Stability of symmetric teleparallel scalar-tensor cosmologies with alternative connections

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In symmetric teleparallel geometry the curvature and torsion tensors are assumed to vanish identically, while the dynamics of gravity is encoded by nonmetricity. Here the spatially homogeneous and isotropic connections that can accompany the flat Friedmann-Lemaître-Robertson-Walker metric come in three sets. As the trivial set has received much attention, we focus on the two alternative sets, which introduce an extra degree of freedom into the equations. Working in the context of symmetric teleparallel scalar-tensor gravity with generic nonminimal coupling and potential, we show that the extra free function in the connection cannot play the role of dark matter nor dark energy, but it drastically alters the scalar field behavior. We determine the restrictions on the model functions which permit the standard cosmological scenario of successive radiation, dust matter, and scalar potential domination eras to be stable. However, the alternative connections also introduce a rather general possibility of the system meeting a singularity in finite time.

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I. INTRODUCTION

The still unresolved origin of the observed accelerated expansion of the Universe known as dark energy, underlined by the significant tension with the cosmic data that the concordance ΛCDM model is evidently facing [1], compels us to look beyond Einstein's theory of general relativity (GR) [2–4]. Perhaps the simplest and most widely studied type of GR extension is to add a nonminimally coupled scalar field to the usual tensor degree of freedom. Such scalar-tensor models are remarkably successful in predicting the correct outcomes of early universe inflation [5,6], but they can also describe late universe dark energy [7,8] with effective "phantom crossing" behavior [9] and the possibility to address the cosmic tension [10]. In the setup where freely falling test particles follow geodesics (the Jordan frame), nonminimal coupling manifests itself by making the Newtonian gravitational constant dependent on the value of the dynamical scalar field. However, large classes of scalar-tensor cosmologies exhibit an "attractor mechanism" whereby the scalar field spontaneously stabilizes since the early matter dominated era [11]. This feature can explain how the theory passes various Solar System, astrophysical, and cosmological tests [12–14], and it has been studied in detail in the literature [15–24].

Remarkably, the Einstein-Hilbert Lagrangian given by the Levi-Civita curvature scalar $\mathring{R}$ is not the only way to derive the dynamics of general relativity [25]. By invoking extra freedom in the connection characterized by torsion or nonmetricity, it is possible to demand that the geometry is teleparallel; i.e., the overall curvature tensor is zero. In the framework where nonmetricity also vanishes, one can then introduce the (metric) teleparallel equivalent of general relativity (TEGR) [26,27] by rewriting the Einstein-Hilbert Lagrangian as $\mathring{R} = -T + B_T$, where T is the torsion scalar and B_T is a boundary term that does not contribute to the field equations. Analogously, when torsion vanishes (and hence the affine connection is symmetric), we get the symmetric teleparallel equivalent of general relativity (STEGR) [28] by writing $\mathring{R} = Q + B_Q$ with nonmetricity scalar Q and the respective boundary term B_Q. While the matter Lagrangian is left unchanged and still includes only the coupling to the metric (or Levi-Civita connection), the dynamics of TEGR and STEGR is equivalent to GR. The extra torsional and nonmetricity bits of connection completely drop out of the field equations and remain arbitrary spectators, with their only role being to keep the curvature zero. However, the freedom to choose the teleparallel connection arbitrarily may become restricted in the situations where the boundary term becomes relevant, like black hole energy or entropy [29–31].

With the same motivation that led to the extensions of GR, it is immediately tempting to introduce modified teleparallel Lagrangians like $f(T)$ [32–36] and $f(Q)$ [36,37], or extend the theory in the scalar-tensor manner by coupling a scalar field nonminimally to the torsion scalar [38,39] or nonmetricity scalar [40]. These extensions differ from their curvature based GR counterparts, which make them really interesting to study in the search for new phenomenology [41]. In contrast to the TEGR and STEGR

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cases, the extra bits of connection are now endowed with their own equation. This raises an issue of how to proceed in solving the combined system of metric and connection equations, with different strategies conceivable [42]. One option that has been fruitful in several situations is to impose an ansatz for the connection which obeys the same set of spacetime symmetries as the metric [43]. For example, in the case of cosmological Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes, the metric teleparallel connection (torsion only) comes in one family for spatial flatness and two families for spatial curvature, including no extra functions besides the scale factor already present in the metric [43,44]. On the other hand, the symmetric teleparallel connection (nonmetricity only) has three distinct options compatible with spatial flatness and one for spatial curvature, all cases endowed with an extra free function of time [45,46]. Different connections imply different field equations and thus different dynamics.

The deeper theoretical discussions about extended teleparallel gravities, in essence, revolve around the issue of how much independent dynamics the extra bits of connection really bring. The connection equation has only first-order derivatives acting on the connection coefficients, and thus it looks more like a constraint equation. Furthermore, there is a gauge freedom in metric teleparallelism to perform a local Lorentz transformation that makes the spin connection vanish (Weitzenböck gauge), usually at the expense of the tetrad assuming a more complicated form [27]; however, in symmetric teleparallelism, a general coordinate transformation make the affine connection vanish (coincident gauge), typically making the metric more involved [47,48]. Thus, the connection itself is a gauge degree of freedom, but it might conspire to hide more items in the tetrad or metric. A well-respected procedure to determine the number of degrees of freedom in a theory is to perform Hamiltonian analysis, but so far there is no consensus on the results, neither in $f(T)$ [49–53] nor $f(Q)$ [54–56] gravity, except that there is something extra in the usual two tensor degrees of freedom of the metric. These extra degrees of freedom are hard to pinpoint precisely, though [57–60]. For instance, it is puzzling that no extra propagating modes show up among the linear perturbations around the cosmological backgrounds in $f(T)$ gravity [61–65], while even the perturbations of the clearly independent scalar field become strangely blocked in simple scalar-torsion cosmology [62,66,67]. Seeking hints of the extra dynamical degrees of freedom, one can go to higher perturbation orders [68] or neglect the spacetime symmetry of the connection [69].

In this context, a closer look at the spatially flat FLRW connections in symmetric teleparallelism may offer new insights. The new function present in the connection could be taken as a clear instance of an extra degree of freedom. To be more precise, in one of the three classes, this function completely decouples from the $f(Q)$ equations [45,46], which reduce to the ones arising from the trivial connection (that vanishes in Cartesian coordinates) [37]. This system actually coincides with the FLRW metric teleparallel equations [40], which have been extensively studied [41]. However, in the two other classes the new function in the connection indeed manages to appear in the $f(Q)$ cosmological equations [45,46]. So far, there have been a few works that considered particular solutions and their properties [70–72], phase space features [73–75], and constraints from observational data [76,77], all for certain specific models of $f(Q)$.

In this paper we investigate spatially flat FLRW cosmology in scalar-nonmetricity gravity, i.e., in the symmetric teleparallel analogue of the scalar-tensor theory [40], utilizing the three classes of connections with respective symmetry [45,46]. We keep the nonminimal coupling function and potential completely generic to cover as wide a class of models as possible. The principal question is, how does the extra function in the connection affect typical cosmological dynamics, from the radiation dominated epoch to matter domination, to the era of dark energy? Can it be a source of dark energy itself or enable or disable the scalar field to behave as one? Can it allow phantom crossing for even a minimally coupled scalar field? Does the cosmological evolution spontaneously stabilize at some field values; i.e., is there an "attractor mechanism" like in the original scalar-tensor theory, or does the connection make the universe unstable? Finding answers to these questions can prepare the groundwork for conducting direct assessments of more interesting types of models through comparisons with observational data. In the end we might even unravel a plausible explanation to the current conundrums in cosmology.

This paper is organized as follows. Section II recalls the geometrical setup of symmetric teleparallelism and the key features of scalar-nonmetricity gravity. All three classes of flat FLRW connections are explained in Sec. III. Section IV presents the cosmological field equations arising for these connections, with a focus on the dimensionality of the phase space and the possibility of emulating dark energy. Section V introduces the expansion scheme for small perturbations, while Sec. VI explores, in detail, the stability of cosmological evolution in the dust matter dominated, radiation dominated, and scalar potential dominated regimes for the three class connections. Finally, Sec. VII provides a discussion of the obtained results.

II. SYMMETRIC TELEPARALLEL GRAVITIES

Symmetric teleparallel gravity assumes a geometric setup where the connection is characterized by identically vanishing curvature and torsion, while only nonmetricity is allowed to deviate from zero. The action can be constructed from the nonmetricity scalar, which is equivalent to the

Levi-Civita Ricci scalar up to a boundary term, thus providing a link to GR.

A. Geometric preliminaries

A generic connection $\tilde{\Gamma}^{\lambda}{}_{\mu\nu}$ with 64 independent components can be decomposed into three parts [78,79],

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \mathring{\Gamma}^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu}, \tag{1}$$

namely, the Levi-Civita connection of the metric $g_{\mu\nu}$,

$$\mathring{\Gamma}^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}), \tag{2}$$

the contortion tensor

$$K^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\beta}(-T_{\mu\beta\nu} - T_{\nu\beta\mu} + T_{\beta\mu\nu}) = -K_{\mu}{}^{\lambda}{}_{\nu}, \tag{3}$$

and the disformation tensor

$$L^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\beta}(-Q_{\mu\beta\nu} - Q_{\nu\beta\mu} + Q_{\beta\mu\nu}) = L^{\lambda}{}_{\nu\mu}. \tag{4}$$

Here, contortion is built from torsion tensors that characterize the antisymmetric part of the connection,

$$T^{\lambda}{}_{\mu\nu} \equiv \tilde{\Gamma}^{\lambda}{}_{\mu\nu} - \tilde{\Gamma}^{\lambda}{}_{\nu\mu}, \tag{5}$$

while disinformation is constructed from nonmetricity tensors that are symmetric in the last two indices,

$$Q_{\rho\mu\nu} \equiv \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \tilde{\Gamma}^{\beta}{}_{\mu\rho} g_{\beta\nu} - \tilde{\Gamma}^{\beta}{}_{\nu\rho} g_{\mu\beta}. \tag{6}$$

These quantities, along with curvature tensor

$$R^{\sigma}{}_{\rho\mu\nu} \equiv \partial_\mu \tilde{\Gamma}^{\sigma}{}_{\nu\rho} - \partial_\nu \tilde{\Gamma}^{\sigma}{}_{\mu\rho} + \tilde{\Gamma}^{\sigma}{}_{\mu\lambda}\tilde{\Gamma}^{\lambda}{}_{\nu\rho} - \tilde{\Gamma}^{\sigma}{}_{\mu\lambda}\tilde{\Gamma}^{\lambda}{}_{\nu\rho}, \tag{7}$$

are the three key properties that characterize any connection. Zero curvature implies that the orientation of vectors does not change under parallel transport along a curve. Zero torsion implies that the connection is symmetric in the lower indices. Hence, the imposition of vanishing curvature and torsion merits the name “symmetric teleparallel,” and we denote it by $\Gamma^{\lambda}{}_{\mu\nu}$.

The connection $\Gamma^{\lambda}{}_{\mu\nu}$ has an interesting property that the scalar curvature of the Levi-Civita part of the connection, $\mathring{R}$, can be expressed as

$$\mathring{R} = Q + \mathring{\nabla}_\mu(\hat{Q}^\mu - Q^\mu), \tag{8}$$

where the nonmetricity scalar and the two independent traces of the nonmetricity tensor are defined as[1]

$$Q \equiv -\frac{1}{4} Q_{\lambda\mu\nu} Q^{\lambda\mu\nu} + \frac{1}{2} Q_{\lambda\mu\nu} Q^{\mu\nu\lambda} + \frac{1}{4} Q_\mu Q^\mu - \frac{1}{2} Q_\mu \hat{Q}^\mu, \tag{9}$$

$$Q_\mu \equiv Q_{\mu\nu}{}^{\nu}, \qquad \hat{Q}_\mu \equiv Q_{\nu\mu}{}^{\nu}. \tag{10}$$

It is also significant that the total Levi-Civita divergence part in (8),

$$B_Q = \mathring{\nabla}_\mu(\hat{Q}^\mu - Q^\mu), \tag{11}$$

becomes a boundary term under a spacetime integral.

The idea behind the symmetric teleparallel equivalent of general relativity is the following. As the Einstein-Hilbert action of GR is given by the Levi-Civita curvature scalar $\mathring{R}$, we can rewrite that action using the nonmetricity scalar Q instead and expect to keep the same dynamical content since the boundary term does not affect the field equations. Various extensions of the symmetric teleparallel theory, like substituting Q in the action by $f(Q)$ [37] or introducing a nonminimal coupling between Q and a scalar field Φ [40], however, lead to theories that are different from their counterparts $f(\mathring{R})$ and scalar-tensor gravity originally formulated in the Riemannian geometry.

B. Scalar-nonmetricity gravity

A symmetric teleparallel analogue of a simple scalar-tensor action can be written as [40]

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}(\mathcal{A}(\Phi)Q - \mathcal{B}(\Phi)g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi - 2\mathcal{V}(\Phi)) + S_{\mathrm{m}}, \tag{12}$$

where $\kappa^2 = 8\pi G$. Like in the usual Riemannian scalar-tensor theory, the nonminimal coupling function $\mathcal{A}$ sets the strength of the effective gravitational constant, $\mathcal{B}$ is the kinetic coupling function, and $\mathcal{V}$ is the scalar potential. We assume that the matter action S_{m} is the same as in GR, i.e., depending on the metric alone.

