$

Especially, $\mathcal{P}^{(\log)rA} = \mathcal{L}_{\vec{Y}} \pi^{(\log)rA}$. Therefore, the parity of $\pi^{(\log)rA}$ is preserved.

The charge $G(\xi, \vec{\xi})$ infinitesimally transforms C_j as[11]

$$\{C_j, G(\xi, \vec{\xi})\} = \{C_j, G(\xi)\} + \{C_j, G(\vec{\xi})\},$$
$$= \partial_j(\xi) H_{\text{phys}} + \mathcal{L}_{\vec{\xi}} C_j,$$
$$= \partial_j(\xi) H_{\text{phys}} + \xi^i D_i C_j + C_i D_j \xi^i + C_j D_i \xi^i. \qquad (5.88)$$

To check the boundary conditions of C_j are preserved by (5.88), the asymptotic Cartesian coordinates are convenient. With (3.16)–(3.18), (3.26), and (3.28), we can rewrite the asymptotic behaviors of C_j and H_{phys} in the asymptotic Cartesian coordinates

$$C_j = \frac{\log r}{r^5} C_j^{(ll)} + O(r^{-5}), \qquad (5.89)$$

$$H_{\text{phys}} = \frac{\log r}{r^4} H^{(\log)} + O(r^{-4}). \qquad (5.90)$$

Additionally, with (4.2), $\xi = O(1)$ and $\xi^i = O(r)$ in the asymptotic Cartesian coordinates. Thus (5.88) gives

$$\{C_j, G(\xi, \vec{\xi})\} = O\left(\frac{\log r}{r^5}\right), \qquad (5.91)$$

which agrees with (5.89). Since H_{phys} commutes with itself [49], $G(\xi, \vec{\xi})$ infinitesimally transforms H_{phys} as

$$\{H_{\text{phys}}, G(\xi, \vec{\xi})\} = \{H_{\text{phys}}, G(\xi)\} + \{H_{\text{phys}}, G(\xi, \vec{\xi})\},$$
$$= \mathcal{L}_{\vec{\xi}} H_{\text{phys}} = D_i(\xi^i H_{\text{phys}}) = \partial_i(\xi^i H_{\text{phys}}). \qquad (5.92)$$

Note that H_{phys} is a density of weight one, D_i can be replaced by ∂_i in the last step. Similar to the discussions above, we have

---

[11]We apply $(D_i D_j - D_j D_i)v^k = -R_{ijl}{}^k v^l$ in the calculation.

$$\{H_{\text{phys}}, G(\xi, \vec{\xi})\} = O\left(\frac{\log r}{r^4}\right), \qquad (5.93)$$

which agrees with (5.90).

VI. ON THE BOOST TRANSFORMATION

In this section we discuss the boost component of the boundary-preserving generators. We will explore some challenges that arise when considering the boost component, which is why we have chosen not to include it in this work.

The vector field of the boost component is [28,29]

$$\xi_b = rb, \qquad (6.1)$$

where $b(\sigma^A)$ is the boost parameter, satisfying

$$\bar{D}_A \bar{D}_B b + \bar{\gamma}_{AB} b = 0. \qquad (6.2)$$

In the traditional spherical coordinates $\{\theta, \varphi\}$, b takes the following formula

$$b = b_1 \sin\theta \cos\varphi + b_2 \sin\theta \sin\varphi + b_3 \cos\theta. \qquad (6.3)$$

Here b_1, b_2, and b_3 are some constants. It is obvious that b has odd parity:

$$b(\pi - \theta, \varphi + \pi) = -b(\pi, \varphi). \qquad (6.4)$$

We formally construct the boost component of the boundary-preserving symmetry generators as

$$G_b = \int_{\mathcal{S}} \mathrm{d}^3 \mathcal{V} \xi_b H_{\text{phys}} + \mathcal{B}_b. \qquad (6.5)$$

To ensure the finiteness of the bulk term of G_b, it is necessary for H_{phys} to decay at a faster rate than $O(r^{-2})$ in the spherical coordinates. We first introduce the higher orders falloff conditions of the canonical variables:

$$g_{rr} = 1 + \frac{1}{r}\bar{h}_{rr} + \frac{\log r}{r^2} h_{rr}^{(\log)} + \frac{1}{r^2} h_{rr}^{(2)} + \frac{\log r}{r^3} h_{rr}^{(ll)}$$
$$+ \frac{1}{r^3} h_{rr}^{(3)} + \frac{\log r}{r^4} h_{rr}^{(lll)} + o\left(\frac{\log r}{r^4}\right), \qquad (6.6)$$

$$g_{rA} = \bar{h}_{rA} + \frac{\log r}{r} h_{rA}^{(\log)} + \frac{1}{r} h_{rA}^{(2)} + \frac{\log r}{r^2} h_{rA}^{(ll)}$$
$$+ \frac{1}{r^2} h_{rA}^{(3)} + \frac{\log r}{r^3} h_{rA}^{(lll)} + o\left(\frac{\log r}{r^3}\right), \qquad (6.7)$$

$$g_{AB} = r^2 \bar{\gamma}_{AB} + r\bar{h}_{AB} + \log(r) h_{AB}^{(\log)} + h_{AB}^{(2)} + \frac{\log r}{r} h_{AB}^{(ll)}$$
$$+ \frac{1}{r} h_{AB}^{(3)} + \frac{\log r}{r^2} h_{AB}^{(lll)} + o\left(\frac{\log r}{r^2}\right), \qquad (6.8)$$

$$\pi^{rr} = \bar{\pi}^{rr} + \frac{\log r}{r} \pi^{(\log)rr} + \frac{1}{r} \pi^{(2)rr} + \frac{\log r}{r^2} \pi^{(ll)rr} + \frac{\pi^{(3)rr}}{r^2}$$
$$+ \frac{\log r}{r^3} \pi^{(lll)rr} + \frac{\pi^{(4)rr}}{r^3} + \frac{\log r}{r^4} \pi^{(llll)rr} + o\left(\frac{\log r}{r^4}\right), \tag{6.9}$$

$$\pi^{rA} = \frac{1}{r} \bar{\pi}^{rA} + \frac{\log r}{r^2} \pi^{(\log)rA} + \frac{1}{r^2} \pi^{(2)rA} + \frac{\log r}{r^3} \pi^{(ll)rA} + \frac{\pi^{(3)rA}}{r^3}$$
$$+ \frac{\log r}{r^4} \pi^{(lll)rA} + \frac{\pi^{(4)rA}}{r^4} + \frac{\log r}{r^5} \pi^{(llll)rA} + o\left(\frac{\log r}{r^5}\right), \tag{6.10}$$

$$\pi^{AB} = \frac{1}{r^2} \bar{\pi}^{AB} + \frac{\log r}{r^3} \pi^{(\log)AB} + \frac{1}{r^3} \pi^{(2)AB} + \frac{\log r}{r^4} \pi^{(ll)AB} + \frac{\pi^{(3)AB}}{r^4}$$
$$+ \frac{\log r}{r^5} \pi^{(lll)AB} + \frac{\pi^{(4)AB}}{r^5} + \frac{\log r}{r^6} \pi^{(llll)AB} + o\left(\frac{\log r}{r^6}\right). \tag{6.11}$$

By (6.6)–(6.11),

$$C = \frac{C^{(1)}}{r} + \frac{\log r}{r^2} C^{(\log)} + \frac{C^{(2)}}{r^2} + \frac{\log r}{r^3} C^{(ll)} + \frac{C^{(3)}}{r^3} + \frac{(\log r)^2}{r^4} \tilde{C}^{(lll)} + \frac{\log r}{r^4} C^{(lll)} + o\left(\frac{\log r}{r^4}\right),$$

$$C_r = \frac{C_r^{(1)}}{r} + \frac{\log r}{r^2} C_r^{(\log)} + \frac{C_r^{(2)}}{r^2} + \frac{\log r}{r^3} C_r^{(ll)} + \frac{C_r^{(3)}}{r^3} + \frac{(\log r)^2}{r^4} \tilde{C}_r^{(lll)} + \frac{\log r}{r^4} C_r^{(lll)}$$
$$+ \frac{C_r^{(4)}}{r^4} + \frac{(\log r)^2}{r^5} \tilde{C}_r^{(llll)} + \frac{\log r}{r^5} C_r^{(llll)} + o\left(\frac{\log r}{r^5}\right),$$

$$C_A = C_A^{(1)} + \frac{\log r}{r} C_A^{(\log)} + \frac{C_A^{(2)}}{r} + \frac{\log r}{r^2} C_A^{(ll)} + \frac{C_A^{(3)}}{r^2} + \frac{(\log r)^2}{r^3} \tilde{C}_A^{(lll)} + \frac{\log r}{r^3} C_A^{(lll)}$$
$$+ \frac{C_A^{(4)}}{r^3} + \frac{(\log r)^2}{r^4} \tilde{C}_A^{(llll)} + \frac{\log r}{r^4} C_A^{(llll)} + o\left(\frac{\log r}{r^4}\right). \tag{6.12}$$

In this case, the boundary conditions for C, C_j are

$$C_A^{(1)} = C_A^{(\log)} = C_A^{(2)} = C_A^{(ll)} = C_A^{(3)} = \tilde{C}_A^{(lll)} = C_A^{(lll)} = C_A^{(4)}$$
$$= C_r^{(1)} = C_r^{(2)} = C_r^{(ll)} = C_r^{(3)} = C_r^{(\log)} = \tilde{C}_r^{(lll)} = C_r^{(lll)} = C_r^{(4)}$$
$$= C^{(1)} = C^{(\log)} = C^{(2)} = C^{(ll)} = 0,$$
$$C^{(3)} \neq 0. \tag{6.13}$$

These boundary conditions ensure

$$N^r = O\left(\frac{(\log r)^2}{r^2}\right), \qquad N^A = O\left(\frac{(\log r)^2}{r^3}\right). \tag{6.14}$$

