Triangular solution to the planar elliptic three-body problem in the parametrized post-Newtonian formalism

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A triangular solution [Phys. Rev. D **107**, 044005 (2023)] has recently been found to the planar circular three-body problem in the parametrized post-Newtonian (PPN) formalism, for which they focus on a class of fully conservative theories characterized by the Eddington-Robertson parameters β and γ. The present paper extends the PPN triangular solution to quasielliptic motion, for which the shape of the triangular configuration changes with time at the PPN order. The periastron shift due to the PPN effects is also obtained.

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I. INTRODUCTION

The three-body problem is among the classical ones in physics, which led to a study of the chaos [1]. Particular solutions, notably Euler's collinear solution and Lagrange's equilateral one [2,3], represent regular orbits, which have attracted a lot of interest, e.g., [4–8].

Nordtvedt [9] pointed out that the position of the triangular points is very sensitive to the ratio between the gravitational mass and the inertial one in gravitational experimental tests, though the post-Newtonian (PN) terms are partly considered.

For the restricted three-body problem in the PN approximation, Krefetz [10] and Maindl [11] found the PN triangular configuration for a general mass ratio between two masses. These studies were extended to the PN three-body problem for general masses [12–17], where the PN counterparts for Euler's collinear [12,13] and Lagrange's equilateral solutions [14,15] were found. It should be noted that the PN triangular solutions are not necessarily equilateral for general mass ratios, and they are equilateral only for either the equal mass case or two test masses. The stability of the PN solution and the radiation reaction at 2.5PN order were also studied [16,17].

In a scalar-tensor theory of gravity, a collinear configuration for three-body problem was discussed [18]. In addition to such fully classical treatments, a possible quantum gravity correction to the Lagrange points was argued [19,20].

Moreover, the recent discovery of a relativistic hierarchical triple system including a neutron star [21] has sparked renewed interest in the relativistic three-body problem and the related gravitational experiments [22–24].

In the parametrized post-Newtonian (PPN) formalism [25], collinear and triangular solutions to the planar circular three-body problem have recently been found [26], where they focus on a class of fully conservative theories characterized by the Eddington-Robertson parameters β and γ, because the two parameters are the most important ones; β measures how much nonlinearity there is in the superposition law for gravity, and γ measures how much space curvature is produced by unit rest mass [27,28]; see, e.g., [29] for the celestial mechanics in this class of PPN theories.

In the Newtonian gravity, triangular solutions are not only to the circular three-body problem but also to the elliptic one [2,30]. Can a (quasi)elliptic orbit of triangular solutions be found in PPN case? A point is that the PPN force seems to be too complicated to admit elliptic orbits for a triple system. The main purpose of the present paper is to find it in the class of fully conservative theories.

This paper is organized as follows. In Sec. II, basic methods and equations are presented. Section III discusses the PPN triangular solution to the planar elliptic three-body problem. Section V summarizes this paper. Throughout this paper, $G = c = 1$. A, B, and $C \in \{1, 2, 3\}$ label three masses.

II. BASIC METHODS AND EQUATIONS

A. Newtonian planar elliptic triangular solution

Let us begin by briefly summarizing the triangular solution to the Newtonian planar elliptic three-body problem [2,30]. A homothetic solution is possible, and it represents the Lagrange equilateral solution in elliptic motion; see, e.g., Sec. 5 of Ref. [2] for more detail. We see that the PN triangular solutions are not necessarily equilateral, mainly because of the velocity-dependent force at the PN order as shown in Sec. III.

The equation of motion (EOM) for three masses (M_A at the position $\boldsymbol{R}_A$) reads

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$$M_A \boldsymbol{a}_A = -\sum_{B=1}^{N} \frac{M_A M_B}{(R_{AB})^2} \boldsymbol{n}_{AB}, \tag{1}$$

where $\boldsymbol{a}_A$ denotes the acceleration of the A-th mass, $\boldsymbol{R}_{AB} \equiv \boldsymbol{R}_A - \boldsymbol{R}_B$, $R_{AB} \equiv |\boldsymbol{R}_{AB}|$, and $\boldsymbol{n}_{AB} \equiv \boldsymbol{R}_{AB}/R_{AB}$.

