Normal modes of Proca fields in AdS_d spacetime

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The normal modes of Proca field perturbations in d-dimensional anti–de Sitter spacetime, AdS_d for short, with reflective Dirichlet boundary conditions, are obtained exactly. Within the Ishibashi-Kodama framework, we decompose the Proca field in scalar-type and vector-type components, according to their tensorial behavior on the $(d-2)$-sphere $\mathcal{S}^{d-2}$. Two of the degrees of freedom of the Proca field are described by scalar-type components, which in general are coupled due to the mass of the field, but in AdS_d we show that they can be decoupled. The other $d-3$ degrees of freedom of the field are described by a vector-type component that generically decouples completely. The normal modes and their frequencies for both the scalar-type and vector-type components of the Proca field are then obtained analytically. Additionally, we analyze the normal modes of the Maxwell field as the massless limit of the Proca field. We find that for scalar-type perturbations in $d=4$ there is a discontinuity in the massless limit, in $d=5$ the massless limit is well-defined using Dirichlet-Neumann rather than Dirichlet boundary conditions, and in $d>5$ the massless limit is completely well-defined, i.e., it is obtained smoothly from the massless limit of the scalar-type perturbations of the Proca field. For vector-type perturbations the Maxwell field limit is obtained smoothly for all d from the massless limit of the vector-type perturbations of the Proca field.

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I. INTRODUCTION

The anti–de Sitter (AdS) spacetime [1–3] is the maximally symmetric vacuum solution to the Einstein field equations with a negative cosmological constant. This spacetime can be obtained by performing the universal cover of the AdS universe. As a result, the AdS spacetime is not a globally hyperbolic spacetime and it possesses a timelike boundary at spatial infinity.

Due to the properties at spatial infinity, there is an intrinsic interest in asymptotically AdS spacetimes as they can describe systems in a gravitational box. In particular, pure AdS spacetime is of special importance in the construction of quantum field theories [4], where two reflective and one transparent boundary conditions to the fields are possible at infinity. AdS spacetime is also essential in the formulation of supergravity theories, where it acts as a natural arena for quantum supersymmetric fields, including the possibility that these might have negative mass [5]. In this connection, AdS plays a fundamental role in the AdS/CFT conjecture, that establishes a duality between supergravity as a low-energy phenomenon of string theory in AdS, and a conformal field theory at its boundary [6].

Given the importance of the AdS spacetime it is relevant to study and understand its stability. To linear perturbations AdS is stable, as we will see below, but to nonlinear perturbations it seems that AdS is unstable. This instability was explored in [7], where the evolution of the Einstein-Klein-Gordon system was considered and a large class of arbitrary small amplitudes of the scalar field was found to have an evolution leading to black hole formation. It was further reinforced by analyzing a complex scalar field [8], yet it does not occur for all classes of initial data as there are islands of stability [9].

Linear stability of a spacetime is also important. To study it, one must analyze linear perturbations which are described by normal modes for the case of pure AdS and quasinormal modes in black hole spacetimes be they asymptotically flat, AdS, or otherwise. In spherical symmetry, the perturbations are classified as scalar-type, vector-type, and tensor-type perturbations, which regards their tensorial behavior on the 2-sphere. This decomposition allows to write the linearized field equations as a radial Schrödinger-like equation with an effective given potential for each type of perturbation. Moreover, to obtain normal modes and quasinormal modes, one must impose boundary conditions at the center or at the black hole horizon if there is one, and at spatial infinity. The linear stability of a Schwarzschild black hole spacetime was solved first trough the Regge-Wheeler formalism by carefully expanding the perturbations [10,11]. In [12] boundary conditions and all

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types of perturbations in the Schwarzschild-AdS black hole were imposed and worked out. In [13] it was analyzed Proca massive vector field perturbations in a Schwarzschild background. In [14] spherical waves of spin-1 particle in anti–de Sitter spacetime were performed. In [15] massive vector fields on the Schwarzschild spacetime were further analyzed. For other boundary conditions based on the vanishing of the energy flux see [16]. In [17] quasinormal modes of Proca fields in a Schwarzschild-AdS spacetime were found. In [18] the normal modes of Proca fields in AdS spacetime were found.

Higher dimensions are important. The universe may be higher dimensional somewhere; higher dimensions may have existed at some time in the very early universe, or perhaps they can be constructed or detected in a future experiment. The study of the physics in higher dimensions also provides a means of understanding what is intrinsic and important to $d = 4$. AdS spacetime can be extended to higher dimensions which is referred to as AdS_d, where d is the number of dimensions. Normal modes and quasinormal modes have also been studied in AdS_d spacetimes. In spherical symmetry in higher dimensions, the perturbations are also classified as scalar-type, vector-type, and tensor-type perturbations, which regards their behavior now on the $(d-2)$-sphere. Again, the decomposition allows us to write the linearized field equations as a radial Schrödinger-like equation for each type of perturbation, whose potential now also depends on the dimension of the spacetime. The normal modes of AdS_d were first obtained in [19] for a scalar field. The problem of linear stability in higher-dimensional spacetimes was considered in the Ishibashi-Kodama formalism by expanding the perturbations in higher-dimensional scalar, vector, and tensor spherical harmonics, which independently form a complete basis on a $(d-2)$-sphere [20]. In [21] it was found that in some cases the Dirichlet boundary conditions are not the only suitable boundary conditions and generalized Robin boundary conditions are also permitted, a general analysis of the equations that the scalar, electromagnetic, and gravitational perturbations obey in AdS_d was made, and a general formula for the eigenfrequencies in a range of parameters of the equation was obtained. Moreover, scalar-type, vector-type and tensor-type gravitational perturbations were studied and the eigenfrequencies given imposing Dirichlet boundary conditions [22]. The result was extended to the scalar-type and vector-type Maxwell electromagnetic perturbations in [23]. The problem of linear stability in higher-dimensional spacetimes was further addressed by expanding perturbations in higher-dimensional scalar, vector and tensor spherical harmonics, which independently form a complete basis on a $(d-2)$-sphere, now with applications to black hole spacetimes [24]. In [25] the wave equation for a Proca field in d-dimensional spherically symmetric black hole spacetimes was obtained with interest in understanding the Hawking radiation for a Proca field in d-dimensions. In [26] massive vector field perturbations on extremal and near-extremal static black holes were analyzed. A study of the Proca perturbations in d-dimensional pure AdS is thus of interest. The mass of the Proca field introduces a coupling between two scalar-type degrees of freedom. For black hole spacetimes in general, a decoupling of these two degrees of freedom does not seem to be analytically allowed. For pure AdS in four dimensions, however, the scalar-type degrees of freedom can be decoupled by making a transformation of the fields, and it is thus of interest to know if this occurs in AdS_d.

In this work we obtain the exact expression for normal modes of linear Proca perturbations in AdS_d background, using the Ishibashi-Kodama formalism. We find that the scalar-type degrees of freedom decouple in AdS_d, by making a linear transformation to the relevant fields. We also study the electromagnetic perturbations in order to understand the $\mu = 0$ limit of the Proca field. We consulted results in [27–29] on $(d-2)$-sphere, and use results of the priceless manual [30].

The work is organized as follows. In Sec. II we introduce the field equations for a Proca field minimally coupled to curved spacetime background. In Sec. III, we obtain the equations for the Proca field perturbations in pure AdS spacetime by introducing the Ishibashi-Kodama formalism and further decomposing the Proca field in scalar-type and vector-type components, according to their tensorial behavior on the $(d-2)$-sphere. We also decouple the scalar-type components by making a linear transformation to the relevant fields. In Sec. IV, we obtain the normal mode eigenfrequencies and eigenfunctions as a solution to the Proca field equations, which can be put into Schrödinger-like equations. In Sec. V, we rederive the normal mode frequencies in AdS_d of the Maxwell electromagnetic perturbations and analyze it in the context of the zero mass limit of the Proca field. In Sec. VI, we conclude. In Appendix A, we review the properties of spherical harmonics on the $(d-2)$-sphere. In Appendix B, we study the solutions of the hypergeometric differential equation.