By introducing the so-called superpotential (or conjugate) tensor

$$P^{\alpha}{}_{\mu\nu} = -\frac{1}{4} Q^{\alpha}{}_{\mu\nu} + \frac{1}{2} Q_{(\mu}{}^{\alpha}{}_{\nu)} + \frac{1}{4} g_{\mu\nu} Q^\alpha - \frac{1}{4}(g_{\mu\nu}\hat{Q}^\alpha + \delta^{\alpha}{}_{(\mu} Q_{\nu)}) \tag{13}$$

[1]Note that some authors define Q with the opposite overall sign, e.g., Ref. [45].

as well as some geometric identities, the field equations arising from the variation of the action (12) with respect to the metric, symmetric teleparallel connection, and scalar field are the following [40,80]:

$$\mathcal{A}(\Phi)\overset{\circ}{G}_{\mu\nu} + 2\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\Phi}P^{\lambda}{}_{\mu\nu}\partial_\lambda\Phi + \frac{1}{2}g_{\mu\nu}(\mathcal{B}(\Phi)g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi + 2\mathcal{V}(\Phi)) - \mathcal{B}(\Phi)\partial_\mu\Phi\partial_\nu\Phi = \kappa^2\mathcal{T}_{\mu\nu}, \tag{14a}$$

$$\left(\frac{1}{2}Q_\beta + \nabla_\beta\right)\left[\partial_\alpha\mathcal{A}\left(\frac{1}{2}Q_\mu g^{\alpha\beta} - \frac{1}{2}\delta^\alpha_\mu Q^\beta - Q_\mu{}^{\alpha\beta} + \delta^\alpha_\mu Q^{\gamma\beta}_\gamma\right)\right] = 0, \tag{14b}$$

$$2\mathcal{B}\overset{\circ}{\Box}\Phi + \frac{\mathrm{d}\mathcal{B}}{\mathrm{d}\Phi}g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi + \frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\Phi}Q - 2\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\Phi} = 0. \tag{14c}$$

Here, $\overset{\circ}{G}_{\mu\nu}$ is the Einstein tensor and $\overset{\circ}{\Box}$ the d'Alembert operator computed from the Levi-Civita part of the connection, while $\mathcal{T}_{\mu\nu}$ is the usual matter energy-momentum tensor. Taking the Levi-Civita covariant divergence of the metric field equations (14), and using the connection equations (14b) as well as the scalar field equation (14c), gives the usual continuity equation of the matter fields [40],

$$\overset{\circ}{\nabla}_\mu\mathcal{T}^\mu{}_\nu = 0. \tag{15}$$

It can be understood as a consequence of the matter part of the action only coupling to the metric (or Levi-Civita connection) whereby there are no matter (hypermomentum) sources in the connection equations (14b). More generally, it is related to the diffeomorphism invariance of the matter action [81]. The same property is shared with the torsion based teleparallel scalar-tensor theory [39]. Another essential point to highlight is that the connection equation (14b) is not independent but intricately linked with the metric equations (14a). This interdependence arises from the Bianchi identity within symmetric teleparallelism, whereby by acting with the Levi-Civita covariant derivative on the metric equation, one can arrive at the connection equation [82]. Consequently, when the metric equations are satisfied, the corresponding connection equations automatically follow. However, we will still keep the connection equations separately, as they express useful information about the fields.

When the scalar field is globally constant, Eq. (14a) reduces to Einstein's equation in GR, with the value of the potential playing the role of the cosmological constant, while (14b) and (14c) immediately vanish. Therefore, the solutions of GR are trivially also the solutions of these scalar-tensor theories, with the scalar field just being constant. If the nonminimal coupling function is fixed to unity, $\mathcal{A}(\Phi) \equiv 1$, and the kinetic and potential terms of the scalar field vanish, $\mathcal{B}(\Phi) \equiv \mathcal{V}(\Phi) \equiv 0$, the theory is reduced to a STEGR. If the nonminimal coupling function is unity but the kinetic term of the scalar field is nontrivial, then we have a theory that is equivalent to a minimally coupled scalar field in GR. With the identifications $\mathcal{A} = f'(Q)$, $\mathcal{B} = 0$, $2\mathcal{V} = Qf'(Q) - f(Q)$, the current model also represents $f(Q)$ gravity [40].

III. SPATIALLY HOMOGENEOUS AND ISOTROPIC FIELD CONFIGURATIONS

Spatially homogeneous and isotropic cosmological spacetimes are characterized by the Killing vectors of translations ζ_{T_i} and rotations ζ_{R_i}, given in spherical coordinates as

$$\zeta^\mu_{T_x} = \begin{pmatrix} 0 & \chi\sin\theta\cos\phi & \frac{\chi}{r}\cos\theta\cos\phi & -\frac{\chi}{r}\frac{\sin\phi}{\sin\theta} \end{pmatrix}, \tag{16a}$$

$$\zeta^\mu_{T_y} = \begin{pmatrix} 0 & \chi\sin\theta\sin\phi & \frac{\chi}{r}\cos\theta\sin\phi & \frac{\chi}{r}\frac{\cos\phi}{\sin\theta} \end{pmatrix}, \tag{16b}$$

$$\zeta^\mu_{T_z} = \begin{pmatrix} 0 & \chi\cos\theta & -\frac{\chi}{r}\sin\theta & 0 \end{pmatrix}, \tag{16c}$$

$$\zeta^\mu_{R_x} = \begin{pmatrix} 0 & 0 & \sin\phi & \frac{\cos\phi}{\tan\theta} \end{pmatrix}, \tag{16d}$$

$$\zeta^\mu_{R_y} = \begin{pmatrix} 0 & 0 & -\cos\phi & \frac{\sin\phi}{\tan\theta} \end{pmatrix}, \tag{16e}$$

$$\zeta^\mu_{R_z} = \begin{pmatrix} 0 & 0 & 0 & -1 \end{pmatrix}, \tag{16f}$$

where $\chi = \sqrt{1-kr^2}$ describes the curvature of the 3-space. In this paper we focus only on the spatially flat case; thus, $k = 0$. Since in the teleparallel context the connection is independent of the metric, imposing the symmetry fully means that the Lie derivatives of the metric and affine connection along these vectors vanish [43],

$$\pounds_\zeta g_{\mu\nu} = 0, \qquad \pounds_\zeta \Gamma^\lambda{}_{\mu\nu} = 0. \tag{17}$$

While it is well known that the metric which satisfies this condition is the Friedmann-Lemaître-Robertson-Walker one, conveniently written as

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2), \tag{18}$$

the symmetric teleparallel connection components with the same spacetime symmetry were worked out only recently, and practically simultaneously, in Refs. [45,46]. They come in three sets and are presented below by adopting the notation of Ref. [70]. First, let us note that matter energy-momentum, consistent with the cosmological symmetry, is given by

$$\mathcal{T}_{\mu\nu} = \begin{pmatrix} \rho(t) & 0 & 0 & 0 \\ 0 & a^2(t)p(t) & 0 & 0 \\ 0 & 0 & r^2a^2(t)p(t) & 0 \\ 0 & 0 & 0 & r^2a^2(t)p(t)\sin^2\theta \end{pmatrix} \tag{19}$$

where we further assume a barotropic equation of state where the pressure is proportional to density, $p = \mathrm{w}\rho$. Similarly, the spatially homogeneous and isotropic scalar field can only depend on time,

$$\Phi = \Phi(t). \tag{20}$$

A. Connection set 1

The first set of spatially homogeneous and isotropic $k = 0$ symmetric teleparallel connections can be presented as

$$\Gamma^{\rho}{}_{\mu\nu} = \left[\begin{bmatrix} \gamma(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{bmatrix} \right] \tag{21}$$

where the four matrices in the columns are labeled by the first index $^{\rho}$, and the entries of the matrices are specified by the last two indices $_{\mu\nu}$. This set was treated as case 1 ($K_2 = K_3 = 0$) with $\gamma = K_1 = -K$ in [45] and as case $\Gamma_Q^{(III)}$ with $\gamma = C_1$ in [46]. There are no extra restrictions on the function $\gamma(t)$. The nonmetricity scalar (9) computed from this connection is

$$Q = -6H^2. \tag{22}$$

B. Connection set 2

The second set of spatially homogeneous and isotropic $k = 0$ symmetric teleparallel connections can be presented as

$$\Gamma^{\rho}{}_{\mu\nu} = \left[\begin{bmatrix} \gamma(t) + \frac{\dot\gamma(t)}{\gamma(t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \gamma(t) & 0 & 0 \\ \gamma(t) & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & \gamma(t) & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ \gamma(t) & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \gamma(t) \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ \gamma(t) & \frac{1}{r} & \cot\theta & 0 \end{bmatrix} \right] \tag{23}$$

where, by definition, $\gamma(t) \neq 0$. This set was called case 3 ($K_2 = 0$, $K_3 \neq 0$) with $\gamma = K_3 = K$ in [45] and case $\Gamma_Q^{(I)}$ with $\gamma = C_3$ in [46]. The nonmetricity scalar characterizing the connection (23) is

$$Q = -6H^2 + 9H\gamma + 3\dot\gamma. \tag{24}$$

C. Connection set 3

The third set of spatially homogeneous and isotropic $k = 0$ symmetric teleparallel connections can be presented as

$$\Gamma^{\rho}{}_{\mu\nu} = \left[\begin{bmatrix} -\frac{\dot\gamma(t)}{\gamma(t)} & 0 & 0 & 0 \\ 0 & \gamma(t) & 0 & 0 \\ 0 & 0 & r^2\gamma(t) & 0 \\ 0 & 0 & 0 & r^2\gamma(t)\sin^2\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{bmatrix} \right] \tag{25}$$

where, by definition, $\gamma(t) \neq 0$. This set was called case 2 ($K_2 \neq 0$, $K_3 = 0$) with $\gamma = K_2 = -a^2K$ in [45] and case $\Gamma_Q^{(II)}$ with $\gamma = C_2$ in [46]. The corresponding nonmetricity scalar is

$$Q = -6H^2 + 9H\bar{\gamma} + 3\dot{\bar{\gamma}} \tag{26}$$

where $\bar{\gamma} = \frac{\gamma(t)}{a(t)^2}$.

IV. SCALAR-TENSOR COSMOLOGY

After these mathematical preliminaries, it is time to write out the cosmological equations and ask whether the extra function $\gamma(t)$ in the connection really constitutes an independent degree of freedom and whether it can mimic dark matter or dark energy in the description of our Universe.

A. Connection set 1

Substituting the metric (18), connection (21), matter (19), and scalar field (20) into the field equations (14) and (15) yields two nontrivial metric equations, a scalar field equation, and a matter continuity equation as follows:

$$6H^2\mathcal{A}(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) - 2\mathcal{V}(\Phi) = 2\kappa^2\rho, \tag{27a}$$

$$-4H\dot{\Phi}\mathcal{A}'(\Phi) - (6H^2 + 4\dot{H})\mathcal{A}(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) + 2\mathcal{V}(\Phi) = 2\kappa^2 \mathrm{w}\rho, \tag{27b}$$

$$-6H^2\mathcal{A}'(\Phi) - (6H\dot{\Phi} + 2\ddot{\Phi})\mathcal{B}(\Phi) - \dot{\Phi}^2\mathcal{B}'(\Phi) - 2\mathcal{V}'(\Phi) = 0, \tag{27c}$$

$$\dot{\rho} + 3H(1+\mathrm{w})\rho = 0. \tag{27d}$$

As first pointed out in Ref. [40], these equations coincide with the scalar-tensor cosmological equations in torsion based teleparallel gravity [39]. They have been studied a great deal in the teleparallel context [38,83–94], and we consider them here mainly for reference in comparison with the respective set 2 and 3 equations. In an analogous notation, the corresponding scalar-tensor equations in the usual Riemannian theory can be found, for instance, in Ref. [23]. The main difference here is that in the (symmetric) teleparallel case, the scalar field equation (27c) lacks matter sources. For minimally coupled ($\mathcal{A}' = 0$) theories, the equations fully coincide.

Notice that for connection set 1 (21), the connection equations (14b) are satisfied identically, and $\gamma(t)$ is left completely arbitrary by the field equations. The remaining four equations above are not independent of each other, as we can take a time derivative of the Friedmann constraint (27a) and derive any of (27b)–(27d) from the rest. Furthermore, the Friedmann constraint (27a) gives an algebraic relation between the variables, making one of them not independent. In other words, the physical dynamics in the four-dimensional phase space of $\{\Phi, \dot{\Phi}, \rho, H\}$ takes place on a three-dimensional hypersurface determined by (27a). In fact, we can explicitly reduce the system (27) to a set of three first-order differential equations that faithfully represent its dynamics. For instance, we may eliminate H and write

$$\dot{\Phi} = \Pi, \tag{28a}$$

$$\dot{\Pi} = -\frac{\Pi^2(\mathcal{A}(\Phi)\mathcal{B}'(\Phi) + \mathcal{B}(\Phi)\mathcal{A}'(\Phi))}{2\mathcal{A}(\Phi)\mathcal{B}(\Phi)} - \frac{\kappa^2\rho\mathcal{A}'(\Phi) + \mathcal{A}(\Phi)\mathcal{V}'(\Phi) + \mathcal{V}(\Phi)\mathcal{A}'(\Phi)}{\mathcal{A}(\Phi)\mathcal{B}(\Phi)} \mp \Pi\sqrt{\frac{3(\Pi^2\mathcal{B}(\Phi) + 2\kappa^2\rho + 2\mathcal{V}(\Phi))}{2\mathcal{A}(\Phi)}}, \tag{28b}$$

$$\dot{\rho} = \mp(1+\mathrm{w})\rho\sqrt{\frac{3(\Pi^2\mathcal{B}(\Phi) + 2\kappa^2\rho + 2\mathcal{V}(\Phi))}{2\mathcal{A}(\Phi)}} \tag{28c}$$

where the upper (lower) sign corresponds to the expanding $H > 0$ (contracting $H < 0$) branch of the Friedmann constraint (27a). Any other combination of the variables like $\{\Phi, \rho, H\}$ or $\{\Phi, \frac{\dot{\Phi}}{H}, \frac{\rho}{H^2}\}$, when properly implemented, would still yield a three-dimensional system. Below we will confirm that for connection sets 2 and 3, the phase space gets an additional independent dimension.