By taking the cross product of $\boldsymbol{R}_1$ and Eq. (1) for $A = 1$, we obtain

$$\boldsymbol{R}_1 \times \boldsymbol{R}_2 \left(\frac{1}{(R_{12})^3} - \frac{1}{(R_{31})^3} \right) = 0, \tag{2}$$

where the coordinate center is chosen as the center of mass of $\sum_A M_A \boldsymbol{R}_A = 0$. For a triangular configuration, $\boldsymbol{R}_1 \nparallel \boldsymbol{R}_2$. From Eq. (2), we thus obtain $R_{12} = R_{23}$. By cyclic arguments, we obtain an equilateral solution [2,30].

In elliptic motion, the arm length R_A becomes $R_1 = af_N\sqrt{\nu_2^2 + \nu_2\nu_3 + \nu_3^2}$, $R_2 = af_N\sqrt{\nu_3^2 + \nu_3\nu_1 + \nu_1^2}$, and $R_3 = af_N\sqrt{\nu_1^2 + \nu_1\nu_2 + \nu_2^2}$, where the total mass is $M \equiv \sum_A M_A$, the mass ratio is defined as $\nu_A \equiv M_A/M$, a is some constant, and f_N denotes the dilation factor [2,14,15,30]. In circular motion, $f_N = 1$, while f_N is a function of time in elliptic motion [2,30].

From the total energy and angular momentum, an elliptic orbit is obtained as [2,30]

$$f_{\mathrm{N}} = \frac{\mathcal{A}_{\mathrm{N}}(1 - e_{\mathrm{N}}^2)}{1 + e_{\mathrm{N}} \cos\theta}, \tag{3}$$

where θ denotes the true anomaly, e_{N} is the eccentricity of the elliptic orbit as $e_{\mathrm{N}} = \sqrt{1 + 2L_{\mathrm{N}}^2 \mathcal{E}_{\mathrm{N}} M^{-2} \mu^{-3}}$ for the total energy $\mathcal{E}_{\mathrm{N}}$, the total angular momentum L_{N} and $\mu \equiv M(\nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1)$, and $\mathcal{A}_{\mathrm{N}} \equiv -\mu M/(2a\mathcal{E}_{\mathrm{N}})$. Here, $\theta = 0$ is chosen as the periastron.

For simplicity, we refer to $A \equiv a\mathcal{A}_{\mathrm{N}}$ as the semimajor axis and $P \equiv a\mathcal{A}_{\mathrm{N}}(1 - e_{\mathrm{N}}^2)$ as the semilatus rectum. For instance, the semimajor axis for the elliptic orbit of M_1 is $a\mathcal{A}_{\mathrm{N}}\sqrt{\nu_2^2 + \nu_2\nu_3 + \nu_3^2}$.

From the total angular momentum, the angular velocity ω_{N} of the triangular configuration is obtained as

$$\omega_{\mathrm{N}} = (1 + e_{\mathrm{N}} \cos\theta)^2 \sqrt{\frac{M}{P^3}}. \tag{4}$$

All the above relations are reduced to Keplerian orbits in the restricted three-body problem (e.g., $\nu_3 \to 0$).

B. EOM in the PPN formalism

In a class of fully conservative theories including only the Eddington-Robertson parameters β and γ, the PPN EOM becomes [27,28]

$$\begin{aligned} \boldsymbol{a}_A = & -\sum_{B\neq A} \frac{M_B}{R_{AB}^2} \boldsymbol{n}_{AB} - \sum_{B\neq A} \frac{M_B}{R_{AB}^2} \left\{ \gamma v_A^2 - 2(\gamma+1)(\boldsymbol{v}_A \cdot \boldsymbol{v}_B) + (\gamma+1) v_B^2 - \frac{3}{2}(\boldsymbol{n}_{AB} \cdot \boldsymbol{v}_B)^2 - (2\gamma + 2\beta + 1)\frac{M_A}{R_{AB}} - 2(\gamma+\beta)\frac{M_B}{R_{AB}} \right\} \boldsymbol{n}_{AB} \\ & + \sum_{B\neq A} \frac{M_B}{R_{AB}^2} \{ \boldsymbol{n}_{AB} \cdot [2(\gamma+1)\boldsymbol{v}_A - (2\gamma+1)\boldsymbol{v}_B] \} (\boldsymbol{v}_A - \boldsymbol{v}_B) + \sum_{B\neq A} \sum_{C\neq A,B} \frac{M_B M_C}{R_{AB}^2} \left[\frac{2(\gamma+\beta)}{R_{AC}} + \frac{2\beta - 1}{R_{BC}} - \frac{1}{2}\frac{R_{AB}}{R_{BC}^2} (\boldsymbol{n}_{AB} \cdot \boldsymbol{n}_{BC}) \right] \boldsymbol{n}_{AB} \\ & - \frac{1}{2}(4\gamma+3) \sum_{B\neq A} \sum_{C\neq A,B} \frac{M_B M_C}{R_{AB} R_{BC}^2} \boldsymbol{n}_{BC} + O(c^{-4}), \end{aligned} \tag{5}$$

where $\boldsymbol{v}_A$ denotes the velocity of the A-th mass.

