

Time delay in the gravitational field of an axisymmetric body with full mass and spin multipole structure

Sven Zschocke 

*Institute of Planetary Geodesy-Lohrmann Observatory,
TUD Dresden University of Technology, Helmholtzstrasse 10, D-01069 Dresden, Germany*

 (Received 1 December 2023; accepted 30 January 2024; published 14 March 2024; corrected 22 March 2024)

The time delay of a light signal that propagates in the gravitational field of an isolated body at rest is considered. The gravitational field is given in the post-Newtonian scheme and in terms of the full set of mass multipoles and spin multipoles of the body. The asymptotic configuration is considered, where source and observer are located at spatial infinity from the massive body. It is found that in this asymptotic limit the higher multipole terms of time delay are related to the higher multipole terms of total light deflection. Furthermore, it is shown that the gauge terms vanish in this asymptotic configuration. In case of an axisymmetric body in uniform rotational motion, the higher multipole terms of time delay can be expressed in terms of Chebyshev polynomials. This fact allows one to determine the upper limits of the time delay for higher multipoles. These upper limits represent a criterion to identify those multipoles that contribute significantly to the time delay for a given accuracy of time measurements. It is found that the first mass multipoles with $l \leq 8$ and the first spin multipoles with $l \leq 3$ are sufficient for an accuracy on the femtosecond scale of accuracy in time measurements.

DOI: [10.1103/PhysRevD.109.064044](https://doi.org/10.1103/PhysRevD.109.064044)

I. INTRODUCTION

In the classical Shapiro time delay [1] one considers the propagation of an electromagnetic signal, for instance, a radar signal or a signal of visible light, in the gravitational field of a spherically symmetric body. Assume the space-time is covered by harmonic four-coordinates $x^\mu = (x^0, x^1, x^2, x^3)$, with time component x^0 and spatial components x^1, x^2, x^3 and the origin of the spatial coordinates is located at the center of mass of the body. Then, the light travel time of a signal, emitted at (t_0, \mathbf{x}_0) and received at (t_1, \mathbf{x}_1) , is in the post-Newtonian (PN) scheme given by [2,3]

$$(t_1 - t_0) = \frac{R}{c} + \frac{2GM}{c^3} \ln \frac{x_1 + \boldsymbol{\sigma} \cdot \mathbf{x}_1}{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0} + \mathcal{O}(c^{-3}), \quad (1)$$

where M is the mass of the body and $\boldsymbol{\sigma}$ is the unit tangent vector along the light ray at minus infinity. The difference between the light travel time $(t_1 - t_0)$ and Euclidean distance, $R = |\mathbf{x}_1 - \mathbf{x}_0|$, divided by the speed of light, is the Shapiro time delay and belongs to the four classical tests of general relativity: perihelion precession of Mercury, light deflection at the Sun, gravitational redshift of light, and light travel time delay. The effect of time delay (1) has been detected in 1968 [4] and 1971 [5], which yields, for the round-trip Earth-Sun-Venus and Venus-Sun-Earth path, up to 251 μs for radar signals grazing the Sun. The most accurate experiments of time-delay measurements have

been performed in 2003 by using the Saturn orbiter *Cassini* as reflector, which amounts, for the round-trip Earth-Sun-Saturn and Saturn-Sun-Earth path, up to 288 μs , where a precision on the nanosecond scale of accuracy has been achieved [6].

Technological developments in time measurements by means of atomic clocks both on the ground as well as in space have made giant progress during recent decades. In particular, the accuracy of the standard deviation of up-to-date optical atomic clocks is $\Delta t/t = 10^{-19}$ [7]. Such an accuracy corresponds to a precision of 0.001 ps for a light signal that travels, for instance, from a giant planet of the Solar System toward an observer located nearby Earth. In fact, there are several mission proposals of the European Space Agency [8–13], aiming at time-delay measurements at the pico- and subpicosecond level of accuracy. Furthermore, the accuracy of time measurements improves by an order of magnitude every seven years [14], and in the near future the frequency standards will arrive at a level of $\Delta t/t = 10^{-20}$ – 10^{-21} , which corresponds to a precision of 0.01 fs for a light signal from a giant planet of the Solar System toward an observer located nearby Earth.

In view of such rapid developments in time measurements, it becomes apparent that the mass-monopole approximation for Solar System bodies is not sufficient anymore, and a more realistic description of gravitational fields of Solar System bodies is necessary. The determination of the gravitational fields of Solar System bodies is achieved by decomposing the metric in terms of mass

multipoles M_L (which describe shape and inner structure of the massive body) and spin multipoles S_L (which describe rotational motions and inner currents of the massive body) of these bodies. Then, the analytical formula for the time delay becomes a complicated function of these multipoles [15],

$$(t_1 - t_0) = \frac{R}{c} + \sum_{l=0}^{\infty} \Delta\tau_{\text{IPN}}^{M_L}(t_1, t_0) + \sum_{l=1}^{\infty} \Delta\tau_{\text{1.5PN}}^{S_L}(t_1, t_0) + \mathcal{O}(c^{-4}), \quad (2)$$

where the first term in the sum ($l = 0$) is the mass-monopole term given by (1). In order to decide which multipoles are relevant for a given accuracy in time measurements, one has to determine the upper limits of the individual terms of the time-delay formula in (2), which means the maximal absolute value of these terms.

In a recent investigation, it was demonstrated that the effect of total light deflection is related to Chebyshev polynomials [16]. This fact has allowed for determining the upper limits of the total light deflection. As in the case of total light deflection, it is therefore the aim of this investigation to obtain an analytical formula for the multipole terms in (2) for the asymptotic configurations, where the source as well as the observer are located at spatial infinity from the massive body. It is found that in the asymptotic limit the multipole terms in (2) are related to Chebyshev polynomials, a fact that allows one to quantify the upper limits of absolute values of these terms to arbitrary multipole order. Using these criteria, one may easily decide which terms in (2) are really relevant for a given goal accuracy in time-delay measurements.

The manuscript is organized as follows: In Sec. II the metric and the geodesic equation are considered. The Shapiro time delay is considered in Sec. III. In Sec. IV it is shown that, in the asymptotic case, where source and observer are located infinitely far from the massive body, the total effect of time delay is related to the total light deflection. These results are applied in Sec. V for the case of an axisymmetric body. A comparison with the literature and some numerical results of the time delay in the gravitational fields of Solar System bodies are given in Secs. VI and VII. A summary is given in Sec. VIII. The used notations are explained in Appendix A, while Appendices B and C contain some further details of the calculations.

II. METRIC TENSOR AND GEODESIC EQUATION

A. Metric tensor

The curved space-time is described by a pair $(\mathcal{M}, g_{\mu\nu})$, where \mathcal{M} is a four-dimensional differentiable manifold and $g_{\mu\nu}$ is the metric tensor of the manifold, and each point $\mathcal{P} \in \mathcal{M}$ represents a possible space-time event; we assume for the metric signature $(-, +, +, +)$. These ten components of the metric tensor $g_{\mu\nu}$ are determined by the ten field

equations of gravity, which are valid in any coordinate system. The Bianchi identities reduce these field equations to only six independent equations. Therefore, four gauge conditions are imposed to fix the coordinates, which cover the physical manifold \mathcal{M} . We adopt harmonic four-coordinates, $z^\mu = (z^0, z^1, z^2, z^3)$, which are imposed by four harmonic gauge conditions [17]

$$\square_g z^\mu = 0, \quad (3)$$

where $\square_g = (-g)^{-1/2} \partial_\mu (-g)^{1/2} g^{\mu\nu} \partial_\nu$ is the general-covariant d'Alembert operator. If the gravitational fields are weak, then it is useful to decompose the metric tensor of the physical space-time [2,3,17,18],

$$g_{\mu\nu}(t, \mathbf{x}) = \eta_{\mu\nu} + h_{\mu\nu}(t, \mathbf{x}), \quad (4)$$

where $\eta_{\mu\nu} = (-1, +1, +1, +1)$ are the components of Minkowskian metric, while $h_{\mu\nu}$ are the metric perturbations that are small corrections to the Minkowskian metric: $|h_{\mu\nu}| \ll 1$. The decomposition (4) implies that the metric perturbations can be thought of as symmetric tensorial fields that propagate in the flat background space-time [2,17,18]. The flat space-time is described by a pair $(\mathcal{M}_0, \eta_{\mu\nu})$, where \mathcal{M}_0 is the flat background manifold. The curved physical manifold \mathcal{M} and the flat background manifold \mathcal{M}_0 are diffeomorphic to each other, which implicates a one-to-one correspondence between the points $\mathcal{P} \in \mathcal{M}$ and the points $\mathcal{Q} \in \mathcal{M}_0$.

The flat background manifold \mathcal{M}_0 is assumed to be covered by harmonic four-coordinates $x^\mu = (x^0, x^1, x^2, x^3)$ that, according to (3), are imposed by the harmonic gauge condition,

$$\square x^\mu = 0, \quad (5)$$

where $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$ is the Lorentz-covariant d'Alembert operator. Then, by inserting the decomposition (4) into the exact field equations of gravity and keeping only terms linear in the metric perturbations, one obtains the linearized field equations of gravity, which in the harmonic gauge read

$$\square \bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu}, \quad (6)$$

where $T^{\mu\nu}$ is the stress-energy tensor of the source of matter. The following relations allow one to deduce the metric perturbations $h_{\mu\nu}$ from the metric density perturbations $\bar{h}_{\mu\nu}$ as soon as the solution of the linearized field equations (6) is found:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}, \quad (7)$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \bar{h} \eta_{\mu\nu}, \quad (8)$$

with $h = h^{\alpha\beta}\eta_{\alpha\beta}$ and $\bar{h} = \bar{h}^{\alpha\beta}\eta_{\alpha\beta}$. These relations are valid at linear order in h . Furthermore, in linearized gravity, the indices of tensors are lowered and raised by the Minkowskian metric.

By imposing the Fock-Sommerfeld boundary condition [19,20], which implies the no-incoming radiation condition as well as the asymptotic flatness of space-time (isolated source of matter), the solution of the differential equation (6) is uniquely given by

$$\bar{h}^{\mu\nu} = \frac{4\pi G}{c^4} \int_V d^3x' \frac{T^{\mu\nu}(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}, \quad (9)$$

where the integral runs over the volume of the compact source of matter (body) and $t' = t - c^{-1}|\mathbf{x} - \mathbf{x}'|$.