Similar to the discussions in Sec. III B, the nondifferentiable terms of $\delta \int_S d^3 \mathcal{V} \xi_b H_{\text{phys}}$ contributed by $\delta \int_S d^3 \mathcal{V} \xi_b N^j C_j$ is then vanished. Furthermore, similar to Sec. IV, the boundary terms $\mathcal{B}_b$ are given by variating $\int_S d^3 \mathcal{V} \xi_b H_{\text{phys}}$ and integrating by part:

$$\delta \int_S d^3 \mathcal{V} \xi_b H_{\text{phys}} = \int_S d^3 \mathcal{V}[\mathcal{A}_{ij} \delta \pi^{ij} - \mathcal{B}^{ij} \delta g_{ij}] + \mathcal{K}_b, \tag{6.15}$$

where

$$\mathcal{A}_{ij} = 2rbg^{-\frac{1}{2}}\left(\pi_{ij} - \frac{1}{2}g_{ij}\pi\right) + \mathcal{L}_{\vec{b}}g_{ij},$$

$$\mathcal{B}^{ij} = -rbg^{\frac{1}{2}}\left(R^{ij} - \frac{1}{2}g^{ij}R\right) + \frac{1}{2}rbg^{-\frac{1}{2}}g^{ij}\left(\pi_{mn}\pi^{mn} - \frac{1}{2}\pi^2\right)$$

$$- 2rbg^{-\frac{1}{2}}\left(\pi^{im}\pi_m^j - \frac{1}{2}\pi^{ij}\pi\right)$$

$$+ g^{\frac{1}{2}}(D^iD^j(rb) - g^{ij}D^mD_m(rb))$$

$$+ \mathcal{L}_{\vec{b}}\pi^{ij} + \frac{rb}{2}H_{\text{phys}}N^iN^j, \tag{6.16}$$

with $\vec{b} := rb\vec{N}$. Before computing the boundary terms of (6.15), we first introduce some useful notations. Following [28], the extrinsic curvature of a 2-sphere reads

$$K_{AB} = \frac{1}{2\lambda}(-\partial_r g_{AB} + D_A\lambda_B + D_B\lambda_A). \tag{6.17}$$

Here $\lambda_A = g_{rA}$. With (3.8)–(3.10), its trace $K := \gamma^{AB}K_{AB}$ asymptotically behaviors as

$$K = -\frac{2}{r} + \frac{\bar{k}}{r^2} + \frac{\log(r)}{r^3}k^{(\log)} + \frac{k^{(2)}}{r^3} + o(r^{-3}). \tag{6.18}$$

Then the boundary terms of (6.15) read

$$\mathcal{K}_b = \pm \lim_{R\to\infty} \oint_{S^2} \mathrm{d}^2S\sqrt{\bar{\gamma}}(-2Rb\delta\bar{k} - 2\log(R)b\delta k^{(\log)}$$

$$- \frac{1}{4}b\delta(\bar{h}^2 + \bar{h}^{AB}\bar{h}_{AB}) - 2b\delta k^{(2)}$$

$$+ (\bar{\lambda}^C\partial_C b\bar{\gamma}^{AB} - b\bar{D}^A\bar{\lambda}^B)\delta\bar{h}_{AB} - \bar{h}b\delta(2\bar{\lambda} + \bar{D}_A\bar{\lambda}^A)). \tag{6.19}$$

The terms $(\bar{\lambda}^C\partial_C b\bar{\gamma}^{AB} - b\bar{D}^A\bar{\lambda}^B)\delta\bar{h}_{AB}$ are obviously not to be differentials, so we introduce an additional boundary $\bar{\lambda}_A = 0$ to make them vanish. To deal with $-\bar{h}b\delta(2\bar{\lambda} + \bar{D}_A\bar{\lambda}^A)$, we require the function f in (4.2) to be phase-space dependent:

$$f = -b\bar{\lambda} - b\bar{k} + T, \tag{6.20}$$

where T is an arbitrary function on S^2. The detailed derivation can be found in [28]. There are a linear divergent term $\mp 2R\oint_{S^2}\mathrm{d}^2S\sqrt{\bar{\gamma}}b\delta\bar{k}$ and a logarithmic divergent term $\mp 2\log(R)\oint_{S^2}\mathrm{d}^2S\sqrt{\bar{\gamma}}b\delta k^{(\log)}$ in (6.15). By imposing the boundary condition $C^{(1)} = 0$, we find that $\mp 2R\oint_{S^2}\mathrm{d}^2S\sqrt{\bar{\gamma}}b\delta\bar{k}$ vanishes with (6.15), see [28]. One way to get rid of $\mp 2\log(R)\oint_{S^2}\mathrm{d}^2S\sqrt{\bar{\gamma}}b\delta k^{(\log)}$ is to assign even parity to $k^{(\log)}$. The parity condition of $k^{(\log)}$ is not preserved by G_b in general but we can assign additional parity conditions to $\pi^{(\log)ij}$ to ensure the parity condition of $k^{(\log)}$ be preserved, which will be shown shortly. Finally we get

$$\mathcal{K}_b = \mp \lim_{R\to\infty} \oint_{S^2} \mathrm{d}^2S\sqrt{\bar{\gamma}}b\delta(2k^{(2)} + \bar{k}^2 + \bar{k}_B^A\bar{k}_A^B - 6\bar{\lambda}\bar{k}). \tag{6.21}$$

Then we can compute the transformations of the canonical variables:

$$\delta_b g_{ij} = 2rbNg^{-\frac{1}{2}}\left(\pi_{ij} - \frac{1}{2}g_{ij}\pi\right) + 2D_{(i}(rbN_{j)}),$$

$$\delta_b\pi^{ij} = -rbNg^{\frac{1}{2}}\left(R^{ij} - \frac{1}{2}g^{ij}R\right) + \frac{1}{2}rbNg^{-\frac{1}{2}}g^{ij}\left(\pi_{mn}\pi^{mn} - \frac{1}{2}\pi^2\right) - 2rbNg^{-\frac{1}{2}}\left(\pi^{im}\pi_m^j - \frac{1}{2}\pi^{ij}\pi\right)$$

$$+ g^{\frac{1}{2}}(D^iD^j(rbN) - g^{ij}D_mD^m(rbN)) + \mathcal{L}_{\vec{b}}\pi^{ij} + \frac{rb}{2}HN^iN^j. \tag{6.22}$$

Here $b^j := rbN^j$. With (6.18), we have

$$k^{(\log)} = h_A^{(\log)A} + 2\lambda^{(\log)} + \bar{D}^A\lambda_A^{(\log)}, \tag{6.23}$$

where $\lambda_A^{(\log)} = h_{rA}^{(\log)}$. We further introduce $C_A^{(llll)} = C_r^{(llll)} = 0$, then $\delta_b g_{ij}$ gives

with

$$\delta_b h_{AB}^{(\log)} = 2b\left(\pi_{AB}^{(\log)} - \frac{1}{2}\bar{\gamma}_{AB}\pi^{(\log)}\right),$$

$$\delta_b\lambda^{(\log)} = 2b\left(\pi_{rr}^{(\log)} - \frac{1}{2}\pi^{(\log)}\right),$$

$$\delta_b\lambda_A^{(\log)} = 2b\pi_{Ar}^{(\log)}, \tag{6.24}$$

$$\pi_{AB}^{(\log)} = \bar{\gamma}_{AC}\bar{\gamma}_{BD}\pi^{(\log)CD}, \qquad \pi_{rr}^{(\log)} = \pi^{(\log)rr},$$

$$\pi_{rA}^{(\log)} = \bar{\gamma}_{AB}\pi^{(\log)rB}, \qquad (6.25)$$

and

$$\pi^{(\log)} = \pi^{(\log)rr} + \bar{\gamma}_{AB}\pi^{(\log)AB}. \qquad (6.26)$$

Therefore,

$$\delta_b k^{(\log)} = \delta_b h_A^{(\log)A} + 2\delta_b\lambda^{(\log)} + \bar{D}_A\delta_b\lambda^{(\log)A},$$
$$= 2b(\pi_A^{(\log)A} + 2\pi_{rr}^{(\log)} - 2\pi^{(\log)} + \bar{D}_A\pi^{(\log)rA}). \qquad (6.27)$$

As mentioned before, $\pi^{(\log)rA}$ has even parity. We assign odd parity to $\pi^{(\log)rr}$ and $\pi^{(\log)AB}$, then the parity condition of $\delta_b k^{(\log)}$ is preserved by G_b.

Although G_b preserves the parity condition of $k^{(\log)}$, it breaks the boundary conditions conditions of C_j. To see this, we first transform (6.12) and (6.13) to the asymptotic Cartesian coordinates

$$C = \frac{C^{(1)}}{r^3} + \frac{\log r}{r^4}C^{(\log)} + \frac{C^{(2)}}{r^4} + \frac{\log r}{r^5}C^{(ll)} + \frac{C^{(3)}}{r^5}$$
$$+ \frac{(\log r)^2}{r^6}\tilde{C}^{(lll)} + \frac{\log r}{r^6}C^{(lll)} + o\left(\frac{\log r}{r^6}\right), \qquad (6.28)$$

$$C_j = \frac{C_j^{(1)}}{r^3} + \frac{\log r}{r^4}C_j^{(\log)} + \frac{C_j^{(2)}}{r^4} + \frac{\log r}{r^5}C_j^{(ll)} + \frac{C_j^{(3)}}{r^5} + \frac{(\log r)^2}{r^6}\tilde{C}_j^{(lll)} + \frac{\log r}{r^6}C_j^{(lll)} + \frac{C_j^{(4)}}{r^6} + \frac{(\log r)^2}{r^7}\tilde{C}_j^{(llll)}$$
$$+ \frac{\log r}{r^7}C_j^{(llll)} + o\left(\frac{\log r}{r^7}\right),$$
$$C_j^{(1)} = C_j^{(2)} = C_j^{(ll)} = C_j^{(3)} = C_j^{(\log)} = \tilde{C}_j^{(lll)} = C_j^{(lll)} = C_j^{(4)}$$
$$= C^{(1)} = C^{(\log)} = C^{(2)} = C^{(ll)} = 0,$$
$$C^{(3)} \neq 0. \qquad (6.29)$$

By (6.29), $C_j = O(\frac{(\log r)^2}{r^7})$ and $H_{\mathrm{phys}} = O(r^{-5})$. From (5.88)

$$\{C_j, G_b\} = \partial_j(rb)H_{\mathrm{phys}},$$
$$= O(r^{-5}). \qquad (6.30)$$

Then $C_j^{(3)}$ becomes nonvanishing, so the boundary conditions of C_j in (6.29) are broken. These boundary conditions are important for ensuring that both $\delta\int_{\mathcal{S}}\mathrm{d}^3\mathcal{V}H_{\mathrm{phys}}$ and $\delta\int_{\mathcal{S}}\mathrm{d}^3\mathcal{V}rbH_{\mathrm{phys}}$ are well defined by adding appropriate boundary terms. Indeed, similar to the discussions in Sec. III B, there are boundary terms $-2\oint_{S^2}\mathrm{d}^2SN^j\delta(\pi^{rk}g_{jk})$ and $-2R\oint_{S^2}\mathrm{d}^2SbN^j\delta(\pi^{rk}g_{jk})$ from $\delta\int_{\mathcal{S}}\mathrm{d}^3\mathcal{V}H_{\mathrm{phys}}$ and $\delta\int_{\mathcal{S}}\mathrm{d}^3\mathcal{V}rbH_{\mathrm{phys}}$, respectively. b has odd parity, so the boundary terms cannot vanish at the same time, no matter what parity condition is assigned to $N^j\delta(\pi^{rk}g_{jk})$. Imposing the boundary conditions in (6.12) can resolve this problem. Since the boundary conditions in (6.12) are broken, G_b is not a boundary-preserving symmetry charge.