III. PPN PLANAR ELLIPTIC TRIANGULAR SOLUTION

A. PPN planar elliptic orbit

In order to obtain a PPN solution as a perturbation around the Newtonian equilateral elliptic solution, we assume a quasicommon dilation as $R_{AB} = af(1 + \varepsilon_{AB})$ for three masses, where ε_{AB} denotes a PPN distortion. The perfectly common dilation occurs at the Newton order, whereas the dilation is not common by ε_{AB}; see also Fig. 1.

In the same way as deriving Eq. (2), we take the cross product of $\boldsymbol{R}_1$ and Eq. (5) for M_1 to obtain

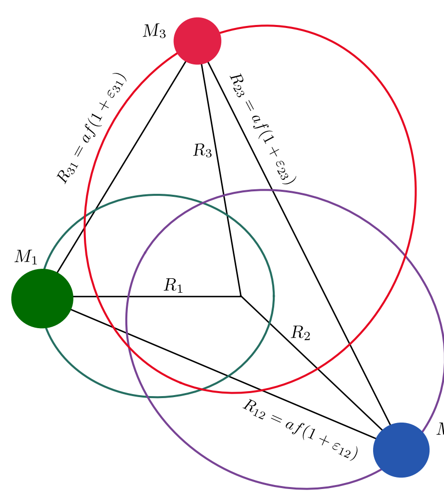

FIG. 1. Schematic figure for the PPN triangular configuration of three masses. The inequilateral triangle is characterized by ε_{AB}. In the Newtonian limit, ε_{AB} vanishes, and R_{AB} becomes af_{N}.

$$
\begin{aligned}
\ell_1^2 \frac{d}{dt}(f^2\omega)(\boldsymbol{\lambda}\times\boldsymbol{\rho}) = (\boldsymbol{\lambda}\times\boldsymbol{\rho})\Bigg\{ & -\frac{\sqrt{3}}{2}\frac{M}{f_{\rm N}a}\nu_2\nu_3\Bigg[3(\varepsilon_{12}-\varepsilon_{31}) + \frac{M}{2a}(\nu_3-\nu_2)\left(\frac{1}{f_{\rm N}}-\frac{1}{\mathcal{A}_{\rm N}}\right) \\
& + \frac{3}{8}a^2\{\dot{f}_{\rm N}(1+3\nu_1) + \sqrt{3}f_{\rm N}\omega_{\rm N}(1-\nu_1-2\nu_2)\}\{\dot{f}_{\rm N}(1-\nu_1-2\nu_2) + \sqrt{3}f_{\rm N}\omega_{\rm N}(1-\nu_1)\} \\
& - \frac{M}{4f_{\rm N}a}(\nu_2-\nu_3)(8\beta-3)\Bigg] - \frac{\sqrt{3}}{4}Ma\nu_2\left(\nu_3\frac{\dot{f}_{\rm N}}{f_{\rm N}} + \frac{\omega_{\rm N}}{\sqrt{3}}(\nu_1-\nu_2-1)\right) \\
& \times((4\gamma+3+\nu_2-\nu_1)f_{\rm N} - \sqrt{3}\nu_3 f_{\rm N}\omega_{\rm N}) + \frac{\sqrt{3}}{4}Ma\nu_3\left(\nu_2\frac{\dot{f}_{\rm N}}{f_{\rm N}} - \frac{\omega_{\rm N}}{\sqrt{3}}(\nu_1-\nu_3-1)\right) \\
& \times((4\gamma+3+\nu_3-\nu_1)\dot{f}_{\rm N} + \sqrt{3}\nu_2 f_{\rm N}\omega_{\rm N})\Bigg\} + O(c^{-4}),
\end{aligned} \tag{6}
$$

where $\ell_1 \equiv a\sqrt{\nu_2^2+\nu_2\nu_3+\nu_3^2}$ [2,15,26,30], and we introduce an orthonormal basis $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$. Here, $\boldsymbol{\lambda} \equiv \boldsymbol{R}_1/R_1$, and $\boldsymbol{\rho}$ is the 90 degree rotation of $\boldsymbol{\lambda}$. It is more convenient to use the orthonormal basis than $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$, because the right-hand side of Eq. (5) relies upon not only the positions but also the velocities. In elliptic motion, the velocity is not always orthogonal to the position vector, though it is in circular motion.

