Cracking and complexity of self-gravitating dissipative compact objects

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The concept of cracking refers to the tendency of a fluid distribution to "split," once it abandons the equilibrium. In this manuscript we develop a general formalism to describe the occurrence of cracking within a dissipative fluid distribution, in comoving coordinates. The role of dissipative processes in the occurrence of cracking is brought out. Next, we relate the occurrence of cracking with the concept of complexity for self-gravitating objects defined in Herrera [Phys. Rev. D **97**, 044010 (2018)], Herrera *et al.* [Phys. Rev. D **98**, 104059 (2018)], and Herrera *et al.* [Eur. Phys. J. C **80**, 631 (2020)]. More specifically we relate the occurrence of cracking with the condition of the vanishing of the scalar function intended to measure the complexity of the fluid distribution (the complexity factor). We also relate the occurrence of cracking with the specific mode of leaving the equilibrium. Thus, we prove that leaving the equilibrium in either the homologous (H) or the quasihomologous regime (QH) prevents the occurrence of cracking. Also, it is shown that imposing the condition of vanishing complexity factor alone (independently of the mode of leaving the equilibrium) prevents the occurrence of cracking in the nondissipative geodesic case, and in the nondissipative isotropic case. These results bring out further the relevance of the complexity factor and its related definition of complexity in the study of self-gravitating systems.

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I. INTRODUCTION

This work deals with the interplay between the concepts of cracking [1] and complexity [2–4] of a fluid distribution in the presence of dissipative processes. These two concepts have been shown to be relevant in the study of self-gravitating systems. Besides, dissipative processes are expected to be present during many phases of the stellar evolution.

The concept of cracking is associated with the tendency of a fluid distribution to "split," once it abandons the equilibrium as a consequence of perturbations. Thus we say that, once the system has abandoned the equilibrium, there is a cracking, whenever its inner part tends to collapse whereas its outer part tends to expand. The cracking takes place at the surface separating the two regions. When the inner part tends to expand and the outer one tends to collapse we say that there is an overturning.

In order to avoid some misunderstandings often found in the literature, we find it useful to stress the following remarks:

(i) The concepts of stability and cracking are different, although they are often confused.

(ii) The term stability refers to the capacity of a given fluid distribution to return to equilibrium once it has been removed from it. The fact that the speeds of sound are not superluminal does not assure in any way the stability of the object, it only ensures causality.

(iii) The cracking only implies the tendency of the system to "split" immediately after leaving the equilibrium, where "immediately" means on a timescale smaller than the hydrostatic timescale, and the thermal relaxation time. Whatever happens next, whether the system enters into a dynamic regime, or returns to equilibrium, is independent of the concept of cracking. Of course the occurrence of cracking will affect the future of the fluid configuration in either case.

(iv) In order to check the occurrence (or not) of cracking one must take the system out of its state of equilibrium. For doing that one submits the system to fluctuations. In the original paper of cracking [1] these fluctuations were assumed to be generic (of an "unspecified" nature). The specific case of fluctuations associated with compression of the fluid has been considered in [5]. In this latter case, the confusion between cracking and stability may appear due to the fact that the adiabatic index is related to the speed of sound and the stability.

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In other words:

(i) In a system which is stable, i.e., a system that, once removed from equilibrium, comes back to it in a timescale of the order of hydrostatic time, cracking may occur or not; or

(ii) After the occurrence of cracking the system may return to equilibrium (the system is stable) or enters into a dynamic regime (the system is unstable).

It is worth mentioning, with respect to the physical relevance of cracking, that cracking might be invoked as the possible origin of quakes in neutron stars [6–8]. In fact, large scale crust cracking in neutron stars and their relevance in the occurrence of glitches and bursts of x rays and gamma rays have been considered in detail in [9].

The notion of complexity of a given fluid distribution involves two different problems. On the one hand the complexity of the structure of the fluid, which is described by the complexity factor. On the other hand, when dealing with systems in the dynamic regime we still need to describe the complexity of its pattern of evolution. The complexity factor is a scalar function (for non–spherical distributions complexity may be described by more than one scalar [10]) intended to measure the degree of complexity of the structure of a self–gravitating fluid distribution. This concept has received a great deal of attention in recent years. The origin of such an interest being the conviction that a variable measuring complexity should be suitable to describe essential aspects of the system. Regarding the complexity of the pattern of evolution, we need to know what is the simplest mode of evolution. In [3,4] two different patterns of evolution were considered as the "simplest" ones, namely: the homologous (H) and the quasi–homologous regime (QH).

Finally, we know that dissipation due to the emission of massless particles (photons and/or neutrinos) is a characteristic process in the evolution of massive stars.

In fact, it seems that the only plausible mechanism to carry away the bulk of the binding energy of the collapsing star, leading to a neutron star or black hole, is neutrino emission [11].

We shall describe dissipation in the diffusion approximation, which applies whenever the energy flux of radiation (as that of thermal conduction) is proportional to the gradient of temperature. This assumption is in general very sensible, since the mean free path of particles responsible for the propagation of energy in stellar interiors is in general very small as compared with the typical length of the object.

In many other circumstances, the mean free path of particles transporting energy may be large enough as to justify the free streaming approximation, however it is a simple matter to include this regime of radiative transport, just by redefining the energy-density and the radial pressure (see below).

The purpose of this work is threefold. On the one hand, since the treatment of cracking, so far, has been handled using noncomoving coordinates, we want to present an approach in comoving coordinates, which many authors consider more suitable for treating numerical problems. In our case, the motivation behind this endeavor is based on the fact that the concept of complexity for dynamical systems has been developed using comoving coordinates [3,4]. It is worth mentioning that although the general idea underlying the concept of cracking remains the same in both frames, it is expressed through variables which are not exactly equivalent.

Next, we want to extend the concept of cracking, as defined in [1], to the dissipative case. More specifically we want to find out what might be the role of dissipative processes in the occurrence of cracking.

Finally, we want to relate the concept of cracking to the concept of complexity (see Refs. [12,13] in relation with this issue). In particular we want to know what constraints on the occurrence of cracking may appear from the vanishing complexity factor condition and/or from conditions on the complexity of the pattern of evolution when leaving the equilibrium. The motivation for such an endeavor becomes intelligible if we notice that the appearance of cracking in a given self-gravitating fluid distribution implies an increasing of complexity in the structure of the fluid, as compared with the situation when cracking is absent.

All the obtained results concerning the three issues mentioned above are discussed in detail in the last section.

Let us start by introducing the notation, conventions, and all the required equations.

II. BASIC EQUATIONS AND VARIABLES

In this section we shall deploy the relevant equations and variables for describing a time dependent, dissipative, spherically symmetric self-gravitating locally anisotropic fluid. To avoid repeating calculations, the procedure to obtain some equations is referred to previous works.