II. PROCA FIELD IN CURVED SPACETIME

The action of a Proca field, i.e., a massive vector field, minimally coupled to the metric field of a generic d-dimensional curved spacetime with negative cosmological constant can be written as

$$S = S_{\mathrm{EH}} + S_{\mathrm{P}}, \tag{1}$$

where

$$S_{\mathrm{EH}} = \int d^d x \sqrt{-g}\, \frac{R - 2\Lambda}{16\pi}, \tag{2}$$

is the Einstein-Hilbert action,

$$S_{\rm P} = -\int d^d x\sqrt{-g}\left(\frac{1}{2}\mu^2 A_\mu A^\mu + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \qquad (3)$$

is the Proca action, g is the determinant of the metric $g_{\mu\nu}$, $R = R_{\mu\nu}g^{\mu\nu}$ is the Ricci scalar defined as the trace of the Ricci tensor R_{ab} composed by the metric itself and its first and second derivatives, $\Lambda = -\frac{(d-1)(d-2)}{2l^2}$ is the cosmological constant with l being the characteristic AdS length, A_μ is the Proca field with mass μ and $F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the Proca field strength. Spacetime indices are denoted by Greek letters, e.g., μ, ν, run from 0 to $d-1$, where 0 is the time index, and 1 to $d-1$ specify the spatial indices. The field equations for the metric and the Proca field are obtained by applying the variational principle to the action given in Eq. (1). The field equations for the metric are then the Einstein equations given by

$$G_{\mu\nu} - \frac{(d-1)(d-2)}{2l^2}g_{\mu\nu} = 8\pi T_{\mu\nu}, \qquad (4)$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor and $T_{\mu\nu}$ is the Proca stress-energy tensor, given by

$$T_{\mu\nu} = g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} + \mu^2 A_\mu A_\nu - g_{\mu\nu}\left(\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{\mu^2}{2}A_\alpha A^\alpha\right). \qquad (5)$$

Moreover, the Einstein tensor obeys the Bianchi identities $\nabla^\mu G_{\mu\nu} = 0$, which in turn imply the conservation law for $T_{\mu\nu}$, i.e., $\nabla^\mu T_{\mu\nu} = 0$. The Proca field equations are obtained either by the conservation law for $T_{\mu\nu}$ or by varying the action with respect to the Proca field A_μ and can be written as

$$\nabla_\nu F^{\mu\nu} + \mu^2 A^\mu = 0. \qquad (6)$$

Due to $F_{\mu\nu}$ being an antisymmetric tensor, one can calculate the divergence of Eq. (6) to obtain a Bianchi identity for A_μ,

$$\nabla^\mu A_\mu = 0. \qquad (7)$$

It must be noted that Eq. (7) is a direct consequence of the Proca field equation, Eq. (6), when $\mu \neq 0$. Thus, A_μ is a physical field and describes $d-1$ degrees of freedom, as one component of the vector field can always be obtained from the others by integrating Eq. (7). For $\mu = 0$, A_μ corresponds to the Maxwell field and the field equation, Eq. (6), becomes invariant under the gauge transformation $A^\mu \to A^\mu + \partial^\mu h$, where h is an arbitrary scalar field. The Bianchi identity for Proca fields, Eq. (7), ceases to be a consequence of the field equation and becomes the usual Lorenz gauge condition of the Maxwell field. Even after imposing this gauge, a residual gauge freedom remains as Eq. (7) is invariant under $A^\mu \to A^\mu + \partial^\mu h$ if $\nabla_\mu \nabla^\mu h = 0$, i.e., if h obeys the Klein-Gordon equation. Hence, the Maxwell field describes $d-2$ degrees of freedom. In $d = 4$, the previous discussion implies that while the Proca field describes three degrees of freedom, corresponding to two transversal polarizations and one longitudinal polarization, the residual gauge freedom of the Maxwell field eliminates the longitudinal polarization, so that in total the Maxwell field describes two degrees of freedom, corresponding to the two transversal polarizations.

To formally describe the spacetime permeated by the Proca field, one would have to solve Eq. (4) for the metric $g_{\mu\nu}$ and Eq. (6) for the Proca field A_μ simultaneously. Here we are only interested in linear perturbations of the Proca field A_μ around the trivial solution $A_\mu = 0$. As a result, we only consider Eqs. (4) and (6) up to first order in perturbations of the Proca field. Since $T_{\mu\nu}$ in Eq. (5) is of second order in A_μ, linear perturbations in the Proca field induce a curvature perturbation on $g_{\mu\nu}$ only at second order and $T_{\mu\nu}$ can be neglected in Eq. (4) at first order, reducing Eq. (4) to the vacuum Einstein equations with negative cosmological constant. Thus, $g_{\mu\nu}$ corresponds to the background metric, as if the Proca field A_μ was absent. Moreover, the linear Proca perturbations obey the Proca equation, Eq. (6), with the connection $\Gamma^\mu_{\nu\sigma}$ being associated to the background metric.

In what follows, we apply the Ishibashi-Kodama formalism to the Proca equation in a d-dimensional AdS spacetime. We firstly write the Proca equation and decompose the A_μ field in the AdS$_d$ background.

III. LINEAR PROCA FIELD PERTURBATIONS IN AdS$_d$

A. The factorization of AdS$_d$ spacetime and the Proca equation

We consider the d-dimensional AdS spacetime which is a solution of the vacuum Einstein equation with a cosmological constant, Eq. (4). The d-dimensional AdS spacetime manifold, $\mathcal{M}^d \equiv \mathrm{AdS}_d$, $d \geq 4$, can be written as a warped product of a submanifold $\mathcal{N}^2$ of dimension two with a $(d-2)$-sphere, $\mathcal{S}^{d-2}$, i.e., $\mathcal{M}^d = \mathcal{N}^2 \times \mathcal{S}^{d-2}$. Its associated line element $g_{\mu\nu}dx^\mu dx^\nu$ in coordinates x^μ, $\mu = 0, \ldots, d-1$, is thus decomposed as

$$g_{\mu\nu}dx^\mu dx^\nu = \tilde{g}_{ab}dy^a dy^b + r^2\hat{g}_{ij}d\theta^i d\theta^j, \qquad (8)$$

where

$$\tilde{g}_{ab}dy^a dy^b = -f(r)dt^2 + \frac{dr^2}{f(r)}, \qquad f(r) = 1 + \frac{r^2}{l^2}, \qquad (9)$$

is the line element of the $\mathcal{N}^2$ submanifold, written in coordinates $y^a = (y^0, y^1) = (t, r)$, with t and r being the time and radial coordinates, respectively, and

$$\hat{g}_{ij} d\theta^i d\theta^j = (d\theta^2)^2 + \sum_{i=3}^{d-1} \prod_{k=2}^{i-1} \sin^2(\theta^k)(d\theta^i)^2, \quad (10)$$

which is the line element of the $(d-2)$-sphere, $\mathcal{S}^{d-2}$, sometimes represented as $(d\Omega^{d-2})^2$, where the θ^i are the angular coordinates and $i, j, k = 2, \ldots, d-1$. Note that in this convention θ^2 would be the usual θ and θ^3 would be the usual ϕ in $d = 4$. As it should be clear by now, to allow one to distinguish between tensors living on the different manifolds $\mathcal{M}^d$, $\mathcal{N}^2$, and $\mathcal{S}^{d-2}$, we use Greek indices $\mu, \nu, \ldots$ for tensors on $\mathcal{M}^d$, latin indices in the range $a, b, \ldots, h$ for tensors on $\mathcal{N}^2$ and latin indices in the range $i, j, \ldots$ for tensors on $\mathcal{S}^{d-2}$.

In order to factorize AdS_d spacetime, one also must separate the connection ∇_μ associated to the manifold $(\mathcal{M}^d, g)$ into the connection $\tilde{\nabla}_a$ associated to the manifold $(\mathcal{N}^2, \tilde{g})$ and the connection $\hat{\nabla}_i$ associated to the manifold $(\mathcal{S}^{d-2}, \hat{g})$. This can be accomplished by using the following relations between the Christoffel symbols associated to each manifold, those are

$$\Gamma^a{}_{bc} = \tilde{\Gamma}^a{}_{bc}, \qquad \Gamma^a{}_{ij} = -r(\partial^a r)\hat{g}_{ij},$$
$$\Gamma^i{}_{aj} = \frac{\partial_a r}{r}\delta^i_j, \qquad \Gamma^i{}_{jk} = \hat{\Gamma}^i{}_{jk}, \quad (11)$$

where $\Gamma^\mu{}_{\nu\rho}$ are the Christoffel symbols associated with the metric $g_{\mu\nu}$, with the greek indices spanning through the indices in $\mathcal{N}^2$ and in $\mathcal{S}^{d-2}$, e.g. $\mu = \{a, i\}$, and $\tilde{\Gamma}^a{}_{bc}$ and $\hat{\Gamma}^i{}_{jk}$ are the Christoffel symbols associated with the metrics $\tilde{g}_{ab}$ and $\hat{g}_{ij}$, respectively.