VII. CONCLUSION AND OUTLOOK

In this paper, we introduce asymptotic boundary conditions for the asymptotic flatness in BK formalism and investigate the symmetries preserving these conditions. A counterterm is added to the symplectic form to make it finite. We add the boundary term to the physical Hamiltonian $\mathbf{H}_{\mathrm{phys}}$ to make $\delta\mathbf{H}_{\mathrm{phys}}$ well defined. The boundary term coincides with the ADM mass.

We define the boundary-preserving symmetry charges on the reduced phase space consisting contributions from both bulk and boundary of the dust space. Unlike the usual formulation in general relativity (on the nonreduced phase space), the bulk terms of these charges do not generally vanish on shell. The Poisson brackets of the charges form a closed Lie algebra up to a central term. The resulting Lie algebra of symmetry charges relates to the BMS algebra by a suitable quotient.

Our work is the first step of investigating the boundary terms in relational formalism. There are several projects we plan to do in future:

(1) We can apply our work to the connection dynamics formalism [50]. In this formalism, the canonical variables are converted to the densitized tetrads and Ashtekar-Barbero connections $\{E^a, A^a\}$. In the canonical formalism of loop quantum gravity, $\{E^a, A^a\}$ are the basic variables in the quantization (e.g., see [2,51,52]). Applying our work in connection dynamics may benefit our understanding of asymptotically flat quantum gravity.

(2) It may be interesting to investigate the BK formalism in asymptotically anti–de Sitter spacetimes. Unlike asymptotically flat spacetimes, the boundary of an asymptotically anti–de Sitter spacetime is a three-dimensional timelike hypersurface [53,54]. This may relate to the bulk/boundary duality.

(3) In this work, we require the asymptotic symmetry preserve the metric on S^2. We may relax this requirement and allow generic diffeomorphisms on S^2 as the asymptotic symmetries. Some existing results along this direction are given in, e.g., [34,55,56].

(4) In general relativity with asymptotically flat spacetime, asymptotic symmetry charges have close relation with Weinberg's soft theorem [57]. For example, Refs. [58,59] show the equivalence between the BMS charges and soft graviton theorem [60,61] at the null infinity by scattering a massless scalar field. There is a subtlety to applying this analysis to BK formalism, since BK formalism can only analyze the structure at the spatial infinity, whereas there is no dynamics at the spatial infinity. Therefore, we expect that the analysis of the scattering process should be carried out in the regime near spatial infinity but still with finite distance. We plan to do this analysis in our future work.

APPENDIX A: THE BOUNDARY TERMS FROM $N\delta C$

Using (1.5.10) in [2], the boundary terms from $N\delta C$ are

$$[N\delta C]_{\text{boundary}} = -\lim_{R\to\infty}\oint_{S^2}\sqrt{g}g^{cd}g^{ef}[(D_cN)(dS_d\delta g_{ef})-(D_eN)(dS_c\delta g_{df})]$$
$$-\lim_{R\to\infty}\oint_{S^2}\sqrt{g}g^{cd}N[-dS_c\delta\Gamma^e_{ed}+dS_e\delta\Gamma^e_{cd}], \tag{A1}$$

with $\sqrt{g}=\lambda\sqrt{\gamma}$. With the results in Appendix A of [28], Eq. (A1) can be expressed in the spherical coordinates:

$$[N\delta C]_{\text{boundary}} = \lim_{R\to\infty}\oint_{S^2}d^2S\sqrt{\gamma}\left(-\frac{1}{\lambda}\gamma^{AB}(D_rN)\delta\gamma_{AB}+\frac{\lambda^A}{\lambda}\gamma^{BC}(D_AN)\delta\gamma_{BC}\right)$$
$$-2\lim_{R\to\infty}\oint_{S^2}d^2S(N\delta K+N\delta(\gamma_{AC})\gamma^{AB}K^C_B)+\lim_{R\to\infty}\oint_{S^2}d^2S\sqrt{\gamma}\frac{N}{\lambda}\delta\left[\left(-\frac{\lambda^A}{\lambda}(\partial_A\lambda+K_{AC}\lambda^C)+D_A\lambda^A\right)-D_A\delta\lambda^A\right]$$
$$-\lim_{R\to\infty}\oint_{S^2}d^2S\sqrt{\gamma}N\frac{\lambda^A}{\lambda}\left(\frac{1}{2}\gamma^{BC}D_A\delta\gamma_{BC}-\delta\left(\frac{\lambda^B}{\lambda}K_{BA}\right)\right)+\lim_{R\to\infty}\oint_{S^2}d^2S\sqrt{\gamma}N\frac{\lambda^A}{\lambda}\delta\left(\frac{1}{\lambda}(\partial_A\lambda+K_{AB}\lambda^B)\right)$$
$$+\lim_{R\to\infty}\oint_{S^2}d^2S\lambda\sqrt{\gamma}\left(-N\frac{\lambda^A\lambda^B}{\lambda^5}\delta K_{AB}+NK_{AB}\frac{\lambda^A\lambda^B}{\lambda^4}\frac{\delta\lambda}{\lambda^2}\right)$$
$$+\lim_{R\to\infty}\oint_{S^2}d^2S\sqrt{\gamma}\left(-ND^A\left(\frac{1}{\lambda}\right)\delta\lambda_A-\frac{\lambda^A}{\lambda}\gamma^{BC}(D_CN)\delta\gamma_{AB}\right). \tag{A2}$$

APPENDIX B: THE CONSERVATION OF THE BOUNDARY CHARGES

This appendix demonstrates the boundary charges defined in (4.14) are conserved. It is convenient to demonstrate this statement in the asymptotically Cartesian coordinates. We first compute the physical time derivative of g_{ij}:

$$\frac{dg_{ij}}{d\tau}=Ng^{-\frac{1}{2}}\left(\pi_{ij}-\frac{1}{2}g_{ij}\pi\right)+\mathcal{L}_{\vec{N}}g_{ij}=O(r^{-2}). \tag{B1}$$

The physical time derivative of the $\bar{h}_{ij}$ in (3.6) is given by expanding (B1) to the boundary, which reads

$$\frac{\mathrm{d}\bar{h}_{ij}}{\mathrm{d}\tau} = 0. \tag{B2}$$

Note that all of the metric variables in (4.14) are the components of $\bar{h}_{ij}$, so their physical time derivatives vanish. Next, we compute the physical time derivative of π^{ij}:

$$\begin{aligned}
\frac{\mathrm{d}\pi^{ij}}{\mathrm{d}\tau} &= -Ng^{\frac{1}{2}}\left(g^{ik}g^{jl(3)}R_{lk} - \frac{1}{2}g^{ij(3)}R\right) \\
&\quad + \frac{1}{2}Ng^{-\frac{1}{2}}g^{ij}\left(\pi_{mn}\pi^{mn} - \frac{1}{2}\pi^2\right) \\
&\quad - 2Ng^{-\frac{1}{2}}\left(\pi^{im}\pi_m^j - \frac{1}{2}\pi^{ij}\pi\right) \\
&\quad + g^{\frac{1}{2}}(D^iD^jN - g^{ij}D_mD^mN) + \frac{1}{2}H_{\text{phys}}N^iN^j + \mathcal{L}_{\vec{N}}\pi^{ij}, \\
&= O(r^{-3}). \tag{B3}
\end{aligned}$$

It turns out that

$$\frac{\mathrm{d}\bar{\pi}^{ij}}{\mathrm{d}\tau} = 0. \tag{B4}$$

Thus, the physical time derivatives of the conjugate momenta with a bar on the top in (4.14) vanish, due to they are the components of $\bar{\pi}^{ij}$. Finally, we deal with the term with $\pi^{(2)rA}$. By (B4),

$$\frac{\mathrm{d}\pi^{(2)rA}}{\mathrm{d}\tau} = \mp \sqrt{\bar{\gamma}}\bar{\gamma}^{AB(3)}R_{rB}^{(1)}. \tag{B5}$$

Here $^{(3)}R_{rA}^{(1)}$ is the coefficient of the leading order of $^{(3)}R_{rA}$, which reads

$$\begin{aligned}
^{(3)}R_{rA}^{(1)} &= \bar{D}_B\bar{D}_A\bar{\lambda}^B - \bar{D}_B\bar{k}_A^B + \frac{1}{2}(\bar{\gamma}^{BC}\partial_A\bar{h}_{BC} - \bar{h}^{BC}\partial_A\bar{\gamma}_{BC}) \\
&\quad - \bar{\lambda}_A + 2\partial_A\bar{\lambda}, \\
&= \bar{D}_A\bar{D}_B\bar{\lambda}^B + {}^{\bar{\gamma}}R_{AB}\bar{\lambda}^B - \bar{D}_B\bar{k}_A^B + \frac{1}{2}\bar{\gamma}^{BC}\bar{D}_A\bar{h}_{BC} \\
&\quad - \bar{\lambda}_A + 2\partial_A\bar{\lambda}, \\
&= \bar{D}_A\bar{D}_B\bar{\lambda}^B - \bar{D}_B\bar{k}_A^B + \frac{1}{2}\bar{\gamma}^{BC}\bar{D}_A\bar{h}_{BC} + 2\partial_A\bar{\lambda}. \tag{B6}
\end{aligned}$$

In the last step, we use the fact that on a unit sphere we have

$$^{\bar{\gamma}}R_{AB} = \bar{\gamma}_{AB}. \tag{B7}$$

Then we find

$$\begin{aligned}
2\oint_{S^2} \mathrm{d}^2S Y^A\bar{\gamma}_{AB}\frac{\mathrm{d}\pi^{(2)rB}}{\mathrm{d}\tau} &= \mp 2\oint_{S^2}\mathrm{d}^2S Y^A\left(\bar{D}_A\bar{D}_B\bar{\lambda}^B - \bar{D}_B\bar{k}_A^B \right. \\
&\quad\left. + \frac{1}{2}\bar{\gamma}^{BC}\bar{D}_A\bar{h}_{BC} + 2\partial_A\bar{\lambda}\right), \\
&= 0. \tag{B8}
\end{aligned}$$

In the last step, we integrate by part and use the Killing equation of Y^A. As the result, we demonstrate that the boundary charges defined in (4.14) are conserved.

APPENDIX C: REVIEW OF (WEAK) SYMPLECTIC FORM

In this appendix, we give a brief review of the (weak) symplectic form. The details can be found in [62].

First, we define the strong and weak nondegenerate bilinear forms of a Banach space.

Definition C.1. Let $\mathcal{V}$ be a Banach space, whose dual space is denoted as $\mathcal{V}^*$. $B:\mathcal{V}\times\mathcal{V}\to\mathbf{R}$ is a continuours bilinear map. B naturally induces a continuous bilinear map $\tilde{B}:\mathcal{V}\to\mathcal{V}^*$, s.t. $\forall\ a, b\in\mathcal{V}$, $\tilde{B}(a)\cdot b = B(a, b)$. Here "·" denotes the inner product. If $\tilde{B}$ is injective, and $\forall\ b\in\mathcal{V}$, $B(a, b) = 0$ gives $a = 0$, then we call B is weakly non-degenerate. If $\tilde{B}$ is an isomorphism, then we call B is nondegenerate or strongly nondegenerate.