A. Einstein equations, physical variables, kinematical variables

We consider spherically symmetric distributions of collapsing fluid, which for the sake of completeness we assume to be locally anisotropic, bounded by a spherical surface Σ, and undergoing dissipation in the form of heat flow (diffusion approximation).

The reason to consider anisotropic fluids is well justified since local anisotropy of pressure may be caused by a large variety of physical phenomena, of the kind we expect in compact objects [14]. More so, as it has been recently shown [15], physical processes expected to play a relevant role in stellar evolution (e.g., dissipation) will always tend to produce pressure anisotropy, even if the system is initially assumed to be isotropic. Since any equilibrium configuration is the final stage of a dynamic regime, there is no reason to think that the acquired anisotropy during this dynamic process would disappear in the final equilibrium state, and therefore the resulting configuration, even if initially had isotropic pressure, should in principle exhibit pressure anisotropy.

Choosing comoving coordinates, the general interior metric can be written

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where A, B, and R are functions of t and r and are assumed positive. We number the coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$. Observe that A and B are dimensionless, whereas R has the same dimension as r.

The energy-momentum tensor $T_{\alpha\beta}$ of the fluid distribution has the form

$$T_{\alpha\beta} = (\mu + P_\perp)V_\alpha V_\beta + P_\perp g_{\alpha\beta} + (P_r - P_\perp)\chi_\alpha\chi_\beta + q_\alpha V_\beta + V_\alpha q_\beta, \quad (2)$$

where μ is the mass-energy density, P_r the radial pressure, $P_\perp$ the tangential pressure, q^α the heat flux, V^α the four-velocity of the fluid, and χ^α a unit four-vector along the radial direction. These quantities satisfy

$$V^\alpha V_\alpha = -1, \quad V^\alpha q_\alpha = 0, \quad \chi^\alpha \chi_\alpha = 1, \quad \chi^\alpha V_\alpha = 0. \quad (3)$$

Or, in the equivalent (canonical) form

$$T_{\alpha\beta} = \mu V_\alpha V_\beta + P h_{\alpha\beta} + \Pi_{\alpha\beta} + q(V_\alpha \chi_\beta + \chi_\alpha V_\beta), \quad (4)$$

with

$$P = \frac{P_r + 2P_\perp}{3}, \qquad h_{\alpha\beta} = g_{\alpha\beta} + V_\alpha V_\beta,$$
$$\Pi_{\alpha\beta} = \Pi\left(\chi_\alpha\chi_\beta - \frac{1}{3}h_{\alpha\beta}\right), \qquad \Pi = P_r - P_\perp,$$

where q is a function of t and r.

Since we are considering comoving observers, we have

$$V^\alpha = A^{-1}\delta_0^\alpha, \qquad q^\alpha = qB^{-1}\delta_1^\alpha, \qquad \chi^\alpha = B^{-1}\delta_1^\alpha. \quad (5)$$

It is worth noticing that both bulk and shear viscosity could be easily introduced to the system through a redefinition of the radial and tangential pressures, P_r and $P_\perp$. Also, dissipation in the free streaming approximation could be introduced by redefining μ, P_r, and q.

The Einstein equations for (1) and (4) are explicitly written in Appendix A.

The acceleration a_α and the expansion Θ of the fluid are given by

$$a_\alpha = V_{\alpha;\beta}V^\beta, \qquad \Theta = V^\alpha{}_{;\alpha}, \quad (6)$$

and its shear $\sigma_{\alpha\beta}$ by

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha}V_{\beta)} - \frac{1}{3}\Theta h_{\alpha\beta}, \quad (7)$$

from which we easily obtain

$$a_1 = \frac{A'}{A}, \qquad a = \sqrt{a^\alpha a_\alpha} = \frac{A'}{AB}, \quad (8)$$

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{R}}{R}\right), \quad (9)$$

$$\sigma_{11} = \frac{2}{3}B^2\sigma, \qquad \sigma_{22} = \frac{\sigma_{33}}{\sin^2\theta} = -\frac{1}{3}R^2\sigma, \quad (10)$$

where

$$\sigma^{\alpha\beta}\sigma_{\alpha\beta} = \frac{2}{3}\sigma^2, \quad (11)$$

with

$$\sigma = \frac{1}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{R}}{R}\right), \quad (12)$$

where the prime stands for r differentiation and the dot stands for differentiation with respect to t.

Next, the mass function $m(t,r)$ introduced by Misner and Sharp [16] reads

$$m = \frac{R^3}{2}R_{23}{}^{23} = \frac{R}{2}\left[\left(\frac{\dot{R}}{A}\right)^2 - \left(\frac{R'}{B}\right)^2 + 1\right], \quad (13)$$

and introducing the proper time derivative D_T given by

$$D_T = \frac{1}{A}\frac{\partial}{\partial t}, \quad (14)$$

we can define the velocity U of the collapsing fluid as the variation of the areal radius with respect to proper time, i.e.,

$$U = D_T R, \quad (15)$$

where R defines the areal radius of a spherical surface inside the fluid distribution (as measured from its area).

Then (13) can be rewritten as

$$E \equiv \frac{R'}{B} = \left(1 + U^2 - \frac{2m}{R}\right)^{1/2}. \quad (16)$$

Using (16) we can express (A6) as

$$4\pi q = E\left[\frac{1}{3}D_R(\Theta - \sigma) - \frac{\sigma}{R}\right], \quad (17)$$

where D_R denotes the proper radial derivative,

$$D_R = \frac{1}{R'}\frac{\partial}{\partial r}. \tag{18}$$

Using (A2)–(A4) with (18) we obtain from (13)

$$D_R m = 4\pi\left(\mu + q\frac{U}{E}\right)R^2, \tag{19}$$

which implies

$$m = 4\pi\int_0^r \left(\mu + q\frac{U}{E}\right)R^2 R' dr, \tag{20}$$

satisfying the regular condition $m(t,0) = 0$.

Integrating (20) we find

$$\frac{3m}{R^3} = 4\pi\mu - \frac{4\pi}{R^3}\int_0^r R^3\left(D_R\mu - 3q\frac{U}{RE}\right)R' dr. \tag{21}$$

B. The complexity factor and the Weyl tensor

As we have already mentioned, in the dynamic case the definition of a quantity measuring the complexity of the system poses two additional problems with respect to the static case.

On the one hand, the definition of the complexity of the structure of the fluid, which in this case also involves dissipative variables, and on the other hand the problem of defining the complexity of the pattern of evolution of the system.

For the static fluid distribution it was assumed in [2] that the scalar function Y_{TF}, appearing in the orthogonal splitting of the Riemann tensor, and named complexity factor, is an appropriate measure of the complexity of the fluid, and therefore was identified as the complexity factor.

As in [3], we shall assume in the dynamic case that Y_{TF} still measures the complexity of the system in what corresponds to the structure of the object.

In order to provide the necessary mathematical expressions for defining Y_{TF}, let us start by finding the expression for the Weyl tensor.