The projections of the Proca equation, Eq. (6), into $\mathcal{N}^2$ and $\mathcal{S}^{d-2}$ are written as

$$\tilde{\nabla}_b F^{ab} + (d-2)\frac{\partial_b r}{r}F^{ab} + \hat{\nabla}_j F^{aj} + \mu^2 A^a = 0, \quad (12)$$

$$\tilde{\nabla}_b F^{ib} + (d-2)\frac{\partial_b r}{r}F^{ib} + \hat{\nabla}_j F^{ij} + \mu^2 A^i = 0, \quad (13)$$

respectively. These equations are supplemented with the Bianchi identity, Eq. (7), which is now

$$\tilde{\nabla}_a A^a + (d-2)\frac{\partial_a r}{r}A^a + \hat{\nabla}_i A^i = 0. \quad (14)$$

B. The decomposition of the Proca field and spherical harmonics expansion

To simplify the field equations for A_μ, one exploits the spherical symmetry of the background metric in Eq. (8). The strategy is to project the field A_μ into components that are orthogonal to $\mathcal{S}^{d-2}$ and components that are tangent to $\mathcal{S}^{d-2}$. The A_μ field can be written as

$$A_\mu dx^\mu = \tilde{A}_a dy^a + \hat{A}_i d\theta^i, \quad (15)$$

where $\tilde{A}_a$ denotes the projection of A_μ orthogonal to the cotangent space of $\mathcal{S}^{d-2}$ and $\hat{A}_i$ denotes the projection of A_μ tangent to the cotangent space of $\mathcal{S}^{d-2}$. The latter can be further decomposed using the Helmoltz-Hodge decomposition [20], which allows one to write uniquely a dual vector field on $\mathcal{S}^{d-2}$, $\hat{A}_i$, as the sum of a scalar field on $\mathcal{S}^{d-2}$, $\hat{A}^{(s)}$, and a transverse covector field on the cotangent space of $\mathcal{S}^{d-2}$, $\hat{A}_i^{(v)}$, in the following way:

$$\hat{A}_i = \hat{A}_i^{(v)} + \hat{\nabla}_i \hat{A}^{(s)}, \qquad \hat{\nabla}_i \hat{A}^{(v)i} = 0. \quad (16)$$

Since $\tilde{A}_a$ and $\hat{A}^{(s)}$ behave as scalars on $\mathcal{S}^{d-2}$, these are called the scalar-type components of A_μ, while $\hat{A}^{(v)i} = \hat{g}^{ij}\hat{A}_j^{(v)}$, a vector on the tangent space of $\mathcal{S}^{d-2}$, is called the vector-type component of A_μ. Moreover, the scalar-type components of the Proca field transform as scalars under the $\mathrm{SO}(d-1)$ rotation group, and the vector-type component of the Proca field transforms as a vectors under the $\mathrm{SO}(d-1)$ rotation group. Since the correspondence between $\hat{A}^{(v)i}$ and $\hat{A}_i^{(v)}$ is one to one, we shall make the abuse of language that $\hat{A}_i^{(v)}$ corresponds to the vector-type component of A_μ as well. In the literature, scalar-type is also referred to as polar- or even-type, whereas vector-type is called axial- or odd-type. This has to do with the transformation properties of these components under parity transformations, see [10,11]. The scalar-type components of A_μ can be expanded in scalar harmonics $Y_{\vec{k}_s}$, where $\vec{k}_s$ is a vector containing the angular momentum number ℓ and the $d-3$ azimuthal numbers, which form a complete basis on $\mathcal{S}^{d-2}$, satisfying

$$(\hat{\Box} + k_s^2)Y_{\vec{k}_s} = 0, \qquad \int d\Omega^{d-2} Y_{\vec{k}_s} Y_{\vec{k}'_s} = \delta_{\vec{k}_s \vec{k}'_s}, \quad (17)$$

where $\hat{\Box} = \hat{g}^{ij}\hat{\nabla}_i\hat{\nabla}_j$,

$$k_s^2 = \ell(\ell + d - 3), \quad \ell = 0, 1, 2, \ldots, \quad (18)$$

and $d\Omega^{d-2}$ is here the volume element over the sphere given by $d\Omega^{d-2} = \sqrt{\hat{g}}d\theta^2 \ldots d\theta^{d-1}$, $\hat{g}$ being the determinant of the metric $\hat{g}_{ij}$. Similarly, the vector-type component of A_μ can be expanded in vector harmonics $Y_{\vec{k}_v i}$ which also form a complete basis on $\mathcal{S}^{d-2}$, where $\vec{k}_v$ is a vector containing the angular momentum number ℓ and $d-3$ azimuthal numbers, that in principle are different from the azimuthal numbers of the scalar harmonics. The vector harmonics $Y_{\vec{k}_v i}$ then satisfy

$$(\hat{\Box} + k_v^2) Y_{\vec{k}_v i} = 0, \qquad \hat{\nabla}_i Y^i_{\vec{k}_v} = 0,$$
$$\int d\Omega^{d-2} \hat{g}^{ij} Y_{\vec{k}_v i} Y_{\vec{k}'_v j} = \delta_{\vec{k}_v \vec{k}'_v}, \tag{19}$$

with

$$k_v^2 = \ell(\ell + d - 3) - 1, \quad \ell = 1, 2, 3, \ldots, \tag{20}$$

see Appendix A for more details.

Thus, the scalar-type components $\{\tilde{A}_a, \hat{A}^{(s)}\}$ can be expanded in terms of the $Y_{\vec{k}_s}(\theta)$, and vector-type component $\{\hat{A}_i^{(v)}\}$ can be expanded in terms of the $Y_{\vec{k}_v i}(\theta)$, where $\theta \equiv (\theta^2 \ldots, \theta^i, \ldots \theta^{d-1})$. Indeed, $\{\tilde{A}_a, \hat{A}^{(s)}\}$ can be expanded as $\tilde{A}_a(y,\theta) = \sum_{\vec{k}_s} \tilde{A}_{\vec{k}_s a}(y) Y_{\vec{k}_s}(\theta)$, where $y \equiv (t, r)$, and $\hat{A}^{(s)}(y,\theta) = \sum_{\vec{k}_s} \hat{A}^{(s)}_{\vec{k}_s}(y) Y_{\vec{k}_s}(\theta)$, respectively, and $\{\hat{A}_i^{(v)}\}$ as $\hat{A}_i^{(v)}(y,\theta) = \sum_{\vec{k}_v} \hat{A}^{(v)}_{\vec{k}_v}(y) Y_{\vec{k}_v i}(\theta)$. Note that $\hat{\nabla}_i \hat{A}^{(s)}(y,\theta) = \sum_{\vec{k}_s} \hat{A}^{(s)}_{\vec{k}_s}(y) \hat{\nabla}_i Y_{\vec{k}_s}(\theta)$. Since $\tilde{A}_{\vec{k}_s a}$, $\hat{A}^{(s)}_{\vec{k}_s}$, and $\hat{A}^{(v)}_{\vec{k}_v}$ are cumbersome symbols to carry along, we define $\psi_{\vec{k}_s a} \equiv \tilde{A}_{\vec{k}_s a}$, $\phi_{\vec{k}_s} \equiv \hat{A}^{(s)}_{\vec{k}_s}$, and $\chi_{\vec{k}_v} \equiv \hat{A}^{(v)}_{\vec{k}_v}$. so that for the scalar-type components $\{\tilde{A}_a, \hat{A}^{(s)}\}$, one writes

$$\tilde{A}_a(y,\theta) = \sum_{\vec{k}_s} \psi_{\vec{k}_s a}(y) Y_{\vec{k}_s}(\theta), \tag{21}$$

$$\hat{A}^{(s)}(y,\theta) = \sum_{\vec{k}_s} \phi_{\vec{k}_s}(y) Y_{\vec{k}_s}(\theta), \tag{22}$$

while for the vector-type component $\{\hat{A}_i\}$ one writes

$$\hat{A}_i^{(v)}(y,\theta) = \sum_{\vec{k}_v} \chi_{\vec{k}_v}(y) Y_{\vec{k}_v i}(\theta). \tag{23}$$

Here, $\psi_{\vec{k}_s a}(y)$ is a vector field on $\mathcal{N}^2$, and $\phi_{\vec{k}_s}(y)$ and $\chi_{\vec{k}_v}(y)$ are scalar fields on $\mathcal{N}^2$. The scalar-type components cover two degrees of freedom, whereas the vector-type component covers $d-3$ degrees of freedom as can be noted from the transverse condition of $Y_{\vec{k}_v i}$ in Eq. (19). As Eq. (7) needs to be satisfied, these variables cover in total $d-1$ degrees of freedom, as expected for a Proca field. In summary, the complete expression of the Proca field decomposed in spherical harmonics is

$$A_\mu dx^\mu = \sum_{\vec{k}_s} \left[\psi_{\vec{k}_s a} Y_{\vec{k}_s} dy^a + \phi_{\vec{k}_s} \hat{\nabla}_i Y_{\vec{k}_s} d\theta^i\right] + \sum_{\vec{k}_v} \chi_{\vec{k}_v} Y_{\vec{k}_v i} d\theta^i. \tag{24}$$

In terms of the expansion in Eqs. (21)–(23), i.e., Eq. (24), the components of the Proca field strength tensor are written as

$$F^{ab} = \sum_{\vec{k}_s} \left[\tilde{\nabla}^a \psi^b_{\vec{k}_s} - \tilde{\nabla}^b \psi^a_{\vec{k}_s}\right] Y_{\vec{k}_s}, \tag{25}$$

$$F^{ai} = \sum_{\vec{k}_s} \left[\frac{1}{r^2}(\nabla^a \phi_{\vec{k}_s} - \psi^a_{\vec{k}_s})\right] \hat{\nabla}^i Y_{\vec{k}_s} + \sum_{\vec{k}_v} \left[\frac{1}{r^2} \nabla^a \chi_{\vec{k}_v}\right] Y^i_{\vec{k}_v}, \tag{26}$$

$$F^{ij} = \sum_{\vec{k}_v} \left[\frac{\chi_{\vec{k}_v}}{r^4}\right] (\hat{\nabla}^i Y^j_{\vec{k}_v} - \hat{\nabla}^j Y^i_{\vec{k}_v}), \tag{27}$$

where the covariant derivative ∇_a can be swapped to a partial derivative ∂_a when acting on scalars.