Without limiting generality, one may assume the harmonic four-coordinates $x^\mu = (x^0, x^1, x^2, x^3)$, which cover the flat background manifold \mathcal{M}_0 , to be Cartesian. However, from the harmonic gauge condition (5), one concludes that the harmonic gauge does not fix these harmonic four-coordinates x^μ uniquely, but allows for smooth deformations [2,17]

$$x_{\text{can}}^\alpha = x^\alpha + w^\alpha(x^\beta), \quad (10)$$

if these gauge vector fields ξ^α satisfy $\square w^\alpha = 0$. The label of these new coordinates $\{x_{\text{can}}\}$ abbreviates the term ‘‘canonical.’’ It is emphasized that the gauge transformations in (10) are tending to 0 at spatial infinity and they are small, $|w^\alpha| \ll |x^\alpha|$, in the sense that the derivatives of the gauge functions w^α with respect to space and time are of the same order as the metric perturbations, $w_{,\mu}^\alpha = \mathcal{O}(h_\mu^\alpha)$, hence $|w_{,\mu}^\alpha| \ll 1$. The residual gauge transformation (10) implies a residual gauge transformation of the metric tensor,

$$g_{\alpha\beta}(t, \mathbf{x}) = \frac{\partial x_{\text{can}}^\mu}{\partial x^\alpha} \frac{\partial x_{\text{can}}^\nu}{\partial x^\beta} g_{\mu\nu}^{\text{can}}(t_{\text{can}}, \mathbf{x}_{\text{can}}). \quad (11)$$

By inserting (10) into (11) and performing a series expansion of the metric tensor on the right-hand side around the old (Cartesian) coordinates $\{x\}$, one obtains $(\partial_\alpha f \equiv f_{,\alpha} \equiv \partial f / \partial x^\alpha)$

$$g_{\alpha\beta}(t, \mathbf{x}) = g_{\alpha\beta}^{\text{can}}(t, \mathbf{x}) + \partial_\alpha w_\beta(t, \mathbf{x}) + \partial_\beta w_\alpha(t, \mathbf{x}). \quad (12)$$

The unique solution in (9) can be expressed in terms of six Cartesian symmetric and trace-free (STF) multipoles \hat{F}_L [Eq. (8.4) in [21] or Eqs. (5.3a)–(5.3c) in [22]],

$$\bar{h}^{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \left[\frac{\hat{F}_L^{\mu\nu}(u)}{r} \right], \quad (13)$$

where $u = t - c^{-1}r$ is the retarded time, $r = |\mathbf{x}|$, the STF multipoles \hat{F}_L are given by Eqs. (5.4a)–(5.4c) in [22], and

$$\hat{\partial}_L = \text{STF}_{i_1 \dots i_l} \frac{\partial}{\partial x^{i_1}} \dots \frac{\partial}{\partial x^{i_l}}, \quad (14)$$

where the ‘‘hat’’ in \hat{F}_L and in $\hat{\partial}_L$ indicates STF operation with respect to the spatial components of the multi-index $L = i_1 \dots i_l$. A detailed proof of this theorem in (13) has been presented in [23].

The solution in (13) is unique and represents the most general solution of the linearized gravity for an isolated compact source of matter. By extensive use of STF Cartesian tensor techniques, it has been demonstrated in [21,22,24] that the general solution in (13) can be written in terms of six STF multipoles $\{\hat{M}_L, \hat{S}_L, \hat{W}_L, \hat{X}_L, \hat{Y}_L, \hat{Z}_L\}$ [21,22,24]. Accordingly, the metric tensor can be written in the following form:

$$h_{\alpha\beta}(t, \mathbf{x}) = h_{\alpha\beta}^{\text{can}}[\hat{M}_L, \hat{S}_L] + \partial_\alpha w_\beta[\hat{W}_L, \hat{X}_L, \hat{Y}_L, \hat{Z}_L] + \partial_\beta w_\alpha[\hat{W}_L, \hat{X}_L, \hat{Y}_L, \hat{Z}_L]. \quad (15)$$

The canonical part of the metric perturbations in (15) depends on two multipoles only, namely, mass multipoles and spin multipoles $\{\hat{M}_L, \hat{S}_L\}$, while the gauge terms of the metric perturbation in (15) depend on four multipoles $\{\hat{W}_L, \hat{X}_L, \hat{Y}_L, \hat{Z}_L\}$. The form of the metric perturbations in (15) depends on the chosen coordinate system. For instance, in canonical coordinates there are no gauge terms at all and only the canonical term $h_{\alpha\beta}^{\text{can}}$ would remain. Here, we will use this most general form in (15) in order to demonstrate that these gauge terms vanish at spatial infinity.

We will consider the metric of bodies with time-independent multipoles and where the center of mass of the body is assumed to be at rest with respect to the harmonic coordinate system. Then, the canonical metric perturbations in (15) are separated into two pieces, $h_{\alpha\beta}^{\text{can}} = h_{\alpha\beta}^{(2)\text{can}} + h_{\alpha\beta}^{(3)\text{can}}$,

$$h_{00}^{(2)\text{can}} = -\frac{2G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{M}_L}{r}, \quad (16)$$

$$h_{0i}^{(3)\text{can}} = +\frac{4G}{c^3} \sum_{l=1}^{\infty} \frac{(-1)^l l}{(l+1)!} \epsilon_{iab} \hat{\partial}_{aL-1} \frac{\hat{S}_{bL-1}}{r}, \quad (17)$$

while $h_{ij}^{(2)\text{can}} = h_{00}^{(2)\text{can}} \delta_{ij}$. The derivatives $\hat{\partial}_L$ are not acting on the multipoles because they are independent of space and time here. These mass multipoles and spin multipoles in (16) and (17) in the stationary case, which means in the case of a time-independent source of matter, are given by

$$\hat{M}_L = \int d^3x \hat{x}_L \Sigma, \quad (18)$$

$$\hat{S}_L = \int d^3x \epsilon_{jk(i_l \hat{x}_{L-1})} x^j \Sigma^k, \quad (19)$$

where $\Sigma = (T^{00} + T^{kk})/c^2$ and $\Sigma^k = T^{0k}/c$ with the stress-energy tensor $T^{\mu\nu}$ of the source of matter [cf. Eq. (6)], and the integration runs over the volume of the body. The time independence of the multipoles implies time independence of the metric.

The gauge functions in (15) have been determined by [21,22,24,25] and read

$$w^0 = + \frac{4G}{c^3} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{W}_L}{r}, \quad (20)$$

$$w^i = - \frac{4G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_{iL} \frac{\hat{X}_L}{r} - \frac{4G}{c^2} \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_{L-1} \frac{\hat{Y}_{iL-1}}{r} - \frac{4G}{c^2} \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \frac{l}{l+1} \epsilon_{iab} \hat{\partial}_{aL-1} \frac{\hat{Z}_{bL-1}}{r}. \quad (21)$$

As mentioned above, these gauge vectors in (20) and (21) represent the most general form of the gauge terms in the metric tensor. The multipoles $\hat{W}_L, \hat{X}_L, \hat{Y}_L, \hat{Z}_L$ of the gauge functions (20) and (21) have been determined in [22,25] and read in the stationary case as follows:

$$\hat{W}_L = + \frac{2l+1}{(l+1)(2l+3)} \int d^3x \hat{x}_{Lj} \Sigma^j, \quad (22)$$

$$\hat{X}_L = + \frac{2l+1}{(2l+2)(l+2)(2l+5)} \int d^3x \hat{x}_{jkL} \Sigma^{jk}, \quad (23)$$

$$\hat{Y}_L = - \int d^3x \hat{x}_L \Sigma^{kk}, \quad (24)$$

$$\hat{Z}_L = - \frac{2l+1}{(l+2)(2l+3)} \int d^3x \epsilon_{jk(i_l \hat{x}_{L-1})km} \Sigma^{jm}, \quad (25)$$

where $\Sigma^{jk} = T^{jk}/c^2$, with T^{jk} being the spatial components of the stress-energy tensor of the body, and the integration runs over the volume of the body.

We will show that the gauge terms in (15) have no impact on the Shapiro time delay in the case of infinite distance of source and observer from the massive body (see Appendix B). This specific case reflects the general fact that $g_{\alpha\beta}$ and $g_{\alpha\beta}^{\text{can}}$ in (12) are physically equivalent, which means they lead to the same observables. This statement is valid in the case where the gauge transformations (10) are small and are tending to 0 at spatial infinity, as emphasized in the text below Eq. (10).

B. Geodesic equation

In flat Minkowskian space-time, assumed to be covered by Cartesian four-coordinates, a light signal, emitted by a source at (t_0, \mathbf{x}_0) in some direction specified by a unit vector $\boldsymbol{\sigma}$, propagates along a straight line, given by

$$\mathbf{x}_N = \mathbf{x}_0 + c(t - t_0)\boldsymbol{\sigma}, \quad (26)$$

where index N stands for Newtonian. The absolute value reads

$$r_N = \sqrt{(x_0)^2 + 2\boldsymbol{\sigma} \cdot \mathbf{x}_0 c(t - t_0) + c^2(t - t_0)^2}. \quad (27)$$

In what follows, we also need the so-called impact vector of the unperturbed light ray,

$$\mathbf{d}_\sigma = \boldsymbol{\sigma} \times (\mathbf{x}_N \times \boldsymbol{\sigma}), \quad (28)$$

with its absolute value

$$d_\sigma = |\boldsymbol{\sigma} \times \mathbf{x}_N|. \quad (29)$$

The impact vector (28) points from the center of mass of the body toward the unperturbed light ray at its closest encounter; for a graphical elucidation, see Fig. 1.

In curved space-time $(\mathcal{M}, g_{\mu\nu})$, the trajectory of a light signal is determined by the geodesic equation, which in terms of coordinate time reads as follows [2,17,26]:

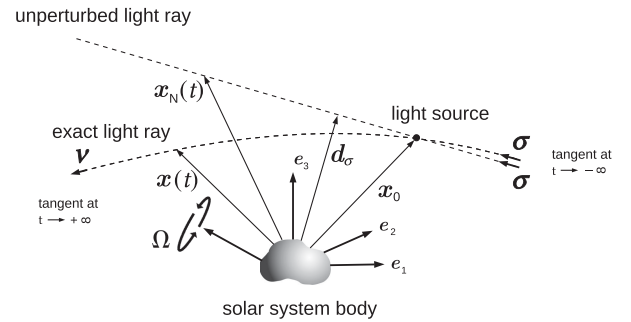


FIG. 1. A geometrical representation of the propagation of a light signal through the gravitational field of a massive Solar System body at rest. The axes of inertia are denoted by e_1, e_2, e_3 . In the general case, the body is of arbitrary shape and can be in arbitrary rotational motion with the vector of angular velocity Ω . In the stationary case the vector of angular velocity is time-independent. The light signal is emitted by the light source at \mathbf{x}_0 and propagates along the exact light trajectory $\mathbf{x}(t)$. The unit tangent vectors along the light trajectory at minus and plus infinity are $\boldsymbol{\sigma}$ and $\boldsymbol{\nu}$. The unperturbed light ray $\mathbf{x}_N(t)$ is given by Eq. (26) and propagates in the direction of $\boldsymbol{\sigma}$ along a straight line through the position of the light source at \mathbf{x}_0 . The impact vector \mathbf{d}_σ of the unperturbed light ray is given by Eq. (28).

$$\frac{\ddot{x}^i(t)}{c^2} + \Gamma_{\mu\nu}^i \frac{\dot{x}^\mu(t)}{c} \frac{\dot{x}^\nu(t)}{c} - \Gamma_{\mu\nu}^0 \frac{\dot{x}^\mu(t)}{c} \frac{\dot{x}^\nu(t)}{c} \frac{\dot{x}^i(t)}{c} = 0, \quad (30)$$

where a dot denotes total derivative with respect to coordinate time, and $\Gamma_{\mu\nu}^\alpha$ are the Christoffel symbols, given by [2,17,26]

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\beta\mu}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right). \quad (31)$$

The geodesic equation is a differential equation of second order of one variable t , thus a unique solution of (30) necessitates two initial-boundary conditions: the spatial position of light source \mathbf{x}_0 and the unit direction $\boldsymbol{\sigma}$ of the light signal at minus infinity [15,20,26–29],

$$\boldsymbol{\sigma} = \frac{\dot{\mathbf{x}}(t)}{c} \Big|_{t=-\infty} \quad \text{with} \quad \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = 1, \quad (32)$$

$$\mathbf{x}_0 = \mathbf{x}(t) \Big|_{t=t_0}. \quad (33)$$

By using the initial-boundary conditions (32) and by inserting the decomposition (4) into (30), which implies weak gravitational fields, the solution of the second integration of geodesic equation (30) is given by

$$\mathbf{x}(t) = \mathbf{x}_0 + c(t - t_0)\boldsymbol{\sigma} + \Delta\mathbf{x}(t, t_0), \quad (34)$$

where $\Delta\mathbf{x}$ are small corrections to the trajectory of the unperturbed light ray (26). Equation (34) represents the trajectory of a light signal propagating in the flat background space-time $(\mathcal{M}_0, \eta_{\mu\nu})$; for a graphical elucidation, see Fig. 1. The solution of the initial value problem (34) implies the following limit:

$$\lim_{t \rightarrow t_0} \Delta\mathbf{x}(t, t_0) = 0, \quad (35)$$

in order to be consistent with the condition (33).