In the spherically symmetric case the Weyl tensor ($C^{\rho}_{\alpha\beta\mu}$) is defined by its "electric" part $E_{\gamma\nu}$ alone, since its "magnetic" part vanishes, with

$$E_{\alpha\beta} = C_{\alpha\mu\beta\nu}V^{\mu}V^{\nu}, \tag{22}$$

where the electric part of the Weyl tensor may also be written as

$$E_{\alpha\beta} = \mathcal{E}\left(\chi_\alpha\chi_\beta - \frac{1}{3}h_{\alpha\beta}\right), \tag{23}$$

with

$$\begin{aligned}\mathcal{E} = {} & \frac{1}{2A^2}\left[\frac{\ddot{R}}{R} - \frac{\ddot{B}}{B} - \left(\frac{\dot{R}}{R} - \frac{\dot{B}}{B}\right)\left(\frac{\dot{A}}{A} + \frac{\dot{R}}{R}\right)\right] \\ & + \frac{1}{2B^2}\left[\frac{A''}{A} - \frac{R''}{R} + \left(\frac{B'}{B} + \frac{R'}{R}\right)\left(\frac{R'}{R} - \frac{A'}{A}\right)\right] \\ & - \frac{1}{2R^2}.\end{aligned} \tag{24}$$

Then, it can be shown that (see Ref. [3] for details)

$$Y_{TF} = \mathcal{E} - 4\pi\Pi. \tag{25}$$

Next, using (A2), (A4), and (A5) with (13) and (24) we obtain

$$\frac{3m}{R^3} = 4\pi(\mu - \Pi) - \mathcal{E}, \tag{26}$$

which combined with (21) and (25) produces

$$Y_{TF} = -8\pi\Pi + \frac{4\pi}{R^3}\int_0^r R^3\left(D_R\mu - 3q\frac{U}{RE}\right)R' dr. \tag{27}$$

Thus the scalar Y_{TF} may be expressed through the Weyl tensor and the anisotropy of pressure or in terms of the anisotropy of pressure, the mass-energy density inhomogeneity and the dissipative variables.

Another useful expression for Y_{TF} may be obtained (see Ref. [17] for details), which reads

$$Y_{TF} \equiv \mathcal{E} - 4\pi\Pi = \frac{a'}{B} - \frac{\dot{\sigma}}{A} + a^2 - \frac{\sigma^2}{3} - \frac{2}{3}\Theta\sigma - a\frac{R'}{RB}. \tag{28}$$

Once the complexity factor for the structure of the fluid distribution has been established, it remains to elucidate what is the simplest pattern of evolution.

From the integration of (A3) one obtains

$$U = \frac{U_\Sigma}{R_\Sigma}R - R\int_r^{r_\Sigma}\left(\frac{4\pi}{E}q + \frac{\sigma}{R}\right)R' dr, \tag{29}$$

where $r = r_\Sigma =$ constant is the equation of the boundary surface of the fluid distribution, and subscript Σ means that the quantity is evaluated on the boundary surface.

If the integral in the above equation vanishes we have that $U \sim R$, which is a reminiscence of the homologous evolution in Newtonian hydrodynamics. This may occur if the fluid is shear-free and nondissipative, or if the two terms in the integral cancel each other.

In the past, two regimes of evolution have been considered as candidates to describe the simplest mode of evolution. One is the relativistic version of homologous evolution (H) characterized by the conditions (see Ref. [3] for details)

$$U = \tilde{a}(t)R, \tag{30}$$

where $\tilde{a} \equiv \frac{U_\Sigma}{R_\Sigma}$, and

$$\frac{R_I}{R_{II}} = \text{constant}, \tag{31}$$

where R_I and R_{II} denote the areal radii of two concentric shells (I, II) described by $r = r_I = \text{constant}$, and $r = r_{II} = \text{constant}$, respectively. In Newtonian hydrodynamics a linear dependence of radial velocity on the radial distance implies a condition similar to (31). However in the relativistic regime, (30) does not imply (31), except in the geodesic case.

However the H condition may be too stringent, ruling out many interesting scenarios from the astrophysical point of view and therefore, another possible (less restrictive) mode of evolution which also could be used to describe the simplest mode of evolution, and which we call quasihomologous (QH), has been proposed [4].

In this case the fluid satisfies condition (30), but not (31).

It follows from (29) that condition (30) implies

$$\frac{4\pi}{R'} Bq + \frac{\sigma}{R} = 0, \tag{32}$$

which is the only condition imposed in the QH regime.

To summarize, the H condition implies (31) and (32), whereas the QH regime only demands (32).

If we impose the H condition, then it can be shown that (see Ref. [2] for details)

$$\frac{\ddot{R}}{R} - \frac{\ddot{B}}{B} = Y_{TF}. \tag{33}$$

If we further assume the fluid to be nondissipative, recalling that in this case the H condition implies the vanishing of the shear, we obtain (see Ref. [2] for details)

$$\frac{\ddot{R}}{R} - \frac{\ddot{B}}{B} = 0 \Rightarrow Y_{TF} = 0. \tag{34}$$

In other words, in this particular case, the H condition already implies the vanishing complexity factor condition.

More so, for the nondissipative case, the H condition not only implies $Y_{TF} = 0$, but also implies that the fluid is shear free, geodesic (nondissipative dust) with homogeneous mass-energy density, and vanishing Weyl tensor, representing the simplest conceivable configuration (Friedman–Robertson–Walker) (see Refs. [18,19]).

Based on all the precedent comments, it seems reasonable to consider the H condition as a good candidate to describe the simplest mode of evolution.

In the dissipative case, we may obtain from (12) and (34)

$$Y_{TF}\frac{R'}{R} = 4\pi Bq\left(\frac{\dot{q}}{q} + 2\frac{\dot{B}}{B} + \frac{\dot{R}}{R}\right). \tag{35}$$

If we assume $Y_{TF} = 0$, then we obtain

$$q = \frac{f(r)}{B^2 R}, \tag{36}$$

where f is an arbitrary integration function.

Taking the time derivative of the above equation and using (9) and (12), it follows at once

$$\dot{q} = -q(\Theta + \sigma). \tag{37}$$

In the above we have assumed the H condition in order to describe the simplest mode of evolution, however as indicated before, such a condition may be too restrictive, and it could be wise to consider less stringent conditions. That's why we shall also consider the QH condition (32) as an alternative to describe the simplest mode of evolution.

In the dissipative case we need to provide a transport equation to describe the evolution and distribution of temperature. Assuming a causal dissipative theory (e.g., the Israel-Stewart theory [20–22]), the transport equation for the heat flux reads

$$\tau h^{\alpha\beta} V^\gamma q_{\beta;\gamma} + q^\alpha = -\kappa h^{\alpha\beta}(T_{,\beta} + T a_\beta) - \frac{1}{2}\kappa T^2 \left(\frac{\tau V^\beta}{\kappa T^2}\right)_{;\beta} q^\alpha, \tag{38}$$

where κ denotes the thermal conductivity, and T and τ denote temperature and relaxation time, respectively.