B. Connection set 2

From the metric (18), connection set 2 (23), matter (19), and scalar field (20), the two nontrivial metric equations (14a), one nontrivial connection equation (14b), the scalar field equation (14c), and the matter continuity equation (15) are

$$3\dot{\Phi}\gamma\mathcal{A}'(\Phi) + 6H^2\mathcal{A}(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) - 2\mathcal{V}(\Phi) = 2\kappa^2\rho, \tag{29a}$$

$$(3\dot{\Phi}\gamma - 4H\dot{\Phi})\mathcal{A}'(\Phi) - (6H^2 + 4\dot{H})\mathcal{A}(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) + 2\mathcal{V}(\Phi) = 2\kappa^2\mathrm{w}\rho, \tag{29b}$$

$$3\gamma(\ddot{\Phi}\mathcal{A}'(\Phi) + 3H\dot{\Phi}\mathcal{A}'(\Phi) + \dot{\Phi}^2\mathcal{A}''(\Phi)) = 0, \tag{29c}$$

$$(-6H^2 + 9H\gamma + 3\dot{\gamma})\mathcal{A}'(\Phi) - (6H\dot{\Phi} + 2\ddot{\Phi})\mathcal{B}(\Phi) - \dot{\Phi}^2\mathcal{B}'(\Phi) - 2\mathcal{V}'(\Phi) = 0, \tag{29d}$$

$$\dot{\rho} + 3H(1+\mathrm{w})\rho = 0. \tag{29e}$$

The nonzero function γ always appears in the equations multiplied by the derivatives of $\mathcal{A}$. Thus, its only effect is in nonminimally coupled theories, i.e., extensions of STEGR. For the minimal couplings ($\mathcal{A}' = \mathcal{A}'' = 0$), the function γ is not present in the field equations and remains completely arbitrary. In the case of minimal couplings, the field equations coincide with those of set 1 and of the minimally coupled scalar field equations in general relativity.

It is interesting to consider whether the connection contribution γ in the equations can mimic dark matter or dark energy. For that, it would need to have an available regime where it can act analogously to ρ with $\mathrm{w}=0$ or -1, respectively. However, this is not possible. If we compare the terms with γ in Eqs. (29a) and (29b) to the terms with ρ, then the effective barotropic index we could assign to the γ term would be $+1$ instead. Thus, even by picking suitable model functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{V}$ that could allow us to rewrite Eq. (29d) as a respective effective continuity equation for γ, it cannot contribute to effective dark matter or dark energy in the cosmological equations.

By construction, only four of the equations (29) are independent since any of them can be expressed as a linear combination of the time derivative of (29a) and the remaining equations. But the actual number of independent variables in the set $\{\Phi,\dot{\Phi},\rho,H,\gamma\}$ is at first not so obvious since the connection equation (29c) does not provide dynamics for the independent connection function γ but rather restrains the scalar field dynamics to

$$\ddot{\Phi}=-3H\dot{\Phi}-\frac{\mathcal{A}''(\Phi)}{\mathcal{A}'(\Phi)}\dot{\Phi}^2. \tag{30}$$

This could give an impression that although there is one extra variable and one extra equation, there is also one extra constraint that allows us to eliminate $\ddot{\Phi}$ from the system and reduce the number of independent variables by one. However, after eliminating $\ddot{\Phi}$, the Friedmann equation (29a) ceases to be a constraint that can lessen the number of independent variables but rather assumes the role of a dynamical equation for Φ. Thus, in effect, we are left with four first-order equations for four independent variables $\{\Phi,\rho,H,\gamma\}$, and the phase space is four dimensional. There is no way we can express any of these four in terms of the others.

Alternatively, we may treat (30) as a dynamical equation and use (29a) to eliminate H, writing the system (29) as

$$\dot{\Phi}=\Pi, \tag{31a}$$

$$\dot{\Pi}=-\frac{\Pi^2\mathcal{A}''(\Phi)}{\mathcal{A}'(\Phi)}\mp\Pi\sqrt{\frac{3(\Pi^2\mathcal{B}(\Phi)-3\Pi\gamma\mathcal{A}'(\Phi)+2\kappa^2\rho+2\mathcal{V}(\Phi))}{2\mathcal{A}(\Phi)}}, \tag{31b}$$

$$\dot{\rho}=\mp(1+\mathrm{w})\rho\sqrt{\frac{3(\Pi^2\mathcal{B}(\Phi)-3\Pi\gamma\mathcal{A}'(\Phi)+2\kappa^2\rho+2\mathcal{V}(\Phi))}{2\mathcal{A}(\Phi)}}, \tag{31c}$$

$$\begin{aligned}\dot{\gamma}=&-\frac{\Pi^2(2\mathcal{A}(\Phi)\mathcal{B}(\Phi)\mathcal{A}''(\Phi)-\mathcal{A}(\Phi)\mathcal{A}'(\Phi)\mathcal{B}'(\Phi)-\mathcal{B}(\Phi)(\mathcal{A}'(\Phi))^2)}{3\mathcal{A}(\Phi)(\mathcal{A}'(\Phi))^2}-\frac{\Pi\gamma\mathcal{A}'(\Phi)}{\mathcal{A}(\Phi)}\\&+\frac{2(\kappa^2\rho\mathcal{A}'(\Phi)+\mathcal{A}(\Phi)\mathcal{V}'(\Phi)+\mathcal{V}(\Phi)\mathcal{A}'(\Phi))}{3\mathcal{A}(\Phi)\mathcal{A}'(\Phi)}\mp\gamma\sqrt{\frac{3(\Pi^2\mathcal{B}(\Phi)-3\Pi\gamma\mathcal{A}'(\Phi)+2\kappa^2\rho+2\mathcal{V}(\Phi))}{2\mathcal{A}(\Phi)}},\end{aligned} \tag{31d}$$

which can be compared to the system (28) of set 1. Again, there are four independent variables $\{\Phi,\dot{\Phi},\rho,\gamma\}$, and for generic model functions, it is not possible to reduce the system to a lower-dimensional one.

Surprisingly, according to Eq. (30), the kinetic coupling $\mathcal{B}(\Phi)$ and potential $\mathcal{V}(\Phi)$ play no direct role in the scalar field dynamics. Here, for an expanding universe, the first term on the rhs dominates at small $\dot{\Phi}$ and acts as friction that slows down the scalar field evolution. The second term on the rhs dominates at larger $\dot{\Phi}$ values and pushes $\dot{\Phi}$ to increasing or decreasing values, depending on the sign of $\frac{\mathcal{A}''(\Phi)}{\mathcal{A}'(\Phi)}$. This means that for an appropriate initial "speed" $\dot{\Phi}$, all solutions will reach a standstill at some arbitrary value of Φ. However, for a sufficiently large initial "speed," the $\dot{\Phi}^2$ term can trigger ever stronger "acceleration" $\ddot{\Phi}$, throwing Φ towards infinity and causing a possible instability. If the coupling function has an extremum ($\mathcal{A}'=0$) at some value Φ_s, then even small perturbations from static Φ can launch such unstable behavior. In the phase space, the scalar field cannot evolve past the value of Φ_s where $\frac{\mathcal{A}''}{\mathcal{A}'}$ becomes singular since, depending on the initial conditions, it is either forced to stop at Φ_s or meets a sudden singularity with $|\dot{\Phi}|\to\infty$ in finite time. Thus, by simply analyzing the field equations, we should become apprehensive about the stability of the solutions with connection set 2.

C. Cosmology of connection set 3

Finally, inserting the metric (18), connection set 3 (25), matter (19), and the scalar field (20) into the field equations yields the two nontrivial metric equations (14a), one nontrivial connection equation (14b), the scalar field equation (14c), and the matter continuity equation (15) as

$$6H^2\mathcal{A}(\Phi) - 3\bar{\gamma}\,\dot{\Phi}\,\mathcal{A}'(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) - 2\mathcal{V}(\Phi) = 2\kappa^2\rho, \tag{32a}$$

$$(\bar{\gamma}\,\dot{\Phi} - 4H\dot{\Phi})\mathcal{A}'(\Phi) - (6H^2 + 4\dot{H})\mathcal{A}(\Phi) - \dot{\Phi}^2\mathcal{B}(\Phi) + 2\mathcal{V}(\Phi) = 2\kappa^2 \mathrm{w}\rho, \tag{32b}$$

$$-6\dot{\bar{\gamma}}\,\dot{\Phi}\,\mathcal{A}'(\Phi) - 3\bar{\gamma}(\ddot{\Phi}\mathcal{A}'(\Phi) + 5H\dot{\Phi}\mathcal{A}'(\Phi) + \dot{\Phi}^2\mathcal{A}''(\Phi)) = 0, \tag{32c}$$

$$(-6H^2 + 9H\bar{\gamma} + 3\dot{\bar{\gamma}})\mathcal{A}'(\Phi) - (6H\dot{\Phi} + 2\ddot{\Phi})\mathcal{B}(\Phi) - \dot{\Phi}^2\mathcal{B}'(\Phi) - 2\mathcal{V}'(\Phi) = 0, \tag{32d}$$

$$\dot{\rho} + 3H(1+\mathrm{w})\rho = 0. \tag{32e}$$

Like in the previous case of set 2, the nonzero function $\bar{\gamma}$ always appears in the equations as multiplied by the derivatives of $\mathcal{A}$ and thus only has an effect in nonminimal theories. For the minimal coupling, the function $\bar{\gamma}$ does not enter the field equations and remains completely arbitrary, while the equations themselves coincide with those of the minimally coupled scalar field equations in general relativity.

Compared to the set 2 case, the $\bar{\gamma}$ term in (32a) comes with an opposite sign. Despite that, $\bar{\gamma}$ cannot play the role of effective dark matter or dark energy since the effective barotropic index we could assign to the $\bar{\gamma}$ terms in (32a)–(32b) is $-\frac{1}{3}$, i.e., analogous to the spatial curvature term. In the remaining equations, the $\bar{\gamma}$ terms do not quite act like the spatial curvature for generic model functions, but perhaps some combinations of $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$, $\mathcal{V}(\Phi)$ may indeed allow it to mimic such behavior even exactly.

Again, by construction, only four of the equations (32) are independent since any of them can be expressed as a linear combination of the time derivative of (32a) and the remaining equations. The connection equation (32c) contains $\ddot{\Phi}$ and $\bar{\gamma}$ and can be used to eliminate either of them, eventually leading to $\{\Phi, \rho, H, \gamma\}$ or $\{\Phi, \dot{\Phi}, \rho, H\}$ as independent variables. For example, we may express, from (32c),

$$\ddot{\Phi} = -\left(\frac{2\dot{\bar{\gamma}}}{\bar{\gamma}} + 5H\right)\dot{\Phi} - \frac{\mathcal{A}''(\Phi)}{\mathcal{A}'(\Phi)}\dot{\Phi}^2 \tag{33}$$

and substitute it in.

Alternatively, we may use (32a) to eliminate H and write the system (32) as

$$\dot{\Phi} = \Pi, \tag{34a}$$

$$\dot{\Pi} = -\frac{\Pi^2\mathcal{A}''(\Phi)}{\mathcal{A}'(\Phi)} - \frac{2\Pi\dot{\bar{\gamma}}}{\bar{\gamma}} \mp 5\Pi\sqrt{\frac{\Pi^2\mathcal{B}(\Phi) + 3\Pi\bar{\gamma}\mathcal{A}'(\Phi) + 2\kappa^2\rho + 2\mathcal{V}(\Phi)}{6\mathcal{A}(\Phi)}}, \tag{34b}$$

$$\dot{\rho} = \mp(1+\mathrm{w})\rho\sqrt{\frac{3(\Pi^2\mathcal{B}(\Phi) + 3\Pi\bar{\gamma}\mathcal{A}'(\Phi) + 2\kappa^2\rho + 2\mathcal{V}(\Phi))}{2\mathcal{A}(\Phi)}}, \tag{34c}$$

$$\begin{aligned}\dot{\bar{\gamma}} = &-\frac{\Pi^2\bar{\gamma}(2\mathcal{A}(\Phi)\mathcal{B}(\Phi)\mathcal{A}''(\Phi) - \mathcal{A}(\Phi)\mathcal{A}'(\Phi)\mathcal{B}'(\Phi) - \mathcal{B}(\Phi)(\mathcal{A}'(\Phi))^2)}{(4\Pi\mathcal{B}(\Phi) + 3\bar{\gamma}\mathcal{A}'(\Phi))\mathcal{A}(\Phi)(\mathcal{A}'(\Phi))} + \frac{3\Pi\bar{\gamma}^2(\mathcal{A}'(\Phi))^2}{(4\Pi\mathcal{B}(\Phi) + 3\bar{\gamma}\mathcal{A}'(\Phi))\mathcal{A}(\Phi)} \\ &+ \frac{2\bar{\gamma}(\kappa^2\rho\mathcal{A}'(\Phi) + \mathcal{A}(\Phi)\mathcal{V}'(\Phi) + \mathcal{V}(\Phi)\mathcal{A}'(\Phi))}{(4\Pi\mathcal{B}(\Phi) + 3\bar{\gamma}\mathcal{A}'(\Phi))\mathcal{A}(\Phi)} \\ &\mp \frac{9\bar{\gamma}^2\mathcal{A}'(\Phi) + 4\Pi\bar{\gamma}\mathcal{B}(\Phi)}{3(4\Pi\mathcal{B}(\Phi) + 3\bar{\gamma}\mathcal{A}'(\Phi))}\sqrt{\frac{3(\Pi^2\mathcal{B}(\Phi) + 3\Pi\bar{\gamma}\mathcal{A}'(\Phi) + 2\kappa^2\rho + 2\mathcal{V}(\Phi))}{2\mathcal{A}(\Phi)}},\end{aligned} \tag{34d}$$

where $\dot{\bar{\gamma}}$ in (34b) should be substituted in from (34d). This can be compared to the systems (28) of set 1 and (31) of set 2. There are four independent variables $\{\Phi, \dot{\Phi}, \rho, \gamma\}$, and for generic model functions, it is not possible to reduce the system to a lower-dimensional one.