The geodesic equation in 1.5PN approximation can be deduced from the exact geodesic equation (30) and is given by [26]

$$\begin{aligned} \frac{\ddot{x}^i(t)}{c^2} = & \frac{1}{2} h_{00,i} - h_{00,j} \sigma^i \sigma^j - h_{ij,k} \sigma^i \sigma^k + \frac{1}{2} h_{jk,i} \sigma^j \sigma^k \\ & - h_{0i,j} \sigma^j + h_{0j,i} \sigma^j - h_{0j,k} \sigma^i \sigma^j \sigma^k + \mathcal{O}(c^{-4}), \end{aligned} \quad (36)$$

where we have omitted all those terms that contain a derivative of the metric perturbations with respect to time because we consider the stationary case, which is the case of the time-independent metric. Note, that in the stationary case, the geodesic equation in 1.5PN approximation of the PN scheme and the geodesic equation in 1PM approximation of the post-Minkowskian (PM) scheme agree with

each other [30]. By inserting the metric perturbation (15) into the geodesic equation (36), one may separate the geodesic equation into a canonical term $\ddot{\mathbf{x}}_{\text{can}}$ plus a gauge term $\ddot{\mathbf{x}}_{\text{gauge}}$ as follows:

$$\frac{\ddot{\mathbf{x}}(t)}{c^2} = \frac{\ddot{\mathbf{x}}_{\text{can}}(t)}{c^2} + \frac{\ddot{\mathbf{x}}_{\text{gauge}}(t)}{c^2} + \mathcal{O}(c^{-4}), \quad (37)$$

where the spatial components of these terms are

$$\begin{aligned} \frac{\ddot{x}_{\text{can}}^i(t)}{c^2} = & h_{00,i}^{(2)\text{can}} - 2h_{00,j}^{(2)\text{can}} \sigma^i \sigma^j - h_{0i,j}^{(3)\text{can}} \sigma^j \\ & + h_{0j,i}^{(3)\text{can}} \sigma^j - h_{0j,k}^{(3)\text{can}} \sigma^i \sigma^j \sigma^k, \end{aligned} \quad (38)$$

$$\frac{\ddot{x}_{\text{gauge}}^i(t)}{c^2} = \partial_j w_{,k}^0 \sigma^i \sigma^j \sigma^k - \partial_j w_{,k}^i \sigma^j \sigma^k. \quad (39)$$

The metric perturbations in (38) are given by (16) and (17), while the gauge functions in (39) are given by (20) and (21). The metric tensor with the metric perturbations (15) is valid in the entire space-time, while in the geodesic equation (37) with (38) and (39), the arguments of the metric perturbations are taken along the light trajectory, which means they have to be replaced by $\mathbf{x} = \mathbf{x}_N + \mathcal{O}(c^{-2})$ and $r = |\mathbf{x}_N| + \mathcal{O}(c^{-2})$.

The integration of (37) yields the light trajectory in 1.5PN approximation, formally given by

$$\mathbf{x}(t) = \mathbf{x}_0 + c(t - t_0)\boldsymbol{\sigma} + \Delta\mathbf{x}_{\text{can}}(t, t_0) + \Delta\mathbf{x}_{\text{gauge}}(t, t_0). \quad (40)$$

In particular, one has to insert the multipole decomposition of the metric perturbations (15) into the geodesic equation (36) and then one has to apply advanced integration methods developed in [15], which yields

$$\begin{aligned} \mathbf{x}(t) = & \mathbf{x}_0 + c(t - t_0)\boldsymbol{\sigma} \\ & + \sum_{l=0}^{\infty} \Delta\mathbf{x}_{\text{1PN}}^{M_L}(t, t_0) + \sum_{l=1}^{\infty} \Delta\mathbf{x}_{\text{1.5PN}}^{S_L}(t, t_0) \\ & + \Delta\mathbf{x}_{\text{gauge}}(t, t_0). \end{aligned} \quad (41)$$

The first line describes the straight trajectory of the unperturbed light ray, the second line represents the perturbations to the unperturbed light ray caused by the canonical terms of the metric tensor, and the third line is the perturbations to the unperturbed light ray caused by the gauge terms of the metric tensor. The canonical terms $\Delta\mathbf{x}_{\text{1PN}}^{M_L}$ and $\Delta\mathbf{x}_{\text{1.5PN}}^{S_L}$ have been obtained for the very first time by advanced integration methods developed in [15], while the gauge term $\Delta\mathbf{x}_{\text{gauge}}$ has been calculated in Appendix B by the same approach developed in [15].

III. TIME DELAY IN FIELD OF AN ARBITRARY BODY

In [15] advanced integration methods have been introduced that one allow to integrate (36) exactly. The basic

ideas of the conception was originally introduced in [15] for bodies with time-independent multipoles and where the center of mass of the body is assumed to be at rest with respect to the harmonic coordinate system. This approach has further been developed for the case of light propagation in the gravitational field of a time-dependent source of matter, where the center of mass of the body was also assumed to be at rest with respect to the harmonic coordinate system [27,31,32]. Later, these mathematical tools developed in [15,27,31,32] have been applied to the case of light propagation in the gravitational fields of N slowly moving bodies with time-dependent multipoles [28,29].

In the approach in [15] two new parameters were introduced,

$$c\tau = \boldsymbol{\sigma} \cdot \mathbf{x}_N, \quad (42)$$

$$\xi^i = P_j^i x_N^j, \quad (43)$$

where $P^{ij} = \delta^{ij} - \sigma^i \sigma^j$ is a projection operator onto the plane perpendicular to vector $\boldsymbol{\sigma}$. The unperturbed light ray (26) expressed in terms of these new variables takes the form

$$\mathbf{x}_N = \boldsymbol{\xi} + c\tau\boldsymbol{\sigma}, \quad (44)$$

with its absolute value

$$r_N = \sqrt{\xi^2 + c^2\tau^2}. \quad (45)$$

In favor of a simpler notation, we will not introduce new notations for the unperturbed light ray (26) given in terms of the standard variables (ct, \mathbf{x}) and for the unperturbed light ray (44) given in terms of the auxiliary variables $(c\tau, \boldsymbol{\xi})$. Similarly, the same notation will be used for their absolute values in (27) and (45).

The three-vector $\boldsymbol{\xi}$ in (43) actually coincides with the impact vector defined by (28). The use of two different notations for the same vector is appropriate for the following reason: the three-vector $\boldsymbol{\xi}$ is laying in the two-dimensional plane perpendicular to $\boldsymbol{\sigma}$, hence only two components are independent, which implies $\partial \xi^i / \partial \xi^j = P_j^i$. However, in practical calculations it is convenient to treat the spatial components of this vector as formally independent, which implies $\partial \xi^i / \partial \xi^j = \delta_j^i$. Therefore, a subsequent projection onto this two-dimensional plane by means of P^{ij} is necessary [27,33]. That is why two different notations for

$\boldsymbol{\xi}$ and \mathbf{d}_σ are in use. Then, for a spatial derivative expressed in terms of these new variables, one obtains

$$\frac{\partial}{\partial x^i} = P_i^j \frac{\partial}{\partial \xi^j} + \sigma_i \frac{\partial}{\partial c\tau}. \quad (46)$$

Using (46) and the binomial theorem, one finds the differential operator in (14) expressed in terms of these new variables,

$$\begin{aligned} \hat{\partial}_L = & \text{STF}_{i_1 \dots i_l} \sum_{p=0}^l \frac{l!}{(l-p)!p!} \sigma_{i_1} \dots \sigma_{i_p} P_{i_{p+1}}^{j_{p+1}} \dots P_{i_l}^{j_l} \\ & \times \frac{\partial}{\partial \xi^{j_{p+1}}} \dots \frac{\partial}{\partial \xi^{j_l}} \left(\frac{\partial}{\partial c\tau} \right)^p, \end{aligned} \quad (47)$$

where the ‘‘wide hat’’ symbol indicates STF operation with respect to the spatial indices $i_1 \dots i_l$. Here we prefer to use the operator as given by Eq. (47) where $\partial \xi^i / \partial \xi^j = \delta_j^i$, while if one applies the operator as given by Eq. (24) in [15] then $\partial \xi^i / \partial \xi^j = P_j^i$. The final results of either these operations are identical. Then, using the basic integral (25) in [15], one finds for the second integration the formula given by Eq. (27) in [15], which leads to the solution for the second integration of geodesic equation (36).

From the solution for the light trajectory in (41), one obtains the time of flight in the gravitational field of a body at rest with full mass- and spin-multipole structure, given by the following formula [15]:

$$\begin{aligned} (t_1 - t_0) = & \frac{R}{c} + \sum_{l=0}^{\infty} \Delta \tau_{1\text{PN}}^{M_L}(t_1, t_0) + \sum_{l=1}^{\infty} \Delta \tau_{1.5\text{PN}}^{S_L}(t_1, t_0) \\ & + \Delta \tau_{\text{gauge}}(t_1, t_0) \end{aligned} \quad (48)$$

up to terms of the order $\mathcal{O}(c^{-4})$, where

$$\Delta \tau_{1\text{PN}}^{M_L}(t_1, t_0) = -\frac{1}{c} \boldsymbol{\sigma} \cdot \Delta \mathbf{x}_{1\text{PN}}^{M_L}(t_1, t_0), \quad (49)$$

$$\Delta \tau_{1.5\text{PN}}^{S_L}(t_1, t_0) = -\frac{1}{c} \boldsymbol{\sigma} \cdot \Delta \mathbf{x}_{1.5\text{PN}}^{S_L}(t_1, t_0), \quad (50)$$

$$\Delta \tau_{\text{gauge}}(t_1, t_0) = -\frac{1}{c} \boldsymbol{\sigma} \cdot \Delta \mathbf{x}_{\text{gauge}}(t_1, t_0). \quad (51)$$

The explicit expressions for the mass-multipole (gravitoelectric) term (49) reads [15]

$$\Delta \tau_{1\text{PN}}^{M_L}(t_1, t_0) = +\frac{2G(-1)^l}{c^3} \frac{\hat{M}_L}{l!} \left(\hat{\partial}_L \ln(r_N + c\tau) \Big|_{\tau=\tau_1} - \hat{\partial}_L \ln(r_N + c\tau) \Big|_{\tau=\tau_0} \right), \quad (52)$$

and the spin-multipole (gravitomagnetic) term (50) reads [15] (see also footnote 3 in [34])

$$\Delta\tau_{1.5\text{PN}}^{S_L}(t_1, t_0) = +\frac{4G}{c^4} \frac{(-1)^l l}{(l+1)!} \epsilon_{abc} \sigma^c \hat{S}_{bL-1} \left(\hat{\partial}_{aL-1} \ln(r_N + c\tau) \Big|_{\tau=\tau_1} - \hat{\partial}_{aL-1} \ln(r_N + c\tau) \Big|_{\tau=\tau_0} \right). \quad (53)$$

The gauge term (51) is determined in Appendix B, where it is shown that this term vanishes at minus and plus infinity,

$$\lim_{\substack{\tau=\tau_0 \rightarrow -\infty \\ \tau=\tau_1 \rightarrow +\infty}} \Delta\tau_{\text{gauge}}(t_1, t_0) = 0. \quad (54)$$

That means the gauge terms have no impact on the Shapiro time delay when source and observer are located at spatial infinity, where the space-time is the flat Minkowski space. Thus, relation (54) is just a specific example of the general fact that observables are independent of the chosen gauge. A similar conclusion is valid for the total light deflection, that is, the bending of light in the case where the source and observer are located at spatial infinity. In [35] it has been shown that the gauge terms have no impact on the total light deflection, which is also an observable.