With the bilinear form defined above, we can define the (weakly) nondegenerate form as

Definition C.2. Let $\mathcal{P}$ be a Banach manifold and $T\mathcal{P}$ is the tangent bundle of $\mathcal{P}$. A two-form field Ω on $\mathcal{P}$ is called a symplectic form if

(i) Ω is closed: $\delta\Omega = 0$ (δ denotes the exterior derivative).

(ii) $\forall\ x\in\mathcal{P}$, the map $\Omega|_x: T_x\mathcal{P}\times T_x\mathcal{P}\to\mathbf{R}$ is non-degenerate.

In particular, if $\Omega|_x$ is weakly nondegenerate, we call Ω a weak symplectic form.

In the following, we choose $(g_{ij}(\vec{\sigma}), \pi^{ij}(\vec{\sigma}))$ as the canonical pairs of the phase space. Given any C^1 function $f(g_{ij}, p^{ij})$ of the phase space, we construct the corresponding dual vector field as

$$(\delta f)_\alpha = \int_{\mathcal{S}}\mathrm{d}^3\sigma\left[\frac{\delta f}{\delta g_{ij}}(\delta g_{ij})_\alpha + \frac{\delta f}{\delta\pi^{ij}}(\delta\pi^{ij})_\alpha\right]. \tag{C1}$$

Here, the greek letters $\alpha, \beta, \gamma, \ldots$ are the indexes of $\mathcal{P}$.

Definition C.3. Given a Banach manifold $\mathcal{P}$ equipped with a weak symplectic form Ω. D is a domain of $\mathcal{P}$: $D\subset\mathcal{P}$. Let $X\in TD$, where $TD\subset T\mathcal{P}$. X is said to be a Hamiltonian vector field if there $\exists$ a C^1 function $h:D\to\mathbf{R}$, s.t.

$$[\Omega_{\alpha\beta}X^\alpha v^\beta]|_x = [(\delta h)_\alpha v^\alpha]|_x, \tag{C2}$$

$\forall\ x\in D$ and $\forall\ v^\alpha\in TD$.

In the following, we denote X_h as the Hamiltonian vector field of h.

Theorem C.1. Suppose X_f^α is a Hamiltonian vector on the domain D, then X_f^α is a (infinitesimal) symmetry:

$$\mathcal{L}_{\vec{X}_f}\Omega_{\alpha\beta} = 0. \tag{C3}$$

Proof.

$$\mathcal{L}_{\vec{X}_f}\Omega_{\alpha\beta} = \delta_\alpha(X_f^\gamma\Omega_{\gamma\beta}) + (\delta\Omega)_{\gamma\alpha\beta}X_f^\gamma = \delta_\alpha(X_f^\gamma\Omega_{\gamma\beta}). \tag{C4}$$

Here we have used $\delta\Omega = 0$ in the last step. Then with (C2), we have

$$X_f^\gamma\Omega_{\gamma\beta} = \delta_\beta f, \tag{C5}$$

thus

$$\mathcal{L}_{\vec{X}_f}\Omega_{\alpha\beta} = \delta_\alpha\delta_\beta f = 0. \tag{C6}$$

∎

Suppose a C^1 function f on D has corresponding Hamiltonian vector field X_f, we can check that

$$X_f^\alpha = \int_{\mathcal{S}} \mathrm{d}^3\sigma\left[\frac{\delta f}{\delta\pi^{ij}}\left(\frac{\delta}{\delta g_{ij}}\right)^\alpha - \frac{\delta f}{\delta g_{ij}}\left(\frac{\delta}{\delta\pi^{ij}}\right)^\alpha\right] \tag{C7}$$

with the (weak) symplectic form

$$\Omega = \int_{\mathcal{S}} \mathrm{d}^3\sigma(\delta g_{ij})_\alpha \wedge (\delta\pi^{ij})_\beta,$$

$$= \int_{\mathcal{S}} \mathrm{d}^3\sigma[(\delta g_{ij})_\alpha(\delta\pi^{ij})_\beta - (\delta g_{ij})_\beta(\delta\pi^{ij})_\alpha]. \tag{C8}$$

To see this, we introduce an arbitrary vector field $v^\alpha \in TD$:

$$v^\alpha = \int_{\mathcal{S}} \mathrm{d}^3\sigma\left[v_{ij}\left(\frac{\delta}{\delta g_{ij}}\right)^\alpha + \bar{v}^{ij}\left(\frac{\delta}{\delta\pi^{ij}}\right)^\alpha\right], \tag{C9}$$

here v_{ij} and $\bar{v}^{ij}$ are the coefficients of v^α. With (C1), and (C7)–(C9), the left-hand side of (C2) reads

$$[\Omega_{\alpha\beta}X_f^\alpha v^\beta]|_x = \left[\int_{\mathcal{S}} \mathrm{d}^3\sigma_1\mathrm{d}^3\sigma_2\mathrm{d}^3\sigma_3[(\delta g_{ij})_\alpha(\delta\pi^{ij})_\beta - (\delta g_{ij})_\beta(\delta\pi^{ij})_\alpha]\right.$$
$$\times \left.\left[\frac{\delta f}{\delta\pi^{kl}}\left(\frac{\delta}{\delta g_{kl}}\right)^\alpha - \frac{\delta f}{\delta g_{kl}}\left(\frac{\delta}{\delta\pi^{kl}}\right)^\alpha\right]\left[v_{mn}\left(\frac{\delta}{\delta g_{mn}}\right)^\beta + \bar{v}^{mn}\left(\frac{\delta}{\delta\pi^{mn}}\right)^\beta\right]\right]\bigg|_x,$$
$$= \left[\int_{\mathcal{S}} \mathrm{d}^3\sigma\left[\frac{\delta f}{\delta\pi^{ij}}\bar{v}^{ij} + \frac{\delta f}{\delta g_{ij}}v_{ij}\right]\right]\bigg|_x, \tag{C10}$$

and the right-hand side of (C2) reads

$$[(\delta f)_\alpha v^\alpha]|_x = \left[\int_{\mathcal{S}} \mathrm{d}^3\sigma\mathrm{d}^3\sigma'\left[\frac{\delta f}{\delta g_{ij}}(\delta g_{ij})_\alpha + \frac{\delta f}{\delta\pi^{ij}}(\delta\pi^{ij})_\alpha\right]\left[v_{mn}\left(\frac{\delta}{\delta g_{mn}}\right)^\alpha + \bar{v}^{mn}\left(\frac{\delta}{\delta\pi^{mn}}\right)^\alpha\right]\right]\bigg|_x,$$
$$= \left[\int_{\mathcal{S}} \mathrm{d}^3\sigma\left[\frac{\delta f}{\delta\pi^{ij}}\bar{v}^{ij} + \frac{\delta f}{\delta g_{ij}}v_{ij}\right]\right]\bigg|_x. \tag{C11}$$

Thus, we have checked (C7). The Poisson bracket between the C^1 functions of the phase space which have the corresponding Hamiltonian vector fields is defined as follows:

Definition C.4. Suppose $\mathcal{P}$ is a Banach manifold equipped with a (weak) symplectic form Ω. $D_1, D_2 \in \mathcal{P}$, and $D_1 \cap D_2 \neq \emptyset$. f is a C^1 functions on D_1, whose Hamiltonian vector field is X_f; g is another C^1 functions on D_2, whose Hamiltonian vector field is X_g. The Poisson bracket $\{f, g\}$ is a map:

$$\{f, g\}: D_1 \cap D_2 \to \mathbf{R}, \tag{C12}$$

given by

$$\{f, g\}(x) = \Omega(X_f(x), X_g(x)), \tag{C13}$$

with $x \in D_1 \cap D_2$.

With (C2) and (C13), we have

$$\{f, g\} = \Omega(X_f, X_g) = (\delta f)_\alpha X_g^\alpha = \mathcal{L}_{\vec{X}_g} f = -\mathcal{L}_{\vec{X}_f} g. \quad \text{(C14)}$$

Next, we prove the Poisson bracket we define satisfies the Jacobi identity. To do this, we first proof the following lemma:

Lemma C.2. Supposed f, g are the C^1 functions with domains D_1, D_2, respectively, and $D_1 \cap D_2 \neq \emptyset$. We have

$$X_{\{f,g\}}^\alpha = -[X_f, X_g]^\alpha, \quad \text{(C15)}$$

where $[\cdot, \cdot]$ denotes the Lie bracket.

Proof. Since $\Omega_{\alpha\beta}$ is (weakly) nondegenerate, we just need to show

$$\Omega_{\alpha\beta}(X_{\{f,g\}}^\alpha + [X_f, X_g]^\alpha) = 0. \quad \text{(C16)}$$

With (C3), we have

$$\Omega_{\alpha\beta}[X_f, X_g]^\alpha = \Omega_{\alpha\beta}\mathcal{L}_{\vec{X}_f} X_g^\alpha = \mathcal{L}_{\vec{X}_f}(\Omega_{\alpha\beta}X_g^\alpha) - \mathcal{L}_{\vec{X}_f}(\Omega_{\alpha\beta})X_g^\alpha,$$
$$= \mathcal{L}_{\vec{X}_f}(\Omega_{\alpha\beta}X_g^\alpha). \quad \text{(C17)}$$

Then with (C2), the left-hand side of (C16) reads

$$\Omega_{\alpha\beta}(X_{\{f,g\}}^\alpha + [X_f, X_g]^\alpha) = \Omega_{\alpha\beta}X_{\{f,g\}}^\alpha + \mathcal{L}_{\vec{X}_f}(\Omega_{\alpha\beta}X_g^\alpha),$$
$$= -\delta_\beta\{f, g\} - \mathcal{L}_{\vec{X}_f}(\delta_\beta g). \quad \text{(C18)}$$

Use (C14), we have

$$-\delta_\beta\{f, g\} - \mathcal{L}_{\vec{X}_f}(\delta_\beta g) = \delta_\beta(\mathcal{L}_{\vec{X}_f} g) - \mathcal{L}_{\vec{X}_f}(\delta_\beta g)$$
$$= \mathcal{L}_{\vec{X}_f}(\delta_\beta g) - \mathcal{L}_{\vec{X}_f}(\delta_\beta g) = 0. \quad \text{(C19)}$$

∎

Thus, we have the following theorem:

Theorem C.3. Given three C^1 functions f, g, h with domains D_1, D_2, D_3, respectively, and $D_1 \cap D_2 \cap D_3 \neq \emptyset$. X_f^α is the Hamiltonian vector field of f. X_g^α is the Hamiltonian vector field of g. X_h^α is the Hamiltonian vector field of h. $\forall\ x \in D_1 \cap D_2 \cap D_3$, we have the Jacobi identity:

$$\{\{g, h\}, f\}(x) + \{\{f, g\}, h\}(x) + \{\{h, f\}, g\}(x) = 0. \quad \text{(C20)}$$