Concerning the scalar field dynamics, Eq. (34b) is structurally rather similar to Eq. (31b) of set 2 analyzed above. The only addition in (34b) is another contribution to friction which depends on $\bar{\gamma}$. Therefore, all the remarks about the possible instabilities also apply to the set 3 connection here, and we should use caution here.

V. LIMIT OF GENERAL RELATIVITY

Since the extra function in the FLRW symmetric teleparallel connections cannot contribute to dark matter or dark energy but rather induces a possibility for instabilities, it would make sense to check whether the usual ΛCDM background behavior is obtainable in the theory and prepare the groundwork to study the dynamics near such regimes.

A. Relaxation to general relativity

The well-known Friedmann equations for a spatially flat universe with a single barotropic fluid matter component in general relativity are

$$3H^2 = 8\pi G_N \rho + \Lambda, \tag{35a}$$

$$2\dot{H} + 3H^2 = -8\pi G_N \mathrm{w}\rho + \Lambda, \tag{35b}$$

$$\dot{\rho} + 3H(1+\mathrm{w})\rho = 0, \tag{35c}$$

where G_N is the Newtonian gravitational constant and Λ the cosmological constant. The scalar-tensor cosmological metric equations for different connections (27a)–(27b), (29a)–(29b), and (32a)–(32b) reduce to (35) when the dynamics of the scalar field stops, i.e., at Φ_* which sustains $\dot{\Phi} = \ddot{\Phi} = 0$. Then, the value of the nonminimal coupling function sets the gravitational constant, $8\pi G_N = \frac{\kappa^2}{\mathcal{A}(\Phi_*)}$, and the value of the potential plays the role of the cosmological constant, $\Lambda = \frac{\mathcal{V}(\Phi_*)}{\mathcal{A}(\Phi_*)}$. Note that the vanishing of $\dot{\Phi}$ also removes the contribution of the connection functions γ and $\bar{\gamma}$ from the metric field equations. Whether the stabilization of the scalar field to a constant value is possible, as well as at which value of Φ it occurs, depends on the model functions $\mathcal{A}(\Phi)$, $\mathcal{B}(\Phi)$, $\mathcal{V}(\Phi)$, and the respective connection and scalar field equations (27c), (29c)–(29d), and (32c)–(32d).

To shorten the expressions in the calculations that follow, it is helpful to remember that, without loss of generality, we can reparametrize the scalar field, i.e., introduce new $\phi(\Phi)$ so that the general form of the action (12) does not change. Since by our starting assumptions the scalar field is only involved in mediating gravity and does not have other interactions with the matter fields, the precise value of the scalar field is unmeasurable and irrelevant. All that matters in the cosmological context is the coupling to the gravity term Q in the action, which sets the effective gravitational constant, and the kinetic and potential terms in the equations, which affect the dynamics. Thus, for the sake of simplicity, we can adopt a particular parametrization of the scalar field whereby the kinetic coupling is canonical, $\mathcal{B}(\phi) \equiv 1$. We can also rewrite the nonminimal coupling to gravity as $\mathcal{A}(\phi) = 1 + f(\phi)$, which splits out a constant part, and denote the potential function in terms of the reparametrized field as $V(\phi)$.

In terms of this parametrization, the solutions of Eqs. (35) for different matter types are easy to write out. In the case of nonrelativistic matter (pressureless dust, $\mathrm{w} = 0$), the Hubble parameter and matter density evolve as

$$H_* = \frac{2}{3(t-t_s)}, \quad \rho_* = \frac{4(1+f_*)}{3\kappa^2(t-t_s)^2}, \quad V_* = 0, \tag{36}$$

in the case of relativistic matter (radiation, $\mathrm{w} = \frac{1}{3}$), they evolve as

$$H_* = \frac{1}{2(t-t_s)}, \quad \rho_* = \frac{3(1+f_*)}{4\kappa^2(t-t_s)^2}, \quad V_* = 0, \tag{37}$$

and finally, in the case where the scalar potential plays the role of dark energy (cosmological constant), the evolution is

$$H_* = \sqrt{\frac{V_*}{3(1+f_*)}}, \quad \rho_* = 0, \quad V_* = \mathrm{const} \neq 0. \tag{38}$$

Here, H_* and ρ_* can evolve in time, but $f_* = f(\phi_*)$ and $V_* = V(\phi_*)$ are constants evaluated at the point where the scalar field stops. The integration constant t_s sets the moment of initial singularity, and we can fix $t_s = 0$. The cosmological standard ΛCDM model involves all three types of matter, but since the densities of the matter types evolve at different rates, the universe goes through a sequence of radiation, dust matter, and dark energy domination eras where the other components can be neglected as subdominant.

B. Expansion around the general relativity limit

In a realistic scenario for the late universe, the eras of radiation, dust, and dark energy domination follow each other as the respective energy densities become dominant in succession. Thus, a basic stability test of a modified gravity model is whether the single fluid cosmological equations are stable against small perturbations; i.e., there is no other phenomenon that could destroy the standard history of the universe. Moreover, the change of the effective gravitational constant has had significant constraints since the time of big bang nucleosynthesis; hence, the scalar must have resided at an almost constant value from the time of the early universe until today [12–14,95,96].

To see whether the limit of general relativity is dynamically stable, let us study the evolution of small perturbations near the value of ϕ_*. Thus, let us expand

$$\begin{aligned} \phi(t) &= \phi_* + x(t), & H(t) &= H_*(t) + h(t), \\ \gamma(t) &= \gamma_*(t) + g(t), & \rho(t) &= \rho_*(t) + r(t), \end{aligned} \tag{39}$$

where $x(t)$, $h(t)$, $g(t)$, and $r(t)$ are small perturbations that we assume to be of roughly the same order. The respective derivatives are

$$\begin{aligned} \dot{\phi}(t) &= \dot{x}(t), & \dot{H}(t) &= \dot{H}_*(t) + \dot{h}(t), \\ \dot{\gamma}(t) &= \dot{\gamma}_*(t) + \dot{g}(t), & \dot{\rho}(t) &= \dot{\rho}_*(t) + \dot{r}(t), \end{aligned} \tag{40}$$

also assumed to be of roughly the same order. We can also expand the functions of the scalar field as

$$f(\phi(t)) = f_* + f'_* x(t) + \frac{f''_*(x(t))^2}{2},$$
$$f'(\phi(t)) = f'_* + f''_* x(t) + \frac{f'''_*(x(t))^2}{2},$$
$$f''(\phi(t)) = f''_* + f'''_* x(t) + \frac{f''''_*(x(t))^2}{2}, \tag{41a}$$

$$V(\phi(t)) = V_* + V'_* x(t) + \frac{V''_*(x(t))^2}{2},$$
$$V'(\phi(t)) = V'_* + V''_* x(t) + \frac{V'''_*(x(t))^2}{2}. \tag{41b}$$

Substituting these definitions into the cosmological equations (27), (29), and (32) yields expressions which can be solved order by order. At the lowest (background) order, for all three sets, the metric and matter equations reduce to (35) as explained above. At the background order, for all the connections, the connection equation is identically satisfied while the scalar field equation is slightly different.

VI. STABILITY OF STANDARD COSMOLOGICAL REGIMES

Now, we are in a position to investigate the stability of the standard dust, radiation, and dark energy dominated regimes near the general relativity limit in the configurations with different symmetric teleparallel cosmological connections.

A. Connection set 1

Applying the parametrization and expansion introduced in Sec. V on the cosmological equations (27) of connection set 1 (21) yields up to first-order small quantities

$$6(1+f_*)H_*^2(t) - 2V_* - 2\kappa^2\rho_*(t) + (12(1+f_*)H_*(t)h(t) + 6f'_*H_*^2(t)x(t) - 2V'_*x(t) - 2\kappa^2 r(t)) = 0, \tag{42a}$$

$$\begin{aligned}4(1+f_*)\dot{H}_*(t) + 6(1+f_*)H_*^2(t) - 2V_* + 2\kappa^2\mathrm{w}\rho_*(t) + (4f'_*x(t)\dot{H}_*(t) + 12(1+f_*)H_*(t)h(t) + 4(1+f_*)\dot{h}(t)\\ + 6f'_*H_*^2(t)x(t) + 4f'_*H_*(t)\dot{x}(t) - 2V'_*x(t) + 2\kappa^2\mathrm{w}r(t)) = 0,\end{aligned} \tag{42b}$$

$$-6f'_*H_*^2(t) - 2V'_* - (2V''_*x(t) + 6f''_*H_*^2(t)x(t) + 12f'_*H_*(t)h(t) + 6H_*(t)\dot{x}(t) + 2\ddot{x}(t)) = 0, \tag{42c}$$

$$\dot{\rho}_*(t) + 3(1+\mathrm{w})H_*(t)\rho_*(t) + (3\mathrm{w}H_*(t)r(t) + 3\mathrm{w}\rho_*(t)h(t) + 3H_*(t)r(t) + 3\rho_*(t)h(t) + \dot{r}(t)) = 0. \tag{42d}$$

The characteristic behavior depends on the dominant matter type. The same equations but without the matter perturbation $r(t)$ were analyzed in Ref. [91]. Therefore, below we add the matter perturbation and summarize the main results.

1. Dust matter domination

In a nonrelativistic ($\mathrm{w} = 0$) matter dominated scenario, when the matter energy density surpasses the potential and we can ignore V_* over $\rho(t)$, the leading-order (background) expressions of Eqs. (42a), (42b), and (42d) are solved by the standard dust dominated background (36). The remaining leading-order part of the scalar field equation (42c) then demands $f'_* = 0$, $V'_* = 0$. This means a necessary condition for the scalar field evolution to stop and the system to relax to the general relativity regime is that the functions of the gravitational coupling $f(\phi)$ and the potential $V(\phi)$ must have coincident critical points at the same value of ϕ_*; otherwise, the scalar field cannot stabilize, and the expansion around ϕ_* is meaningless.

Provided a scalar-tensor model has such model functions, we can substitute the expressions $H_*(t)$ and $\rho_*(t)$ in (36) into the first-order perturbed parts of Eqs. (42a)–(42d), and solving those, we find the small perturbations of the Hubble parameter and matter density decay in time,

$$h(t) \sim t^{-2}, \qquad r(t) \sim t^{-3}. \tag{43}$$

Upon substituting the background value of $H_*(t)$ into the perturbed scalar field equation (42c), we get a Klein-Gordon equation in curved spacetime,

$$\ddot{x}(t) + \frac{2}{t}\dot{x}(t) + \left(V''_* + \frac{4f''_*}{3t^2}\right)x(t) = 0. \tag{44}$$

For large t the only surviving mass term is V''_*, which predicts oscillatory solutions for the minimum of the potential, $V''_* > 0$, and exponential growth for the maximum of the potential, $V''_* < 0$. The term linear in $\dot{x}$ modifies this overall dynamics by adding extra frictional damping. In fact, the above equation (44) can be recognized as a Bessel equation and solved exactly in terms of the Bessel functions or modified Bessel functions. Ignoring the oscillating factors in the solutions, the leading time dependence of the scalar perturbation turns out to be [91]

$$x(t) \sim \begin{cases} t^{-1} & V_*'' > 0 \\ t^{-\frac{1}{2}} & V_*'' = 0 \quad v_* \text{ imaginary} \\ t^{-\frac{1}{2}(1-v_*)} & V_*'' = 0 \quad v_* \text{ real} \\ t^{-1} e^{\sqrt{-V_*''}t} & V_*'' < 0, \end{cases}$$

$$v_* = \sqrt{1 - \frac{16 f_*''}{3}}. \tag{45}$$

Thus, in summary, for the perturbations to converge and the general relativistic dust dominated regime to be stable, the model functions must have the value ϕ_* which corresponds to either a simultaneous local minimum of the potential and a critical point of the gravitational coupling function ($V_*' = 0, V_*'' > 0, f_*' = 0$), or to a local minimum of the gravitational coupling function and an inflection point of the potential ($f_*' = 0, f_*'' > 0, V_*' = 0, V_*'' = 0$).