From the PPN total angular momentum, we find

$$
\frac{d}{dt}(f^2\omega) = -\frac{M\dot{f}_{\rm N}\omega_{\rm N}}{4a}\frac{13\nu_1\nu_2\nu_3 - 8\{(\gamma+1)\eta-\zeta\}}{\eta} + O(c^{-4}), \tag{7}
$$

where the dot denotes the time derivative, and we denote $\eta \equiv \nu_1\nu_2+\nu_2\nu_3+\nu_3\nu_1$ and $\zeta \equiv \nu_1^2\nu_2^2+\nu_2^2\nu_3^2+\nu_3^2\nu_1^2$. Equation (7) is reduced to $d(f_N\omega_{\rm N}^2)/dt = 0$ in the Newtonian limit, which recovers the Newtonian case of the planar elliptic triangular solution. It follows that Eq. (7) can be derived also from the sum of Eq. (6) for $A = 1, 2, 3$.

By substituting Eq. (7) into the left-hand side of Eq. (6), we obtain

$$
\begin{aligned}
\varepsilon_{12}-\varepsilon_{31} = & \frac{M}{8A}(\nu_3-\nu_2)(3\nu_1+1) - \frac{M}{12P}(\nu_3-\nu_2)(9\nu_1+8\beta-2)(1+e_{\rm N}\cos\theta) \\
& + \frac{M}{4P}(\nu_3-\nu_2)(3\nu_1-1)(1+e_{\rm N}\cos\theta)^2 - \frac{\sqrt{3}e_{\rm N}M}{72\nu_2\nu_3 P}\sin\theta(1+e_{\rm N}\cos\theta) \\
& \times\Bigg[34\nu_1\nu_2\nu_3 + 16\nu_1(\nu_2^2+\nu_3^2) + 9\nu_2\nu_3\{1-3\nu_1^2+(\nu_2-\nu_3)^2\} \\
& - \frac{4(\nu_2^2+\nu_2\nu_3+\nu_3^2)(13\nu_1\nu_2\nu_3+8\zeta)}{\eta}\Bigg] + O(c^{-4}),
\end{aligned} \tag{8}
$$

where Eq. (3) is used for f_N. By cyclic arguments, $\varepsilon_{23}-\varepsilon_{12}$ and $\varepsilon_{31}-\varepsilon_{23}$ are obtained.

Following Ref. [15], the gauge fixing is chosen as $\varepsilon_{12}+\varepsilon_{23}+\varepsilon_{31} = 0$, for which the PN triangular area remains the same as the Newtonian one. From this gauge fixing and Eq. (8), we obtain

$$
\begin{aligned}
\varepsilon_{12} = & \frac{M}{24A}[3\{\nu_1(\nu_3-2\nu_2)+\nu_3(1+\nu_2)\}-1] \\
& - \frac{M}{36P}[2(4\beta-1)(3\nu_3-1) + 9\{\nu_1(\nu_3-\nu_2)+\nu_2(\nu_3-\nu_1)\}](1+e_{\rm N}\cos\theta) \\
& + \frac{M}{12P}[1-3(\nu_3^2+2\nu_1\nu_2)](1+e_{\rm N}\cos\theta)^2 - \frac{e_{\rm N}M\sqrt{3}}{108P}(\nu_1-\nu_2)\sin\theta(1+e_{\rm N}\cos\theta) \\
& \times\left[\frac{8\nu_3(1-\nu_3)}{\nu_1\nu_2} + 27\nu_3 - 1 + \frac{2(\nu_1\nu_2-\nu_3^2)(13\nu_1\nu_2\nu_3+8\zeta)}{\nu_1\nu_2\nu_3\eta}\right] + O(c^{-4}).
\end{aligned} \tag{9}
$$

It is worthwhile to mention that γ makes no contribution to Eq. (9), while β is included in it. This means that the PPN nonliearity parameter β affects the asymmetric shape of the PN triangle, whereas γ does not affect the asymmetry.