Proof. Note that $\delta\Omega = 0$,

$$[(\delta\Omega)_{\alpha\beta\gamma}X_f^\alpha X_g^\beta X_h^\gamma]|_x = 0. \quad \text{(C21)}$$

Use the properties of the exterior derivative, (C21) gives

$$0 = [X_f^\alpha \delta_\alpha(\Omega_{\beta\gamma}X_g^\beta X_h^\gamma) - X_g^\alpha \delta_\alpha(\Omega_{\beta\gamma}X_f^\beta X_h^\gamma) + X_h^\alpha \delta_\alpha(\Omega_{\beta\gamma}X_f^\beta X_g^\gamma)$$
$$- \Omega_{\alpha\beta}[X_f, X_g]^\alpha X_h^\beta + \Omega_{\alpha\beta}[X_f, X_h]^\alpha X_g^\beta - \Omega_{\alpha\beta}[X_g, X_h]^\alpha X_f^\beta]|_x. \quad \text{(C22)}$$

With (C14), we have

$$X_f^\alpha \delta_\alpha(\Omega_{\beta\gamma}X_g^\beta X_h^\gamma) = X_f^\alpha \delta_\alpha\{g, h\} = \{\{g, h\}, f\}. \quad \text{(C23)}$$

With Lemma C.2 we have

$$-\Omega_{\alpha\beta}[X_f, X_g]^\alpha X_h^\beta = \Omega_{\alpha\beta}X_{\{f,g\}}^\alpha X_h^\beta = \{\{f, g\}, h\}. \quad \text{(C24)}$$

Thus, (C22) reads

$$0 = [X_f^\alpha \delta_\alpha(\Omega_{\beta\gamma}X_g^\beta X_h^\gamma) - X_g^\alpha \delta_\alpha(\Omega_{\beta\gamma}X_f^\beta X_h^\gamma) + X_h^\alpha \delta_\alpha(\Omega_{\beta\gamma}X_f^\beta X_g^\gamma) - \Omega_{\alpha\beta}[X_f, X_g]^\alpha X_h^\beta + \Omega_{\alpha\beta}[X_f, X_h]^\alpha X_g^\beta - \Omega_{\alpha\beta}[X_g, X_h]^\alpha X_f^\beta]|_x,$$
$$= [\{\{g, h\}, f\} - \{\{f, h\}, g\} + \{\{f, g\}, h\} + \{\{f, g\}, h\} - \{\{f, h\}, g\} + \{\{g, h\}, f\}]|_x,$$
$$= 2[\{\{g, h\}, f\} - \{\{f, h\}, g\} + \{\{f, g\}, h\}]|_x,$$
$$= 2[\{\{g, h\}, f\} + \{\{h, f\}, g\} + \{\{f, g\}, h\}]|_x. \quad \text{(C25)}$$

∎

We consider a simple example. Suppose $D_1 = D_2$ and the Ω takes the form of (C8), we have

$$\{f, g\}(x) = \left[\left[\int_{\mathcal{S}} d^3\sigma_1 d^3\sigma_2 d^3\sigma_3 [(\delta g_{ij})_\alpha(\delta\pi^{ij})_\beta - (\delta g_{ij})_\beta(\delta\pi^{ij})_\alpha] \right. \right.$$
$$\left. \times \left[\frac{\delta f}{\delta\pi^{kl}}\left(\frac{\delta}{\delta g_{kl}}\right)^\alpha - \frac{\delta f}{\delta g_{kl}}\left(\frac{\delta}{\delta\pi^{kl}}\right)^\alpha \right] \left[\frac{\delta g}{\delta\pi^{ml}}\left(\frac{\delta}{\delta g_{ml}}\right)^\beta - \frac{\delta g}{\delta g_{ml}}\left(\frac{\delta}{\delta\pi^{ml}}\right)^\beta \right] \right] \Big|_x ,$$
$$= \left[\int_{\mathcal{S}} d^3\sigma \left[\frac{\delta f}{\delta g_{ij}}\frac{\delta g}{\delta\pi^{ij}} - \frac{\delta g}{\delta g_{ij}}\frac{\delta f}{\delta\pi^{ij}} \right] \right] \Big|_x . \quad \text{(C26)}$$

This result agrees with Eq. (3.31).

It is proven that the symplectic form defined in (3.29) is strong [2]. Also note that the variation of the boundary-preserving symmetry charges $G(\xi, \vec{\xi})$ construct in Sec. IV is well defined. Therefore, for any $G(\xi, \vec{\xi})$, there is corresponding Hamiltonian vector field. Consequently, the symplectic form is preserved the transformations generated by $G(\xi, \vec{\xi})$.

APPENDIX D: PROPERTIES OF $(\hat{\xi}, \hat{\xi}^i)$

From (5.17), we have

$$\hat{\xi} = \hat{f} + O(r^{-1}), \qquad \hat{\xi}^r = \hat{W} + O(r^{-1}),$$

$$\hat{\xi}^A = \hat{Y}^A + \frac{1}{r}\hat{I}^A + O(r^{-2}), \tag{D1}$$

$$\hat{Y}^A = Y_1^B \bar{D}_B Y_2^A - (1 \leftrightarrow 2),$$

$$\hat{f} = Y_1^A \partial_A f_2 - (1 \leftrightarrow 2),$$

$$\hat{W} = Y_1^A \partial_A W_2 - (1 \leftrightarrow 2),$$

$$\hat{I}^A = Y_1^B \bar{D}_B I_2^A + I_1^B \bar{D}_B Y_2^A - (1 \leftrightarrow 2). \tag{D2}$$

Since f, W are functions of S^2 and Y^A is the Killing vector of S^2, $\hat{f}$, $\hat{W}$ are also functions of S^2. Similarly, $\hat{I}^A$ is the vector field of S^2. Note that Y_1^A, Y_2^A are the Killing vector fields on S^2, their commutator $\hat{Y}^A := [Y_1, Y_2]^A$ is also the Killing vector field on S^2. Therefore, $(\hat{\xi}, \hat{\xi}^j)$ satisfies (4.2).

APPENDIX E: DETAILS OF SECTION 5.1

This subsection gives some detailed calculations of Sec. VA. We first give the details of (5.11). By (2.31) and (2.30), (5.11) yields

$$\begin{aligned}
\{G(\vec{\xi}_1), G(\xi_2)\} &= \int_{\mathcal{S}} d^3\mathcal{V}\xi_1^i D_i(N\xi_2)C + \int_{\mathcal{S}} d^3\mathcal{V}[\xi_1^i D_i(N^j\xi_2) - N^i\xi_2 D_i\xi_1^j]C_j \\
&\quad - \int_{\mathcal{S}} d^3\mathcal{V}\xi_2 H_{\text{phy}} N^j N^k D_j(\xi_1)_k - \delta_{\vec{\xi}_1}\tilde{\mathcal{B}}(\tilde{\xi}_2) - \lim_{r\to\infty}\oint_{S^2} d^2 S\xi_1^r\tilde{\xi}_2 C + 2\lim_{r\to\infty}\oint_{S^2} d^2 S[\vec{\xi}_1, \vec{\tilde{\xi}}_2]^j \pi_j^r, \\
&= \int_{\mathcal{S}} d^3\mathcal{V}\left[\pm\xi_1^i \frac{g^{jk}N_j D_i N_k}{\sqrt{1 + g^{jk}N_j N_k}}\xi_2 C + \xi_1^i D_i(\xi_2)\frac{C^2}{H_{\text{phys}}}\right] \\
&\quad + \int_{\mathcal{S}} d^3\mathcal{V}\left[\xi_1^i D_i(N^j)\xi_2 - \frac{C^j}{H_{\text{phys}}}\xi_1^i D_i(\xi_2) + \frac{C^i}{H_{\text{phys}}}\xi_2 D_i\xi_1^j\right]C_j \\
&\quad - \int_{\mathcal{S}} d^3\mathcal{V}\xi_2 \frac{C^j C^k}{H_{\text{phys}}}D_j(\xi_1)_k - \delta_{\vec{\xi}_1}\tilde{\mathcal{B}}(\tilde{\xi}_2) - \lim_{r\to\infty}\oint_{S^2} d^2 S\xi_1^r\tilde{\xi}_2 C + 2\lim_{r\to\infty}\oint_{S^2} d^2 S[\vec{\xi}_1, \vec{\tilde{\xi}}_2]^j \pi_j^r, \\
&= \int_{\mathcal{S}} d^3\mathcal{V}[\xi_1^i N_j D_i(N^j)H_{\text{phy}}\xi_2 + \xi_1^i D_i(N^j)\xi_2 C_j] + \int_{\mathcal{S}} d^3\mathcal{V}\xi_1^i D_i(\xi_2)\frac{C^2 - g^{jk}C_j C_k}{H_{\text{phys}}} \\
&\quad + \int_{\mathcal{S}} d^3\mathcal{V}\xi_2\left[\frac{C^j C^k}{H_{\text{phys}}}\xi_2 D_j(\xi_1)_k - \frac{C^j C^k}{H_{\text{phys}}}D_j(\xi_1)_k\right] - \delta_{\vec{\xi}_1}\tilde{\mathcal{B}}(\tilde{\xi}_2) - \oint_{S^2} d^2 S\xi_1^r\tilde{\xi}_2 C + 2\oint_{S^2} d^2 S[\vec{\xi}_1, \vec{\tilde{\xi}}_2]^j \pi_j^r, \\
&= \int_{\mathcal{S}} d^3\mathcal{V}\xi_1^i D_i(\xi_2)H_{\text{phys}} - \delta_{\vec{\xi}_1}\tilde{\mathcal{B}}(\tilde{\xi}_2) - \lim_{r\to\infty}\oint_{S^2} d^2 S\xi_1^r\tilde{\xi}_2 C + 2\lim_{r\to\infty}\oint_{S^2} d^2 S[\vec{\xi}_1, \vec{\tilde{\xi}}_2]^j \pi_j^r. \tag{E1}
\end{aligned}$$