In the nonrelativistic regime the above equation leads to the Cattaneo-type equation [23]

$$\tau \frac{\partial \vec{q}}{\partial t} + \vec{q} = -\kappa \vec{\nabla} T, \tag{39}$$

which in turn produces a hyperbolic equation for the temperature (the telegraph equation) [24]

$$\frac{\kappa}{\tau\gamma}\nabla^2 T = \frac{\partial^2 T}{\partial t^2} + \frac{1}{\tau}\frac{\partial T}{\partial t}, \tag{40}$$

where γ denotes the heat capacity per volume unit.

In the spherically symmetric case under consideration, the transport equation has only one independent component, which may be obtained from (38) by contracting with the unit spacelike vector χ^α, producing

$$\tau V^\alpha q_{,\alpha} + q = -\kappa(\chi^\alpha T_{,\alpha} + T a) - \frac{1}{2}\kappa T^2 \left(\frac{\tau V^\alpha}{\kappa T^2}\right)_{;\alpha} q. \tag{41}$$

III. SETUP OF THE PROBLEM

We consider a fluid distribution which is initially (say at $t=0$) in equilibrium, and then at $t=0$, due to perturbations, it is forced to leave the equilibrium state. We shall evaluate the system in the time interval $(0, \tilde{t})$, such that $\tilde{t}$ is smaller than the hydrostatic time and the thermal relaxation time. Therefore in that time interval, we have

$$\dot{R} = \dot{B} = U = \Theta = \sigma = q = 0,$$
$$\ddot{R} \neq 0, \qquad D_T U \neq 0, \qquad D_T q \neq 0, \tag{42}$$

$$B = B_0, \qquad R = R_0, \qquad A = A_0, \tag{43}$$

where the subscript 0 indicates the value of the quantity in the equilibrium, and (A4) and (B5) have been used.

To summarize: at the timescale considered here, the metric variables conserve the same value they have before the perturbation, and their first order time derivatives vanish. Also, all kinematical variables vanish, but not so their first time derivatives.

We say that there is a cracking (overturning) at some value of r (say $r = r_{cr}$) whenever $D_T U$ vanishes at $r = r_{cr}$, being positive (negative) for $r > r_{cr}$ and negative (positive) for $r < r_{cr}$.

We shall denote by $F \equiv (\mu + P_r) D_T U$ the total force applied to any fluid element immediately after leaving the equilibrium. Then from (B6) (evaluated at the timescale mentioned above) we may write

$$F = -(\mu + P_r)\left[\frac{m}{R^2} + 4\pi P_r R\right] - E^2\left[D_R P_r + 2(P_r - P_\perp)\frac{1}{R}\right] - E D_T q. \tag{44}$$

A. Nondissipative isotropic fluid

Let us first consider an isotropic fluid in equilibrium, whose energy density is given by

$$\mu = \xi / R^2, \tag{45}$$

where ξ is a constant.

Integrating (44) for $F = P_r - P_\perp = D_T q = 0$, we obtain for P_r

$$P_r = \frac{3\xi}{r^2}\frac{\left(1 - \frac{r}{r_\Sigma}\right)}{\left(9 - \frac{r}{r_\Sigma}\right)}, \tag{46}$$

with $\xi = \frac{3}{56\pi}$, and $r = r_\Sigma$ denotes the boundary surface of the fluid distribution.

Thus our a static solution is characterized by (45), (46), and

$$R = r, \qquad B = \frac{1}{\sqrt{1 - 8\pi\xi}} = \frac{\sqrt{7}}{2}, \qquad A = \sqrt{r}(9r_\Sigma - r). \tag{47}$$

It is worth mentioning that such a solution belongs to the type VI Tolman class [25], whose equation of state for large values of μ approaches that for a highly compressed Fermi gas. Since it is singular at $r=0$, the center should be excluded from consideration.

Let us now remove our system from equilibrium by perturbing the parameter ξ, assuming

$$\xi = \frac{3}{56\pi} + \epsilon, \tag{48}$$

where $|\epsilon| \ll 1$. It is important to stress that such a perturbation concerns only the physical variables, the metric functions remaining the same as for the static situation.

Then feeding (48) back into (44) we obtain

$$D_T U = -28\pi\epsilon \frac{\left(3 - \frac{r}{r_\Sigma}\right)}{r\left(9 - \frac{r}{r_\Sigma}\right)}, \tag{49}$$

where we have neglected terms of order $|\epsilon^2|$ and higher, and we have assumed that the system abandons the equilibrium without dissipation.

As is apparent from (49), $D_T U$ does not change its sign in the whole interval $(0, r_\Sigma)$, implying that the system does not endure a cracking.

B. Nondissipative anisotropic fluid

Let us now consider the anisotropic case. For doing that we shall assume for the anisotropic factor the expression

$$P_r - P_\perp = \frac{\chi}{r^2}, \tag{50}$$

where $\chi = \frac{\xi}{4}$ and as before $\xi = \frac{3}{56\pi}$.

Integrating (44) we obtain for P_r

$$P_r = \frac{\xi}{r^2}\frac{(\sqrt{r_\Sigma} - \sqrt{r})}{(\sqrt{r_\Sigma} - \frac{3}{7}\sqrt{r})}. \tag{51}$$

This solution is characterized by (45), (50), (51), and $R = r$.

Then perturbing the system by $\xi \to \xi + \epsilon$ and $\chi \to \frac{\xi}{4}(1 + \omega)$, where $|\epsilon|$, $|\omega| \ll 1$, we obtain from (B6)

$$D_T U = \frac{2\pi\epsilon}{3r}\frac{\left(-17r_\Sigma + \frac{186}{7}\sqrt{r_\Sigma}\sqrt{r} - \frac{489}{49}r\right)}{\left(\sqrt{r_\Sigma} - \frac{5}{7}\sqrt{r}\right)\left(\sqrt{r_\Sigma} - \frac{3}{7}\sqrt{r}\right)} - \frac{\omega}{7r}\frac{\left(\sqrt{r_\Sigma} - \frac{3}{7}\sqrt{r}\right)}{\left(\sqrt{r_\Sigma} - \frac{5}{7}\sqrt{r}\right)}, \tag{52}$$

which we will rewrite as

$$W = \epsilon\left(-17 + \frac{186}{7}\sqrt{x} - \frac{489}{49}x\right) - \omega\left(1 - \frac{3}{7}\sqrt{x}\right)^2, \tag{53}$$

where $W \equiv x(1 - \frac{5}{7}\sqrt{x})(1 - \frac{3}{7}\sqrt{x})r_\Sigma D_T U$ is non-negative in all the range $x \in [0, 1]$, with $x \equiv \frac{r}{r_\Sigma}$, and the parameters ϵ and ω have been reparametrized as $\frac{2\pi\epsilon}{3} \to \epsilon$, $\frac{\omega}{7} \to \omega$.

