

SO(2, n)-compatible embeddings of conformally flat n-dimensional submanifolds in \mathbb{R}^{n+2} E. Huguet^{1,*}, J. Queva^{2,†} and J. Renaud^{3,‡}¹*Université Paris Cité, APC-Astroparticule et Cosmologie (UMR-CNRS 7164), Batiment Condorcet, 10 Rue Alice Domon et Léonie Duquet, F-75205 Paris Cedex 13, France*²*Université de Corse—CNRS UMR 6134 SPE, Campus Grimaldi BP 52, 20250 Corte, France*³*Université Gustave Eiffel, APC-Astroparticule et Cosmologie (UMR-CNRS 7164), Batiment Condorcet, 10 Rue Alice Domon et Léonie Duquet, F-75205 Paris Cedex 13, France*

(Received 7 December 2023; accepted 8 February 2024; published 18 March 2024)

We describe embeddings of n -dimensional Lorentzian manifolds, including Friedmann-Lemaître-Robertson-Walker spaces, in \mathbb{R}^{n+2} such that the metrics of the submanifolds are inherited by a restriction from that of \mathbb{R}^{n+2} , and the action of the linear group $\text{SO}(2, n)$ of the ambient space reduces to conformal transformations on the submanifolds.

DOI: [10.1103/PhysRevD.109.064054](https://doi.org/10.1103/PhysRevD.109.064054)**I. INTRODUCTION**

It is well known that de Sitter space of dimension n can be obtained as a hyperboloid embedded in a \mathbb{R}^{n+1} Minkowski space. This short note presents an observation to realize more general cosmological Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes in a similar fashion in \mathbb{R}^{n+2} , compatible with the action of the conformal group $\text{SO}(2, n)$.

Embeddings of conformally flat n -dimensional Lorentzian manifolds in higher-dimensional flat spacetimes have been considered in the past in various contexts [1–7]. More often than not, these embeddings use convenient coordinate systems, in which the metric tensor of the embedded manifold can be extracted from the ambient metric by applying suitably chosen constraints. The advantage of such a method is that it is fitted for specific practical applications. By contrast, some aspects, such as the action of the linear conformal group (when applicable) or a global geometric view, are much less apparent.

In this paper, in a coordinate-free approach, we build the embedding in \mathbb{R}^{n+2} of n -dimensional conformally flat spaces, hereafter denoted by W , including FLRW spacetimes as special cases. While explicit embedding formulas date back to FLRW space's infancy [8,9] and were rediscovered unknowingly later [10], here the embedding is as natural as possible, meaning that the metric on W is the restriction of the \mathbb{R}^{n+2} metric, and the action of the linear group $\text{SO}(2, n)$ of \mathbb{R}^{n+2} reduces to conformal transformations on W . The present work originates from

the need of a geometric (coordinate-free) framework for the generalization of previous works regarding the restriction to submanifolds of differential operators—in particular, the Laplace operator [11].

The geometric setting with its definitions and conventions is exposed in Sec. II. The embeddings follow from Proposition 1, proved in Sec. III. The inverse problem, extensions, and relations to previous works are discussed in Sec. IV.

II. GEOMETRIC SETTING

Let \mathbb{R}^{n+2} be a pseudo-Euclidean space of dimension $n + 2$ equipped with the metric $\eta = + - \cdots - +$. Throughout the paper, $\mu, \nu, \dots = 0, \dots, n - 1$ are related to n -dimensional manifolds, and $\alpha, \beta, \dots = 0, \dots, n + 1$ to \mathbb{R}^{n+2} . The canonical coordinates of a point y of \mathbb{R}^{n+2} are denoted $\{y^\alpha\}$, and the associated Cartesian orthonormal basis is denoted $\{\partial_\alpha\}$. We denote by D the dilation operator; in y coordinates it reads $D = y^\alpha \partial_\alpha$.

We will denote by X_h the n -dimensional manifold obtained as the intersection of the $(n + 1)$ -dimensional null cone of \mathbb{R}^{n+2} : $\mathcal{C} := \{y \in \mathbb{R}^{n+2}; C(y) := y^\alpha y_\alpha = 0\}$, and the surface $P_h := \{y \in \mathbb{R}^{n+2}; h(y) = 1\}$, h being a homogeneous function of degree one.

III. EMBEDDINGS

The $\text{SO}(2, n)$ -compatible embedding of (n -dimensional) FLRW spaces in \mathbb{R}^{n+2} is a special case of the following theorem:

Theorem 1. Let \mathbb{R}^{n+2} be the $n + 2$ dimensional real space with the pseudo-Euclidean metric η with signature $(2, n)$. Let f and l be two homogeneous functions of degrees one and zero, respectively, and $k = e^{-l}f$. Let g^f be the induced metric from \mathbb{R}^{n+2} to X_f , with g^k having the same relation

*huguet@apc.univ-paris7.fr

†queva@univ-corse.fr

‡jacques.renaud@u-pem.fr

with respect to X_k . Then, $g^k = e^{2l}\tilde{g}^f$, where \tilde{g}^f is a metric on X_k which is isometric to g^f . Moreover, the elements of the linear group $\text{SO}(2, n)$ on \mathbb{R}^{n+2} act on X_k and X_f as conformal transformations.