Next, we show that $\mathcal{B}$ in (5.13) is vanished. By (2.31) and (2.30),

$$\begin{aligned}
\mathcal{B} &= \int_{\mathcal{S}} d^3\mathcal{V}[N\xi_1 D^j(N\xi_2) - N\xi_2 D^j(N\xi_1)]C_j + \int_{\mathcal{S}} d^3\mathcal{V}[N^j\xi_1 D_j(N\xi_2) - N^j\xi_2 D_j(N\xi_1)]C \\
&\quad + \int_{\mathcal{S}} d^3\mathcal{V}[N^i\xi_1 D_i(N^j\xi_2) - N^i\xi_2 D_i(N^j\xi_1)]C_j + \frac{1}{2}\int_{\mathcal{S}} d^3\mathcal{V}H_{\text{phys}}N^i N^j\xi_1 N\xi_2 g^{-\frac{1}{2}}\left(\pi_{ij} - \frac{1}{2}\pi g_{ij}\right) \\
&\quad - \frac{1}{2}\int_{\mathcal{S}} d^3\mathcal{V}H_{\text{phys}}N^i N^j\xi_2 N\xi_1 g^{-\frac{1}{2}}\left(\pi_{ij} - \frac{1}{2}\pi g_{ij}\right) + \int_{\mathcal{S}} d^3\mathcal{V}H_{\text{phys}}N^i N^j\xi_1 D_i(N_j\xi_2) \\
&\quad - \int_{\mathcal{S}} d^3\mathcal{V}H_{\text{phys}}N^i N^j\xi_2 D_i(N_j\xi_1),
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathcal{S}} d^3\mathcal{V}(N^2\xi_1 D^j\xi_2 - N^2\xi_2 D^j\xi_1)C_j + \int_{\mathcal{S}} d^3\mathcal{V}(NN^j\xi_1 D_j\xi_2 - NN^j\xi_2 D_j\xi_1)C \\
&\quad + \int_{\mathcal{S}} d^3\mathcal{V}(\xi_1 N^i N^j D_i\xi_2 - \xi_2 N^i N^j D_i\xi_1)C_j + \int_{\mathcal{S}} d^3\mathcal{V} H_{\text{phys}} N^i N^j \xi_1 N_j D_i\xi_2 \\
&\quad - \int_{\mathcal{S}} d^3\mathcal{V} H_{\text{phys}} N^i N^j \xi_2 N_j D_i\xi_1, \\
&= \int_{\mathcal{S}} d^3\mathcal{V}\left(\frac{C^2}{H_{\text{phys}}^2}\xi_1 C_j D^j\xi_2 - \frac{C^2}{H_{\text{phys}}^2}\xi_2 C_j D^j\xi_1 \right) - \int_{\mathcal{S}} d^3\mathcal{V}\left(\frac{C^2}{H_{\text{phys}}^2}\xi_1 C_j D^j\xi_2 - \frac{C^2}{H_{\text{phys}}^2}\xi_2 C_j D^j\xi_1 \right) \\
&\quad + \int_{\mathcal{S}} d^3\mathcal{V}(\xi_1 N^i N^j C_j D_i\xi_2 - \xi_2 N^i N^j C_j D_i\xi_1) - \int_{\mathcal{S}} d^3\mathcal{V}(N^i N^j \xi_1 C_j D_i\xi_2 - N^i N^j \xi_2 C_j D_i\xi_1), \\
&= 0. \tag{E2}
\end{aligned}
$$

APPENDIX F: PROOF OF (5.34)

$$
\begin{aligned}
\bar{D}_A(\mathcal{L}_{\vec{Y}}\bar{\lambda}_B) &= \bar{D}_A(Y^C \bar{D}_C\bar{\lambda}_B + \bar{\lambda}_C \bar{D}_B Y^C), \\
&= Y^C \bar{D}_A\bar{D}_C\bar{\lambda}_B + \bar{D}_A(Y^C)\bar{D}_C\bar{\lambda}_B + \bar{D}_A(\bar{\lambda}_C)\bar{D}_B Y^C + \bar{\lambda}_C \bar{D}_A\bar{D}_B Y^C, \\
&= Y^C \bar{D}_C\bar{D}_A\bar{\lambda}_B + Y^{C\bar{\gamma}} R_{ACB}{}^D \bar{\lambda}_D + \bar{D}_A(Y^C)\bar{D}_C\bar{\lambda}_B + \bar{D}_A(\bar{\lambda}_C)\bar{D}_B Y^C + \bar{\lambda}_C \bar{D}_A\bar{D}_B Y^C. \tag{F1}
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\mathcal{L}_{\vec{Y}}(\bar{D}_A\bar{\lambda}_B) &= Y^C \bar{D}_C\bar{D}_A\bar{\lambda}_B + \bar{D}_C(\bar{\lambda}_B)\bar{D}_A Y^C \\
&\quad + \bar{D}_A(\bar{\lambda}_C)\bar{D}_B Y^C. \tag{F2}
\end{aligned}
$$

To prove (5.34), we need to show

$$
Y^{C\bar{\gamma}} R_{ACB}{}^D \bar{\lambda}_D + \bar{\lambda}_C \bar{D}_A\bar{D}_B Y^C = 0. \tag{F3}
$$

For a Killing vector field on the spacetime ξ^μ, we have

$$
\nabla_\mu \nabla_\nu \xi_\sigma = -R_{\nu\sigma\mu}{}^\rho \xi_\rho, \tag{F4}
$$

then

$$
\begin{aligned}
\bar{\lambda}_C \bar{D}_A\bar{D}_B Y^C &= \bar{\lambda}^C \bar{D}_A\bar{D}_B Y_C, \\
&= -\bar{\lambda}^{C\bar{\gamma}} R_{BCA}{}^D Y_D. \tag{F5}
\end{aligned}
$$

We have

$$
\begin{aligned}
&Y^{C\bar{\gamma}} R_{ACB}{}^D \bar{\lambda}_D + \bar{\lambda}_C \bar{D}_A\bar{D}_B Y^C \\
&= Y^{C\bar{\gamma}} R_{ACB}{}^D \bar{\lambda}_D - \bar{\lambda}^{C\bar{\gamma}} R_{BCA}{}^D Y_D, \\
&= Y^C \bar{\lambda}^{D\bar{\gamma}} R_{ACBD} - Y^D \bar{\lambda}^{C\bar{\gamma}} R_{BCAD}, \\
&= Y^C \bar{\lambda}^{D\bar{\gamma}} R_{ACBD} - Y^C \bar{\lambda}^{D\bar{\gamma}} R_{ACBD}, \\
&= 0. \tag{F6}
\end{aligned}
$$

APPENDIX G: CENTRAL CHARGES

We first give the definition of the central extension of a Lie algebra L_g, which can be find in, e.g., [63]. For a Lie algebra $L\mathfrak{g}$, its central extension $\hat{g}$ is the direct sum of itself and a one-dimensional complex vector space with the basis c

$$
L\mathfrak{g} \oplus \mathbf{C}c, \tag{G1}
$$

satisfying the following requirements:

$$
\begin{aligned}
[\xi + \alpha c, \eta + \beta c] &= [\xi, \eta] + \omega(\xi, \eta)c, \\
\xi, \eta &\in L\mathfrak{g}, \qquad \alpha, \beta \in \mathbf{C}, \\
[c, \xi] &= 0. \tag{G2}
\end{aligned}
$$

Here the square bracket denotes the Lie bracket of the algebra, and the ω is a bilinear form, $\omega : L\mathfrak{g} \times L\mathfrak{g} \to \mathbf{C}$ satisfying the 2-cocycle condition:

$$
\begin{aligned}
\omega(x, y) &= -\omega(y, x), \\
\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) &= 0, \tag{G3}
\end{aligned}
$$

with $x, y, z \in L\mathfrak{g}$.

With the definition above, we can check that the $\mathcal{C}(\hat{\xi}, \hat{\vec{\xi}})$ in (5.40) provides a central extension of the algebra. Introduce $\xi_1^\mu := (\xi_1, \vec{\xi}_1)$, $\xi_2^\mu := (\xi_2, \vec{\xi}_2)$, and $\xi_3^\mu := (\xi_3, \vec{\xi}_3)$, which satisfy (4.2). Define their commutators with Poisson bracket:

$$\{G(\xi_1^\mu), G(\xi_2^\mu)\} = G([\xi_1^\mu, \xi_2^\mu]) + \mathcal{C}([\xi_1^\mu, \xi_2^\mu]). \tag{G4}$$

Here $[\xi_1^\mu, \xi_2^\mu]$ is the commutator of ξ_1^μ and ξ_2^μ defined by the Poisson bracket. In our case, $\xi^\mu \in L\mathfrak{g}$. The complex vector space **C** is restricted to be the real vector **R**. Especially, $c = 1$. The ω is given by

$$\omega(\xi_1^\mu, \xi_1^\mu) \equiv \mathcal{C}([\xi_1^\mu, \xi_2^\mu]).$$

By (5.39),

$$\omega([\xi_1^\mu, \xi_2^\mu]) = -\omega([\xi_2^\mu, \xi_1^\mu]). \tag{G5}$$

By (5.18) and (5.39),

$$\omega([\xi_1^\mu, \xi_2^\mu], \xi_3^\mu) = \pm 2 \oint_{S^2} \mathrm{d}^2 S \sqrt{\bar\gamma} [(Y_1^A \bar{D}_A f_2 - Y_2^A \bar{D}_A f_1)(\bar{D}^B \bar{D}_B W_3 - \bar{D}_B I_3^B)$$
$$- f_3 \bar{D}^A \bar{D}_A (Y_1^B \bar{D}_B W_2 - Y_2^B \bar{D}_B W_1) + f_3 \bar{D}_A (Y_1^B \bar{D}_B I_2^A + I_1^B \bar{D}_B Y_2^A - Y_2^B \bar{D}_B I_1^A - I_2^B \bar{D}_B Y_1^A)]. \tag{G6}$$

Since

$$\bar{D}_A (Y_1^B \bar{D}_B I_2^A + I_1^B \bar{D}_B Y_2^A - Y_2^B \bar{D}_B I_1^A - I_2^B \bar{D}_B Y_1^A)$$
$$= \bar{D}_A (Y_1^B) \bar{D}_B I_2^A + Y_1^B \bar{D}_A \bar{D}_B I_2^A + \bar{D}_A (I_1^B) \bar{D}_B Y_2^A + I_1^B \bar{D}_A \bar{D}_B Y_2^A$$
$$- \bar{D}_A (Y_2^B) \bar{D}_B I_1^A - Y_2^B \bar{D}_A \bar{D}_B I_1^A - \bar{D}_A (I_2^B) \bar{D}_B Y_1^A - I_2^B \bar{D}_A \bar{D}_B Y_1^A,$$
$$= \bar{D}_A (Y_1^B) \bar{D}_B I_2^A + Y_1^B \bar{D}_B \bar{D}_A I_2^A + Y_1^B R_{BAC}{}^A I_2^C + \bar{D}_A (I_1^B) \bar{D}_B Y_2^A + I_1^B \bar{D}_B \bar{D}_A Y_2^A + I_1^B R_{BAC}{}^A Y_2^C$$
$$- \bar{D}_A (Y_2^B) \bar{D}_B I_1^A - Y_2^B \bar{D}_B \bar{D}_A I_1^A - Y_2^B R_{BAC}{}^A I_1^C - \bar{D}_A (I_2^B) \bar{D}_B Y_1^A - I_2^B \bar{D}_B \bar{D}_A Y_1^A - I_2^B R_{BAC}{}^A Y_1^C,$$
$$= \bar{D}_A (Y_1^B) \bar{D}_B I_2^A + Y_1^B \bar{D}_B \bar{D}_A I_2^A + Y_1^B R_{BC} I_2^C + \bar{D}_A (I_1^B) \bar{D}_B Y_2^A + I_1^B \bar{D}_B \bar{D}_A Y_2^A + I_1^B R_{BC} Y_2^C$$
$$- \bar{D}_A (Y_2^B) \bar{D}_B I_1^A - Y_2^B \bar{D}_B \bar{D}_A I_1^A - Y_2^B R_{BC} I_1^C - \bar{D}_A (I_2^B) \bar{D}_B Y_1^A - I_2^B \bar{D}_B \bar{D}_A Y_1^A - I_2^B R_{BC} Y_1^C,$$
$$= \bar{D}_A (Y_1^B) \bar{D}_B I_2^A + Y_1^B \bar{D}_B \bar{D}_A I_2^A + \bar{D}_A (I_1^B) \bar{D}_B Y_2^A - \bar{D}_A (Y_2^B) \bar{D}_B I_1^A - Y_2^B \bar{D}_B \bar{D}_A I_1^A - \bar{D}_A (I_2^B) \bar{D}_B Y_1^A,$$
$$= Y_1^B \bar{D}_B \bar{D}_A I_2^A - Y_2^B \bar{D}_B \bar{D}_A I_1^A, \tag{G7}$$