2. Radiation domination

In a relativistic ($\mathrm{w} = \frac{1}{3}$) matter dominated case, when we can ignore V_* in the presence of $\rho(t)$, the leading-order expressions of Eqs. (42a), (42b), and (42d) are solved by the standard radiation dominated background (37). The remaining leading-order part of the scalar field equation (42c) again demands $f_*' = 0$, $V_*' = 0$. Provided the scalar-tensor model has such suitable model functions, we can substitute the expressions $H_*(t)$ and $\rho_*(t)$ from (37) into the first-order perturbed parts of Eqs. (42a)–(42d) and find that the small perturbations of the Hubble parameter and matter density decay in time,

$$h(t) \sim t^{-2}, \qquad r(t) \sim t^{-3}. \tag{46}$$

After substituting the background value $H_*(t)$ into Eq. (42c), we again get a Klein-Gordon type equation,

$$\ddot{x}(t) + \frac{3}{2t}\dot{x}(t) + \left(V_*'' + \frac{3 f_*''}{4t^2}\right)x(t) = 0. \tag{47}$$

The overall stability is principally determined by the sign of V_*'', i.e., whether we are at the maximum or minimum of the potential, while the $\dot{x}$ term adds an extra friction effect. The full solutions to Eq. (47) can again be found in the form of the Bessel functions, whereby, ignoring the oscillating factors, the leading time dependence of the scalar perturbation turns out to be [91]

$$x(t) \sim \begin{cases} t^{-\frac{3}{4}} & V_*'' > 0 \\ t^{-\frac{1}{4}} & V_*'' = 0 \quad v_* \text{ imaginary} \\ t^{-\frac{1}{4}(1-v_*)} & V_*'' = 0 \quad v_* \text{ real} \\ t^{-\frac{3}{4}} e^{\sqrt{-V_*''}t} & V_*'' < 0, \end{cases}$$

$$v_* = \sqrt{1 - 12 f_*''}. \tag{48}$$

Thus, in summary, just like in the dust matter case, for the perturbations to converge and the general relativistic radiation dominated regime to be stable, the model functions must have the value ϕ_* which corresponds to either a simultaneous local minimum of the potential and a critical point of the gravitational coupling function ($V_*' = 0, V_*'' > 0, f_*' = 0$), or to a local minimum of the gravitational coupling function and an inflection point of the potential ($f_*' = 0, f_*'' > 0, V_*' = 0, V_*'' = 0$).

3. Potential domination

In the era when the scalar potential dominates over the matter energy density, and we can drop $\rho(t)$ in comparison with $V(\phi_*)$ [but keep dust matter perturbations $r(t)$ with $\mathrm{w} = 0$], the leading-order expressions of Eqs. (42a), (42b), and (42d) are solved by the standard dark energy (cosmological constant) dominated background (38). The remaining leading-order part of the scalar field equation (42c) can then be satisfied in two ways.

First, the background scalar field equation can be solved by $f_*' = 0$, $V_*' = 0$. Then, the first-order perturbed metric and matter equations give

$$h(t) \sim e^{-3H_* t}, \qquad r(t) \sim e^{-3H_* t}. \tag{49}$$

The first-order perturbed scalar field equation

$$\ddot{x}(t) + 3H_*\dot{x}(t) + (V_*'' + 3H_*^2 f_*'')x(t) = 0 \tag{50}$$

is again a Klein-Gordon type with a friction term. The overall stability, i.e., exponentially damped oscillations or exponential growth, depends on the sign of the third term (effective mass squared). Dropping the factor of oscillations in the solutions, the time dependence of the scalar field perturbation can be expressed as [91]

$$x(t) \sim \begin{cases} e^{-\frac{3H_* t}{2}} & v_* \text{ imaginary} \\ e^{-\frac{3H_* t}{2}(1-v_*)} & v_* \text{ real}, \end{cases}$$

$$v_* = \sqrt{1 - \frac{4}{9H_*^2}(V_*'' + 3H_*^2 f_*'')}. \tag{51}$$

Thus, to realize a stable dark energy era in this scenario, the model functions must have the value ϕ_* which corresponds to either a local minimum of the potential and simultaneously to a local minimum of the gravitational coupling function ($V_*' = 0, V_*'' > 0, f_*' = 0, f_*'' > 0$), or at least to a simultaneous critical point of the potential and the gravitational coupling ($f_*' = 0, V_*' = 0$) with an additional condition $V_* f_*'' + (1 + f_*)V_*'' > 0$.

The background scalar field equation (42c) can also be satisfied by a "balanced" configuration where the scalar field resides not at a critical point of the potential or of the gravitational coupling function but at a value which satisfies $V_* f'_* = -V'_*(1+f_*)$ whereby Eq. (38) implies $H_* = \sqrt{-\frac{V'_*}{3f'_*}}$. In this case the perturbed part of the matter continuity equation gives [Eq. (42d)]

$$r(t) \sim e^{-3H_* t}, \tag{52}$$

while the remaining perturbed equations show that, taking the leading order, the Hubble and scalar field perturbations are proportional to each other, $h(t) \sim x(t)$. The latter evolves according to a Klein-Gordon type equation

$$\ddot{x}(t) + 3H_* \dot{x}(t) + \left(V''_* + 3H_*^2 f''_* + 2\frac{(V'_*)^2}{V_*} \right) x(t) = 0. \tag{53}$$

The behavior of the solutions to this equation again depends on the sign of the last term and can be summarized as [91]

$$h(t) \sim x(t) \sim \begin{cases} e^{-\frac{3H_* t}{2}} & v_* \text{ imaginary} \\ e^{-\frac{3H_* t}{2}(1-v_*)} & v_* \text{ real}, \end{cases} \qquad v_* = \sqrt{1 - \frac{4}{9H_*^2}\left(V''_* + 3H_*^2 f''_* + \frac{2(V'_*)^2}{V_*} \right)}. \tag{54}$$

The perturbations die off in time, and this configuration is stable if $V_* f''_* + (1+f_*)V'' + 2f'_* V'_* > 0$. However, in constructing a realistic cosmic history, we might prefer this configuration to be unstable, like a saddle point, and play the role of a launching point for an inflationary trajectory [93]. If it is unstable, it can trigger an early epoch of inflationary expansion, making the scalar field roll to a value where $f'_* = 0$, $V'_* = 0$. If the latter point is stable, inflation would give way to radiation, dust matter, and dark energy eras, whereas the scalar field just relaxes in damped oscillations around that second point.

B. Connection set 2

Applying the parametrization and expansion introduced in Sec. V on the cosmological equations (29) of connection set 2 yields

$$6(1+f_*)H_*^2(t) - 2V_* - 2\kappa^2\rho_*(t) + (12(1+f_*)H_*(t)h(t) + 6f'_* H_*^2(t)x(t) + 3f'_*\gamma_*(t)\dot{x}(t) - 2V'_* x(t) - 2\kappa^2 r(t)) = 0, \tag{55a}$$

$$\begin{aligned} &4(1+f_*)\dot{H}_*(t) + 6(1+f_*)H_*^2(t) - 2V_* + 2\kappa^2 \mathrm{w}\rho_*(t) + (4f'_* x(t)\dot{H}_*(t) + 12(1+f_*)H_*(t)h(t) + 4(1+f_*)\dot{h}(t) \\ &\quad + 6f'_* H_*^2(t)x(t) + 4f'_* H_*(t)\dot{x}(t) - 3f'_*\gamma_*(t)\dot{x}(t) - 2V'_* x(t) + 2\kappa^2 \mathrm{w} r(t)) = 0, \end{aligned} \tag{55b}$$

$$\begin{aligned} &(9f'_* H_*(t)\gamma_*(t)\dot{x}(t) + 3f'_*\gamma_*(t)\ddot{x}(t)) + (9f''_* H_*(t)\gamma_*(t)x(t)\dot{x}(t) + 3f''_*\gamma_*(t)x(t)\ddot{x}(t) + 3f''_*\gamma_*(t)(\dot{x}(t))^2 \\ &\quad + 9f'_* H_*(t)g(t)\dot{x}(t) + 9f'_*\gamma_*(t)h(t)\dot{x}(t) + 3f'_* g(t)\ddot{x}(t)) = 0, \end{aligned} \tag{55c}$$

$$\begin{aligned} &3f'_*(\dot{\gamma}_*(t) - 2H_*^2(t) + 3H_*(t)\gamma_*(t)) - 2V'_* - (2V''_* x(t) + 6f''_* H_*^2(t)x(t) - 9f''_* H_*(t)\gamma_*(t)x(t) - 3f''_* x(t)\dot{\gamma}_*(t) \\ &\quad + 12f'_* H_*(t)h(t) - 9f'_*\gamma_*(t)h(t) - 9f'_* H_*(t)g(t) - 3f'_*\dot{g}(t) + 6H_*(t)\dot{x}(t) + 2\ddot{x}(t)) = 0, \end{aligned} \tag{55d}$$

$$\dot{\rho}_*(t) + 3(1+\mathrm{w})H_*(t)\rho_*(t) + (3\mathrm{w}H_*(t)r(t) + 3\mathrm{w}\rho_*(t)h(t) + 3H_*(t)r(t) + 3\rho_*(t)h(t) + \dot{r}(t)) = 0. \tag{55e}$$

We will consider the stability in the case of different matter types separately, going through the dust matter calculations in detail, and presenting the main results in the other cases.

1. Dust matter domination

The regime of dust matter domination means we take $\mathrm{w} = 0$ and neglect V_* in comparison with $\rho(t)$. At the background level (without perturbations), the metric and matter equations (55a), (55b), and (55e) coincide with the GR cosmological equations (35) and are solved by the standard background evolution (36). The background part of the connection equation (55c) is identically zero, while in solving the scalar field equation (55d) at the background level,

$$3f'_*\left(\dot{\gamma}_*(t) + \frac{2\gamma_*(t)}{t} - \frac{8}{9t^2}\right) - 2V'_* = 0, \tag{56}$$

there are three options: either $f'_* \neq 0$, $V'_* \neq 0$ and the evolution of $\gamma_*(t)$ is given by

$$\gamma_*(t) = \frac{2V'_* t}{9f'_*} + \frac{8}{9t} + \frac{c_1}{t^2} \tag{57}$$

which leaves the value of ϕ_* arbitrary; $f'_* \neq 0$, $V'_* = 0$ in (56) and (57); or the model functions $f(\phi)$ and $V(\phi)$ both have an extremum at the same value of ϕ and the respective derivatives vanish,

$$f'_* = 0, \qquad V'_* = 0, \tag{58}$$

which fixes ϕ_* but leaves $\gamma_*(t)$ undetermined.

In the first case, substituting the background values (36) and (57) into the first-order perturbed equations, i.e., keeping only terms that are linear in $x(t)$, $h(t)$, $g(t)$, and $r(t)$ in (55), gives

$$\frac{4(1+f_*)h(t)}{t} - \frac{(3V'_*t^2 - 4f'_*)x(t)}{3t^2} + \frac{(2V'_*t^3 + f'_*(9c_1 + 8t))\dot{x}(t)}{6t^2} = \kappa^2 r(t), \tag{59a}$$

$$-4(1+f_*)\dot{h}(t) - \frac{8(1+f_*)h(t)}{t} + \frac{(2V'_*t^3 + 9c_1 f'_*)\dot{x}(t)}{3t^2} + 2V'_*x(t) = 0, \tag{59b}$$

$$\frac{(2V'_*t^3 + 9c_1 f'_* + 8f'_*t)}{3t^3}(t\ddot{x}(t) + 2\dot{x}(t)) = 0. \tag{59c}$$

$$f'_*\left(3\dot{g}(t) + \frac{6g(t)}{t}\right) + \frac{(2V'_*t^3 + 9c_1 f'_*)h(t)}{t^2} - \frac{2(V''_*f'_* - V'_*f''_*)x(t)}{f'_*} - 2\ddot{x}(t) - \frac{4}{t}\dot{x}(t) = 0, \tag{59d}$$

$$\dot{r}(t) + \frac{2r(t)}{t} + \frac{4(1+f_*)h(t)}{\kappa^2 t^2} = 0. \tag{59e}$$

Here, we can first integrate Eq. (59c) to get

$$x(t) = \frac{c_2}{t} + c_3. \tag{60}$$

The integration constant $c_3 = 0$ since we have defined that, in the end, the scalar field stops at ϕ_*, not at $\phi_* + c_3$. Thus, as time passes, the perturbation in the scalar field diminishes, which ensures that the $\gamma(t)\dot{\phi}(t)$ term in the Friedmann equation (55a) does not grow to spoil the dust domination regime, despite $\gamma(t)$ [Eq. (57)] increasing in time. Substituting (60) into (59b), we get an equation for $h(t)$, which is solved by

$$h(t) = \frac{c_2 V'_*}{6(1+f_*)} + \frac{c_4}{t^2} + \frac{3c_1 c_2 f'_*}{4(1+f_*)t^3}. \tag{61}$$

Then, we can substitute (60) and (61) into (59d) and solve to find

$$g(t) = -\frac{c_2(V'_*)^2 t^2}{36 f'_*(1+f_*)} + \frac{c_2 V''_* f'_* - c_2 V'_* f''_* - c_4 V'_* f'_*}{3(f'_*)^2} - \frac{c_1 c_2 V'_*}{(1+f_*)t} + \frac{c_5}{t^2} + \frac{3c_1 c_4}{t^3} + \frac{9c_1^2 c_2 f'_*}{8(1+f_*)t^4}. \tag{62}$$

Finally, the solutions (60) and (61) can be substituted into (59e), where it is possible to algebraically express

$$r(t) = -\frac{2c_2 V'_*}{3\kappa^2 t} + \frac{4c_4(1+f_*)}{3\kappa^2 t^3} + \frac{3c_1 c_2 f'_*}{2\kappa^2 t^4}. \tag{63}$$

The latter expression, along with (61), also solves (59e). Thus, we see that if we perturb around an arbitrary scalar field value, where neither $V'_* \neq 0$ nor $f'_* \neq 0$, then, to the leading order in time, the quantities evolve as

$$\gamma_*(t) \sim t, \qquad x(t) \sim t^{-1}, \qquad h(t) \sim t^0, \qquad g(t) \sim t^2, \qquad r(t) \sim t^{-1}, \tag{64}$$

and the configuration cannot be considered stable. Although the evolution of ϕ slows to a stop and ρ converges to its general relativity regime, and even the effects of the growing connection function are suppressed by decreasing $\dot{\phi}$, the connection function perturbations $g(t)$ grow bigger in time and eventually spoil the approximation which assumes the perturbations to be small.