Some remarks are in order at this point:

(i) There is no cracking (overturning) if $\epsilon = 0$ or $\omega = 0$.
(ii) There is no cracking (overturning) if $\epsilon = \omega$.
(iii) There is no cracking (overturning) if ϵ and ω have the same sign.

The occurrence of cracking may be easily illustrated in this case by assuming $\omega = -\delta\epsilon$, where δ is a positive real number. In this case (53) becomes

$$W = \epsilon\left[\left(-17 + \frac{186}{7}\sqrt{x} - \frac{489}{49}x\right) + \delta\left(1 - \frac{3}{7}\sqrt{x}\right)^2\right]. \tag{54}$$

Figure 1 depicts function W for six different values of δ in the range (2.5, 15). As illustrated by the figure, cracking occurs for all values of δ in the indicated range. Also as it is apparent from this figure, greater values of δ are associated with cracking closer to the center.

Thus, while the isotropic fluid considered above leaves the equilibrium without the appearance of cracking, its anisotropic version may exhibit the occurrence of cracking when both the radial pressure and the anisotropic factor are perturbed. Similar conclusions were already obtained in [1].

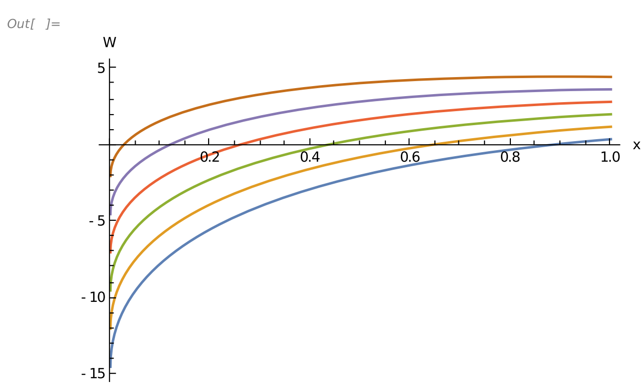

FIG. 1. W as function of x for six values of δ (2.5, 5, 7.5, 10, 12.5, 15). Curves from the bottom to the top correspond to increasing values of δ.

C. Dissipative isotropic fluid

We shall now turn to the case when the system leaves the equilibrium allowing the presence of dissipative processes. Since as we have just seen, pressure anisotropy (at least in the example examined above) may produce cracking, we shall consider the isotropic pressure case, in order to isolate the effects of dissipation in the possible occurrence of cracking.

Thus evaluating the system immediately after leaving the equilibrium ("immediately" in the sense explained above), we obtain from the transport Eqs. (41) and (B6)

$$D_T U(1-\alpha)(\mu + P_r) = -(1-\alpha)(\mu + P_r)\left(\frac{m}{R^2} + 4\pi P_r R\right) - \frac{E^2 P_r'}{R'} + \frac{E\kappa T'}{\tau B}, \tag{55}$$

where

$$\alpha = \frac{\kappa T}{(\mu + P_r)\tau}. \tag{56}$$

The last term in the above equation brings out the possible role of dissipative processes in the occurrence of cracking. However, in order to isolate the dissipative effects on the occurrence of cracking, we need to resort to a specific fluid distribution. For doing that we shall consider the isotropic toy model (45), (46), and (47).

In this case the above equation becomes

$$D_T U = \frac{1}{(1-\alpha)}\left[\frac{(3r_\Sigma - r)}{r(9r_\Sigma - r)}\left(-28\pi\epsilon + \frac{6}{7}\alpha + 16\pi\epsilon\alpha\right) + \frac{E\alpha T'}{BT}\right], \tag{57}$$

where we have assumed $\alpha \neq 1$.

Before proceeding further we need to make some rough estimations on the possible values of α defined by (56). First of all it is worth noticing that the range of possible values of energy density, with respect to P_r, lie between $\mu \gg P_r$ and $\mu \approx P_r$. Therefore we can neglect P_r in (56), since, at most, it would change α by a factor $1/2$.

So, we shall evaluate

$$\alpha \approx \frac{\kappa T}{\tau\mu}. \tag{58}$$

Going back from relativistic units to cgs units we have

$$\kappa T = \frac{G}{c^5}(\kappa)_{\text{c.g.s.}}(T)_{\text{c.g.s.}}, \tag{59}$$

where $G = 6.67 \times 10^{-8}\ \text{g}^{-1}\,\text{cm}^3\,\text{s}^{-2}$ is the gravitational constant, c is the light speed, and $(\kappa)_{\text{c.g.s.}}, (T)_{\text{c.g.s.}}$ denote the values of κ and T, expressed in $\mathrm{erg} \times \mathrm{cm}^{-1}\,\mathrm{s}^{-1}\,\mathrm{K}^{-1}$ and K (Kelvin degrees) respectively.

Also

$$\tau = c(\tau)_{\rm c.g.s.} \qquad \mu = \frac{G}{c^2}(\mu)_{\rm c.g.s.} \tag{60}$$

where $(\tau)_{\rm c.g.s.}$ and $(\mu)_{\rm c.g.s.}$ denote the values of τ and μ, expressed in seconds and $\mathrm{g/cm^3}$ respectively.

With all the above we may write

$$\frac{\kappa T}{\tau\mu} \approx 10^{-42}\frac{(\kappa)_{\rm c.g.s.}(T)_{\rm c.g.s.}}{(\tau)_{\rm c.g.s.}(\mu)_{\rm c.g.s.}}. \tag{61}$$

Next, in the high frequency limit of the thermal wave, we have from the telegraph equation (40)

$$\tau \approx \frac{\kappa}{v^2\gamma}, \tag{62}$$

where v and γ denote the speed of the thermal wave and the heat capacity per volume, respectively.

If the thermal conductivity is dominated by degenerate electrons, then we may assume for κ [26,27]

$$\kappa \approx 10^{23}[\mu/(10^{14}\ \mathrm{g/cm^3})][10^8\ \mathrm{K/T}]\,\mathrm{erg\,s^{-1}\,cm^{-1}\,K^{-1}}. \tag{63}$$

On the other hand,

$$c_v \equiv \gamma V = \beta T, \tag{64}$$

where c_v is the specific heat, V is the volume and for the coefficient β which is model dependent we assume the value proposed by Shibazaki and Lamb [28]

$$\beta \approx 10^{29}\ \mathrm{erg\,K^{-2}}. \tag{65}$$

Feeding back (63), (64), and (65) into (62) we obtain (for densities of the order $[\mu] \approx 10^{14}$, and the radius of the degenerate core $\approx$10 Km)

$$\tau \approx \frac{10^{20}}{[T^2][v^2]}\ \mathrm{s}, \tag{66}$$

where $[\mu]$, $[T]$, and $[v]$ denote the numerical values of density, temperature, and velocity of the thermal wave in $\mathrm{g/cm^3}$, Kelvin degrees, and cm/s respectively.