With these decompositions, it is possible to decouple only partially the equations in general. We will see that the scalar-type perturbations are completely decoupled from the vector-type perturbations. Moreover, the separation of the Proca equations are achieved due to the separation of the fields into functions that purely depend on the coordinates (t, r) and the spherical harmonics that depend only on the spherical angles.

C. Separation of the Proca equations

1. Proca equations after spherical harmonics expansion

The expansion on spherical harmonics of the Proca field in Eqs. (21)–(23), i.e., Eq. (24), and the corresponding expansion of the strength field tensor in Eqs. (25)–(27) can be inserted into Eqs. (12)–(14), which allows the separation of the Proca equations in AdS$_d$. In this way, the Proca equations are separated into three sums, one in terms of the spherical harmonics $Y_{\vec{k}_s}$ over each $\vec{k}_s$, i.e., $\sum_{\vec{k}_s} \tilde{W}^a_{\vec{k}_s} Y_{\vec{k}_s}$, another in terms of the gradients of the spherical harmonics $\hat{\nabla}_i Y_{\vec{k}_s}$ over each $\vec{k}_s$, i.e., $\sum_{\vec{k}_s} \hat{W}^{(s)}_{\vec{k}_s} \hat{\nabla}^i Y_{\vec{k}_s}$, and the remaining sum in terms of the vector spherical harmonics $Y_{\vec{k}_v i}$ over each $\vec{k}_v$, i.e., $\sum_{\vec{k}_v} \hat{W}^{(v)}_{\vec{k}_v} Y^i_{\vec{k}_v}$, where the coefficients of all the three sums only depend on the coordinates t and r, i.e., $\tilde{W}^a_{\vec{k}_s} = \tilde{W}^a_{\vec{k}_s}(t, r)$, $\hat{W}^{(s)}_{\vec{k}_s} = \hat{W}^{(s)}_{\vec{k}_s}(t, r)$, and $\hat{W}^{(v)}_{\vec{k}_v} = \hat{W}^{(v)}_{\vec{k}_v}(t, r)$. We must note that $Y_{\vec{k}_s}$, $\hat{\nabla}_i Y_{\vec{k}_s}$, and $Y_{\vec{k}_v i}$ are orthogonal between each other. An argument can be made that these span different representations of the rotation group $SO(d-1)$, for $d > 4$, while for $d = 4$, the scalar and vector spherical harmonics span different representations of the rotation group $O(3)$, see [21] and also Appendix A. Therefore, the Proca equations separate into the equations $\tilde{W}^a_{\vec{k}_s} = 0$, $W^{(s)}_{\vec{k}_s} = 0$, and $W^{(v)}_{\vec{k}_v} = 0$, described only by the coordinates t and r. Since the Proca equation is a linear differential equation in A_μ, there is no mixing between the different $\vec{k}_v$ and $\vec{k}_s$ modes. Therefore, without loss of generality and for convenience, we drop the sum on all the $\vec{k}_s$ and $\vec{k}_v$. In the rest of the section, we treat the scalar spherical harmonics as Y and the vector spherical harmonics as Y_i.

We now show the equations obtained from the separation of the Proca equations. The projection of the Proca equation into $\mathcal{N}^2$, Eq. (12), can be written in terms of a sum of spherical harmonics Y, in which the associated coefficients must satisfy $\tilde{W}^a_{\vec{k}_s} = 0$, or explicitly,

$$2\tilde{\nabla}_b\tilde{\nabla}^{[a}\psi^{b]} + 2(d-2)\frac{\tilde{\nabla}_b r}{r}\tilde{\nabla}^{[a}\psi^{b]} + \left(\frac{\ell(\ell+d-3)}{r^2}+\mu^2\right)\psi^a - \frac{\ell(\ell+d-3)}{r^2}\partial^a\phi = 0, \tag{28}$$

for each $\vec{k}_s$, where it was used that $k_s^2 = \ell(\ell+d-3)$, $\tilde{\nabla}^{[a}\psi^{b]} = \frac{1}{2}(\tilde{\nabla}^a\psi^b - \tilde{\nabla}^b\psi^a)$, and that $\hat{\nabla}_i Y^i = 0$, which avoids the appearance of χ in Eq. (28). The projection of the Proca equation into $\mathcal{S}^{d-2}$, Eq. (13), can be written in terms of two sums. One sum is in terms of the gradient of the spherical harmonics $\hat{\nabla}_i Y$, whose coefficients satisfy $\hat{W}^{(s)}_{\vec{k}_s} = 0$ or explicitly

$$\tilde{\Box}\phi + (d-4)\frac{\partial_b r}{r}\partial^b\phi - \mu^2\phi - \tilde{\nabla}_b\psi^b - (d-4)\frac{\partial_b r}{r}\psi^b = 0, \tag{29}$$

for each $\vec{k}_s$, where $\tilde{\Box} = \tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b$. The other sum is in terms of the vector spherical harmonics Y_i, whose coefficients must satisfy $\hat{W}^{(v)}_{\vec{k}_v} = 0$, or explicitly

$$\tilde{\Box}\chi + \frac{d-4}{r}(\partial^b r)(\partial_b\chi) - \left(\frac{\ell(\ell+d-3)+d-4}{r^2}+\mu^2\right)\chi = 0, \tag{30}$$

for each $\vec{k}_v$, where $k_v^2 = \ell(\ell+d-3)-1$ was used. Furthermore, to obtain Eq. (30), the commutator $2\hat{\nabla}_{[j}\hat{\nabla}_{i]}Y^j = \hat{R}_{mi}Y^m$ was used, where $\hat{R}_{mi} = (d-3)\hat{g}_{mi}$ is the Ricci tensor of $\mathcal{S}^{d-2}$. Finally, the Bianchi identity Eq. (14) can be written as a sum of spherical harmonics Y, whose coefficients satisfy

$$\tilde{\nabla}_b\psi^b + (d-2)\frac{\partial_b r}{r}\psi^b - \frac{\ell(\ell+d-3)}{r^2}\phi = 0, \tag{31}$$

for each $\vec{k}_s$, where again $k_s^2 = \ell(\ell+d-3)$ and $\hat{\nabla}_i Y^i = 0$ were used. We see that Eqs. (28)–(30) form a set of four equations, two component equations in Eq. (28) and two equations in Eqs. (29) and (30). There is a fifth equation, the Bianchi identity for Proca fields given in Eq. (31). So in total we can play with five equations.

2. Proca equations decoupled: The important equations for quasinormal modes

The equations for the scalar-type components of the Proca field, i.e., Eqs. (28) and (29) are coupled in ψ_t, ψ_r, ϕ, and Eq. (30) for the vector-type component represented by χ is decoupled. The Bianchi identity Eq. (31) is also coupled. By defining new variables q_0, q_1, q_2, and q_3 as functions of ψ_t, ψ_r, ϕ, and χ it is possible to decouple the equations. We give first the result and then show the steps to obtain it. So, Eqs. (28)–(30) in the new variables are

$$\hat{\mathcal{D}}_\ell q_0 + \frac{2r}{l^2}(\partial_t q_1 + \partial_t q_2 - \partial_{r_*} q_0) = 0, \tag{32}$$

$$\hat{\mathcal{D}}_{j_k} q_k = 0, \tag{33}$$

and the Bianchi identity Eq. (31) is now

$$\partial_t q_0 - \partial_{r_*}(q_1 + q_2) = \frac{f}{r}\left(\frac{d-2}{2} - (\ell+d-3)\right)q_2 + \frac{f}{r}\left(\ell + \frac{d-2}{2}\right)q_1, \tag{34}$$

where here $k \in \{1,2,3\}$, $j_k = (j_1, j_2, j_3)$, $j_1 = \ell+1$ with $\ell \in \mathbb{N}_0$, $j_2 = \ell - 1$ with $\ell \in \mathbb{N}$, and $j_3 = \ell$ with $\ell \in \mathbb{N}$, r_* is defined as $r_* = l\arctan(\frac{r}{l})$, the operator $\hat{\mathcal{D}}_\ell$ is defined as

$$\hat{\mathcal{D}}_\ell = -\partial_t^2 + \partial_{r_*}^2 - f\left[\frac{\ell(\ell+d-3)}{r^2} + \mu^2 + \frac{(d-2)(d-4)}{4l^2}\left(1 + \frac{l^2}{r^2}\right)\right], \tag{35}$$

and $q_0(t,r)$, $q_1(t,r)$, $q_2(t,r)$, and $q_3(t,r)$ are defined by

$$q_0(t,r) = \psi_t(t,r) r^{\frac{d}{2}-1}, \tag{36}$$

$$q_1(t,r) = \frac{(\ell-d-3)\psi_r(t,r)f(r) - \ell(\ell+d-3)\frac{\phi(t,r)}{r}}{2\ell+d-3} r^{\frac{d}{2}-1}, \tag{37}$$

$$q_2(t,r) = \frac{\ell\psi_r(t,r)f(r) + \ell(\ell+d-3)\frac{\phi(t,r)}{r}}{2\ell+d-3} r^{\frac{d}{2}-1}, \tag{38}$$

$$q_3(t,r) = \frac{\chi(t,r)}{r} r^{\frac{d}{2}-1}. \tag{39}$$

We see that Eqs. (32)–(33) provide four equations, the number we had originally, and there is still the Bianchi identity for Proca fields given in Eq. (34), yielding five equations in total.