If the perturbation takes place around $f'_* \neq 0$, $V'_* = 0$, mentioned as the second case above, then for $V''_* \neq 0$

$$\gamma_*(t) \sim t^{-1}, \qquad x(t) \sim t^{-1}, \qquad h(t) \sim t^{-2}, \qquad g(t) \sim t^0, \qquad r(t) \sim t^{-3}. \tag{65}$$

Although the connection function perturbations do not decrease, they do not increase either, and the configuration may be considered marginally stable. If the second derivatives of the potential are also zero (like for a cosmological constant), then

$$\gamma_*(t) \sim t^{-1}, \qquad x(t) \sim t^{-1}, \qquad h(t) \sim t^{-2}, \qquad g(t) \sim t^{-2}, \qquad r(t) \sim t^{-3}, \tag{66}$$

and a stable regime is possible.

When we consider perturbations around a simultaneous critical point of the potential and gravitational coupling functions, Eq. (58), the scalar field equation is identically solved at the background level and $\gamma_*(t)$ remains undetermined. The lowest-order perturbed equations (55) are now

$$\frac{4(1+f_*)h(t)}{t} = \kappa^2 r(t), \tag{67a}$$

$$-2(1+f_*)\left(\dot{h}(t) + \frac{2h(t)}{t}\right) = 0, \tag{67b}$$

$$x(t)\ddot{x}(t)+\dot{x}(t)^2+\frac{2x(t)\dot{x}(t)}{t}=0, \tag{67c}$$

$$(9f''_*t^2\dot{\gamma}_*(t)-6V''_*t^2+18f''_*t\gamma_*(t)-8f''_*)\frac{x(t)}{3t^2} \\ -2\ddot{x}(t)-\frac{4}{t}\dot{x}(t)=0, \tag{67d}$$

$$\dot{r}(t)+\frac{2r(t)}{t}+\frac{4(1+f_*)h(t)}{\kappa^2t^2}=0. \tag{67e}$$

Note that at the first (linear) perturbation level, the connection equation is identically satisfied, and (67c) represents the second (nonlinear) perturbation level as the leading nonzero order. The equation for the Hubble perturbation (67b) can be integrated to

$$h(t)=\frac{c_2}{t^2}, \tag{68}$$

and Eq. (67a) then gives

$$r(t)=\frac{4c_2(1+f_*)}{\kappa^2t^3}. \tag{69}$$

We cannot tackle the remaining first-order perturbed scalar field equation (67d) directly since it contains two unknown functions $\gamma_*(t)$ and $x(t)$. However, it is still possible to proceed by taking the next order perturbation of the connection equation which contains second-order small quantities (67c). Interestingly, the structure of that equation parallels the perturbed scalar field equation near the general relativity limit of the usual curvature based scalar-tensor cosmology with dust matter or potential [20–22]. However, the signs are different, and the solutions behave differently here. Equation (67c) can be solved easily by

$$x(t)=\pm\sqrt{\frac{c_6}{t}+c_7}. \tag{70}$$

Despite its appearance, this expression actually harbors a singularity, related to the feature briefly discussed already at the end of Sec. IV B. From (70), we can compute

$$\dot{x}(t)=\mp\frac{c_6}{2t^2\sqrt{\frac{c_6}{t}+c_7}} \tag{71}$$

and express the integration constants in terms of initial conditions x_0, $\dot{x}_0$ at t_0 as

$$c_6=-2\dot{x}_0x_0t_0^2, \qquad c_7=x_0^2+2\dot{x}_0x_0t_0. \tag{72}$$

Both integration constants are real. At a finite time $t_*=-\frac{c_6}{c_7}$, the scalar perturbation $x(t)$ goes to zero, but the speed $\dot{x}(t)$ becomes singular and the approximation of perturbations being small breaks down. The singularity occurs in the physical time $t>0$ if c_6 and c_7 have opposite signs. We can write

$$\frac{t_0}{t_*}=-\frac{c_7t_0}{c_6}=1+\frac{x_0}{2\dot{x}_0t_0} \tag{73}$$

and analyze the situation in conjunction with the phase portrait in Fig. 1(a) as follows. First, in regions (I), where x_0 and $\dot{x}_0$ are of the same sign, $t_*<t_0$ and the singularity occurs before the solution, i.e., the solution which is specified by some initial conditions at the moment t_0 emerges from a singularity at a finite time. These solutions evolve away from ϕ_* and could be classified as unstable. Asymptotically, they would reach $|x(t\to\infty)|\to\sqrt{c_7}$; however, first, the current approximation breaks down, and they should fall under the purview of the more general case that follows from (57) which was considered before. Second, in regions (II), where

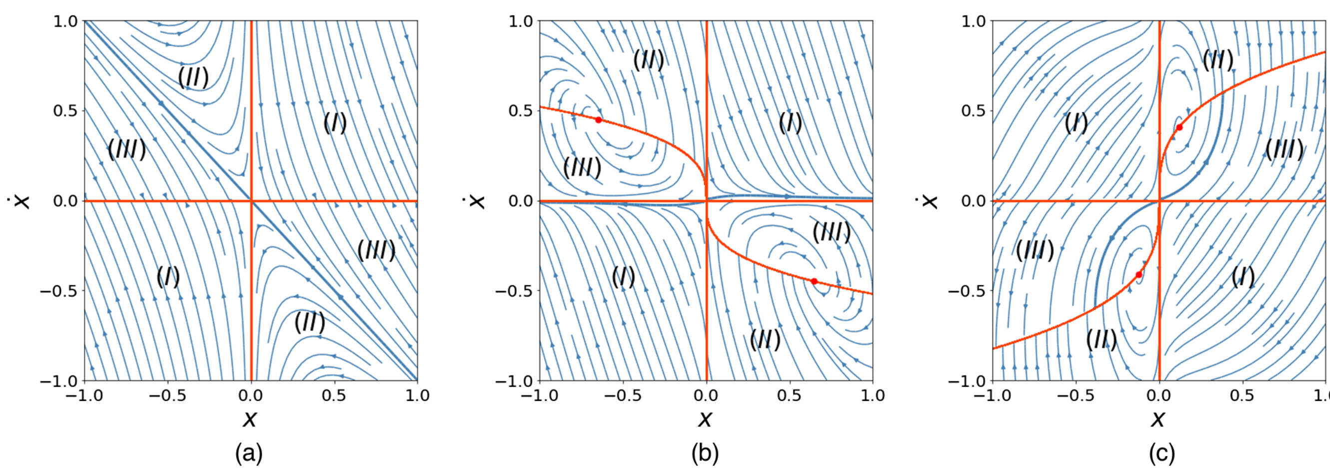

FIG. 1. Sketch of the phase space where singular solutions occur. The portrait of (a) set 2, dust dominated, Eq. (67c) at $t=1$, populated by the solutions (70); (b) set 3, dust dominated, Eq. (97) for $V''_*=1$, $f''_*=1$, $c_9=0.5$ at $t=1$; (c) set 3, dust dominated, Eq. (97) for $V''_*=1$, $f''_*=1$, $c_9=-1$ at $t=1$.

x_0 and $\dot{x}_0$ are of opposite signs and $|\dot{x}_0| > \frac{|x_0|}{2t_0}$, then $t_* > t_0$ and the singularity happens in the future of the solution. Definitely the approximation scheme breaks down as $\dot{x}$ diverges, but quite likely the full system hits a singularity. Third, in regions (III), where x_0 and $\dot{x}_0$ are of opposite signs but $|\dot{x}_0| < \frac{|x_0|}{2t_0}$, the integration constants are of the same sign, and the solutions have a tendency to slow down and arrive at $|x(t \to \infty)| \to \sqrt{c_7}$. However, in a strict sense, they would again belong to the more general case considered above since the point of eventual stability for ϕ does not satisfy the condition (58). Finally, there is a particular set of solutions specified by $x_0 = -2\dot{x}_0 t_0$ so that $c_7 = 0$. Running between regions (II) and (III), they come from $x > 0$ and $x < 0$. Only this type of solution manages to asymptotically reach $|x(t \to \infty)| \to 0$ while avoiding the singularity as well as the fate of stopping before that value. To complete the analysis of this particular case, we can substitute (70) with $c_7 = 0$ into (67e) and find the background evolution of the connection function to be

$$\gamma_*(t) = \frac{2V''_* t}{9f''_*} + \frac{8}{9t} - \frac{1}{6f''_* t} + \frac{c_8}{t^2}. \tag{74}$$

For nonzero V''_*, the connection function diverges linearly in time, but the $\gamma_*\dot{\phi}$ term in the Friedmann equation is still suppressed. Although the Hubble, matter, and scalar field perturbations diminish in time, we cannot safely conclude that this particular type of solution is convergent since the connection perturbation $g(t)$ remains undetermined.

At this point, an astute reader may raise a concern as to whether it was consistent to consider quantities quadratic in the perturbations only in the perturbed connection equation (67c) but not in the others of (67). The reason is that for each equation, we are interested only in the leading dominant behavior that is relevant for the stability of the system, i.e., whether the solutions converge to or diverge from the general relativity limit. Although in the other equations the subdominant quadratic and higher terms are also present, under the assumption of smallness, they have less influence and do not decide the issue of stability. If we want to know the higher corrections to the time dependence of the solutions, then we would also need to include higher-order small perturbations in the expansions (39). However, most likely, the system of equations would then become even more complicated and harder to solve.

In any case, for large enough initial velocities, the system meets a singularity in finite time. It is remarkable that the limit $V'_* = 0$, $f'_* = 0$, which gives a stable standard history in the case of connection set 1, is unstable for a large range of initial conditions in the dust dominant case of set 2. The strange fact that the background equations fail to determine the connection function $\gamma_*(t)$ in this limit while the derivative of the scalar field perturbation becomes singular may indicate that the scalar-nonmetricity theory is problematic and does not reduce to general relativity in a smooth manner here. Alternatively, we may interpret this feature as an indication that connection set 2 is unphysical and should be discarded in favor of set 1 where the GR limit is smooth. In view of the remark at the end of Sec. IV B, a more thorough investigation of the instability of the full equations might shed light on this issue, but it is beyond the scope of the present work.

2. Radiation domination

The regime of radiation domination allows us to neglect V_* in comparison with $\rho(t)$, which is characterized by $\mathrm{w} = \frac{1}{3}$. The calculations can be performed analogously to the previous subsection, and we summarize the key results. At the background level, the metric and matter equations (55a), (55b), and (55e) coincide with the GR cosmological equations (35) and are solved by the standard background evolution (37). The background part of the connection equation (55c) is identically zero, while solving the scalar field equation (55d) at the background level,

$$3f'_*\left(\dot{\gamma}_*(t) + \frac{3\gamma_*(t)}{2t} - \frac{1}{2t^2}\right) - 2V'_* = 0, \tag{75}$$

gives three options.

First, for an arbitrary ϕ_* whereby $f'_* \neq 0$, $V'_* \neq 0$, the evolution of $\gamma_*(t)$ in (75) is given by

$$\gamma_*(t) = \frac{4V'_* t}{15f'_*} + \frac{1}{t} + \frac{c_1}{t^{\frac{3}{2}}}. \tag{76}$$

Substituting the background values (37) and (76) into the first-order perturbed parts of (55) gives a system of equations to determine the behavior of perturbations, which, at leading order, turn out to be

$$x(t) \sim t^{-\frac{1}{2}}, \quad h(t) \sim t^{\frac{1}{2}}, \quad g(t) \sim t^{\frac{5}{2}}, \quad r(t) \sim t^{-\frac{1}{2}}. \tag{77}$$

Even the perturbations of the Hubble parameter grow, and the configuration is unstable.

Second, if ϕ_* is fixed by $V'_* = 0$, but $f'_* \neq 0$, then the scalar field equation (75) is solved by the subleading terms in (76). Then, the first-order perturbations at leading order behave as

$$\begin{aligned} &\gamma_*(t) \sim t^{-1}, \qquad x(t) \sim t^{-\frac{1}{2}}, \qquad h(t) \sim t^{-2}\ln t, \\ &g(t) \sim t^{\frac{1}{2}}, \qquad r(t) \sim t^{-3}\ln t, \end{aligned} \tag{78}$$

and the configuration is still unstable because the connection function perturbations diverge. If the second derivative of the potential is also zero, we find better stability properties,

$$\gamma_*(t) \sim t^{-1}, \qquad x(t) \sim t^{-\frac{1}{2}}, \qquad h(t) \sim t^{-2}\ln t,$$
$$g(t) \sim t^{-\frac{3}{2}}, \qquad r(t) \sim t^{-3}\ln t, \tag{79}$$

and the solutions converge to general relativity.

Finally, if the model functions $f(\phi)$ and $V(\phi)$ both have an extremum at the same value of ϕ and the respective derivatives vanish, $f'_* = 0$, $V'_* = 0$, the background equations leave $\gamma_*(t)$ undetermined. We can integrate the first-order perturbed metric and matter equations to find

$$h(t) \sim t^{-2}, \qquad r(t) \sim t^{-3}. \tag{80}$$

As the first-order perturbed connection equation is identically satisfied, we can invoke the second-order perturbed connection equation, which is solved by

$$x(t) = \pm\sqrt{\frac{c_6}{\sqrt{t}} + c_7}. \tag{81}$$

Like in the dust matter case, a significant class of solutions with large enough initial velocities, $|\dot{x}_0| > \frac{|x_0|}{4t_0}$, reaches ϕ_* (i.e., $x = 0$) with diverging speed $\dot{x}$, and the system experiences a singularity. Only the solutions with the integration constant $c_7 = 0$ converge asymptotically to ϕ_* without meeting a singularity. For those solutions, we can solve the first-order scalar field equation by

$$\gamma_*(t) = \frac{4V''_* t}{15 f''_*} + \frac{1}{t} - \frac{1}{12 f''_* t} + \frac{c_8}{t^{\frac{3}{2}}}, \tag{82}$$

but the question of stability remains without a decisive answer since the connection perturbations $g(t)$ remain undetermined.