Thus, Eqs. (52) and (53) represent the effect of time delay and were also given in the textbook [33]. In (52) and (53), the differentiations have to be performed. Afterward, one has to substitute the unperturbed light ray (44) and its absolute value (45) by the standard expressions as given by (26) and (27). At the very end of these differentiations, the sublabels in (52) and (53) are replaced by

$$c\tau_0 = \boldsymbol{\sigma} \cdot \mathbf{x}_N(t_0), \quad (55)$$

$$c\tau_1 = \boldsymbol{\sigma} \cdot \mathbf{x}_N(t_1), \quad (56)$$

and $P^{ij}\xi_j$ by the spatial components of the impact vector d_σ^i . Further details about this approach can be found in [15,33,36].

The time delay in 1.5PN approximation, given by Eq. (48), is valid for finite distances of source and observer from the gravitating body. According to (40), the spatial coordinates of source \mathbf{x}_0 and observer \mathbf{x}_1 are related to the spatial coordinates of the unperturbed light signal at time of emission t_0 and time of observation t_1 as follows:

$$\mathbf{x}_0 = \mathbf{x}_N(t_0), \quad (57)$$

$$\mathbf{x}_1 = \mathbf{x}_N(t_1) + \mathcal{O}(c^{-2}). \quad (58)$$

Therefore, after performing the differentiations, one may replace these awkward terms $\mathbf{x}_N(t_0)$ and $\mathbf{x}_N(t_1)$ in the arguments of (52) and (53) just by the exact positions of source and observer.

As in the case of total light deflection [15,16], we will determine the total effect of time delay (52) and (53). That means, we will consider astrometric configurations, where both the source and the observer are located at spatial

infinity from the gravitating body. In particular, we will determine the time delay for asymptotic configurations of source and observer, where the limits are

$$\boldsymbol{\sigma} \cdot \mathbf{x}_0 \rightarrow -\infty, \quad (59)$$

$$\boldsymbol{\sigma} \cdot \mathbf{x}_1 \rightarrow +\infty. \quad (60)$$

Roughly speaking, these conditions represent configurations where the massive body is located somewhere between source and observer, as shown in Fig. 2. In view of (55)–(60), these asymptotic limits (59) and (60) in terms of the arguments of the time delay in (52) and (53) read

$$c\tau_0 \rightarrow -\infty, \quad (61)$$

$$c\tau_1 \rightarrow +\infty. \quad (62)$$

The time delay (52) and (53) depends on the impact vector of the unperturbed light ray, which is constant for a given light ray. Therefore, these limits in (61) and (62) have to be taken along the unperturbed light trajectory with constant impact vector, as elucidated by Fig. 2. In particular, one finds the following limits:

$$\lim_{\tau_0 \rightarrow -\infty} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N(t_0)}{r_N(t_0)} = -1, \quad (63)$$

$$\lim_{\tau_1 \rightarrow +\infty} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N(t_1)}{r_N(t_1)} = +1, \quad (64)$$

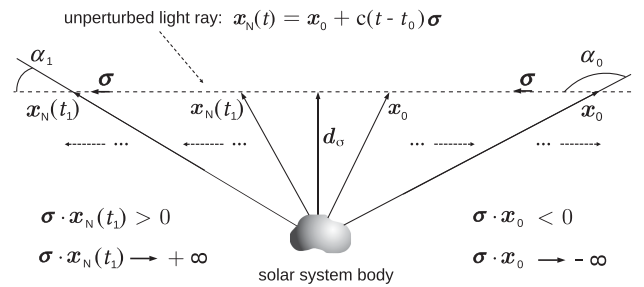


FIG. 2. This diagram elucidates the limits in (61) and (62). The angles are $\alpha_0 = \delta(\boldsymbol{\sigma}, \mathbf{x}_0)$ and $\alpha_1 = \delta(\boldsymbol{\sigma}, \mathbf{x}_N(t_1))$. The dashed arrows to the right and left show the directions to which the spatial positions of source, $\mathbf{x}_0 = \mathbf{x}_N(t_0)$, and the spatial position of the unperturbed light signal $\mathbf{x}_N(t_1)$ at time of observation are shifted up to infinity. The limits in (61) and (62) implicitly assume that the impact vector of the unperturbed light ray remains constant: $d_\sigma = \text{const}$.

where $r_N(t_0) = |\mathbf{x}_N(t_0)|$ and $r_N(t_1) = |\mathbf{x}_N(t_1)|$. Thus, the angles are $\delta(\boldsymbol{\sigma}, \mathbf{x}_0) \rightarrow \pi$ and $\delta(\boldsymbol{\sigma}, \mathbf{x}_N(t_1)) \rightarrow 0$ in these limits, a result that is shown by the graphical representation in Fig. 2.

The mass-monopole term [$l = 0$ in (52)] has already been given by Eq. (1), which is logarithmically divergent. Therefore, we consider the asymptotic limit only for mass-multipole terms with $l \geq 1$ and for spin-multipole terms with $l \geq 1$. We use the following notation for these asymptotic limits:

$$\Delta\tau_{\text{IPN}}^{M_L} = \lim_{\substack{\tau_0 \rightarrow -\infty \\ \tau_1 \rightarrow +\infty}} \Delta\tau_{\text{IPN}}^{M_L}(t_1, t_0), \quad (65)$$

$$\Delta\tau_{\text{1.5PN}}^{S_L} = \lim_{\substack{\tau_0 \rightarrow -\infty \\ \tau_1 \rightarrow +\infty}} \Delta\tau_{\text{1.5PN}}^{S_L}(t_1, t_0), \quad (66)$$

where $\Delta\tau_{\text{IPN}}^{M_L}(t_1, t_0)$ and $\Delta\tau_{\text{1.5PN}}^{S_L}(t_1, t_0)$ were given by Eqs. (52) and (53). In Appendix C the following results for these limits are shown:

$$\hat{\partial}_L \ln(r_N + c\tau) \Big|_{\tau=\tau_0 \rightarrow -\infty} = 2\hat{\partial}_L \ln|\boldsymbol{\xi}|, \quad (67)$$

$$\hat{\partial}_L \ln(r_N + c\tau) \Big|_{\tau=\tau_1 \rightarrow +\infty} = 0, \quad (68)$$

which are valid for $l \geq 1$. These limits in (67) and (68) can also nicely be verified for the first few orders in l just by explicit computation. Let us notice that on the left-hand side of these relations one has to, first of all, perform the differentiations by using the differential operator in (47) and afterward one has to calculate the limits. Then, on the right-hand side of these relations, only the term with $p = 0$ in the differential operator (47) contributes, which is given by (C3) in Appendix C; see also the comment in the text below Eq. (C12).

By inserting (67) and (68) into (52) and (53), one obtains for the time delay in the asymptotic limit, where source and observer are located at spatial infinity, the following expressions:

$$\Delta\tau_{\text{IPN}}^{M_L} = -\frac{4G}{c^3} \frac{(-1)^l}{l!} \hat{M}_L \hat{\partial}_L \ln|\boldsymbol{\xi}|, \quad (69)$$

$$\Delta\tau_{\text{1.5PN}}^{S_L} = -\frac{8G}{c^4} \frac{(-1)^l}{(l+1)!} \epsilon_{abc} \sigma^c \hat{S}_{bL-1} \hat{\partial}_{aL-1} \ln|\boldsymbol{\xi}|, \quad (70)$$

which are valid for $l \geq 1$. In order to determine the time delay by means of the expressions (69) and (70), one has to calculate the term

$$\hat{\partial}_L \ln|\boldsymbol{\xi}| = \text{STF}_{i_1 \dots i_l} P_{i_1}^{j_1} \dots P_{i_l}^{j_l} \frac{\partial}{\partial \xi^{j_1}} \dots \frac{\partial}{\partial \xi^{j_l}} \ln|\boldsymbol{\xi}|. \quad (71)$$

In our investigation [16], it has been shown that this term is given by the following expression:

$$\hat{\partial}_L \ln|\boldsymbol{\xi}| = \frac{(-1)^{l+1}}{|\boldsymbol{\xi}|^l} \text{STF}_{i_1 \dots i_l} \sum_{n=0}^{\lfloor l/2 \rfloor} G_n^l P_{i_1 i_2} \dots P_{i_{2n-1} i_{2n}} \times \frac{\xi_{i_{2n+1}} \dots \xi_{i_l}}{|\boldsymbol{\xi}|^{l-2n}}, \quad (72)$$

which is valid for any natural number $l \geq 1$, and the coefficients are given by [16]

$$G_n^l = (-1)^n 2^{l-2n-1} \frac{l! (l-n-1)!}{n! (l-2n)!}. \quad (73)$$

Inserting (72) into (69) and (70) completes the calculation of the time delay in the gravitational field of a massive body with full (time-independent) mass- and spin-multipole structure, if the source and the observer are at infinite distance from the massive body.