We shall need next to provide some possible values for the velocity of the thermal wave.

If we take the upper limit for v ($\approx 3\times 10^{10}$ cm/s), then assuming $[T] \approx 10^2$ we obtain

$$\tau \approx 10^{-4}\ \mathrm{s}. \tag{67}$$

However, the above is probably a too low value for the temperature, corresponding to the latest phases of the evolution of a neutron star (see Ref. [28]), and a too high value of v.

For a much more reasonable value of v such as

$$v \approx 10^3\ \mathrm{cm/s}, \tag{68}$$

corresponding to the value of the second sound in superfluid helium, we obtain $\tau \approx 10^{-4}$ s for temperatures of the order of $\approx 10^9$ K, or $\tau \approx 10^2$ s for $T \approx 10^6$ K.

If instead we take the temperature proposed by Harwit [29] ($T \approx 10^7$ K), we obtain

$$\tau \approx 1\ \mathrm{s}. \tag{69}$$

To summarize, for the conditions considered above the relaxation time is in the range $(10^{-4}\ \mathrm{s}, 10^2\ \mathrm{s})$.

On the other hand, feeding back (63) into (61) we obtain

$$\frac{\kappa T}{\tau\mu} \approx 10^{-25}\frac{1}{[\tau]}, \tag{70}$$

or, using (66)

$$\alpha \approx \frac{\kappa T}{\tau\mu} \approx 10^{-45}[T^2][v^2]. \tag{71}$$

Using the above expression, we obtain for the extreme values $[T] \approx 10^{13}$ and $[v] \approx 10^9$

$$\alpha \approx 0.1. \tag{72}$$

An alternative scenario corresponds to the early stages of a supernova during the neutronization process. In this case the temperature may be in the range $(10^{11}\ \mathrm{K}, 10^{13}\ \mathrm{K})$, and the density is about $10^{15}\ \mathrm{g/cm^3}$ at the center and 10^{13} on the surface. Under these conditions τ may be in the range $(10^{-6}\ \mathrm{s}, 10^{-4}\ \mathrm{s})$ [30], in which case α lies within the range $(10^{-4}, 10^2)$.

Although some arguments based on causality and stability conditions (see Ref. [31]) seem to prohibit values of $\alpha \geq 1$, suggesting that the value of α, most likely, lies within the range $(10^{-4}, < 1)$, there is not a conclusive proof about this issue. Accordingly we shall also consider the possibility of $\alpha > 1$.

In order to remove the fluid from the state of thermodynamic equilibrium we have to perturb the value of the temperature gradient corresponding to equilibrium.

Accordingly, we shall write

$$T = T_{\rm eq}(1+\psi) \rightarrow T' = T'_{\rm eq}(1+\psi), \quad \psi \ll 1, \tag{73}$$

where the subscripts eq denotes the value in equilibrium.

The condition of thermal equilibrium reads [32]

$$(T_{\rm eq}A)' = 0, \tag{74}$$

implying

$$T' = -\frac{A'}{A}T_{\rm eq}(1+\psi). \tag{75}$$

Using (47) and (75), we may write

$$\frac{\alpha T' E}{BT} \approx -\frac{6\alpha(3r_\Sigma - r)}{7r(9r_\Sigma - r)}(1 - 7\pi\epsilon), \tag{76}$$

where the expressions

$$B \approx \frac{\sqrt{7}}{2}, \qquad E \approx \frac{2}{\sqrt{7}}(1 - 7\pi\epsilon), \qquad \frac{T_{\rm eq}}{T} \approx 1 - \psi, \tag{77}$$

have been used, and terms of order ψ^2 have been neglected.

Feeding back (76) into (57) we obtain

$$D_T U = \frac{2\epsilon\pi}{(1-\alpha)}\frac{(3r_\Sigma - r)}{r(9r_\Sigma - r)}[-14 + 11\alpha], \tag{78}$$

where terms of order ϵ^2 have been neglected, and we recall that α is assumed to be different from 1.

For cracking to occur at some point (say $r = r_{ck}$), the expression within the square bracket in (78) should vanish at $r = r_{ck}$, being positive (negative) for $r > r_{ck}$ ($r < r_{ck}$), for any positive small value of ϵ (for negative values of ϵ the same argument applies for the inverse sign of the term within the square bracket).

Then for any sign of ϵ, for values of α in the range $[10^{-4}, < 14/11]$, neither cracking nor overturning will occur.

However, if we allow the value of α to rise from some value within the range $[14/11 > \alpha > 1]$ for $r < r_{ck}$, to the value $14/11$ for $r = r_{ck}$, increasing for $r > r_{ck}$, then we may observe a cracking produced by dissipative processes, for negative values of ϵ, whereas an overturning will happen for $\epsilon > 0$.

Inversely, if $\epsilon > 0$, then if we allow values of α larger than $14/11$ for $r < r_{ck}$, decreasing to $14/11$ for $r = r_{ck}$, cracking will also happen, whereas overturning will be observed for $\epsilon < 0$. We shall discuss these results with more detail in the last section.

IV. CRACKING AND COMPLEXITY

We shall next bring out the link between the restrictions imposed on the complexity of the fluid distribution and the occurrence (or not) of cracking (overturning). As mentioned before, the analysis of the complexity of an evolving fluid implies two different (though related) issues: on the one hand the complexity of the fluid distribution, described by the scalar Y_{TF} and on the other hand, the complexity of the mode of evolution.

Regarding the complexity of the mode of evolution we shall consider the H evolution described by (30) and (31), and the QH evolution, described only by (30), as the two modes of evolution to be considered as the simplest ones.

So, the question we want to answer to here is: are the vanishing complexity factor condition and the H or the QH evolution compatible with the appearance of cracking (overturning)?

We shall tackle this problem by considering separately nondissipative and dissipative fluids.

A. H condition, $\dot{q} = 0$

Let us first consider a nondissipative fluid, and let us assume that immediately after perturbation, the system abandons the equilibrium satisfying the H condition. Then we obtain from (30)

$$\dot{U} = \dot{U}_\Sigma \frac{R}{R_\Sigma}, \tag{79}$$

where the fact that the system is evaluated at the timescale defined by (42) has been used.

From the above expression it is evident that no cracking (overturning) occurs, since the sign of $\dot{U}$ will be the same as that of $\dot{U}_\Sigma$, i.e., it will be the same for any value of r. On the other hand we know that the H condition in the nondissipative case implies that $Y_{TF} = 0$ (see Ref. [3] for details). Thus, the H condition alone implies in the nondissipative case the vanishing of the complexity factor and prevents the occurrence of cracking.