In summary, the only radiation dominated configuration which is stable for perturbations around the standard general relativistic cosmological scenario is given by $f_* \neq 0$, $V'_* = V''_* = 0$.

3. Potential domination

In the era when the scalar potential dominates over the matter energy density, and we can drop $\rho(t)$ in comparison with $V(\phi_*)$ [but keep dust matter perturbations $r(t)$ with $\mathrm{w} = 0$], the leading-order expressions of Eqs. (55a), (55b), and (55e) are solved by the standard dark energy (cosmological constant) dominated background (38) where $H_* = \sqrt{\frac{V_*}{3(1+f_*)}}$. The background part of the connection equation (55c) is identically zero, while the remaining leading-order part of the scalar field equation (42c) can then be satisfied in three ways.

First, for an arbitrary ϕ_* whereby $f'_* \neq 0$, $V'_* \neq 0$, the evolution of $\gamma_*(t)$ in (75) is given by

$$\gamma_*(t) = c_1 e^{-3H_* t} + \frac{2H_*}{3} + \frac{2V'_*}{9H_* f'_*}. \tag{83}$$

Substituting the background values (38) and (83) into the first-order perturbed parts of (55) gives a system of equations to determine the behavior of perturbations, which, at leading order, turn out to be

$$x(t) \sim e^{-3H_* t}, \qquad h(t) \sim e^{-3H_* t},$$
$$g(t) \sim t e^{-3H_* t}, \qquad r(t) \sim e^{-3H_* t}. \tag{84}$$

This regime is stable.

Second, if ϕ_* is fixed by $V'_* = 0$, but $f'_* \neq 0$, the time dependence of the connection function $\gamma_*(t)$ remains as in (83), and the first-order perturbations at leading order behave exactly as in (84).

Finally, if the model functions $f(\phi)$ and $V(\phi)$ both have an extremum at the same value of ϕ and the respective derivatives vanish, $f'_* = 0$, $V'_* = 0$, the background equations leave $\gamma_*(t)$ undetermined. We can integrate the first-order perturbed metric and matter equations to find

$$h(t) \sim e^{-3H_* t}, \qquad r(t) \sim e^{-3H_* t}. \tag{85}$$

As the first-order perturbed connection equation is identically satisfied, we can invoke the second-order perturbed connection equation, which is solved by

$$x(t) = \pm\sqrt{c_6 e^{-3H_* t} + c_7}. \tag{86}$$

Analogously to the previous cases of dust and radiation, the trajectories encounter a singularity with diverging $\dot{x}$ at a finite future moment if the initial values x_0 and $\dot{x}_0$ are of the opposite sign and $|\dot{x}_0| > 9H_*|x_0|$. Only the class of solutions with $c_7 = 0$ converges to ϕ_*. For those solutions, we can solve the perturbed scalar field equation to find

$$\gamma_*(t) = c_1 e^{-3H_* t} + \frac{2H_*}{3} - \frac{H_*}{2f''_*} + \frac{2V''_*}{9H_* f''_*}, \tag{87}$$

but the perturbations $g(t)$ remain undetermined and the stability of the system unclear.

Thus, combining the analysis results of all three eras, the cosmic history with connection set 2 can be stable only if there exists ϕ_* which satisfies $f'_* \neq 0$, $V'_* = V''_* = 0$. In such a scenario, the function $\gamma(t)$ decreases in time, and the solutions will converge to their respective GR behaviors. Otherwise, the presence of the extra connection function diverts the cosmic evolution from the standard path. In particular, if the extrema of the model functions $f(\phi)$ and potential $V(\phi)$ coincide, a wide class of initial conditions will lead to a singular behavior of the scalar field perturbations, which would probably be detrimental to the background dynamics.

C. Connection set 3

Applying the parametrization and expansion introduced in Sec. V on the cosmological equations (32) of connection set 3 yields

$$6(1+f_*)H_*^2(t)-2V_*-2\kappa^2\rho_*(t)+(12(1+f_*)H_*(t)h(t)+6f'_*H_*^2(t)x(t)-3f'_*\gamma_*(t)\dot{x}(t)-2V'_*x(t)-2\kappa^2r(t))=0, \tag{88a}$$

$$\begin{aligned}&4(1+f_*)\dot{H}_*(t)+6(1+f_*)H_*^2(t)-2V_*+2\kappa^2\mathrm{w}\rho_*(t)+(4f'_*x(t)\dot{H}_*(t)+12(1+f_*)H_*(t)h(t)+4(1+f_*)\dot{h}(t)\\&\quad+6f'_*H_*^2(t)x(t)+4f'_*H_*(t)\dot{x}(t)-f'_*\gamma_*(t)\dot{x}(t)-2V'_*x(t)+2\kappa^2\mathrm{w}r(t))=0,\end{aligned} \tag{88b}$$

$$\begin{aligned}&15f'_*H_*(t)\gamma_*(t)\dot{x}(t)+3f'_*\gamma_*(t)\ddot{x}(t)+6f'_*\dot{\gamma}_*(t)\dot{x}(t)+(15f''_*H_*(t)\gamma_*(t)x(t)\dot{x}(t)+3f''_*\gamma_*(t)x(t)\ddot{x}(t)\\&\quad+3f''_*\gamma_*(t)(\dot{x}(t))^2+6f''_*x(t)\dot{\gamma}_*(t)\dot{x}(t)+15f'_*H_*(t)g(t)\dot{x}(t)+15f'_*\gamma_*(t)h(t)\dot{x}(t)+3f'_*g(t)\ddot{x}(t)+6f'_*\dot{g}(t)\dot{x}(t))=0,\end{aligned} \tag{88c}$$

$$\begin{aligned}&3f'_*(\dot{\gamma}_*(t)-2H_*^2(t)+3H_*(t)\gamma_*(t))-2V'_*-(2V''_*x(t)+6f''_*H_*^2(t)x(t)-9f''_*H_*(t)\gamma_*(t)x(t)\\&\quad-3f''_*x(t)\dot{\gamma}_*(t)-9f'_*H_*(t)g(t)+12f'_*H_*(t)h(t)-9f'_*\gamma_*(t)h(t)-3f'_*\dot{g}(t)+6H_*(t)\dot{x}(t)+2\ddot{x}(t))=0,\end{aligned} \tag{88d}$$

$$\dot{\rho}_*(t)+3(1+\mathrm{w})H_*(t)\rho_*(t)+(3\mathrm{w}H_*(t)r(t)+3\mathrm{w}\rho_*(t)h(t)+3H_*(t)r(t)+3\rho_*(t)h(t)+\dot{r}(t))=0. \tag{88e}$$

As before, we will consider the stability of the equations in the case of different matter types separately. The calculations are rather similar to the dust matter case in the previous section, and we will just present the main results here.

1. Dust matter domination

In the regime of dust matter domination, we take $\mathrm{w}=0$ and assume V_* is negligible in comparison with $\rho(t)$. At the background level, the metric and matter equations (88a), (88b), and (88e) coincide with the GR cosmological equations (35) and are solved by the standard background evolution (36). The background part of the connection equation (88c) is identically zero, while in solving the scalar field equation (88d) at the background level,

$$3f'_*\left(\dot{\bar{\gamma}}_*+\frac{2\bar{\gamma}_*}{t}-\frac{8}{9t^2}\right)-2V'_*=0, \tag{89}$$

there are three options.

First, if $f'_*\neq 0$ and $V'_*\neq 0$, then to leading order the background connection function obeys

$$\bar{\gamma}_*=\frac{2V'_*t}{9f'_*}+\frac{8}{9t}+\frac{c_1}{t^2}. \tag{90}$$

Substituting this into the first-order small equations, we find that the perturbations evolve as

$$h\sim t^{-2},\qquad r\sim t^{-3},\qquad x\sim t^{-\frac{13}{3}},\qquad g\sim t^0, \tag{91}$$

and the configuration is marginally stable. Although $\bar{\gamma}_*$ increases in time, its effects are not visible since $\dot{\phi}$ decreases and, in the Friedmann equation, the combined term evolves as $\bar{\gamma}\dot{\phi}\sim t^{-\frac{13}{3}}$.

Second, if $f'_*\neq 0$ but $V'_*=0$, then to leading order

$$\bar{\gamma}_*\sim t^{-1}, \tag{92}$$

and the first-order small equations give

$$h\sim t^{-\frac{4}{3}},\qquad r\sim t^{-\frac{7}{3}},\qquad x\sim t^{-\frac{1}{3}},\qquad g\sim t^{\frac{2}{3}}. \tag{93}$$

This situation is unstable due to the growth of connection function perturbations $g(t)$, which, at some moment, would spoil the assumption that all perturbations are small. However, if, in addition, $V''_*=0$, then the leading-order solution becomes

$$h\sim t^{-\frac{4}{3}},\qquad r\sim t^{-\frac{7}{3}},\qquad x\sim t^{-\frac{1}{3}},\qquad g\sim t^{-\frac{4}{3}}, \tag{94}$$

and the regime can be considered to be stable instead.

Third, if the model functions allow a scalar field value where simultaneously $f'_*=0$ and $V'_*=0$, then the connection field equation is automatically satisfied at leading as well as first perturbative order, and we cannot determine $\bar{\gamma}_*$ from there. The other first-order small equations give

$$h\sim t^{-2},\qquad r\sim t^{-3}. \tag{95}$$

To find $\bar{\gamma}_*(t)$ and $x(t)$, we have the first perturbation of the scalar field equation (88d) and the second perturbation of the connection equation (88c) at our disposal. It is a coupled system for $\bar{\gamma}_*(t)$ and $x(t)$, and straightforward integration seems difficult. However, incidentally, the connection equation (88c), with $h(t)$ and $r(t)$ substituted in, can be solved by

$$\bar{\gamma}_*=\frac{c_9}{t^{\frac{5}{3}}\sqrt{|x||\dot{x}|}}. \tag{96}$$

We can substitute that expression into the remaining equation (88d), which yields

$$\ddot{x} = -\frac{\dot{x}(3c_9 f''_*(3t\dot{x} - 2x) + 4t^{\frac{2}{3}}(3V''_* t^2 x + 4f''_* x + 6t\dot{x})\sqrt{|x||\dot{x}|})}{3t(3c_9 f''_* x + 4t^{\frac{5}{3}}\dot{x}\sqrt{|x||\dot{x}|})}. \tag{97}$$

This is by far a more complicated equation than the corresponding equation (67c) which emerged for connection set 2. It is rather hard to solve analytically, but we can still discern the main characteristics by studying the equation in different limits, complemented by the sample phase portraits in Fig. 1. Since Eq. (97) depends explicitly on time t, the actual phase space is three dimensional. Furthermore, it depends on the values of the parameters V''_*, f''_*, and c_9. However, to gain a glimpse of the principal features of the dynamics, Figs. 1(b) and 1(c) present two illustrative phase portraits for fixed values of parameters and time that capture the qualitative behavior of the available cases.

In the limit $x \to 0$, we can expand (97) to get

$$\ddot{x} = -\frac{3c_9 f_*{}'' \sqrt{|\dot{x}|}}{4t^{\frac{5}{3}}\sqrt{x}} + \mathcal{O}(x^0), \tag{98}$$

and we see that the force diverges. Hence, the solutions experience a singularity, depicted by a red line between the regions (I) and (II) in Figs. 1(b) and 1(c). The direction of the force (accelerating or decelerating) depends on the signs of the parameters c_9 and f''_*. Next, in the limit $\dot{x} \to 0$, we get from (97)

$$\ddot{x} = \frac{2\dot{x}}{3t} + \mathcal{O}(\dot{x}^{\frac{3}{2}}), \tag{99}$$

which tells us that the standstill state $\dot{x} = 0$ is unstable and any small deviation from it will meet a force ("antifriction") pushing the solutions away. This is marked by a red line between the regions (I) and (III) in Figs. 1(b) and 1(c), and it is a common feature for all values of parameters. Thus, contrary to the set 2 case in Fig. 1(a), the solutions cannot become stabilized at some value of the scalar field. Finally, we notice that the expression (97) also has another string of singularities at

$$x_s = -\frac{16t^{\frac{10}{3}}\dot{x}^3}{9c_9^2(f''_*)^2}\,\mathrm{sign}(c_9 f''_*) \tag{100}$$

where the denominator on the rhs vanishes. This is shown as the red line between the regions (II) and (III) on the plots. This string of singularities can act as a source or sink for the neighboring trajectories, depending on the sign of the numerator in Eq. (97). The point where the numerator vanishes, and this singular curve switches between repeller and attractor behaviors, is marked by an enlarged dot on the plots.

In summary, for the case in Fig. 1(b), the available classes of solutions can be summarized as follows. First, if the initial conditions x_0 and $\dot{x}_0$ are of the same sign, i.e., regions (I), the solutions either started from an initial singularity at $x = 0$ or from an unstable state at $\dot{x} = 0$, and consistently flow away from the value ϕ_*. Alternatively, if x_0 and $\dot{x}_0$ are of opposite signs, either in region (II) with $|x| < |x_s|$ or region (III) with $|x| > |x_s|$, the solutions inevitably crash into a singularity where $\ddot{x}$ diverges. Thus, none of the solutions can actually manage to reach the point ϕ_* and stabilize there ($x = 0$, $\dot{x} = 0$).

For the case in Fig. 1(c), the available classes of solutions can be summarized in a similar manner. First, if the initial conditions x_0 and $\dot{x}_0$ are of the opposite sign, i.e., regions (I), the solutions started from the unstable state at $\dot{x} = 0$ and flowed towards the singularity of $|x| \to 0$, $|\dot{x}| \to \infty$. Alternatively, if x_0 and $\dot{x}_0$ are of the same sign, either in region (II) with $|x| < |x_s|$ or region (III) with $|x| > |x_s|$, the solutions inevitably crash into another singularity where $\ddot{x}$ diverges. Hence, none of the solutions can actually manage to reach the point ϕ_* and stabilize there. Therefore, we do not need to consider the scenario of $f'_* = 0$, $V'_* = 0$ any further [estimating $\bar{\gamma}(t)$, $g(t)$] but just conclude that this regime is unstable.