IV. RELATION BETWEEN TIME DELAY AND TOTAL LIGHT DEFLECTION

The tangent vector of the light trajectory at minus infinity $\boldsymbol{\sigma}$ has been defined by Eq. (32), and the tangent vector of the light trajectory at plus infinity $\boldsymbol{\nu}$ is defined by

$$\boldsymbol{\nu} = \frac{\dot{\mathbf{x}}(t)}{c} \Big|_{t=+\infty} \quad \text{with} \quad \boldsymbol{\nu} \cdot \boldsymbol{\nu} = 1. \quad (74)$$

A graphical representation of the three-vector in (74) is given in Fig. 1. The angle of total light deflection is defined as angle $\delta(\boldsymbol{\sigma}, \boldsymbol{\nu})$ between these tangent vectors,

$$\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}) = \arcsin |\boldsymbol{\sigma} \times \boldsymbol{\nu}|. \quad (75)$$

The tangent vector $\boldsymbol{\nu}$ can be expanded in terms of mass multipoles \hat{M}_L and spin multipoles \hat{S}_L [16],

$$\boldsymbol{\nu} = \boldsymbol{\sigma} + \sum_{l=0}^{\infty} \boldsymbol{\nu}_{\text{IPN}}^{M_L} + \sum_{l=1}^{\infty} \boldsymbol{\nu}_{\text{1.5PN}}^{S_L} + \mathcal{O}(c^{-4}), \quad (76)$$

where the individual terms are given by Eqs. (48) and (49) in [16]. By inserting (76) into (75) one finds that the angle of total light deflection is also expanded in terms of mass multipoles and spin multipoles,

$$\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}) = \sum_{l=0}^{\infty} \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\text{IPN}}^{M_L}) + \sum_{l=1}^{\infty} \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\text{1.5PN}}^{S_L}), \quad (77)$$

up to terms of the order $\mathcal{O}(c^{-4})$. The individual terms are given by [15,16]

$$\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\text{IPN}}^{M_L}) = -\frac{4G}{c^2} \frac{1}{|\boldsymbol{\xi}|} \frac{(-1)^l}{(l-1)!} \hat{M}_L \hat{\partial}_L \ln|\boldsymbol{\xi}|, \quad (78)$$

$$\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^S) = -\frac{8G}{c^3} \frac{1}{|\boldsymbol{\xi}|} \frac{(-1)^l l^2}{(l+1)!} \epsilon_{abc} \sigma^c \hat{S}_{bL-1} \hat{\partial}_{aL-1} \ln |\boldsymbol{\xi}|, \quad (79)$$

which are valid for $l \geq 1$. By comparing (78) and (79) with Eqs. (69) and (70), one recovers the following remarkable relations between the multipole terms of the total effect of time delay and the multipole terms of the angle of total light deflection:

$$\Delta\tau_{1\text{PN}}^{M_L} = \frac{1}{l} \frac{|\boldsymbol{\xi}|}{c} \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{M_L}), \quad (80)$$

$$\Delta\tau_{1.5\text{PN}}^{S_L} = \frac{1}{l} \frac{|\boldsymbol{\xi}|}{c} \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^{S_L}), \quad (81)$$

which are valid for multipoles of order $l \geq 1$ and $|\boldsymbol{\xi}| = d_\sigma$ is identical with the impact parameter (29) here, because the differentiations are performed. These relations (80) and (81) are strictly valid for light signals that propagate in the gravitational fields of a body at rest, having the full set of mass multipoles and spin multipoles. That means the body can be of arbitrary shape, inner structure and can be in arbitrary but uniform rotational motions and stationary inner currents.

V. TIME DELAY IN FIELD OF AN AXISYMMETRIC BODY

The largest effect of the Shapiro effect is expected from the Sun and the giant planets of the Solar System. In order to determine the Shapiro time delay, one needs the explicit form for mass multipoles (18) and for spin multipoles (19). For an estimation of the individual terms in (69) and (70), one may approximate the Sun and the giant planets by a rigid axisymmetric body with radial-dependent mass distribution and in uniform rotational motion with angular velocity Ω around the symmetry axis of the body, which is aligned with the x^3 axis of the coordinate system. A graphical representation of this configuration is given by Fig. 3. Then, the higher mass multipoles and spin multipoles for such a body are given by [16,36]

$$\hat{M}_L = -M(P)^l J_l \delta_{(i_1}^3 \dots \delta_{i_l)}^3, \quad (82)$$

$$\hat{S}_a = +\kappa^2 M P^2 \Omega \delta_{3a}, \quad (83)$$

$$\hat{S}_L = -M(P)^{l+1} \Omega J_{l-1} \frac{l+1}{l+4} \delta_{(i_1}^3 \dots \delta_{i_l)}^3, \quad (84)$$

where (82) is valid for any natural number of $l \geq 2$, while (84) is valid for any natural number of $l \geq 3$. The mass-dipole term, that is, $l = 1$ in (82), vanishes in the case where the origin of the coordinate system is located at the center of mass of the body (i.e., $J_1 = 0$) and will therefore not be considered in what follows. Here, M is the Newtonian mass of the body, P is its equatorial radius, J_l are the actual zonal

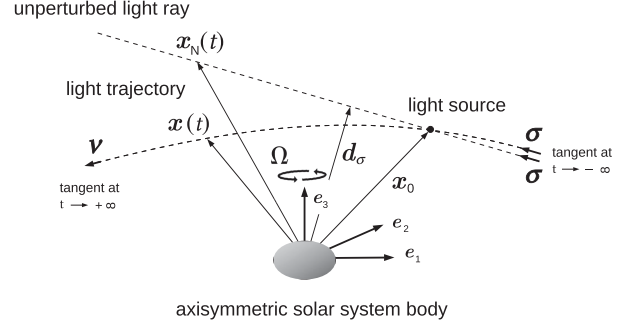


FIG. 3. A geometrical representation of the propagation of a light signal through the gravitational field of an axisymmetric Solar System body at rest. The axes of inertia are denoted by e_1, e_2, e_3 . The body is in uniform rotational motion with angular velocity Ω around the axis of symmetry e_3 . The light signal is emitted by the light source at x_0 and propagates along the exact light trajectory $x(t)$. The unit tangent vectors along the light trajectory at minus and plus infinity are σ and ν . The unperturbed light ray $x_N(t)$ is given by Eq. (26) and propagates in the direction of σ along a straight line through the position of the light source at x_0 . The impact vector d_σ of the unperturbed light ray is given by Eq. (28).

harmonic coefficients of index l , κ^2 is the dimensionless moment of inertia, Ω is the angular velocity of the rotating body, and $\delta_{(i_1}^3 \dots \delta_{i_l)}^3 = \text{STF}_{i_1 \dots i_l} \delta_{3i_1} \dots \delta_{3i_l}$ denotes products of Kronecker symbols that are symmetric and traceless with respect to indices $i_1 \dots i_l$. It is noticed that the definition of the angular velocity Ω has an unambiguous meaning only at linear order around flat space-time.

For reason of completeness, we notice the upper limit of the Shapiro time delay caused by the mass-monopole (1), which reads [3]

$$\left| \Delta\tau_{1\text{PN}}^{M_0} \right| \leq \frac{2GM}{c^3} \ln \frac{4x_0 x_1}{(d_\sigma)^2}. \quad (85)$$

Now we consider the upper limits of the higher mass multipoles and spin multipoles, which means the upper limits of the absolute values in the asymptotic limits (65) and (66). In order to get the time delay in the gravitational field of an axisymmetric body in uniform rotational motion, one has to insert the multipoles (82)–(84) into Eqs. (69) and (70) and taking account of relation (72). However, in view of the relations (80) and (81), which are valid for bodies of arbitrary shape, inner structure and uniform rotational motions, it is easier to use the results for the total light deflection angle, which has been determined for axisymmetric bodies in [16]. In particular, in our investigation [16] it has been found that the angle of total light deflection is related to Chebyshev polynomials. This relation has been established for mass multipoles by Eq. (114) in [16] and for spin multipoles by Eq. (121) in [16]. Hence, one obtains for the time delay of light signals in the field of an

axisymmetric body the following expressions in the asymptotic limit:

$$\Delta\tau_{1\text{PN}}^{M_L} = -\frac{4GM J_l}{c^3} \left(\frac{P}{d_\sigma}\right)^l \left[1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2\right]^{[l/2]} T_l(x) \quad (86)$$

for the mass-multipole terms, which is valid for $l \geq 2$. For the effect of time delay caused by the spin-dipole, one obtains, by means of Eq. (120) in [16], the following expression in the asymptotic limit:

$$\Delta\tau_{1.5\text{PN}}^{S_1} = -\frac{4GM}{c^4} P\Omega\kappa^2 J_0 \left(\frac{P}{d_\sigma}\right) \frac{(\boldsymbol{\sigma} \times \mathbf{d}_\sigma) \cdot \mathbf{e}_3}{d_\sigma}, \quad (87)$$

with $J_0 = -1$, and for the effect of time delay caused by higher spin multipoles,

$$\Delta\tau_{1.5\text{PN}}^{S_l} = -\frac{8GM}{c^4} P\Omega J_{l-1} \left(\frac{P}{d_\sigma}\right)^l \frac{(\boldsymbol{\sigma} \times \mathbf{d}_\sigma) \cdot \mathbf{e}_3}{d_\sigma} \frac{1}{l+4} \times \left[1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2\right]^{[l/2]} U_{l-1}(x), \quad (88)$$

which is valid for $l \geq 3$. The variable in (86) and (88) reads [16]

$$x = \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2\right)^{-1/2} \left(\frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{d_\sigma}\right), \quad (89)$$

which is a real number. It has already been shown in [16] that the interval of the argument is, in fact, given by $-1 \leq x \leq +1$. The power representation of Chebyshev polynomials of the first kind reads [37]

$$T_l(x) = \frac{l}{2} \sum_{n=0}^{[l/2]} \frac{(-1)^n (l-n-1)!}{n! (l-2n)!} (2x)^{l-2n}, \quad (90)$$

where $l \geq 1$. The power representation of Chebyshev polynomials of the second kind reads [37]

$$U_l(x) = \sum_{n=0}^{[l/2]} \frac{(-1)^n (l-n)!}{n! (l-2n)!} (2x)^{l-2n}, \quad (91)$$

where $l \geq 0$. The remarkable feature that the total effect of time delay is given in terms of Chebyshev polynomials allows for a straightforward determination of the upper limit of the total effect of time delay, because the upper limits of Chebyshev polynomials are given by

$$|T_l| \leq 1 \quad \text{and} \quad |U_{l-1}| \leq l. \quad (92)$$

Inserting (92) into (86) and (88) yields for the absolute values of the Shapiro time delay induced by the mass multipoles and spin multipoles the following inequalities: for the mass-multipole terms ($l \geq 2$) one obtains

$$\left|\Delta\tau_{1\text{PN}}^{M_L}\right| \leq \frac{4GM |J_l|}{c^3} \frac{1}{l} \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2\right)^{[l/2]} \left(\frac{P}{d_\sigma}\right)^l, \quad (93)$$

for the spin-dipole term ($l = 1$) one finds

$$\left|\Delta\tau_{1.5\text{PN}}^{S_1}\right| \leq \frac{4GM}{c^4} P\Omega\kappa^2 \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2\right)^{[l/2]} \left(\frac{P}{d_\sigma}\right), \quad (94)$$

and for the spin-multipole terms ($l \geq 3$) one obtains

$$\left|\Delta\tau_{1.5\text{PN}}^{S_L}\right| \leq \frac{8GM}{c^4} P\Omega \frac{|J_{l-1}|}{l+4} \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2\right)^{[l/2]} \left(\frac{P}{d_\sigma}\right)^l. \quad (95)$$

These upper limits (93) and (95) represent a criterion to identify those multipoles that contribute significantly to the time delay for a given accuracy of time measurements. By taking into account that $|1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2| \leq 1$ one may derive simpler expressions for these upper limits, namely, for the mass multipoles ($l \geq 2$),

$$\left|\Delta\tau_{1\text{PN}}^{M_L}\right| \leq \frac{4GM |J_l|}{c^3} \frac{1}{l} \left(\frac{P}{d_\sigma}\right)^l, \quad (96)$$

for the spin dipole ($l = 1$),

$$\left|\Delta\tau_{1.5\text{PN}}^{S_1}\right| \leq \frac{4GM}{c^4} P\Omega\kappa^2 \left(\frac{P}{d_\sigma}\right), \quad (97)$$

and for the spin multipoles ($l \geq 3$),

$$\left|\Delta\tau_{1.5\text{PN}}^{S_L}\right| \leq \frac{8GM}{c^4} P\Omega \frac{l}{l+4} |J_{l-1}| \left(\frac{P}{d_\sigma}\right)^l. \quad (98)$$

These upper limits have also been presented by Eqs. (39) and (41) in [36]. However, the coefficients in front of Eqs. (39) and (41) in [36] were only given for the very few first multipoles for ellipsoidal bodies, while here these upper limits in (96) and (98) are valid for any multipole order and for the more general case of axisymmetric bodies.