Eq. (G6) becomes

$$\omega([\xi_1^\mu, \xi_2^\mu], \xi_3^\mu) = \pm 2 \oint_{S^2} \mathrm{d}^2 S \sqrt{\bar\gamma} [(Y_1^A \bar{D}_A f_2 - Y_2^A \bar{D}_A f_1)(\bar{D}^B \bar{D}_B W_3 - \bar{D}_B I_3^B)$$
$$- f_3 \bar{D}^A \bar{D}_A (Y_1^B \bar{D}_B W_2 - Y_2^B \bar{D}_B W_1) + f_3 \bar{D}_A (Y_1^B \bar{D}_B \bar{D}_A I_2^A - Y_2^B \bar{D}_B \bar{D}_A I_1^A)]. \tag{G8}$$

Similarly

$$\omega([\xi_3^\mu, \xi_1^\mu], \xi_2^\mu) = \pm 2 \oint_{S^2} \mathrm{d}^2 S \sqrt{\bar\gamma} [(Y_3^A \bar{D}_A f_1 - Y_1^A \bar{D}_A f_3)(\bar{D}^B \bar{D}_B W_2 - \bar{D}_B I_2^B)$$
$$- f_2 \bar{D}^A \bar{D}_A (Y_3^B \bar{D}_B W_1 - Y_1^B \bar{D}_B W_3) + f_2 \bar{D}_A (Y_3^B \bar{D}_B \bar{D}_A I_1^A - Y_1^B \bar{D}_B \bar{D}_A I_3^A)], \tag{G9}$$

and

$$\omega([\xi_2^\mu, \xi_3^\mu], \xi_1^\mu) = \pm 2 \oint_{S^2} \mathrm{d}^2 S \sqrt{\bar\gamma} [(Y_2^A \bar{D}_A f_3 - Y_3^A \bar{D}_A f_2)(\bar{D}^B \bar{D}_B W_1 - \bar{D}_B I_1^B)$$
$$- f_1 \bar{D}^A \bar{D}_A (Y_2^B \bar{D}_B W_3 - Y_3^B \bar{D}_B W_2) + f_1 \bar{D}_A (Y_2^B \bar{D}_B \bar{D}_A I_3^A - Y_3^B \bar{D}_B \bar{D}_A I_2^A)]. \tag{G10}$$

With the results above, we can show that $\omega(\xi^\mu)$ satisfies the Jacobi identity:

$$\omega([\xi_1^\mu, \xi_2^\mu], \xi_3^\mu) + \omega([\xi_3^\mu, \xi_1^\mu], \xi_2^\mu) + \omega([\xi_2^\mu, \xi_3^\mu], \xi_1^\mu) = 0. \tag{G11}$$

First

$$\omega([\xi_1^\mu, \xi_2^\mu], \xi_3^\mu) + \omega([\xi_3^\mu, \xi_1^\mu], \xi_2^\mu) + \omega([\xi_2^\mu, \xi_3^\mu], \xi_1^\mu)$$

$$= \pm 2 \oint_{S^2} d^2 S \sqrt{\bar{\gamma}} [(Y_1^A \bar{D}_A f_2 - Y_2^A \bar{D}_A f_1)(\bar{D}^B \bar{D}_B W_3 - \bar{D}_B I_3^B)$$
$$- f_3 \bar{D}^A \bar{D}_A (Y_1^B \bar{D}_B W_2 - Y_2^B \bar{D}_B W_1) - f_3 \bar{D}_A (Y_1^B \bar{D}_B \bar{D}_A I_2^A - Y_2^B \bar{D}_B \bar{D}_A I_1^A)$$
$$+ (Y_3^A \bar{D}_A f_1 - Y_1^A \bar{D}_A f_3)(\bar{D}^B \bar{D}_B W_2 - \bar{D}_B I_2^B)$$
$$- f_2 \bar{D}^A \bar{D}_A (Y_3^B \bar{D}_B W_1 - Y_1^B \bar{D}_B W_3) - f_2 \bar{D}_A (Y_3^B \bar{D}_B \bar{D}_A I_1^A - Y_1^B \bar{D}_B \bar{D}_A I_3^A)$$
$$+ (Y_2^A \bar{D}_A f_3 - Y_3^A \bar{D}_A f_2)(\bar{D}^B \bar{D}_B W_1 - \bar{D}_B I_1^B)$$
$$- f_1 \bar{D}^A \bar{D}_A (Y_2^B \bar{D}_B W_3 - Y_3^B \bar{D}_B W_2) - f_1 \bar{D}_A (Y_2^B \bar{D}_B \bar{D}_A I_3^A - Y_3^B \bar{D}_B \bar{D}_A I_2^A)]. \tag{G12}$$

Next since

$$\bar{D}^A \bar{D}_A (Y_B \bar{D}^B W) = \bar{D}^A (\bar{D}_A(Y_B) \bar{D}^B W + Y_B \bar{D}_A \bar{D}^B W),$$
$$= \bar{D}^A (-\bar{D}_B(Y_A) \bar{D}^B W + Y_B \bar{D}_A \bar{D}^B W),$$
$$= -\bar{D}^A (\bar{D}_B Y_A) \bar{D}^B W - \bar{D}_{[B}(Y_{A]}) \bar{D}^{(A} \bar{D}^{B)} W + \bar{D}_{[A}(Y_{B]}) \bar{D}^{(A} \bar{D}^{B)} W + Y_B \bar{D}^A \bar{D}_A \bar{D}^B W,$$
$$= -\bar{D}_B(\bar{D}_A Y^A) \bar{D}^B W - Y^C R_{BC} \bar{D}^B W + Y^A \bar{D}_A \bar{D}_B \bar{D}^B W + Y^A R_{AC} \bar{D}^C W,$$
$$= Y^A \bar{D}_A \bar{D}_B \bar{D}^B W, \tag{G13}$$

we have

$$\pm 2 \oint_{S^2} d^2 S \sqrt{\bar{\gamma}} [(Y_1^A \bar{D}_A f_2 - Y_2^A \bar{D}_A f_1) \bar{D}^B \bar{D}_B W_3 - f_3 \bar{D}^A \bar{D}_A (Y_1^B \bar{D}_B W_2 - Y_2^B \bar{D}_B W_1)$$
$$+ (Y_3^A \bar{D}_A f_1 - Y_1^A \bar{D}_A f_3) \bar{D}^B \bar{D}_B W_2 - f_2 \bar{D}^A \bar{D}_A (Y_3^B \bar{D}_B W_1 - Y_1^B \bar{D}_B W_3)$$
$$+ (Y_2^A \bar{D}_A f_3 - Y_3^A \bar{D}_A f_2) \bar{D}^B \bar{D}_B W_1 - f_1 \bar{D}^A \bar{D}_A (Y_2^B \bar{D}_B W_3 - Y_3^B \bar{D}_B W_2)]$$
$$= \pm 2 \oint_{S^2} d^2 S \sqrt{\bar{\gamma}} [(Y_1^A \bar{D}_A f_2 - Y_2^A \bar{D}_A f_1) \bar{D}^B \bar{D}_B W_3 - f_3 (Y_1^A \bar{D}_A \bar{D}_B \bar{D}^B W_2 - Y_2^A \bar{D}_A \bar{D}_B \bar{D}^B W_1)$$
$$+ (Y_3^A \bar{D}_A f_1 - Y_1^A \bar{D}_A f_3) \bar{D}^B \bar{D}_B W_2 - f_2 (Y_3^A \bar{D}_A \bar{D}_B \bar{D}^B W_1 - Y_1^A \bar{D}_A \bar{D}_B \bar{D}^B W_3)$$
$$+ (Y_2^A \bar{D}_A f_3 - Y_3^A \bar{D}_A f_2) \bar{D}^B \bar{D}_B W_1 - f_1 (Y_2^A \bar{D}_A \bar{D}_B \bar{D}^B W_3 - Y_3^A \bar{D}_A \bar{D}_B \bar{D}^B W_2)]$$
$$= \pm 2 \oint_{S^2} d^2 S \sqrt{\bar{\gamma}} [-(Y_1^A f_2 \bar{D}_A \bar{D}^B \bar{D}_B W_3 - Y_2^A f_1 \bar{D}_A \bar{D}^B \bar{D}_B W_3)$$
$$- (Y_1^A f_3 \bar{D}_A \bar{D}_B \bar{D}^B W_2 - Y_2^A f_3 \bar{D}_A \bar{D}_B \bar{D}^B W_1) - (Y_3^A f_1 \bar{D}_A \bar{D}^B \bar{D}_B W_2 - Y_1^A f_3 \bar{D}_A \bar{D}^B \bar{D}_B W_2)$$
$$- (Y_3^A f_2 \bar{D}_A \bar{D}_B \bar{D}^B W_1 - Y_1^A f_2 \bar{D}_A \bar{D}_B \bar{D}^B W_3) - (Y_2^A f_3 \bar{D}_A \bar{D}^B \bar{D}_B W_1 - Y_3^A f_2 \bar{D}_A \bar{D}^B \bar{D}_B W_1)$$
$$- (Y_2^A f_1 \bar{D}_A \bar{D}_B \bar{D}^B W_3 - Y_3^A f_1 \bar{D}_A \bar{D}_B \bar{D}^B W_2)]$$
$$= 0. \tag{G14}$$

Furthermore,

$$\pm 2 \oint_{S^2} \mathrm{d}^2 S \sqrt{\bar{\gamma}} [-(Y_3^A \bar{D}_A f_1 - Y_1^A \bar{D}_A f_3) \bar{D}_B I_2^B - (-f_2(Y_3^B \bar{D}_B \bar{D}_A I_1^A - Y_1^B \bar{D}_B \bar{D}_A I_3^A))$$

$$- (Y_1^A \bar{D}_A f_2 - Y_2^A \bar{D}_A f_1) \bar{D}_B I_3^B - (-f_3(Y_1^B \bar{D}_B \bar{D}_A I_2^A - Y_2^B \bar{D}_B \bar{D}_A I_1^A))$$

$$- (Y_2^A \bar{D}_A f_3 - Y_3^A \bar{D}_A f_2) \bar{D}_B I_1^B - (-f_1(Y_2^B \bar{D}_B \bar{D}_A I_3^A - Y_3^B \bar{D}_B \bar{D}_A I_2^A))]$$

$$= \pm 2 \oint_{S^2} \mathrm{d}^2 S \sqrt{\bar{\gamma}} [(Y_3^A f_1 \bar{D}_A \bar{D}_B I_2^B - Y_1^A f_3 \bar{D}_A \bar{D}_B I_2^B) + (f_2 Y_3^B \bar{D}_B \bar{D}_A I_1^A - f_2 Y_1^B \bar{D}_B \bar{D}_A I_3^A)$$

$$+ (Y_1^A f_2 \bar{D}_A \bar{D}_B I_3^B - Y_2^A f_1 \bar{D}_A \bar{D}_B I_3^B) + (f_3 Y_1^B \bar{D}_B \bar{D}_A I_2^A - f_3 Y_2^B \bar{D}_B \bar{D}_A I_1^A)$$

$$+ (Y_2^A f_3 \bar{D}_A \bar{D}_B I_1^B - Y_3^A f_2 \bar{D}_A \bar{D}_B I_1^B) + (f_1 Y_2^B \bar{D}_B \bar{D}_A I_3^A - f_1 Y_3^B \bar{D}_B \bar{D}_A I_2^A)]$$

$$= 0. \tag{G15}$$

As the result, we verify (G11).