3. Proof of the decoupling of Proca equations

Now we show how to obtain Eqs. (32)–(34) together with the definitions Eqs. (35)–(39). The equations for the scalar-type components of the Proca field, i.e., Eqs. (28) and (29) are coupled in ψ_t, ψ_r, and ϕ. The Bianchi identity for Proca fields given in Eq. (31) is also coupled in ψ_t, ψ_r, and ϕ. We now show that in the case of AdS_d spacetime, it is indeed possible to manipulate these equations and decouple these components through further transformations. We start by differentiating Eq. (31) to obtain

$$\ell(\ell+d-3)\partial^a\phi = \tilde{\nabla}^a(r^2\tilde{\nabla}_b\psi^b) + (d-2)\tilde{\nabla}^a(r\psi^b\partial_b r). \tag{40}$$

Using Eq. (40) in Eq. (28) we obtain a coupled equation for the ψ^a components given by

$$\tilde{\Box}\psi^a - \tilde{R}^a_b\psi^b + (d-2)\frac{\partial^b r}{r}\tilde{\nabla}_b\psi^a - \left(\frac{\ell(\ell+d-3)}{r^2}+\mu^2\right)\psi^a + (d-2)\tilde{\nabla}^a\left(\frac{\partial_b r}{r}\right)\psi^b + \frac{2\partial^a r}{r}\left(\tilde{\nabla}_b\psi^b + (d-2)\frac{\partial_b r}{r}\psi^b\right) = 0, \tag{41}$$

where $\tilde{R}^a_b = -\frac{f''}{2}\delta^a_b$ is the Ricci tensor of $\mathcal{N}^2$ and the commutator $2\tilde{\nabla}_{[a}\tilde{\nabla}_{b]}\psi^a = \tilde{R}_{ab}\psi^a$ was used. Now, Eqs. (29) (31), and (41) can be further simplified into

$$\hat{\mathcal{D}}_\ell u_0 + \frac{2r}{l^2}(\partial_t u_1 - \partial_{r_*}u_0) = 0, \tag{42}$$

$$\hat{\mathcal{D}}_\ell u_1 - \frac{2f}{r^2}\left(\frac{d-2}{2}u_1 - u_2\right) = 0, \tag{43}$$

$$\hat{\mathcal{D}}_\ell u_2 + \frac{2f}{r^2}\left(\left(\frac{d}{2}-2\right)u_2 + \ell(\ell+d-3)u_1\right) = 0, \tag{44}$$

$$\partial_t u_0 - \partial_{r_*}u_1 = \frac{f}{r}\left(\frac{d-2}{2}u_1 - u_2\right), \tag{45}$$

where again r_* is defined such as $\frac{dr_*}{dr} = \frac{1}{f(r)}$ with $f(r) = 1 + \frac{r^2}{l^2}$, i.e., $r_* = l\arctan(\frac{r}{l})$, $\hat{\mathcal{D}}_\ell$ is the operator already given in Eq. (35), i.e., $\hat{\mathcal{D}}_\ell = -\partial_t^2 + \partial_{r_*}^2 - f[\frac{\ell(\ell+d-3)}{r^2} + \mu^2 + \frac{(d-2)(d-4)}{4l^2}(1+\frac{l^2}{r^2})]$, $u_0(t,r)$, $u_1(t,r)$, and $u_2(t,r)$ are defined by

$$u_0(t,r) = \psi_t(t,r)r^{\frac{d}{2}-1}, \tag{46}$$

$$u_1(t,r) = \psi_r(t,r)f(r)r^{\frac{d}{2}-1}, \tag{47}$$

$$u_2(t,r) = \frac{\phi(t,r)}{r}\ell(\ell+d-3)r^{\frac{d}{2}-1}, \tag{48}$$

and we also have used Eq. (45) in the last two terms of Eq. (43). Finally, one can notice that the coupling terms in Eqs. (43) and (44) are constants multiplied by $\frac{2f}{r^2}$. Therefore, it is possible to further decouple Eqs. (43) and (44) by making the transformations

$$u_0 = q_0, \tag{49}$$

$$u_1 = q_1 + q_2, \tag{50}$$

$$u_2 = (\ell+d-3)q_2 - \ell q_1. \tag{51}$$

Inserting Eqs. (49)–(51) into Eqs. (42)–(44) yield Eqs. (32)–(33) with $k = 1, 2$, and inserting Eqs. (49)–(51) into Eq. (45) yields Eq. (34). The vector-type component of the Proca field is described by Eq. (30) and it is completely decoupled from the scalar-type components of the Proca field. Equation (30) can be further simplified into

$$\hat{\mathcal{D}}_\ell u_3 = 0, \tag{52}$$

where

$$u_3 = \frac{\chi}{r}r^{\frac{d}{2}-1}. \tag{53}$$

Defining trivially

$$u_3 = q_3. \tag{54}$$

yields Eq. (33) with $k = 3$. Thus, Eqs. (32)–(34) together with the definitions Eqs. (35)–(39) have been obtained.

We confirm the results obtained in [25,26], for the particular case of pure AdS, where the vector-type component of the Proca field yields $d-3$ degrees of freedom, whereas the scalar-type component describes two degrees of freedom, which are coupled by the mass of the field. We find that, similarly to what was found in [18] for AdS_4, the scalar-type degrees of freedom decouple in AdS_d, by making a particular linear transformation to the relevant fields.

IV. NORMAL MODES OF PROCA PERTURBATIONS IN THE AdS_d BACKGROUND

A. Initial considerations and boundary conditions

The normal modes in AdS_d are dynamical solutions of the Proca equations described by q_0, q_1, q_2, and q_3 in Eqs. (32)–(34). Moreover, the Bianchi identity in Eq. (34) can be used to describe q_0 in terms of q_1, q_2, and q_3, which in turn obey Eq. (33). With the relation given by Eq. (34), the equation for q_0, Eq. (32), is satisfied and so the picture is consistent. The normal modes can then be obtained solely by solving Eq. (33) for the q_1, q_2, and q_3, with the appropriate boundary conditions at the origin and at spatial infinity. By making an extension to the complex numbers, one can assume for the $q_k(t,r)$ an ansatz of the form $q_k(t,r) = q_k(r)e^{-i\omega_k t}$, where ω_k is the normal mode frequency of the mode q_k, which is analogous to performing a Fourier transformation from the time domain to the frequency domain. Of course, when one wants to treat the real field, one must project the complex field into the real axis. The equations given in Eq. (33) can then be written for the $q_k(r)$ as

$$\partial^2_{r_*} q_k + (\omega_k^2 - V_{j_k})q_k = 0, \quad (55)$$

$$V_{j_k} = f\left[\frac{4j_k(j_k+d-3)+(d-2)(d-4)}{4r^2} + \mu^2 + \frac{(d-2)(d-4)}{4l^2}\right], \quad (56)$$

for $k \in \{1,2,3\}$. Equation (55) is a Schrödinger-like equation for the normal modes, q_k, with associated normal mode frequencies, ω_k. A general class of Schrödinger-like equations governing the dynamics of fields in AdS with all their possible boundary conditions at spatial infinity, were analyzed in [21]. Here, we provide the analysis for the Proca field equation given in Eq. (55) and its solutions. We start by studying the behavior of the solutions near the origin and near spatial infinity. We first deal with the behavior at $r = 0$. One can pinpoint the boundary conditions by checking when the q_k are square-integrable in the sense of $\int \bar{q}_k q_k dr_*$, where $\bar{q}_k$ means complex conjugate of q_k. This implies that the regularity conditions are such that $q_k \propto r^s$ as $r \to 0$, for some s with $s > -\frac{1}{2}$. The functions q_k near $r = 0$ have the behavior

$$q_k = \alpha_k^{r=0} r^{j_k + \frac{d-2}{2}} + \beta_k^{r=0} r^{-j_k - \frac{d-4}{4}}, \quad (57)$$