Finally, we notice the upper limits for grazing rays, for the mass-multipole terms ($l \geq 2$) given by

$$\left|\Delta\tau_{1\text{PN}}^{M_L}\right| \leq \frac{4GM |J_l|}{c^3} \frac{1}{l}, \quad (99)$$

for the spin dipole ($l = 1$) given by

$$\left|\Delta\tau_{1.5\text{PN}}^{S_1}\right| \leq \frac{4GM}{c^4} P\Omega\kappa^2, \quad (100)$$

and for the spin-multipole terms ($l \geq 3$) one obtains

$$\left|\Delta\tau_{1.5\text{PN}}^{S_L}\right| \leq \frac{8GM}{c^4} P\Omega \frac{l}{l+4} |J_{l-1}|. \quad (101)$$

Numerical values of these limits (99)–(101) are given by Table II for Solar System bodies.

VI. COMPARISON WITH THE LITERATURE

The upper limits of Shapiro time delay for mass monopole (85) and spin dipole (100) are well known and agree, for instance, with Eq. (40.13) in [2] and Eq. (75) in [38], respectively.

Upper limits for the effect of time delay caused by higher mass multipoles and spin multipoles of the massive body are rare and have, thus far, only been presented in our previous investigation in [36]. In order to compare the upper limits in (96) and (98) with those upper limits presented by Eqs. (39) and (41) in [36], we rewrite (96) and (98) in the following form:

$$\left| \Delta\tau_{\text{IPN}}^{M_L} \right| \leq A_l \frac{GM}{c^3} |J_l| \left(\frac{P}{d_\sigma} \right)^l \quad \text{with} \quad A_l = \frac{4}{l}, \quad (102)$$

which is valid for $l \geq 2$, and for the spin-multipole terms one obtains

$$\left| \Delta\tau_{1.5\text{PN}}^{S_L} \right| \leq B_l \frac{GM}{c^4} P\Omega |J_{l-1}| \left(\frac{P}{d_\sigma} \right)^l \quad \text{with} \quad B_l = \frac{8l}{l+4}, \quad (103)$$

which is valid for $l \geq 3$. These coefficients, for the first few mass multipoles and spin multipoles, read

$$\begin{aligned} A_2 &= 2, & A_4 &= 1, & A_6 &= \frac{2}{3}, & A_8 &= \frac{1}{2}, \\ A_{10} &= \frac{2}{5}, & B_3 &= \frac{24}{7}. \end{aligned} \quad (104)$$

In Eqs. (42) and (43) in [36], the following coefficients were presented:

$$\begin{aligned} A_2 &= \frac{11}{5}, & A_4 &= \frac{7}{6}, & A_6 &= \frac{3}{5}, & A_8 &= \frac{3}{10}, \\ A_{10} &= \frac{3}{20}, & B_3 &= \frac{7}{6}. \end{aligned} \quad (105)$$

The coefficients in (105) have been obtained for the case of finite spatial distances of source and observer from the massive body, while the coefficients in (104) are valid for infinite spatial distances of source and observer from the massive body. Since the general case of finite distances contains also the specific case of infinite distances, one would expect that the coefficients in (105) are slightly larger than the coefficients in (104). In fact, this is the case for the multipole orders $l = 2$ and $l = 4$, while the coefficients for $l = 6$ in (105) and (104) are almost equal. These facts are also reflected by the numerical values presented here in Table II and in Table II in our previous investigation in [36].

TABLE I. Numerical parameters for Schwarzschild radius GM/c^2 , equatorial radius P , and actual zonal harmonic coefficients J_l of the Sun, Jupiter, and Saturn. The values for GM/c^2 and P are taken from [39]. The values for J_l for the Sun are from [40] and references therein. The values J_l with $n = 2, 4, 6$ for Jupiter and Saturn are taken from [41], while J_l with $n = 8$ for Jupiter and Saturn come from [42,43], respectively. The angular velocities $\Omega = 2\pi/T$ (with rotational period T) are presented by NASA planetary fact sheets. The dimensionless moment of inertia κ^2 is defined by Eq. (106) and their values have been determined in [39].

Parameter	Sun	Jupiter	Saturn
GM/c^2 (m)	1476.8	1.41	0.42
P (m)	696×10^6	71.5×10^6	60.3×10^6
J_2	1.7×10^{-7}	14.696×10^{-3}	16.291×10^{-3}
J_4	9.8×10^{-7}	-0.587×10^{-3}	-0.936×10^{-3}
J_6	4×10^{-8}	0.034×10^{-3}	0.086×10^{-3}
J_8	-4×10^{-9}	-2.5×10^{-6}	-10.0×10^{-6}
Ω (sec $^{-1}$)	2.865×10^{-6}	1.758×10^{-4}	1.638×10^{-4}
κ^2	0.059	0.254	0.210

On the other side, the upper limits of mass-multipole terms of order $l = 8, 10$ and the upper limit of the spin-multipole term of order $l = 3$, presented in our previous investigation in [36], are too small. These deviations have been analyzed and were caused by an inaccuracy in the analytical calculations in [36], which were assisted by computer algebra systems because of the involved algebraic structure of the expressions for the Shapiro time delay, especially in the case of higher multipoles.

VII. NUMERICAL VALUES OF TIME DELAY IN THE SOLAR SYSTEM

For the numerical values of time delay we take the parameter of Solar System bodies as given by Table I.

The parameter κ^2 in (83) is defined by [39] [see also Eqs. (B60)–(B62) in [36]]

$$\kappa^2 = \frac{I}{MP^2}, \quad (106)$$

where I is the moment of inertia of the real Solar System body under consideration, which is related to the body's angular momentum via $|\mathbf{S}| = I\Omega$. For a spherically symmetric body with uniform density, $\kappa^2 = 2/5$ [39], while for real Solar System bodies, $\kappa^2 < 2/5$ because the mass densities are increasing toward the center of the massive bodies. The values of κ^2 are given in Table I for the Sun and giant planets of the Solar System bodies.

Numerical values of the upper limits in (99)–(101) are presented in Table II for the first mass multipoles and spin multipoles in the case of grazing rays at the Sun and the giant planets of the Solar System.

TABLE II. The effect of (one-way) Shapiro time delay caused by the mass-multipole $\Delta\tau_{1\text{PN}}^{M_l}$ and spin-multipole terms $\Delta\tau_{1.5\text{PN}}^{S_l}$ in the gravitational field of the Sun and giant planets of the Solar System according to the upper limits (99)–(101). The time delay is given in units of picoseconds: 1 ps = 10^{-12} sec. A blank entry means less than 0.001 ps. The values are given for grazing rays (impact parameter d_σ equals body's equatorial radius P). The numerical values should be compared with the assumed goal accuracy of 0.001 ps in time-delay measurements.

Object	$\Delta\tau_{1\text{PN}}^{M_2}$	$\Delta\tau_{1\text{PN}}^{M_4}$	$\Delta\tau_{1\text{PN}}^{M_6}$	$\Delta\tau_{1\text{PN}}^{M_8}$	$\Delta\tau_{1.5\text{PN}}^{S_1}$	$\Delta\tau_{1.5\text{PN}}^{S_3}$
Sun	1.68	4.83	0.13	0.01	7.73	
Jupiter	138.24	2.76	0.11	0.01	0.20	0.010
Saturn	45.65	1.31	0.08	0.01	0.04	0.003

VIII. SUMMARY

In this investigation, the time delay in the gravitational field of a body at rest with full multipole structure has been considered. In particular, the impact of higher mass-multipoles and spin-multipoles on time delay has been determined. Two main results were found:

- (1) If the source and the observer are located at spatial infinity from the massive body, then the individual multipole terms of time delay are related to the individual multipole terms of total light deflection. These relations are given by (80) and (81), which are valid for multipoles of order $l \geq 1$.
- (2) In the case of an axisymmetric massive body, these individual multipole terms of time delay can be expressed in terms of Chebyshev polynomials: mass-multipole terms of time delay are related to Chebyshev polynomials of the first kind (86) and spin-multipole terms of time delay are related to Chebyshev polynomials of the second kind (88).

These remarkable facts allow one to determine strict upper limits for the absolute value of the multipole terms in this asymptotic configuration, where the source and the observer are located at spatial infinity from the massive body. These strict upper limits are given by Eqs. (99) and (101), which represent a criterion to identify those multipoles that contribute significantly to the time delay for a given accuracy of time measurements. The coincidence between the total effect of time delay and total light deflection for higher multipoles, as demonstrated by relations (80) and (81), must have a deep reason, which represents a problem for subsequent investigations.

Numerical values of the Shapiro time delay, based on these upper limits, have been calculated and presented in Table II. These numerical results show that the impact of the first mass multipoles with $l \leq 8$ and the first spin multipoles with $l \leq 3$ on the effect of time delay are relevant for an accuracy on the femtosecond scale of

accuracy in time measurements and might, in principle, be detected with present-day atomic clocks or, at least, with the next generation of atomic clocks.

The more realistic astrometric configurations, where source and observer are located at finite spatial distances from the massive body, will be considered in a subsequent investigation. It is, however, expected that the upper limits of higher multipoles on the effect of time delay for such scenarios will not much be different from the asymptotic limit that has been considered in this investigation.

ACKNOWLEDGMENTS

This work was funded by the German Research Foundation (Deutsche Forschungsgemeinschaft DFG) under Grant No. 447922800. Sincere gratitude is expressed to Prof. Sergei A. Klioner, Prof. Michael H. Soffel, Prof. Ralf Schützhold, Prof. William G. Unruh, Priv.-Doz. Günter Plunien, Dr. Alexey Butkevich, Dipl.-Inf. Robin Geyer, Prof. Burkhard Kämpfer, and Prof. Laszlo Csernai for inspiring discussions about astrometry and general theory of relativity.

APPENDIX A: NOTATION

The following notation is in use:

- (i) Newtonian constant of gravitation: G .
- (ii) Vacuum speed of light in flat space-time: c .
- (iii) Newtonian mass of the body: M .
- (iv) Equatorial radius of the body: P .
- (v) Angular velocity of the body: Ω .
- (vi) Zonal harmonic coefficients of the body: J_l .
- (vii) $\eta_{\alpha\beta} = \eta^{\alpha\beta}$ is the metric tensor of flat space-time.
- (viii) $g^{\alpha\beta}$ and $g_{\alpha\beta}$ are the contravariant and covariant components of the metric.
- (ix) $g = \det(g_{\mu\nu})$ is the determinant of the metric.
- (x) $n! = n(n-1)(n-2) \cdots 2 \cdot 1$ is the factorial; by definition: $0! = 1$.
- (xi) Lowercase Greek indices take values 0–3.
- (xii) The contravariant components of four-vectors: $a^\mu = (a^0, a^1, a^2, a^3)$.
- (xiii) Lowercase Latin indices take values 1–3.
- (xiv) The three-dimensional coordinate quantities (three-vectors) referring to the spatial axes of the reference system are in boldface: \mathbf{a} .
- (xv) The contravariant components of three-vectors: $a^i = (a^1, a^2, a^3)$.
- (xvi) The absolute value of a three-vector: $a = |\mathbf{a}| = \sqrt{a^1 a^1 + a^2 a^2 + a^3 a^3}$.
- (xvii) The scalar product of two three-vectors: $\mathbf{a} \cdot \mathbf{b} = \delta_{ij} a^i b^j = a^i b^i$ with Kronecker delta δ_{ij} .
- (xviii) The angle between two three-vectors \mathbf{a} and \mathbf{b} is designated as $\delta(\mathbf{a}, \mathbf{b})$.