B. QH condition, $\dot{q} = 0$

If we relax the homologous condition, assuming that immediately after perturbation, the system abandons the equilibrium under the quasihomologous regime, then we obtain from (30)

$$\dot{U} = \dot{U}_\Sigma \frac{R}{R_\Sigma}, \tag{80}$$

implying again that no cracking (overturning) occurs. However, in this case Y_{TF} does not necessarily vanish.

C. H condition, $\dot{q} \neq 0$

Let us next assume that the system abandons the equilibrium in the homologous regime, but now we allow for dissipative processes to be present. In such a case it follows from (35) that $Y_{TF} = 0$, implies $\dot{q} = 0$, i.e., dissipative processes require nonvanishing Y_{TF} at the timescale under consideration.

Of course no cracking (overturning) occurs in this case since condition (79) is satisfied.

D. QH condition, $\dot{q} \neq 0$

If we assume instead that the system abandons the equilibrium in the QH regime with the presence of dissipative processes, no cracking (overturning) occurs since condition (79) is satisfied, but again Y_{TF} does not necessarily vanish.

E. $Y_{TF} = 0$, $\dot{q} = 0$, $\Pi = 0$

If we assume the vanishing complexity factor condition, without imposing any restriction on the mode of evolution, then in the isotropic nondissipative case, we obtain that the Weyl tensor vanishes, which implies (see Eq. (78) in [19]) that the shear and its time derivatives of all orders vanish. In such a case we obtain from (29) again

$$\dot{U} = \dot{U}_\Sigma \frac{R}{R_\Sigma}. \tag{81}$$

Thus, no cracking (overturning) is observed in this case either.

F. $Y_{TF} = 0$, $\dot{q} \neq 0$, $\Pi = 0$

If we assume the vanishing complexity factor condition, without imposing any restriction on the mode of evolution, then in the isotropic dissipative case, we obtain that the Weyl tensor vanishes, however in this case we do not obtain (81) from (29), implying that in principle cracking (overturning) is allowed.

G. $Y_{TF} = 0$, $\dot{q} = 0$, $a = 0$

If we assume the vanishing complexity factor condition and the geodesic condition $a = 0$, we see from (28) that (at the timescale we are working with) the shear and its time derivatives of all orders vanish. In such a case we obtain from (29) again

$$\dot{U} = \dot{U}_\Sigma \frac{R}{R_\Sigma}. \tag{82}$$

Thus, no cracking (overturning) is observed in this case either.

H. $Y_{TF} = 0$, $\dot{q} \neq 0$, $a = 0$

If we assume the vanishing complexity factor condition, with dissipation, then in the geodesic case $a = 0$, we see from (28) that (at the timescale we are working with) the shear and its time derivatives of all orders vanish. However, in such a case we do not obtain (82) from (29), implying that cracking may occur in this case.

All these results as well as those obtained in the previous section are summarized in Tables I and II.

V. DISCUSSION

We started this work with three main objectives in mind. First, we wanted to develop a general method to treat the problem of cracking in comoving coordinates. Such an approach was set up and it allowed us to tackle our second goal, namely, to study the influence of dissipative processes on the occurrence (or not) of cracking. Finally, using the presented approach we were able to establish the link between the occurrence (or not) of cracking and different restrictions imposed on the complexity of the fluid distribution.

Let us start the discussion by making some general remarks on the formalism here presented:

(i) Although the basic idea underlying the concept of cracking is independent on the frame (comoving or noncomoving), the variables used to describe it are different in both frames.

(ii) As for the case of noncomoving coordinates, the occurrence of cracking is described for specific solutions. However, the link between cracking and complexity was analyzed in general, without any reference to an explicit solution.

(iii) The influence of dissipation on the occurrence (or not) of cracking is also highly model dependent. Accordingly the example examined here is just a guide to proceed in each specific case.

(iv) In order to force the system to leave the equilibrium, we shall perturb some of the parameters of the physical variables corresponding to the solution under consideration. Then, the system is analyzed on a timescale which is smaller than the hydrostatic time and the relaxation time. In such a case we may safely assume that on this timescale, the metric functions remain the same as those before perturbation, as well as their first time derivatives, while physical variables are perturbed.

(v) In relation with the comment above, particular care should be exercised with the mass function. Indeed, as defined by (13), it depends only on metric functions (and first derivatives) and then one could (wrongly) conclude that it should not be perturbed. On the other hand the mass function may also be defined through physical variables as in (20). This apparent contradiction is easily resolved if we remember that expressions (13) and (20) are equivalent modulo field equations. However the metric variables after perturbation are not solutions to the field equations for the perturbed physical variables (they represent solutions for nonperturbed physical variables). Therefore, the mass function should be perturbed according to its expression (20).

(vi) The concept of cracking adopted here is based on the definition of "velocity" as given by U (the areal velocity). However alternative definitions of "velocity" exist (see Ref. [4]) which could be used instead of U, giving rise to different definitions of cracking.

Once the general setup of the problem was well defined, we proceeded to analyze first the role of dissipative processes in the occurrence of cracking. For doing that we started by considering a toy model describing a static solution for an isotropic fluid defined by (45)–(47). This solution is then removed from equilibrium, by perturbing the parameter ξ, and we took a "snapshot" of the system after perturbation on a timescale smaller than the hydrostatic time and the thermal relaxation time. We did that assuming that no dissipative processes are allowed ensuing the perturbation. Then, it was shown that no cracking appears as a consequence of the perturbation. Next, we generalized the toy model to the case where the pressure is anisotropic, such a toy model is characterized by (45), (50), and (51). In this case we observe the appearance of cracking if both parameters ξ and χ are perturbed and furthermore ϵ and ω are different and have different signs. This situation is illustrated in Fig. 1.

Next, we considered the toy model described above for the isotropic fluid, but we perturbed it allowing dissipation to be present when the system abandons the equilibrium. In this case we were led to (78), where the variable α plays a fundamental role.

Based on some likely astrophysical scenarios, we started by making some rough estimations about the possible values of α, as a result of which we established as a reasonable range $[10^{-4}, 10^{2}]$. However as mentioned before, some arguments based on stability seem to rule out values of $\alpha \geq 1$ (see Ref. [31] for a discussion on this issue). Nevertheless such arguments are not conclusive, and furthermore some numerical models with good physical behavior and $\alpha > 1$ have been described in the literature (see Ref. [33]). Accordingly we have considered also the possibility of $\alpha > 1$.

Equation (78) brings out the influence of dissipation on the occurrence or not of cracking. Thus, for all values of α in the range $[\approx 0, < 14/11]$, no cracking occurs. However, cracking (or overturning) may occur for $\alpha \approx 14/11$, for conditions summarized in Table I.

Next, we focused on our third goal, namely, to find out how the occurrence of cracking is related to restrictions imposed on the complexity of the fluid distribution. These restrictions involve the complexity factor Y_{TF} and/or restrictions on the mode in which the system leaves the equilibrium (H or QH). It is important to stress here that while in the treatment of the second problem (bringing out the relevance of dissipation on the occurrence of cracking) we resorted to a specific toy model in this third problem and we obtained general results without reference to any specific solution.