Proof. Let

$$\begin{aligned} \Lambda: \mathbb{R}^{n+2} &\rightarrow \mathbb{R}^{n+2} \\ y &\mapsto e^{l(y)}y. \end{aligned}$$

We clearly have $\Lambda X_f = X_k$, since, for $y \in X_f$, one has $e^{l(y)}y = e^{l(y)}y/f(y) = y/k(y)$ and $k(y/k(y)) = 1$. Moreover, Λ induces a diffeomorphism Λ_r between X_f and X_k .

In order to determine the relation between g^f and g^k , the metrics induced from the ambient space on X_f and on X_k , respectively, let us consider $V \in T_y\mathbb{R}^{n+2}$. A straightforward calculation from the definition of the push forward leads to

$$\Lambda_*V = e^l(V + \langle dl, V \rangle D).$$

This map induces the map Λ_{r*} between TX_f and TX_k , with the same expression on a vector field of TX_f .

Now, let $U' = \Lambda_{r*}U$, $V' = \Lambda_{r*}V$, where $U, V \in TX_f$, and let m and n be the canonical injections from X_f and X_k , respectively, in \mathbb{R}^{n+2} . Then, noting that $\eta(D, U) = \eta(D, V) = \eta(D, D) = 0$, one has successively

$$\begin{aligned} g_{\Lambda(y)}^k(U', V') &= (n^*\eta)_{\Lambda(y)}(U', V') \\ &= \eta(\Lambda_{r*}U, \Lambda_{r*}V) \\ &= \eta(e^l(V + \langle dl, V \rangle D), e^l(V + \langle dl, V \rangle D)) \\ &= e^{2l}\eta(U, V) \\ &= e^{2l}\eta(m_*U, m_*V) \\ &= e^{2l}m^*\eta(U, V) \\ &= e^{2l}g^f(U, V). \end{aligned}$$

The map Λ_{r*} is thus an isometry between $(X_f, e^{2l}g^f)$ and (X_k, g^k) , proving the first assertion of the theorem.

We now consider the $\text{SO}(2, n)$ action on X_k . Let the action of $\text{SO}(2, n)$ on the set $X_k \subset \mathbb{R}^{n+2}$ be

$$\alpha^k(y) = \frac{\alpha \cdot y}{k(\alpha \cdot y)} \in X_k, \quad \alpha \in \text{SO}(2, n),$$

where $\alpha \cdot y$ is the natural $\text{SO}(2, n)$ action in \mathbb{R}^{n+2} . We claim that this action on X_k is a conformal transformation.

In fact, the action $(\alpha^k)'$ on X_k is the tangent one. For $V \in TX_k$, one has

$$(\alpha^k)'(y)(V) = \frac{1}{k(\alpha \cdot y)}\alpha \cdot V - \frac{k'(\alpha \cdot y)(\alpha \cdot V)}{k^2(\alpha \cdot y)}(\alpha \cdot y).$$

Since α is isometric with respect to η , and y , viewed as a vector of \mathbb{R}^{n+2} , is perpendicular to $V \in TX_k$ —that is, $\eta(y, V) = \eta(y, y) = 0$ —one has

$$\begin{aligned} \eta((\alpha^k)'(y)(V_1), (\alpha^k)'(y)(V_2)) &= \frac{1}{k^2(\alpha \cdot y)}\eta(V_1, V_2), \\ V_1, V_2 &\in TX_k. \end{aligned}$$

That is, α^k acts as a conformal transformation on X_k . This completes the proof of the theorem. ■

We now return to the problem of the natural embedding of FLRW spaces in \mathbb{R}^{n+2} . We begin with the n -dimensional de Sitter (dS) space Σ , with metric g^{dS} and scalar (Ricci) curvature $R = -n(n-1)H^2$. Let us define the FLRW space W through the scale factor a such that $g^W = e^{2a}g^{dS}$.

The de Sitter space Σ is first realized, as usual [12], as the hypersphere of the pseudo-Euclidean space \mathbb{R}^{n+1} through the equation

$$y^\mu y_\mu - (y^n)^2 = -H^{-2}.$$

We then identify \mathbb{R}^{n+1} with the (hyper)plane of \mathbb{R}^{n+2} defined through $f(y) = Hy^{n+1} = 1$. This realizes an isometric embedding of Σ in \mathbb{R}^{n+2} (see Ref. [11] for a more general approach). The key point is that Σ is now defined as the intersection of the null cone of \mathbb{R}^{n+2} and the plane $f = 1$:

$$\begin{cases} C(y) = y^\mu y_\mu - (y^n)^2 + (y^{n+1})^2 = 0, \\ f(y) = Hy^{n+1} = 1. \end{cases}$$

In other words, $\Sigma = X_f$.