APPENDIX B: PROOF OF EQ. (54)

In this appendix, we will demonstrate the limit (54). The gauge terms in the geodesic equation (37) are given by Eq. (39), which consists of two pieces,

$$\ddot{\mathbf{x}}_{\text{gauge}} = \ddot{\mathbf{x}}_{\text{g1}} + \ddot{\mathbf{x}}_{\text{g2}}. \quad (\text{B1})$$

Their spatial components are given by

$$\frac{\ddot{x}_{\text{g1}}^i(t)}{c^2} = +\partial_j w_{,k}^0 \sigma^i \sigma^j \sigma^k, \quad (\text{B2})$$

$$\frac{\ddot{x}_{\text{g2}}^i(t)}{c^2} = -\partial_j w_{,k}^i \sigma^j \sigma^k, \quad (\text{B3})$$

where the gauge vectors are given by Eqs. (20) and (21). Let us consider the first term (B2). Using $(r^{-1})_{,jk} = 3x_j x_k / r^5 - \delta_{jk} / r^3$, one obtains

$$\begin{aligned} \frac{\ddot{\mathbf{x}}_{\text{g1}}(t)}{c^2} = & + \frac{8G}{c^3} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{W}_L}{(r_N)^3} \boldsymbol{\sigma} \\ & - \frac{12G}{c^3} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{W}_L}{(r_N)^5} (d_\sigma)^2 \boldsymbol{\sigma}. \end{aligned} \quad (\text{B4})$$

In (B4) the replacements $\mathbf{x} \rightarrow \mathbf{x}_N$ and $r \rightarrow r_N = |\mathbf{x}_N|$ have been performed [cf. text below Eq. (39)], where the unperturbed light ray \mathbf{x}_N is given by Eq. (26) and its absolute value by Eq. (27). In addition, the relation $(\boldsymbol{\sigma} \cdot \mathbf{x}_N)^2 = (r_N)^2 - (d_\sigma)^2$ has been used.

The expression (B4) has to be integrated over the time variable. To apply the advanced integration methods developed by [15], we have to transform (B4) from (ct, \mathbf{x}) into terms of two new variables, $c\tau = \boldsymbol{\sigma} \cdot \mathbf{x}_N$ and $\xi^i = P^{ij} x_N^j$, which are independent of each other, and obtain [note that, as always after differentiations are performed, $\boldsymbol{\xi} = \mathbf{d}_\sigma$, hence $(d_\sigma)^2 = \boldsymbol{\xi} \cdot \boldsymbol{\xi} = \xi^2$]

$$\frac{\ddot{\mathbf{x}}_{\text{g1}}(t)}{c^2} = + \frac{4G}{c^3} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{W}_L \hat{\partial}_L \left(\frac{2}{(r_N)^3} - \frac{3(\xi^2)}{(r_N)^5} \right) \boldsymbol{\sigma}, \quad (\text{B5})$$

where the double overdot in (B5) means twice the total derivative with respect to variable τ , and r_N is the absolute value of the unperturbed light ray in terms of these new variables (45). Let us note that the left-hand side in (B5) depends on the variable of coordinate time, because in (B5) the differentiations have to be performed, and afterward one has to replace $c\tau$ by $\boldsymbol{\sigma} \cdot \mathbf{x}_N(t)$ and $P^{ij} \xi_j$ by d_σ^i ; see text below Eq. (61). The differential operator (B5) has been given by Eq. (47). To get the coordinate velocity of the light signal, one has to integrate (B5) over variable $c\tau$ and obtain for the spatial components

$$\begin{aligned} \frac{\Delta \dot{x}_{\text{g1}}^i(t)}{c} &= - \frac{4G}{c^3} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{W}_L \hat{\partial}_L \frac{c\tau}{(r_N)^3} \sigma^i \\ &= + \frac{4G}{c^3} \frac{\partial}{\partial c\tau} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{W}_L}{r_N} \sigma^i. \end{aligned} \quad (\text{B6})$$

A similar calculation can be performed for the second gauge term (B3), which yields

$$\begin{aligned} \frac{\Delta \dot{x}_{\text{g2}}^i(t)}{c} &= - \frac{4G}{c^2} \frac{\partial}{\partial c\tau} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_{iL} \frac{\hat{X}_L}{r_N} \\ &\quad - \frac{4G}{c^2} \frac{\partial}{\partial c\tau} \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_{L-1} \frac{\hat{Y}_{iL-1}}{r_N} \\ &\quad - \frac{4G}{c^2} \frac{\partial}{\partial c\tau} \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \frac{l}{l+1} \epsilon_{iab} \hat{\partial}_{aL-1} \frac{\hat{Z}_{bL-1}}{r_N}. \end{aligned} \quad (\text{B7})$$

The second integration over variable $c\tau$, from lower integration limit $c\tau_0$ to upper integration limit $c\tau_1$, can be performed immediately and yields

$$\Delta x_{\text{g1}}^i(t_1, t_0) = \Delta x_{\text{g1}}^i(t_1) - \Delta x_{\text{g1}}^i(t_0), \quad (\text{B8})$$

$$\Delta x_{\text{g2}}^i(t_1, t_0) = \Delta x_{\text{g2}}^i(t_1) - \Delta x_{\text{g2}}^i(t_0), \quad (\text{B9})$$

with

$$\Delta x_{\text{g1}}^i(t) = + \frac{4G}{c^3} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{W}_L}{r_N} \sigma^i, \quad (\text{B10})$$

$$\begin{aligned} \Delta x_{\text{g2}}^i(t) &= - \frac{4G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_{iL} \frac{\hat{X}_L}{r_N} \\ &\quad - \frac{4G}{c^2} \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_{L-1} \frac{\hat{Y}_{iL-1}}{r_N} \\ &\quad - \frac{4G}{c^2} \epsilon_{iab} \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \frac{(-1)^l}{l!} \hat{\partial}_{aL-1} \frac{\hat{Z}_{bL-1}}{r_N}. \end{aligned} \quad (\text{B11})$$

The time dependence of the gauge terms (B8) and (B9) are just via the spatial positions of the unperturbed light ray at the time of emission and reception, $\mathbf{x}_N(t_0)$ and $\mathbf{x}_N(t_1)$. Furthermore, in line with (B1), the terms (B8) and (B9) are added together, so we arrive at

$$\Delta \mathbf{x}_{\text{gauge}}(t_1, t_0) = \Delta \mathbf{x}_{\text{g1}}(t_1, t_0) + \Delta \mathbf{x}_{\text{g2}}(t_1, t_0). \quad (\text{B12})$$

By inserting the differential operator (47) into these solutions (B10) and (B11) one finds that these terms in (B10) and (B11) vanish separately in the asymptotic limit,

$$\lim_{\tau=\pm\infty} \Delta \mathbf{x}_{\text{g1}}(\mathbf{x}_N(t)) = 0, \quad (\text{B13})$$

$$\lim_{\tau=\pm\infty} \Delta \mathbf{x}_{g^2}(\mathbf{x}_N(t)) = 0, \quad (\text{B14})$$

with $c\tau = \boldsymbol{\sigma} \cdot \mathbf{x}_N(t)$. Equations (B13) and (B14) imply for (B12) the limits

$$\lim_{\substack{\tau=\tau_0 \rightarrow -\infty \\ \tau=\tau_1 \rightarrow +\infty}} \Delta \mathbf{x}_{\text{gauge}}(t_1, t_0) = 0, \quad (\text{B15})$$

which is more general than the asserted relation in (54).

APPENDIX C: PROOF OF EQS. (67) AND (68)

In this appendix, we will show relations (67) and (68). That means, we consider the limits

$$\lim_{\substack{\tau=\tau_0 \rightarrow -\infty \\ \tau=\tau_1 \rightarrow +\infty}} \hat{\partial}_L \ln(r_N + c\tau), \quad (\text{C1})$$

where $r_N = \sqrt{\xi^2 + c^2\tau^2}$ with $\xi^2 = \boldsymbol{\xi} \cdot \boldsymbol{\xi}$, while the differential operator is given by Eq. (47), and $c\tau_0$ and $c\tau_1$ are defined by Eqs. (55) and (56), respectively.

First of all, we consider the first derivative of (C1) with respect to variable $c\tau$,

$$\lim_{\substack{\tau=\tau_0 \rightarrow -\infty \\ \tau=\tau_1 \rightarrow +\infty}} \frac{\partial}{\partial c\tau} \ln(r_N + c\tau) = \lim_{\substack{\tau=\tau_0 \rightarrow -\infty \\ \tau=\tau_1 \rightarrow +\infty}} \frac{1}{\sqrt{\xi^2 + c^2\tau^2}} = 0. \quad (\text{C2})$$

Clearly, any further derivative of this term, with respect to either variable $\boldsymbol{\xi}$ or $c\tau$, increases the inverse power of $c\tau$ at least by one order. Therefore, all those terms in the differential operator (47) that contain at least one derivative with respect to variable $c\tau$ will vanish in these limits. Hence, one has only to consider the term with $p = 0$ of the differential operator in Eq. (47), given by

$$\hat{\partial}_L^{p=0} = \text{STF}_{i_1 \dots i_l} P_{i_1}^{j_1} \dots P_{i_l}^{j_l} \frac{\partial}{\partial \xi^{j_1}} \dots \frac{\partial}{\partial \xi^{j_l}}, \quad (\text{C3})$$

in line with the comment below Eq. (68).