[1] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 2010).

[2] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, Cambridge, England, 2008).

[3] V. Mukhanov, *Physical Foundations of Cosmology* (Cambridge University Press, Cambridge, England, 2005).

[4] S. Dodelson, *Modern Cosmology* (Elsevier, New York, 2003).

[5] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Courier Corporation, New York, 2001), Vol. 2.

[6] C. Rovelli, What is observable in classical and quantum gravity?, Classical Quantum Gravity **8**, 297 (1991).

[7] C. Rovelli, Quantum reference systems, Classical Quantum Gravity **8**, 317 (1991).

[8] C. Rovelli, Time in quantum gravity: An hypothesis, Phys. Rev. D **43**, 442 (1991).

[9] C. Rovelli, Quantum mechanics without time: A model, Phys. Rev. D **42**, 2638 (1990).

[10] C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, England, 2004).

[11] C. Rovelli and L. Smolin, The physical Hamiltonian in nonperturbative quantum gravity, Phys. Rev. Lett. **72**, 446 (1994).

[12] K. V. Kuchař and J. D. Romano, Gravitational constraints that generate a lie algebra, Phys. Rev. D **51**, 5579 (1995).

[13] B. Dittrich, Partial and complete observables for Hamiltonian constrained systems, Gen. Relativ. Gravit. **39**, 1891 (2007).

[14] B. Dittrich, Partial and complete observables for canonical general relativity, Classical Quantum Gravity **23**, 6155 (2006).

[15] J. Tambornino *et al.*, Relational observables in gravity: A review, SIGMA **8**, 017 (2012).

[16] A. Ashtekar and P. Singh, Loop quantum cosmology: A status report, Classical Quantum Gravity **28**, 213001 (2011).

[17] M. Domagala, K. Giesel, W. Kaminski, and J. Lewandowski, Gravity quantized: Loop quantum gravity with a scalar field, Phys. Rev. D **82**, 104038 (2010).

[18] K. Giesel, S. Hofmann, T. Thiemann, and O. Winkler, Manifestly gauge-invariant general relativistic perturbation theory: I. foundations, Classical Quantum Gravity **27**, 055005 (2010).

[19] K. Giesel, J. Tambornino, and T. Thiemann, LTB spacetimes in terms of dirac observables, Classical Quantum Gravity **27**, 105013 (2010).

[20] G. Domènech and M. Sasaki, Hamiltonian approach to second order gauge invariant cosmological perturbations, Phys. Rev. D **97**, 023521 (2018).

[21] J.-c. Hwang, D. Jeong, and H. Noh, Gauge dependence of gravitational waves generated from scalar perturbations, Astrophys. J. **842**, 46 (2017).

[22] K. Tomikawa and T. Kobayashi, Gauge dependence of gravitational waves generated at second order from scalar perturbations, Phys. Rev. D **101**, 083529 (2020).

[23] V. De Luca, G. Franciolini, A. Kehagias, and A. Riotto, On the gauge invariance of cosmological gravitational waves, J. Cosmol. Astropart. Phys. 03 (2020) 014.

[24] C. Yuan, Z.-C. Chen, and Q.-G. Huang, Scalar induced gravitational waves in different gauges, Phys. Rev. D **101**, 063018 (2020).

[25] H. Noh and J.-c. Hwang, Second-order perturbations of the friedmann world model, Phys. Rev. D **69**, 104011 (2004).

[26] J. D. Brown and K. V. Kuchař, Dust as a standard of space and time in canonical quantum gravity, Phys. Rev. D **51**, 5600 (1995).

[27] T. Regge and C. Teitelboim, Role of surface integrals in the Hamiltonian formulation of general relativity, Ann. Phys. (N.Y.) **88**, 286 (1974).

[28] M. Henneaux and C. Troessaert, BMS group at spatial infinity: The Hamiltonian (ADM) approach, J. High Energy Phys. 03 (2018) 147.

[29] R. Beig and N óMurchadha, The poincaré group as the symmetry group of canonical general relativity, Ann. Phys. (N.Y.) **174**, 463 (1987).

[30] R. B. Mann and D. Marolf, Holographic renormalization of asymptotically flat spacetimes, Classical Quantum Gravity **23**, 2927 (2006).

[31] G. Compere and F. Dehouck, Relaxing the parity conditions of asymptotically flat gravity, Classical Quantum Gravity **28**, 245016 (2011).

[32] H. Bondi, M. G. J. Van der Burg, and A. Metzner, Gravitational waves in general relativity, VII. Waves from axisymmetric isolated system, Proc. R. Soc. A **269**, 21 (1962).

[33] R. K. Sachs, Gravitational waves in general relativity VIII. Waves in asymptotically flat space-time, Proc. R. Soc. A **270**, 103 (1962).

[34] G. Barnich and C. Troessaert, Aspects of the BMS/CFT correspondence, J. High Energy Phys. 05 (2010) 062.

[35] G. Barnich and C. Troessaert, BMS charge algebra, J. High Energy Phys. 12 (2011) 105.

[36] D. Christodoulou, The global initial value problem in general relativity, in *The Ninth Marcel Grossmann Meeting: On Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories (In 3 Volumes)* (World Scientific, Singapore, 2002), pp. 44–54.

[37] H. Friedrich, Peeling or not peeling–is that the question?, Classical Quantum Gravity **35**, 083001 (2018).

[38] M. Henneaux, Corvino-schoen theorem and supertranslations at spatial infinity, arXiv:2306.12505.

[39] K. Giesel and T. Thiemann, Algebraic quantum gravity (AQG): IV. Reduced phase space quantization of loop quantum gravity, Classical Quantum Gravity **27**, 175009 (2010).

[40] M. Han and H. Liu, Semiclassical limit of new path integral formulation from reduced phase space loop quantum gravity, Phys. Rev. D **102**, 024083 (2020).

[41] D. Harlow and J.-Q. Wu, Covariant phase space with boundaries, J. High Energy Phys. 10 (2020) 146.

[42] L. Andersson and A. Y. Burtscher, On the asymptotic behavior of static perfect fluids, in *Annales Henri Poincaré* (Springer, New York, 2019), Vol. 20, pp. 813–857.

[43] R. Beig and B. G. Schmidt, Einstein's equations near spatial infinity, Commun. Math. Phys. **87**, 65 (1982).

[44] C. Troessaert, The BMS4 algebra at spatial infinity, Classical Quantum Gravity **35**, 074003 (2018).

[45] A. Ashtekar and R. O. Hansen, A unified treatment of null and spatial infinity in general relativity. I. Universal structure, asymptotic symmetries, and conserved quantities at spatial infinity, J. Math. Phys. (N.Y.) **19**, 1542 (1978).

[46] H. Friedrich, Gravitational fields near space-like and null infinity, J. Geom. Phys. **24**, 83 (1998).

[47] H. Friedrich and J. Kannar, Calculating asymptotic quantities near space-like and null infinity from cauchy data, Ann. Phys. (Berlin) **512**, 321 (2000).

[48] H. Friedrich and J. Kannar, Bondi-type systems near spacelike infinity and the calculation of the Newman–Penrose constants, J. Math. Phys. (N.Y.) **41**, 2195 (2000).

[49] K. Giesel, S. Hofmann, T. Thiemann, and O. Winkler, Manifestly gauge-invariant general relativistic perturbation theory: III. FRW background and second order (to be published).

[50] A. Ashtekar, New variables for classical quantum gravity, Phys. Rev. Lett. **57**, 2244 (1986).

[51] M. Han, Y. Ma, and W. Huang, Fundamental structure of loop quantum gravity, Int. J. Mod. Phys. D **16**, 1397 (2007).

[52] A. Ashtekar and J. Lewandowski, Background independent quantum gravity: A status report, Classical Quantum Gravity **21**, R53 (2004).

[53] A. Ashtekar and S. Das, Asymptotically anti-de Sitter spacetimes: Conserved quantities, Classical Quantum Gravity **17**, L17 (2000).

[54] D. Marolf, W. Kelly, and S. Fischetti, Conserved charges in asymptotically (locally) AdS spacetimes, in *Springer Handbook of Spacetime* (Springer, New York, 2014), pp. 381–407.

[55] L. Freidel, R. Oliveri, D. Pranzetti, and S. Speziale, The Weyl BMS group and Einstein's equations, J. High Energy Phys. 07 (2021) 170.

[56] L. Freidel, R. Oliveri, D. Pranzetti, and S. Speziale, Extended corner symmetry, charge bracket and Einstein's equations, J. High Energy Phys. 09 (2021) 083.

[57] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory* (Princeton University Press, Princeton, NJ, 2018).

[58] A. Strominger, On BMS invariance of gravitational scattering, J. High Energy Phys. 07 (2014) 152.

[59] T. He, V. Lysov, P. Mitra, and A. Strominger, BMS supertranslations and Weinberg's soft graviton theorem, J. High Energy Phys. 05 (2015) 151.

[60] S. Weinberg, Infrared photons and gravitons, Phys. Rev. **140**, B516 (1965).

[61] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, England, 1995), Vol. 2.

[62] P. R. Chernoff and J. E. Marsden, *Some Basic Properties of Infinite Dimensional Hamiltonian Systems* (1975).

[63] T. Kohno, *Conformal Field Theory and Topology* (American Mathematical Society, Providence, 2002).

Correction: The "Corresponding author" label was missing from a byline footnote and has been inserted.