The first important point to mention is that the restrictions on the mode of evolution (as the system leaves the equilibrium) appear to be more relevant (concerning the occurrence of cracking) than the restrictions on Y_{TF}.

Thus, the sole imposition of H or QH regime, in both the nondissipative and the dissipative cases, rules out the possibility of cracking. Furthermore in the former case the complexity factor vanishes. This brings out further the link between H or QH regime and the complexity of the mode of evolution, if we recall that the occurrence of cracking may be regarded as a factor enhancing the complexity of the system.

We also were able to prove that the vanishing of the complexity factor alone (without any imposition on the mode of leaving the equilibrium) rules out the occurrence of cracking in the geodesic nondissipative case, and in the nondissipative isotropic case. These results are summarized in Table II.

Finally, we would like to say few words about the case $\alpha = 1$. As it is apparent from (55), one of the effects of dissipative processes consists of decreasing the inertial mass density (the factor multiplying $D_T U$) and (as a consequence of the equivalent principle) the passive gravitational mass density [the factor multiplying the first square bracket on the right of (55)], by a factor $1 - \alpha$. This strange effect which was discovered and discussed in [34] implies that the effective inertial mass density vanishes for $\alpha = 1$. Until now, in spite of long discussions about this point, no definitive answer has been reached concerning the physical meaning (if any) of this strange effect, and the possibility of reaching the above mentioned critical value in a real physical system.

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APPENDIX A: EINSTEIN EQUATIONS

Einstein's field equations for the interior spacetime (1) are given by

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{A1}$$

and its nonzero components with (1), (2), and (41) become

$$8\pi T_{00}=8\pi\mu A^2=\left(2\frac{\dot B}{B}+\frac{\dot R}{R}\right)\frac{\dot R}{R}-\left(\frac{A}{B}\right)^2\left[2\frac{R''}{R}+\left(\frac{R'}{R}\right)^2-2\frac{B'}{B}\frac{R'}{R}-\left(\frac{B}{R}\right)^2\right], \tag{A2}$$

$$8\pi T_{01}=-8\pi qAB=-2\left(\frac{\dot R'}{R}-\frac{\dot B}{B}\frac{R'}{R}-\frac{\dot R}{R}\frac{A'}{A}\right), \tag{A3}$$

$$8\pi T_{11}=8\pi P_r B^2=-\left(\frac{B}{A}\right)^2\left[2\frac{\ddot R}{R}-\left(2\frac{\dot A}{A}-\frac{\dot R}{R}\right)\frac{\dot R}{R}\right]+\left(2\frac{A'}{A}+\frac{R'}{R}\right)\frac{R'}{R}-\left(\frac{B}{R}\right)^2, \tag{A4}$$

$$8\pi T_{22}=\frac{8\pi}{\sin^2\theta}T_{33}=8\pi P_\perp R^2=-\left(\frac{R}{A}\right)^2\left[\frac{\ddot B}{B}+\frac{\ddot R}{R}-\frac{\dot A}{A}\left(\frac{\dot B}{B}+\frac{\dot R}{R}\right)+\frac{\dot B}{B}\frac{\dot R}{R}\right]$$
$$+\left(\frac{R}{B}\right)^2\left[\frac{A''}{A}+\frac{R''}{R}-\frac{A'}{A}\frac{B'}{B}+\left(\frac{A'}{A}-\frac{B'}{B}\right)\frac{R'}{R}\right]. \tag{A5}$$

The component (A3) can be rewritten with (9) and (11) as

$$4\pi qB=\frac{1}{3}(\Theta-\sigma)'-\sigma\frac{R'}{R}. \tag{A6}$$

APPENDIX B: DYNAMICAL EQUATIONS

The nontrivial components of the Bianchi identities, $T^{\alpha\beta}_{;\beta}=0$, from (A1) yield

$$T^{\alpha\beta}_{;\beta}V_\alpha=-\frac{1}{A}\left[\dot\mu+(\mu+P_r)\frac{\dot B}{B}+2(\mu+P_\perp)\frac{\dot R}{R}\right]-\frac{1}{B}\left[q'+2q\frac{(AR)'}{AR}\right]=0, \tag{B1}$$

$$T^{\alpha\beta}_{;\beta}\chi_\alpha=\frac{1}{A}\left[\dot q+2q\left(\frac{\dot B}{B}+\frac{\dot R}{R}\right)\right]+\frac{1}{B}\left[P_r'+(\mu+P_r)\frac{A'}{A}+2(P_r-P_\perp)\frac{R'}{R}\right]=0, \tag{B2}$$

or, by using (8), (9), (14), (18), and (16), they become, respectively,

$$D_T\mu+\frac{1}{3}(3\mu+P_r+2P_\perp)\Theta+\frac{2}{3}(P_r-P_\perp)\sigma+ED_Rq+2q\left(a+\frac{E}{R}\right)=0, \tag{B3}$$

$$D_Tq+\frac{2}{3}q(2\Theta+\sigma)+ED_RP_r+(\mu+P_r)a+2(P_r-P_\perp)\frac{E}{R}=0. \tag{B4}$$

This last equation may be further transformed as follows, the acceleration D_TU of an infalling particle can be obtained by using (8), (A4), (13), and (16), producing

$$D_TU=-\frac{m}{R^2}-4\pi P_rR+Ea, \tag{B5}$$

and then, substituting a from (B5) into (B4), we obtain

$$(\mu+P_r)D_TU=-(\mu+P_r)\left[\frac{m}{R^2}+4\pi P_rR\right]-E^2\left[D_RP_r+2(P_r-P_\perp)\frac{1}{R}\right]-E\left[D_Tq+2q\left(2\frac{U}{R}+\sigma\right)\right]. \tag{B6}$$

TABLE I. Cracking and complexity.

$H, QH, Y_{TF}\backslash\dot q$	0	$\neq 0$
H	No cracking, no overturning, $Y_{TF}=0$	No cracking, no overturning, $Y_{TF}\neq 0$
$QH\to Y_{TF}\neq 0$	No cracking, no overturning	No cracking, no overturning
$Y_{TF}=0, \Pi=0$	No cracking, no overturning	Cracking or overturning are allowed
$Y_{TF}=0, a=0$	No cracking, no overturning	Cracking or overturning are allowed

TABLE II. Cracking and dissipation.

$\epsilon \backslash \alpha$	$[\alpha \approx 10^{-4}, \alpha < 14/11]$	$[14/11 > \alpha > 1, \alpha > 14/11]$ increasing	$[\alpha > 14/11, 14/11 > \alpha > 1]$ decreasing
$\epsilon > 0$	No cracking, no overturning	Overturning	Cracking
$\epsilon < 0$	No cracking, no overturning	Cracking	Overturning

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