Proposition 1. Let l be a homogeneous function of degree zero on \mathbb{R}^{n+2} whose restriction to Σ is the function a . Let $W = e^l\Sigma$, then the induced metric on W by η of \mathbb{R}^{n+2} is the Weyl rescaling $g^W = e^{2a}g^{dS}$ of the de Sitter metric g^{dS} . Moreover, the elements of the linear group $\text{SO}(2, n)$ on \mathbb{R}^{n+2} act on W as conformal transformations.

Proof. This proposition is nothing but a specialization of the above theorem with $\Sigma = X_f$ and $W = X_k$. ■

One can ask why we could not specialize l so that it does not depend on y^{n+1} , eliminate that variable, and then obtain an embedding in \mathbb{R}^{n+1} . In that case, the metric on W would no longer be induced by that of \mathbb{R}^{n+1} , and $\text{SO}(2, n)$ invariance would be lost. Indeed, defining W as a subset of the \mathbb{R}^{n+2} cone is key to the proof through isotropic vectors.

Note that the same result can be obtained by continuous deformation of Minkowski or anti-de Sitter (AdS) spaces instead of de Sitter space. This amounts to choosing $f = Hy^n$ or $f = H(y^n + y^{n+1})/2$, respectively.

IV. CONCLUDING REMARKS

In the previous section, Sec. III, we built k , and thus the manifold X_k , from f and the scale factor a . One can ask about the converse problem—that is, starting from a function k , homogeneous of degree one, is the manifold X_k the continuous deformation of some de Sitter space X_f , where f is a homogeneous polynomial of degree one? Globally, the answer is most likely negative. Locally, however, X_k can be obtained as a continuous deformation of dS or AdS spaces whose related defining planes P_f are tangent to X_k . Precisely, let us consider a point y_o of X_k and set $K_o := \nabla k(y_o)$: we assume that $(K_o)^2 \neq 0$, and the tangent plane at y_o is defined through $f(y) = K_o \cdot (y - y_o) + k(y_o)$. Then, from Theorem 1, the metric g^k is, locally, a continuous deformation of a dS or AdS metric depending on

the sign of $(K_o)^2$, and hence on the point y_o considered on the manifold.

Concerning the scale factor a , it can be chosen as any continuous function of $y \in \Sigma$. As a consequence, by contrast with the scale factor appearing in the metric of FLRW spacetimes in (four-dimensional) cosmology (see, for instance, [13] about different forms of the RW metric), it can describe more general conformally flat spacetimes—in particular, those with less symmetry than the FLRW space.

In the most general case, the need to embed FLRW spaces in \mathbb{R}^{n+2} , in contrast to simply \mathbb{R}^{n+1} , has been pointed out in [6]. Here, in our works, considering \mathbb{R}^{n+2} to boot stems from the need to track how the SO(2, n) group acts.

-
- [1] M. Pavsic and V. Tapia, Resource letter on geometrical results for embeddings and branes, [arXiv:gr-qc/0010045](https://arxiv.org/abs/gr-qc/0010045).
 - [2] S. Seahra Sanjeev and S. Wesson Paul, The structure of the big bang from higher dimensional embeddings, *Classical Quantum Gravity* **19**, 1139 (2002).
 - [3] I. E. Gulamov and M. N. Smolyakov, Submanifolds in five-dimensional pseudo-Euclidean spaces and four-dimensional FRW universes, *Gen. Relativ. Gravit.* **44**, 703 (2012).
 - [4] S. A. Paston and A. A. Sheykin, Embeddings for solutions of Einstein equations, *Theor. Math. Phys.* **175**, 806 (2015).
 - [5] J. Ponce de Leon, Embeddings for general relativity, *Classical Quantum Gravity* **32**, 195018 (2013).
 - [6] M. M. Akbar, Embedding FLRW geometries in pseudo-Euclidean and anti-de Sitter spaces, *Phys. Rev. D* **95**, 064058 (2017).
 - [7] M. Dunajski and P. Tod, Conformally isometric embeddings and Hawking temperature, *Classical Quantum Gravity* **36**, 125005 (2019).
 - [8] H. P. Robertson, LXXXVI. On relativistic cosmology, *Philos. Mag.* **5**, 835 (1928).
 - [9] H. P. Robertson, Relativistic cosmology, *Rev. Mod. Phys.* **5**, 62 (1933).
 - [10] J. Rosen, Embedding of various relativistic Riemannian spaces in pseudo-Euclidean spaces, *Rev. Mod. Phys.* **37**, 204 (1965).
 - [11] E. Huguet, J. Queva, and J. Renaud, Restriction of Laplace operator on one-forms: From \mathbb{R}^{n+2} and \mathbb{R}^{n+1} ambient spaces to embedded (A)dS_n submanifolds, *J. Math. Phys. (N.Y.)* **63**, 07230 (2022).
 - [12] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 2023), [10.1017/9781009253161](https://doi.org/10.1017/9781009253161).
 - [13] M. Ivison, On the conformal forms of the Robertson-Walker metric, *J. Math. Phys. (N.Y.)* **48**, 122501 (2007).