The logarithm in (C1) can be written in the form

$$\ln(r_N + c\tau) = \ln|c\tau| + \ln\left(\sqrt{1 + \xi^2/c^2\tau^2} \pm 1\right), \quad (\text{C4})$$

where the plus sign in the argument of the logarithm is for $c\tau > 0$, while the minus sign in the argument of the logarithm is for $c\tau < 0$. The first logarithm on the right-hand side of (C4) can be omitted because

$$\frac{\partial}{\partial \xi^{j_{p+1}}} \dots \frac{\partial}{\partial \xi^{j_l}} \ln|c\tau| = 0. \quad (\text{C5})$$

Thus, one only needs to consider the second logarithmic term on the right-hand side of (C4). In order to evaluate these limits in (C1), it is appropriate to introduce the dimensionless three-vector,

$$\zeta^a = \frac{\xi^a}{c\tau} \rightarrow \frac{\partial}{\partial \xi^a} = \frac{1}{c\tau} \frac{\partial}{\partial \zeta^a}, \quad (\text{C6})$$

where the absolute value of this three-vector, $\zeta = \sqrt{\boldsymbol{\zeta} \cdot \boldsymbol{\zeta}}$, is a small quantity in the limits $c\tau \rightarrow \pm\infty$, that means $\xi \rightarrow 0$, because the impact vector $\boldsymbol{\xi}$ remains constant; see also Fig. 2. The differential operator in (C3) in terms of the three-vector ζ^a transforms into

$$\hat{\partial}_L^{p=0} = \frac{1}{(c\tau)^l} \text{STF}_{i_1 \dots i_l} P_{i_1}^{j_1} \dots P_{i_l}^{j_l} \frac{\partial}{\partial \zeta^{j_1}} \dots \frac{\partial}{\partial \zeta^{j_l}}. \quad (\text{C7})$$

Let us consider the term with the plus sign in (C1), which in terms of the three-vector (C6) reads

$$\lim_{\tau=\tau_1 \rightarrow +\infty} \hat{\partial}_L \ln(r_N + c\tau) = \lim_{\tau=\tau_1 \rightarrow +\infty} \frac{1}{(c\tau)^l} \times \text{STF}_{i_1 \dots i_l} P_{i_1}^{j_1} \dots P_{i_l}^{j_l} \frac{\partial}{\partial \zeta^{j_1}} \dots \frac{\partial}{\partial \zeta^{j_l}} \ln(\sqrt{1 + \zeta^2} + 1). \quad (\text{C8})$$

The term in the second line of (C8) is finite, even for any values of ζ^2 . Thus, in view of the prefactor $(c\tau)^{-l}$, the term (C8) vanishes in the limit $c\tau \rightarrow +\infty$ and one obtains

$$\lim_{\tau=\tau_1 \rightarrow +\infty} \hat{\partial}_L \ln(r_N + c\tau) = 0. \quad (\text{C9})$$

Let us consider the term with the minus sign in (C1), which in terms of the three-vector (C6) reads

$$\lim_{\tau=\tau_0 \rightarrow -\infty} \hat{\partial}_L \ln(r_N + c\tau) = \lim_{\tau=\tau_0 \rightarrow -\infty} \frac{1}{(c\tau)^l} \text{STF}_{i_1 \dots i_l} \times P_{i_1}^{j_1} \dots P_{i_l}^{j_l} \frac{\partial}{\partial \zeta^{j_1}} \dots \frac{\partial}{\partial \zeta^{j_l}} \ln(\sqrt{1 + \zeta^2} - 1). \quad (\text{C10})$$

The term in the second line of (C10) diverges in the limit $\zeta \rightarrow 0$. To determine this limit, a series expansion of the logarithm in (C10) is performed for $\zeta \ll 1$, which reads

$$\ln(\sqrt{1 + \zeta^2} - 1) = -\ln(2) + \ln(\zeta^2) - \frac{\zeta^2}{4} + \mathcal{O}(\zeta^4). \quad (\text{C11})$$

The constant $\ln(2)$ does not contribute, because a derivative of a constant with respect to variable ζ in (C10) vanishes. Similarly, the third term of power ζ^2 and also terms of higher powers $\mathcal{O}(\zeta^4)$ on the right-hand side in (C11) vanish in the limit $\zeta \rightarrow 0$ and, in addition, also in view of the prefactor $(c\tau)^{-l}$ in (C10). Thus, only the logarithmic term remains, keeping in mind that the differential operator (C7) is acting on this logarithm. By transforming (C11) from three-vector $\boldsymbol{\zeta}$ back into three-vector $\boldsymbol{\xi}$ and by accounting for the statements below Eq. (C11), one arrives at

$$\lim_{\tau \rightarrow \tau_0 \rightarrow -\infty} \hat{\partial}_L \ln(r_N + c\tau) = 2\hat{\partial}_L^{p=0} \ln|\xi|. \quad (\text{C12})$$

By relations (C9) and (C12) the validity of Eqs. (67) and (68) has been shown. Let us notice again that the differential operator $\hat{\partial}_L$ on the left-hand side in (C9) and (C12) is given

by Eq. (47), while on the right-hand side the differential operator $\hat{\partial}_L^{p=0}$ is given by (C3). However, one may keep the differential operator $\hat{\partial}_L$ on the right-hand side in (C9) and (C12), because derivatives with respect to variable $c\tau$ in the differential operator (47) would not contribute anyway.

-
- [1] I. I. Shapiro, Fourth test of general relativity, *Phys. Rev. Lett.* **13**, 789 (1964).
- [2] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Palgrave Macmillan, New York, 1973).
- [3] C. M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, United Kingdom, 1993).
- [4] I. I. Shapiro, G. H. Pettengill, M. E. Ash, M. L. Stone, W. B. Smith, R. P. Ingalls, and R. A. Brockelman, Fourth test of general relativity: Preliminary results, *Phys. Rev. Lett.* **20**, 1265 (1968).
- [5] I. I. Shapiro, M. E. Ash, R. P. Ingalls, W. B. Smith, D. B. Campbell, R. B. Dyce, R. F. Jurgens, and G. H. Pettengill, Fourth test of general relativity: New radar result, *Phys. Rev. Lett.* **26**, 1132 (1971).
- [6] B. Bertotti, L. Iess, and P. Tortora, A test of general relativity using radio links with the Cassini spacecraft, *Nature (London)* **425**, 374 (2003).
- [7] S. M. Brewer, J. S. Chen, A. M. Hankin, E. R. Clements, C. W. Chou, D. J. Wineland, D. B. Hume, and D. R. Leibbrandt, $27^{\text{A}}I^+$ quantum-logic clock with a systematic uncertainty below 10^{-18} , *Phys. Rev. Lett.* **123**, 033201 (2019).
- [8] Wei-Tou Ni, Astrod and Astrod 1—overview and progress, *Int. J. Mod. Phys. D* **17**, 921 (2008).
- [9] J. E. Plowman and R. W. Hellings, LATOR covariance analysis, *Classical Quantum Gravity* **23**, 309 (2006).
- [10] B. Christophe, P. H. Andersen, J. D. Anderson *et al.*, Odyssey: A Solar System mission, *Exp. Astron.* **23**, 529 (2009).
- [11] P. Wolf, Ch. J. Bordé, A. Clairon *et al.*, Quantum physics exploring gravity in the outer Solar System: The Sagas project, *Exp. Astron.* **23**, 651 (2009).
- [12] E. Samain, One way laser ranging in the Solar System: Tipo, *Geowiss. Mitt.* **69**, 80 (2004).
- [13] S. Schiller, G. M. Tino, and P. Gill, Einstein gravity explorer: A medium-class fundamental physics mission, *Exp. Astron.* **23**, 573 (2009).
- [14] D. W. Allan and N. Ashby, Coordinate time in the vicinity of the Earth in *Relativity in Celestial Mechanics and Astrometry*, edited by J. Kovalevsky and V. A. Brumberg (1986), pp. 299–313.
- [15] S. M. Kopeikin, Propagation of light in the stationary field of multipole gravitational lens, *J. Math. Phys. (N.Y.)* **38**, 2587 (1997).
- [16] S. Zschocke, Total light deflection in the gravitational field of an axisymmetric body at rest with full mass and spin multipole structure, *Phys. Rev. D* **107**, 124055 (2023).
- [17] S. Kopeikin, M. Efroimsky, and G. Kaplan, *Relativistic Celestial Mechanics of the Solar System* (Wiley-VCH, Singapore, 2012).
- [18] S. Carroll, *Spacetime and geometry: An introduction to general relativity* (Pearson New International, Edinburgh Gate, UK, 2013).
- [19] L. Blanchet, S. M. Kopeikin, and G. Schäfer, Gravitational radiation theory and light propagation, *Lect. Notes Phys.* **562**, 141 (2001).
- [20] S. A. Klioner and S. M. Kopeikin, Microarcsecond astrometry in space: Relativistic effects and reduction of observations, *Astron. J.* **104**, 897 (1992).
- [21] K. S. Thorne, Multipole expansions of gravitational radiation, *Rev. Mod. Phys.* **52**, 299 (1980).
- [22] T. Damour and B. R. Iyer, Multipole analysis for electromagnetism and linearized gravity with irreducible Cartesian tensors, *Phys. Rev. D* **43**, 3259 (1991).
- [23] S. Zschocke, A detailed proof of the fundamental theorem of STF multipole expansion in linearized gravity, *Int. J. Mod. Phys. D* **23**, 1450003 (2014).
- [24] L. Blanchet and T. Damour, Radiative gravitational fields in general relativity: I. General structure of the field outside the source, *Phil. Trans. R. Soc. A* **320**, 379 (1986).
- [25] L. Blanchet, On the multipole expansion of the gravitational field, *Classical Quantum Gravity* **15**, 1971 (1998).
- [26] V. A. Brumberg, *Essential Relativistic Celestial Mechanics* (Adam Hilder, Bristol, 1991).
- [27] S. M. Kopeikin, G. Schäfer, C. R. Gwinn, and T. M. Eubanks, Astrometric and timing effects of gravitational waves from localized sources, *Phys. Rev. D* **59**, 084023 (1999).
- [28] S. Zschocke, Light propagation in the gravitational field of N arbitrarily moving bodies in 1PN approximation for high-precision astrometry, *Phys. Rev. D* **92**, 063015 (2015).
- [29] S. Zschocke, Light propagation in the gravitational field of N arbitrarily moving bodies in the 1.5PN approximation for high-precision astrometry, *Phys. Rev. D* **93**, 103010 (2016).
- [30] S. A. Klioner and M. Peip, Numerical simulations of the light propagation in the gravitational field of moving bodies, *Astron. Astrophys.* **410**, 1063 (2003).
- [31] S. Kopeikin, P. Korobkov, and A. Polnarev, Propagation of light in the field of stationary and radiative gravitational multipoles, *Classical Quantum Gravity* **23**, 4299 (2006).
- [32] S. Kopeikin and P. Korobkov, General relativistic theory of light propagation in the field of radiative gravitational multipoles, arXiv:gr-qc/0510084v2.

- [33] M. Soffel and W. B. Han, *Applied General Relativity* (Springer, Cham, Switzerland, 2019).
- [34] I. Ciufolini, S. Kopeikin, B. Mashhoon, and F. Ricci, On the gravitomagnetic time delay, *Phys. Lett. A* **308**, 101 (2003).
- [35] S. Zschocke, Total light deflection in the gravitational field of Solar System bodies, *Proceeding Conference Les Journées: Time and General Relativity* (Université Côte d'Azur, Nice, France, 2023), [arXiv:gr-qc/2310.17345](https://arxiv.org/abs/gr-qc/2310.17345).
- [36] S. Zschocke, Time delay in the quadrupole field of a body at rest in 2PN approximation, *Phys. Rev. D* **106**, 104052 (2022).
- [37] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists*, 4th ed. (Academic Press, London, 1995).
- [38] S. A. Klioner, Influence of the quadrupole field and rotation of objects on light propagation, *Sov. Astron.* **35**, 523 (1991).
- [39] H. Essen, The physics of rotational flattening and the point core model, *Int. J. Geosci.* **5**, 555 (2014).
- [40] T. J. Lydon and S. Sofia, A measurement of the shape of the solar disk: The solar quadrupole moment, the solar octopole moment, and the advance of perihelion of the planet Mercury, *Phys. Rev. Lett.* **76**, 177 (1996).
- [41] I. de Pater and J. J. Lissauer, *Planetary Science*, 2nd ed. (Cambridge University Press, Cambridge, England, 2015).
- [42] W. B. Hubbard and B. Militzer, A preliminary Jupiter model, *Astron. J.* **820**, 80 (2016).
- [43] J. D. Anderson and G. Schubert, Saturn's gravitational field, internal rotation, and interior structure, *Science* **317**, 1384 (2007).
- Correction:* The second and third sentences after Eq. (1) contained incorrect units for time and have been fixed.