where $\alpha_k^{r=0}$ and $\beta_k^{r=0}$ are constants, and where the equality is valid in first order near $r = 0$. In all the cases except the case $d = 4$, $k = 2$, and $j_2 = 0$, the functions q_k are not square integrable if $\beta_k^{r=0}$ is finite. Therefore, the regularity condition is $\beta_k^{r=0} = 0$. In the case of $d = 4$, $k = 2$, and $j_2 = 0$, the function q_2 is square integrable if $\beta_2^{r=0}$ is finite, since $q_2 = \alpha_2^{r=0} r + \beta_2^{r=0}$. Still, this solution means that the Proca field goes as $A_\mu \sim \frac{1}{r}$ and it is rather a solution to the Proca equations but with a delta dirac distribution as a source term, see Appendix B. For this reason, this particular solution cannot be considered and the regularity condition $\beta_k^{r=0} = 0$ is maintained. Such argument for the regularity condition is also present for the scalar field in AdS_4 with $\ell = 0$, see [21]. Note also that in the case of the scalar-type Proca field with $d = 4$ and $j_2 = 0$, the asymptotic expansion at $r = 0$ of the integrand of the energy E defined as $E = \int_t T_{\mu\nu}\xi^\mu t^\nu \frac{r^{d-2}}{\sqrt{f}} dr d\Omega$, for a constant t slice, where t^μ is its normal vector, ξ^μ is the timelike Killing vector, and $d\Omega$ is the line element of the unit 2-sphere, seems to have divergent terms r^{-1}, r^{-2}, and r^{-4}, which only vanish if $\beta_2^{r=0} = 0$.

Now, we analyze the boundary conditions at spatial infinity, $r \to \infty$. To do that, one looks at the behavior of the Proca field near $r \to +\infty$. For the functions q_k to be square integrable they must behave as $q_k \sim r^s$, for some s with $s < \frac{1}{2}$. From the Proca field equations, we get a behavior of the q_k near $r \to +\infty$ as

$$q_k = \alpha_k^{r=\infty} r^{-\frac{1}{2}\left(1+\sqrt{(d-3)^2+4\mu^2 l^2}\right)} + \beta_k^{r=\infty} r^{-\frac{1}{2}\left(1-\sqrt{(d-3)^2+4\mu^2 l^2}\right)}, \quad (58)$$

where $\alpha_k^{r=\infty}$ and $\beta_k^{r=\infty}$ are constants, and where the equality is valid in first order near $r = \infty$. For the case $d > 4$, the functions q_k are only square integrable if $\beta_k^{r=\infty} = 0$, which is the Dirichlet boundary condition. For $d = 4$ and $(\mu l)^2 \geq \frac{3}{4}$, the same rationale applies. However, for $d = 4$ and $0 < (\mu l)^2 < \frac{3}{4}$, the functions q_k are square integrable for finite $\beta_k^{r=\infty}$. This is an interesting case because the potential does diverge for positive μ^2. According to [21], it is then possible to impose a one parameter family of boundary conditions. Still, by the calculation of the usual energy of the Proca field, the condition $\beta_k^{r=\infty} = 0$ is the only condition that ensures that the energy is finite and does not diverge on t constant slices. Although the usual definition of the energy has been chosen, we note that there are different valid definitions of the energy function for the fields where the energy is finite and conserved, see [21]. Nevertheless, we admit Dirichlet boundary conditions for the q_k, which means that $q_k(r \to +\infty) = 0$ for every possible case.

B. Solutions of the Proca equations and normal mode frequencies

The Proca equations in Eq. (55) can be put in the form,

$$\partial^2_{r_*} q_k + \left(\omega_k^2 - \frac{G_k}{\sin^2(\frac{r_*}{l})} - \frac{H_k}{\cos^2(\frac{r_*}{l})}\right) q_k = 0, \quad (59)$$

where $k \in \{1,2,3\}$, $r_* = l \arctan(\frac{r}{l})$, $G_k = \frac{4j_k(j_k+d-3)+(d-2)(d-4)}{4l^2}$ and $H_k = \frac{(d-2)(d-4)+4\mu^2 l^2}{4l^2}$. This is a second-order partial differential equation which is linear and it has three regular singularity points at $\frac{r_*}{l} = -\frac{\pi}{2}$, $\frac{r_*}{l} = 0$, and $\frac{r_*}{l} = +\frac{\pi}{2}$. Therefore, this Fuchsian differential equation can be transformed into an hypergeometric equation, see details in Appendix B. The solutions of Eq. (59) that satisfy the regularity conditions at $r = 0$ and the Dirichlet boundary conditions at $r \to +\infty$ are

$$q_k = a_k \left(\frac{r}{l}\right)^{j_k+\frac{d-2}{2}} \left(1+\frac{r^2}{l^2}\right)^{n-\frac{\omega_k l}{2}} \times {}_2F_1\left[-n+\omega_k l, -n, j_k+\frac{d-1}{2}; \frac{\frac{r^2}{l^2}}{1+\frac{r^2}{l^2}}\right], \quad (60)$$

where $k \in \{1,2,3\}$, a_k is a constant, and with the normal mode frequencies ω_k being

$$\omega_k l = 2n + j_k + \frac{d-1}{2} + \frac{1}{2}\sqrt{(d-3)^2 + 4\mu^2 l^2}, \quad (61)$$

where $n \in \mathbb{N}_0$, $j_1 = \ell + 1$ with $\ell \in \mathbb{N}_0$, $j_2 = \ell - 1$ with $\ell \in \mathbb{N}$, and $j_3 = \ell$ with $\ell \in \mathbb{N}$. Notice that the monopole case of the Proca field is described by q_1 with $j_1 = 1$ or $\ell = 0$. Setting $d = 4$, the expression of the normal mode frequencies given in Eq. (61) agrees with the expression given in [18].

Although we have not analyzed the case of negative $\mu^2 l^2$, the asymptotic behavior in Eq. (58) and the mode frequencies in Eq. (61) indicate that the eigenvalue problem is well-defined even for negative field masses, as along as they obey the inequality,

$$(\mu l)^2 \geq -\frac{(d-3)^2}{4}. \quad (62)$$

The bound given in Eq. (62) is the Proca field analog of the Breitenlohner-Freedman bound. In $d = 4$ the Breitenlohner-Freedman bound for Proca fields in AdS$_4$ is $(\mu l)^2 \geq -\frac{1}{4}$. The Breitenlohner-Freedman bound was found originally for a massive scalar field in pure AdS and is given by $(\mu l)^2 \geq -\frac{9}{4}$ in $d = 4$ [5], and is given by $(\mu l)^2 \geq -\frac{(d-1)^2}{4}$ for generic d [21].

V. NORMAL MODES OF THE MAXWELL FIELD IN AdS$_d$ AS THE $\mu = 0$ LIMIT OF PROCA NORMAL MODES

A. Maxwell equations and boundary conditions

1. Maxwell equations

Maxwellian electromagnetic perturbations, also called Maxwell perturbations, can be viewed as the limit of Proca perturbations when $\mu = 0$. However, in Maxwell theory, the field equations are gauge invariant and the field loses one physical degree of freedom, which becomes a pure gauge one. Thus, one cannot simply set $\mu = 0$ for A_μ in the results above, as one of the degrees of freedom becomes spurious. Indeed, the identity on the field A_μ, Eq. (7), no longer follows directly from the field equations, becoming simply a gauge choice. One finds that the gauge freedom in A_μ is only scalar, with the vector-type sector of the Maxwell field being gauge invariant. We display first the final equations that are of interest here and then we show how to obtain them from the Proca equations.

The Maxwell field is described by $d-1$ components. The time component $q_0(t,r) = q_0(r)e^{-i\omega_{12}t}$, the scalar-type component $q_{12}(t,r) = q_{12}(r)e^{-i\omega_{12}t}$ and the $d-3$ vector-type components $q_3(t,r) = q_3(r)e^{-i\omega_3 t}$, where ω_{12} is a normal mode frequency of q_{12} and ω_3 is a normal mode frequency of q_3. The component q_0 is given by

$$q_0(r) = \frac{i}{\omega_{12} r^{\frac{d}{2}-2}} \partial_{r_*}(q_{12} r^{\frac{d}{2}-2}), \quad (63)$$

and is completely determined by the scalar-type component q_{12}, which in turn obeys

$$\partial_{r_*}^2 q_{12} + (\omega_{12}^2 - V_{12}(r)) q_{12} = 0, \quad (64)$$

$$V_{12}(r) = f\left[\frac{4\ell(\ell+d-3) + (d-4)(d-2)}{4r^2} + \frac{(d-4)(d-6)}{4l^2}\right]. \quad (65)$$

This scalar-type component of the Maxwell field has been denominated as q_{12} since it is the corresponding mode to the Proca q_1 and q_2 modes when the mass of the field is zero $\mu = 0$. The vector-type components covers the same $d-3$ degrees of freedom as in the massive case and so they are governed by Eq. (59), for $k = 3$ and $\mu = 0$, i.e.,

$$\partial_{r_*}^2 q_3 + (\omega_3^2 - V_3) q_3 = 0, \quad (66)$$

$$V_3 = f\left[\frac{4\ell(\ell+d-3) + (d-2)(d-4)}{4r^2} + \frac{(d-2)(d-4)}{4l^2}\right], \quad (67)$$

where these vector-type components of the Maxwell field have been denominated as q_3 since they are the corresponding modes to the Proca q_3 modes when the mass of the field is zero $\mu = 0$. Surely, there is no possibility of confusion, now we are dealing with Maxwell modes.