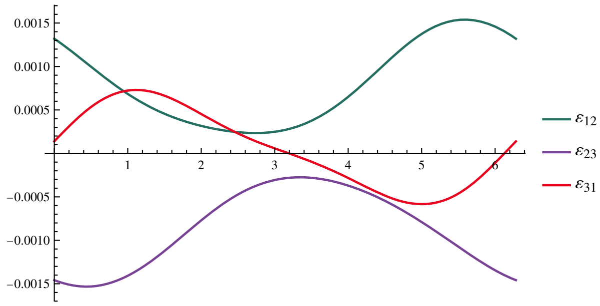

FIG. 2. ε_{12}, ε_{23}, ε_{31} for $\nu_1 = 1/2$, $\nu_2 = 1/3$, $\nu_3 = 1/6$ and $M/a = 0.01$ in elliptic motion with $e_{\rm N} = 0.5$. The horizontal axis denotes θ from a periastron ($\theta = 0$) to the next periastron ($\theta = 2\pi$).

We can obtain also ε_{23} and ε_{31} cyclically. We thus obtain the PPN triangular quasielliptic solution. See Fig. 2 for $\varepsilon_{12}, \varepsilon_{23}$ and ε_{31} for an elliptic orbit. Note that this solution does not follow a perfectly elliptic motion owing to the periastron shift as shown below, whereas the Newtonian counterpart is elliptic. For instance, the periastron position in the PPN orbit moves significantly in a long time scale. Namely, the obtained solution represents an osculating orbit [2,30].

B. Periastron shift

After direct calculations, the PPN expression of the total energy [31,32] for the PPN planar quasielliptic triangular solution can be rewritten as

$$\left(\frac{du}{d\theta}\right)^2 + G(u)\frac{du}{d\theta} = F(u), \tag{10}$$

where $u \equiv 1/f$. $F(u)$ and $G(u)$ are functions of u, which are too long to write down in this paper.

The periastron shift is

$$\theta_{\rm PPN} = \int_{u_{\rm min}}^{u_{\rm max}} du \frac{1}{\left(\frac{du}{d\theta}\right)} - \pi, \tag{11}$$

where $u_{\rm max}$ and $u_{\rm min}$ correspond to the apoapsis and periapsis, respectively.

In the same way as the post-Newtonian calculations of the periastron shift [27,28], by using Eq. (10) for Eq. (11), we obtain the periastron shift at the PPN order as

$$\theta_{\rm PPN} = \frac{\pi M}{36 P\eta}[18\nu_1\nu_2\nu_3(9-2\beta) + \eta(65 - 44\beta + 72\gamma) + 36\zeta] + O(c^{-4}). \tag{12}$$

The periastron shift per orbital period is $2\theta_{\rm PPN}$. In General Relativity ($\beta = \gamma = 1$), Eq. (12) becomes

$$\theta_{\rm PPN} = \frac{\pi M}{36 P\eta}(126\nu_1\nu_2\nu_3 + 93\eta + 36\zeta) + O(c^{-4}). \tag{13}$$

In the test particle limit of a third mass ($\nu_3 \to 0$), Eq. (12) disagrees with that of a binary system, because the restricted three-body dynamics does not equal to the binary dynamics [2,15,30]; see, e.g., Eq. (66) in Ref. [27] and Eq. (13.51) in [28] for the PPN periastron shift formula of a binary case.

IV. CONCLUSION

We found a PPN triangular solution to the planar elliptic three-body problem in a class of fully conservative theories. The distortion function ε_{AB} of a triangular solution depends on β but not on γ. It follows that, in the circular limit, the present solution recovers the PPN triangular circular solution in Ref. [26]. In the limit of $e_{\rm N} \to 0$, Eq. (9) agrees with Eq. (41) in [26].

The periastron shift of the PPN triangular solution was also obtained. Because of the three-body interactions, the periastron shift in the PPN triangular solution is different from that of a binary system.

There are potential observational tests for the above models. One is the monitoring of an artificial satellite at (or around) L_4 (or L_5) of the Sun-Jupiter system (or Sun-Earth system), if such a satellite is launched. It could allow to test the relativistic three-body gravity through the measurement of β and γ, though it is technically difficult.

The other is to find a hypothetical object of a relativistic triangular system composed from, e.g., two black holes and a neutron star. If the two black holes are much heavier than the neutron star, the triple system is likely to be stable, though its formation process is unclear.

It is left for future to study the stability of the present solution.

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