In summary, the dust matter dominated regime for connection set 3 can be stable if $f'_* \neq 0$ and $V'_* = V''_* = 0$. However, if a model allows a value ϕ_* where simultaneously $f'_* = 0$ and $V'_* = 0$, then a large class of solutions will likely face a singularity in finite time.

2. Radiation domination

In the radiation domination regime, we take the barotropic index $\mathrm{w} = \frac{1}{3}$ and assume V_* is negligible in comparison with $\rho(t)$. At the background level, the metric and matter equations (88a), (88b), and (88e) coincide with the GR cosmological equations (35) and are solved by the standard background evolution (37). The background part of the connection equation (88c) is identically zero, while in solving the scalar field equation (88d) at the background level,

$$3f'_*\left(\dot{\bar{\gamma}}_* + \frac{3\bar{\gamma}_*}{2t} - \frac{1}{2t^2}\right) - 2V'_* = 0, \tag{101}$$

there are again three possible cases.

First, if ϕ_* is arbitrary in the sense that $f'_* \neq 0$ and $V'_* \neq 0$, then the connection background solution is given by

$$\bar{\gamma}_*(t) = \frac{4V'_* t}{15 f'_*} + \frac{1}{t} + \frac{c_1}{t^{\frac{3}{2}}}, \tag{102}$$

while, taking into account only the leading orders, the perturbations evolve as

$$x(t) \sim t^{-\frac{7}{2}}, \quad h(t) \sim t^{-2}, \quad r(t) \sim t^{-3}, \quad g(t) \sim t^0. \tag{103}$$

This regime is only marginally stable.

Second, if $f'_* \neq 0$ but $V'_* = 0$, then

$$\bar{\gamma}_*(t) \sim t^{-1}, \tag{104}$$

and the perturbations solve the equations at leading order as

$$x(t) \sim t^{\frac{1}{2}}, \quad h(t) \sim t^{-2}, \quad r(t) \sim t^{-3}, \quad g(t) \sim t^{\frac{3}{2}}, \tag{105}$$

which is unstable. If, in addition, $V''_* = 0$, then

$$x(t) \sim t^{\frac{1}{2}}, \quad h(t) \sim t^{-2}, \quad r(t) \sim t^{-3}, \quad g(t) \sim t^{-\frac{1}{2}}, \tag{106}$$

but the regime is still unstable as the scalar field perturbations are not under control.

Third, if there exists ϕ_* such that $f'_* = 0$ and $V'_* = 0$, then we obtain

$$h(t) \sim t^{-2}, \qquad r(t) \sim t^{-3}. \tag{107}$$

Like in the dust case, the first-order perturbation of the connection equation (88c) is identically satisfied. To proceed, we turn to the second-order perturbation of the connection equation, which can be solved by

$$\bar{\gamma}_* = \frac{c_9}{t^{\frac{5}{4}} \sqrt{|x||\dot{x}|}}. \tag{108}$$

Substituting this into the first-order scalar field equation (88d) again yields a highly nonlinear equation

$$\ddot{x}(t) = -\frac{\dot{x}(3c_9 f''_*(2t\dot{x} - x) + 2t^{\frac{1}{4}}(3f''_* x + 4V''_* t^2 x + 6t\dot{x})\sqrt{|x||\dot{x}|})}{2t(3c_9 f''_* + 4t^{\frac{5}{4}}\dot{x}\sqrt{|x||\dot{x}|})}. \tag{109}$$

This is structurally analogous to Eq. (97) from the dust matter dominated case with only slightly differing numerical factors and one power of t. Thus, the phase portraits are qualitatively similar to Figs. 1(b) and 1(c), and so are the results of the analysis of the solutions. The conclusion is that the dynamics around $f'_* = 0$ and $V'_* = 0$ is unstable and can lead to a singular behavior.

In summary, we see that the radiation dominated regime is unstable, even the case of $f'_* \neq 0$ but $V'_* = V''_* = 0$. Again, if a model allows a value ϕ_* where simultaneously $f'_* = 0$ and $V'_* = 0$, then a large class of solutions will likely face a singularity in finite time.

3. Dark energy domination

In the dark energy domination regime, we assume V_* is much larger than $\rho(t)$ but keep the dust matter perturbations $r(t)$ with $\mathrm{w} = 0$. At the background level, the metric and matter equations (88a), (88b), and (88e) coincide with the GR cosmological equations (35) and are solved by the standard background evolution (38) where $H_* = \sqrt{\frac{V_*}{3(1+f_*)}}$. The background part of the connection equation (88c) is identically zero, while solving the scalar field equation (88d) at the background level,

$$3f'_*(\dot{\bar{\gamma}}_* + 3H_*\bar{\gamma}_* - 2H_*^2) - 2V'_* = 0, \tag{110}$$

again allows three possible cases.

First, if $f'_* \neq 0$ and $V'_* \neq 0$, then Eq. (110) is solved by

$$\bar{\gamma}_*(t) = c_1 e^{-3H_* t} + \frac{2H_*}{3} + \frac{2V'_*}{9H_* f'_*}, \tag{111}$$

and up to leading order, the perturbations evolve as

$$\begin{aligned} x(t) &\sim e^{-5H_* t}, \qquad h(t) \sim e^{-3H_* t}, \\ r(t) &\sim e^{-3H_* t}, \qquad g(t) \sim t e^{-3H_* t}, \end{aligned} \tag{112}$$

which is a stable situation.

Second, if $f'_* \neq 0$ but $V'_* = 0$, then some terms in the equations drop out, but the leading-order behavior of $\bar{\gamma}_*$ and the perturbations does not change compared to the above. Similarly, taking also $V''_* = 0$ retains the picture of convergence.

Third, if $f'_* = 0$ and $V'_* = 0$, we can solve the equations by the same means as in the analogous case of set 3 dust matter domination to find that the leading behavior is

$$r(t) \sim e^{-3H_* t}, \qquad h(t) \sim e^{-3H_* t}. \tag{113}$$

To obtain $\gamma_*(t)$, since the background connection equation (88c) is identically satisfied, we need the second-order small connection equation

$$\gamma_*(t) = \frac{c_9}{e^{\frac{5H_* t}{2}} \sqrt{|x||\dot{x}|}}. \tag{114}$$

Using the above equation (114) in the first-order small scalar field equation (88d), we again obtain a highly nonlinear differential equation

$$\ddot{x}(t) = -\frac{\dot{x}(3c_9 f_*''(\dot{x} - H_* x) + 4e^{\frac{5H_* t}{2}}(3f_*'' H_*^2 x + V_*'' x + 3H_*\dot{x})\sqrt{|x||\dot{x}|})}{3c_9 f_*'' x + 4e^{\frac{5H_* t}{2}}\dot{x}\sqrt{|x||\dot{x}|}}. \tag{115}$$

Apart from the details of the explicit time factors, this equation is analogous to Eqs. (97) and (109) in the dust and radiation cases. A closer analysis reveals the same key features of dynamical behavior. It is not possible for the system to relax at ϕ_* where $f_*' = 0$ and $V_*' = 0$, but the solutions either flow away from this value or the system encounters a singularity where $\ddot{x}$ diverges and the approximation of small perturbations breaks down.

In summary, the potential dominated regime is stable if $f_*' \neq 0$. On the other hand, for $f_*' = 0$ and $V_*' = 0$, large classes of solutions meet a finite singularity. In the end, combining the analysis results of all three eras, it seems a cosmic history from radiation to dust to dark energy domination cannot be realized in a stable manner in the vicinity of some fixed scalar field value ϕ_*. Dust matter and potential domination eras are stable around $f_*' \neq 0$, $V_*' = V_*'' = 0$, but this is not the case in the radiation domination epoch.

VII. DISCUSSION

In this paper, we have explored the cosmological implications of alternative FLRW connections that become available in symmetric teleparallel geometry, focusing on the analogue of scalar-tensor gravity where a scalar field is nonminimally coupled to the nonmetricity scalar in the action (12). Demanding that the independent connection with zero curvature and zero torsion obeys the symmetries of spatial homogeneity and isotropy was recently shown to yield three sets of connections which involve an extra free function of time [45,46]. In the first set, this extra function actually drops out of the cosmological equations, and the system reduces to the one already known to correspond to a trivial connection and also to coincide with the scalar-torsion case in metric teleparallelism, which has been studied a great deal. In the two other classes, however, the extra function is present in the equations and could be interpreted as an instance of an extra degree of freedom that has been notoriously difficult to pinpoint in extended teleparallel gravity before.

In Sec. IV, we presented the cosmological field equations arising from these connections and confirmed that the new function does indeed increase the number of independent phase space dimensions by one; thus, it is not a constraint in disguise. We also observed, from the Friedmann equations, that the new function cannot itself take the role of dark energy or dark matter; rather, it behaves as a stiff fluid or spatial curvature in connection set 2 or 3, respectively. Furthermore, in the Friedmann equations, the extra function only appears if the nonminimal coupling $\mathcal{A}(\Phi)$ between the scalar field and nonmetricity is not constant and the time derivative of the scalar field is not zero. Hence, it has no effect in the case of minimally coupled fields or when the dynamics of the scalar field has stopped, i.e., in the cases when the model is equivalent to GR. In addition, we also found that the extra function drastically modifies the scalar field dynamics since the connection equation can be viewed as a dynamical equation for the scalar field, albeit without any contribution from the kinetic coupling $\mathcal{B}(\Phi)$ or scalar potential $\mathcal{V}(\Phi)$; see Eqs. (30) and (33), which look quite puzzling. Moreover, these equations could potentially push the scalar field into a singular state (infinite $\ddot{\Phi}$) when the nonminimal coupling function has an extremum ($d\mathcal{A}/d\Phi = 0$), again a rather problematic feature, although possible to mitigate with monotonic functions. Though difficult to state in full generality for arbitrary model functions, in the cosmological equations, it is not obvious how the extra function in the connection could offer new options to generate dark energy, besides the well-known regime of a slowly evolving scalar field with a positive potential acting similarly to a cosmological constant.

Owing to the strange features that the alternative connections introduce into the system, we proceeded in Secs. V and VI to study whether and under which conditions the standard cosmological eras are stable in the model, i.e., that the succession of eras is not disturbed by factors other than the densities of different matter components decreasing at different rates as the universe expands. In the scalar-tensor context, the scalar field value should not change too much either, in order to satisfy the observational constraints. The vague expectation is that the current tension in the data could eventually be explained by suitable convergence processes or oscillations around the standard scenario. Thus, we expanded the scalar-nonmetricity FLRW equations around the ΛCDM background with the radiation domination, dust matter domination, and potential domination assumptions, solved them explicitly, and determined the asymptotic behavior of all quantities. In a stable situation, all perturbations should decrease in time. In essence, the question is whether the "attractor mechanism" that is known in Riemannian scalar-tensor cosmology [11,15–24] has an analogue in the symmetric teleparallel counterpart.

For connection set 1, the investigation can be carried over from the metric teleparallel case which has the same equations [91]. In the parametrization where $\mathcal{A}=1+f(\phi)$, $\mathcal{B}=1$, $\mathcal{V}=V(\phi)$, all three eras are stable if the model functions allow a value ϕ_* which corresponds to a simultaneous minimum of the gravitational coupling function ($f'_*=0$, $f''_*>0$) and a minimum or at least an inflection point of the potential ($V'_*=0, V''_*\geq 0$). It is notable that, while in the Riemannian scalar-tensor case during the radiation era the scalar field stabilizes to an arbitrary value which can be very different from the value it will be drawn to in the matter and potential domination eras (thus possibly causing a large drift in the gravitational constant at the beginning of matter domination), in the symmetric teleparallel case of connection set 1, the scalar field value would then remain stable since the radiation era.

For the alternative connection sets, our results are new and can be summarized as follows. The cosmic history with connection set 2 can be stable through all three eras only if there exists ϕ_* which satisfies $f'_*\neq 0$, $V'_*=V''_*=0$. In such a scenario, the extra connection function decreases in time, and the solutions converge to their respective GR behaviors. Otherwise, the presence of the extra connection function diverts the cosmic evolution from the standard path. In contrast, for connection set 3, the dust matter and potential domination eras are stable around $f'_*\neq 0$, $V'_*=V''_*=0$, but there is no stable configuration for the radiation domination epoch. At face value, these stability properties seem adequate, although in comparison with connection set 1, it is unusual that the minima of the coupling function or potential do not figure into the stability conditions. Further investigation, however, uncovers a deeper problem. The gradient of the potential that would normally act as a force term in the evolution equation has no role in determining the scalar field dynamics; see Eqs. (30) and (33). The scalar field only experiences friction and antifriction types of influence and can only rather randomly end up at the point of stability. On the contrary, the scalar field can experience a singularity if the extrema of the model functions coincide, $f'(\phi_s)=V'(\phi_s)$. This feature is illustrated in the phase portraits of Fig. 1 and probably means that the other quantities become singular as well.

In conclusion, the alternative FLRW connections cannot be deemed outright pathological and do not make the universe definitely unstable, but they have a very strange and possibly dangerous influence on the scalar field dynamics nevertheless. Further studies are needed to understand whether this influence is overwhelmingly harmful or whether it could be useful in describing some phenomena in the end. We might find that, although compatible with FLRW symmetry, the alternative connections are eventually ruled out when we learn how to correctly implement the boundary term in the actions of extended teleparallel gravities to fix the connection.

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