Let us derive the above equations. In Maxwell theory, the field equations are gauge invariant which means that the Proca field loses one physical degree of freedom when $\mu = 0$. Thus, in the perturbed quantities there is one mode that becomes nonphysical. In order to distinguish between the physical degrees of freedom and the pure gauge ones, it is useful to work with gauge-invariant variables rather than with A_μ. Under the gauge transformation, using the notation of Eqs. (15) and (16), the fields transform as

$$\tilde{A}_a \to \tilde{A}_a + \partial_a h, \quad \hat{A}_i = \hat{A}_i^{(v)} + \hat{\nabla}_i \hat{A}^{(s)} \to \hat{A}_i^{(v)} + \hat{\nabla}_i(\hat{A}^{(s)} + h), \quad (68)$$

for some gauge function h. One sees that the gauge freedom in A_μ is only scalar, with the vector-type sector of the Maxwell field being gauge invariant. We start to play with the scalar-type components. For the scalar-type components, one has to go back to Eqs. (28) and (29). Setting $\mu = 0$ in them one has

$$\tilde{\nabla}_b \tilde{\nabla}^{[a}\psi^{b]} + (d-2)\frac{\partial_b r}{r}\tilde{\nabla}^{[a}\psi^{b]} + \frac{k_s^2}{2r^2}(\psi^a - \partial^a \phi) = 0, \quad (69)$$

$$\tilde{\nabla}_b(\psi^b - \partial^b\phi) + (d-4)\frac{\partial_b r}{r}(\psi^b - \partial^b\phi) = 0. \quad (70)$$

This motivates the definition of the field ζ^a given by

$$\zeta^a = \psi^a - \partial^a\phi, \quad (71)$$

which is gauge invariant. Indeed, one can expand the gauge function h as $h(y,\theta) = h(y)Y(\theta)$, so that, under a gauge transformation, one has $\psi_a \to \psi_a + \partial_a h$ and $\phi \to \phi + h$, where the $\vec{k}_s$ indices were omitted for convenience. In terms of ζ^a the equations of motion, Eqs. (69) and (70) become

$$2\tilde{\nabla}_b\tilde{\nabla}^{[a}\zeta^{b]} + 2(d-2)\frac{\partial_b r}{r}\tilde{\nabla}^{[a}\zeta^{b]} + \frac{k_s^2}{r^2}\zeta^a = 0, \quad (72)$$

$$\tilde{\nabla}_b(r^{d-4}\zeta^b) = 0. \quad (73)$$

Note that this transformation completely removes a pure gauge degree of freedom from the system, as $\{\psi^a, \phi\} \to \{\zeta^a\}$. This only happens in the massless case, where ζ^a factorizes. In the background of AdS_d spacetime, substituting Eq. (73) in the r component of Eq. (72), and further making the transformation

$$q_{12}(r) = \frac{\zeta_r(t,r)}{r} f r^{\frac{d}{2}-1} e^{i\omega_{12}t}, \quad (74)$$

yields the equations, $\partial_{r_*}^2 q_{12} + (\omega_{12}^2 - V_{12}(r))q_{12} = 0$, with $V_{12}(r) = f[\frac{4\ell(\ell+d-3)+(d-4)(d-2)}{4r^2} + \frac{(d-4)(d-6)}{4l^2}]$, which correspond to Eqs. (64) and (65). Making the transformation

$$q_0(r) = \frac{\zeta_t(t,r)}{r} r^{\frac{d}{2}-1} e^{i\omega_{12}t}, \quad (75)$$

yields $q_0(r) = \frac{i}{\omega_{12} r^{\frac{d}{2}-2}}\partial_{r_*}(q_{12} r^{\frac{d}{2}-2})$, which corresponds to Eq. (63) and is completely determined from the scalar-type component q_{12}. Finally, since the gauge freedom in A_μ is only scalar, with the vector-type sector of the Maxwell field being gauge invariant, see Eq. (68), this means that, in the $\mu = 0$ limit, the vector-type component of A_μ covers the same $d-3$ degrees of freedom as in the massive case and so they are governed by Eq. (59), for $k = 3$ and $\mu = 0$, i.e., $\partial_{r_*}^2 q_3 + (\omega_3^2 - V_3)q_3 = 0$ with $V_3 = f[\frac{4\ell(\ell+d-3)+(d-2)(d-4)}{4r^2} + \frac{(d-2)(d-4)}{4l^2}]$, which corresponds to Eqs. (66) and (67).

All this agrees with the Maxwell field having $d-2$ degrees of freedom. Indeed, while the vector-type component covers $d-3$ degrees of freedom, the scalar-type component only covers one degree of freedom, which in this case was chosen to be $\zeta_r(t,r)$ or $q_{12}(r)$. Note also that we are treating only dynamical solutions of the Maxwell's equations, and so the frequencies ω must be nonzero.

2. Boundary conditions

In summary, the Maxwell field is comprised of a scalar-type component q_{12} that satisfies the Maxwell equation Eq. (64). and of vector-type components q_3 that satisfy the Maxwell equation Eq. (66).

With respect to the regularity conditions, both q_{12} and q_3 have the same behavior as the Proca q_k in Eq. (57) near $r = 0$. Thus,

$$q_i = \alpha_i^{r=0} r^{\ell+\frac{d-2}{2}} + \beta_i^{r=0} r^{-\ell-\frac{d-4}{2}}, \quad (76)$$

where now $i \in \{12, 3\}$, and $\alpha_i^{r=0}$ and $\beta_i^{r=0}$ are constants. For all cases of the electromagnetic field, the q_i are square integrable if $\beta_i^{r=0} = 0$, and so we admit this condition as the regularity condition.

In relation to the boundary conditions at spatial infinity, some care must be taken. In the massive case, both scalar-type and vector-type perturbations have the same effective mass. This contrasts with the massless case, where the effective mass for the scalar-type component, $\mu_{\mathrm{eff}}^2 = \frac{(d-4)(d-6)}{4l^2}$, and the effective mass for the vector-type component, $\mu_{\mathrm{eff}}^2 = \frac{(d-2)(d-4)}{4l^2}$, are functionally different, being only the same in $d = 4$. This means that the asymptotic behavior at spatial infinity is different for the scalar-type component and for the vector-type components. Near spatial infinity, the electromagnetic fields behave as

$$q_{12} = \begin{cases} \frac{\alpha_{12}^{r=\infty}}{r} + \beta_{12}^{r=\infty} & \text{for } d = 4, \\ \frac{\alpha_{12}^{r=\infty}}{\sqrt{r}} + \frac{\beta_{12}^{r=\infty}\ln r}{\sqrt{r}} & \text{for } d = 5, \\ \frac{\alpha_{12}^{r=\infty}}{r^{\frac{d}{2}-2}} + \beta_{12}^{r=\infty} r^{\frac{d}{2}-3} & \text{for } d \geq 6, \end{cases} \quad (77)$$

$$q_3 = \frac{\alpha_3^{r=\infty}}{r^{\frac{d}{2}-1}} + \beta_3^{r=\infty} r^{\frac{d}{2}-2} \quad \text{for } d \geq 4, \quad (78)$$

where $\alpha_i^{r=\infty}$ and $\beta_i^{r=\infty}$ are constants, with $i \in \{12, 3\}$.

For the scalar-type perturbation, q_{12}, in $d = 4$ the field is square integrable for nonzero and finite β_i when $r \to \infty$. The requirement that the usual definition of the energy is finite and independent of the t constant slice allows for the two reflective boundary conditions, the Dirichlet and the Neumann, with the Dirichlet imposing $\beta_i^{r=\infty} = 0$ and the Neumann imposing $\alpha_i^{r=\infty} = 0$. Also, other boundary conditions for these cases are also possible, see [16,21]. Nevertheless, we impose $\beta_i^{r=\infty}=0$, which corresponds to the Dirichlet boundary condition. For the scalar-type perturbation, q_{12}, in $d=5$, the field is square integrable for every $\alpha_{12}^{r=\infty}$, $\beta_{12}^{r=\infty}$. Moreover, the Dirichlet boundary condition, which imposes the field to vanish at $r\to+\infty$, does not restrict the asymptotic coefficients and leaves the eigenvalue problem ill-defined. In order to have well-defined dynamics for the field in the sense of [21], one needs to choose $\alpha_{12}^{r=\infty}$ and $\beta_{12}^{r=\infty}$ carefully. The boundary condition that keeps the usual definition of the energy to be finite and time independent is that $\beta_{12}^{r=\infty}=0$, which removes the dominant logarithmic term in Eq. (77), and in this case, since it involves the field and first derivatives of the field, is a Dirichlet-Neumann boundary condition. For the scalar-type perturbation, q_{12}, in $d=6$, the field is square integrable for nonzero and finite $\beta_i^{r=\infty}$, as in the case $d=4$. Here, it seems that the requirement that the usual definition of the energy is finite and independent of the t constant slice only allows the Dirichlet boundary condition. Also notice that other boundary conditions for this cases are also possible, see [16,21]. Nevertheless, we impose $\beta_i^{r=\infty}=0$, which corresponds to the Dirichlet boundary condition. For the scalar-type perturbations, q_{12}, in $d\geq 7$, the field is square integrable only if $\beta_i^{r=\infty}=0$. Therefore, Dirichlet boundary conditions must be imposed.

For the vector-type perturbation, q_3, in $d=4$, the field is square integrable for nonzero and finite $\beta_i^{r=\infty}$. The requirement that the usual definition of the energy is finite and independent of the t constant slice allows also for the two reflective boundary conditions, the Dirichlet and the Neumann, with the Dirichlet imposing $\beta_i^{r=\infty}=0$ and the Neumann imposing $\alpha_i^{r=\infty}=0$. As well, other boundary conditions for these cases are possible, see [16,21]. Nevertheless, we impose $\beta_i^{r=\infty}=0$, which corresponds to the Dirichlet boundary condition. For the vector-type perturbations, q_3, in $d\geq 5$, the fields are only square integrable if $\beta_i^{r=\infty}=0$. Therefore, Dirichlet boundary conditions must be imposed.

With these considerations and for consistency in order to compare with the Proca field normal modes, we still apply to the Maxwell field the Dirichlet boundary conditions for both vector-type and scalar-type perturbations, for every case, i.e., $q_i(r\to\infty)=0$.

B. Solutions of the Maxwell equations and normal mode frequencies

To obtain the normal modes, one can put the equations obeyed by q_{12} and q_3 in the form

$$\partial_{r_*}^2 q_i+\left(\omega_i^2-\frac{G_i}{\sin^2(\frac{r_*}{l})}-\frac{H_i}{\cos^2(\frac{r_*}{l})}\right)q_i=0, \qquad (79)$$

where $i\in\{12,3\}$, and the constants are $G_i=\frac{4\ell(\ell+d-3)+(d-2)(d-4)}{4l^2}$, $H_{12}=\frac{(d-4)(d-6)}{4l^2}$, and $H_3=\frac{(d-2)(d-4)}{4l^2}$.

Imposing the regularity and the Dirichlet boundary conditions for $d=4$ and $d\geq 6$, and Dirichlet-Neumann boundary conditions for $d=5$, one obtains that the solutions are described by the hypergeometric functions, see Appendix B, and they yield,

$$q_i=a_i\left(\frac{r}{l}\right)^{\ell+\frac{d-2}{2}}\left(1+\frac{r^2}{l^2}\right)^{n-\frac{\omega_i l}{2}} \times {}_2F_1\left[-n+\omega_i l,-n,\ell+\frac{d-1}{2};\frac{\frac{r^2}{l^2}}{1+\frac{r^2}{l^2}}\right], \qquad (80)$$

for $i\in\{12,3\}$, with the normal mode frequencies

$$\omega_{12}=\begin{cases}2n+\ell+2 & \text{for } d=4,\\ 2n+\ell+2 & \text{for } d=5,\\ 2n+\ell+d-3 & \text{for } d\geq 6,\end{cases} \qquad (81)$$

$$\omega_3=2n+\ell+d-2 \quad \text{for } d\geq 4, \qquad (82)$$

with $n\in\mathbb{N}_0$ and $\ell\in\mathbb{N}$, where $\mathbb{R}^+$ are the positive real numbers.

It is interesting to reflect on the normal modes of the Maxwell electromagnetic field as the massless limit of the Proca field. The scalar-type perturbations need to be treated with some care as we have seen. For $d=4$, the scalar-type perturbation has the same frequency as the vector-type. Since the massless limit of the Proca field yields different frequencies for the scalar and vector perturbations, this discrepancy indicates some discontinuity in the massless limit, which indeed happens in the frequencies of the scalar perturbations. For $d=5$, the normal mode frequencies of the scalar-type perturbation of the electromagnetic field follow from the massless limit of the Proca field. It must be noted however that the frequencies for the scalar-type perturbation of the electromagnetic field were obtained using Dirichlet-Neumann conditions, rather than Dirichlet boundary conditions that were imposed to the original Proca field. Indeed, the Dirichlet boundary conditions leave the eigenvalue problem ill-defined. For $d\geq 6$, the electromagnetic scalar-type normal modes follow from the massless limit of the modes in Eq. (61) for $k=2$ and $j_2=\ell-1$, indicating that q_2 describes the electromagnetic mode, but now extended to the massive case, and that q_1 describes the scalar field degree of freedom of the Proca fields, which in the massless case can be removed by the gauge freedom. On the other hand, the vector-type perturbation of the electromagnetic follows directly from the massless limit of the vector-type perturbation of the Proca field and so one sets directly $\mu=0$ in Eq. (61) for $k=3$ and $j_3=\ell$, and obtains the normal frequencies in Eq. (82).

We have obtained the normal modes for the Maxwell electromagnetic perturbations by working in all detail the massless limit of the Proca perturbations. Our results agree with those found in [21], where a direct analysis of the Maxwell field from a master equation in AdS_d was performed. The results also conform to the Maxwell electromagnetic perturbations for AdS_d found in [22,23], and they recover the $\mu = 0$ limit of the Proca field for AdS_4 [18], see also [12].

VI. CONCLUSIONS

In this work, the normal modes of linear Proca perturbations in AdS_d background were obtained analytically, using the Ishibashi-Kodama formalism. The Proca field was decomposed into different components according to its tensorial behavior on the sphere, yielding scalar-type components, covering two degrees of freedom of the field, and vector-type components, covering the remaining $d - 3$ degrees of freedom. In general, while in the scalar-type sector the two degrees of freedom are coupled, due to the mass of the field, in the vector-type sector the modes are completely decoupled. In AdS_d, the scalar-type perturbations can be indeed decoupled in the equations and they are covered independently by two fields, q_1 and q_2. The vector-type perturbations can be covered by one field only, q_3.

The usual regularity boundary conditions at the center and Dirichlet boundary conditions at infinity were imposed to the Proca equations. We used a Dirichlet condition because it is the condition that ensures that the usual definition of the energy through the conservation of $\xi^\mu T_{\mu\nu}$, where ξ^μ is the timelike Killing vector and $T_{\mu\nu}$ is the stress-energy tensor, is finite and is independent of the t constant hypersurface. It would be interesting to investigate the eigenfrequency problem for other possible boundary conditions. Using the regularity and the Dirichlet conditions, the solutions for the Proca field were obtained and found to be described by hypergeometric functions. The normal mode frequencies were obtained for any value of the Proca mass μ and for $d \geq 4$. For the $d = 4$ case, the expression of the frequencies agrees with previous works. We have also found an explicit expression for Breitenlohner-Freedman bound of the Proca field in AdS_d.

The normal modes of Maxwell electromagnetic perturbations in AdS_d were also obtained analytically through the $\mu = 0$ limit of Proca perturbations. In this case, by working with gauge-invariant variables, it was possible to separate the physical modes from the nonphysical ones. For consistency, the regularity and Dirichlet boundary conditions were used, except for the case of $d = 5$ where the Dirichlet-Neumann boundary condition was used, in order to analyze the massless limit of the Proca field. We have thus recovered the normal mode frequencies of the Maxwell field found in previous works, with the $d = 4$ case regarding the scalar perturbations having to be treated with much care, in particular, the massless limit of the Proca scalar perturbations in $d = 4$ does not lead directly to the Maxwell field scalar perturbation modes.

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APPENDIX A: SPHERICAL HARMONICS ON THE $(d - 2)$-SPHERE

1. Initial considerations

The decomposition of fields in spherical harmonics is important to study the structure of their perturbations and to isolate the physical degrees of freedom of the fields themselves. Here, we discuss in some detail the properties of these special functions.

The approach we adopt here to construct spherical harmonics on the $(d - 2)$-sphere, $\mathcal{S}^{d-2}$, follows [27], where spherical harmonics are constructed by embedding $\mathcal{S}^{d-2}$ in a $(d - 1)$-Euclidean space, $\mathbb{R}^{d-1}$. Firstly, we introduce some useful concepts: (a) a polynomial $h_\ell\colon \mathbb{R}^{d-1} \to \mathbb{C}$ is homogeneous of degree ℓ in $\mathbb{R}^{d-1}$ if $h_\ell(\lambda x^\mu) = \lambda^\ell h_\ell(x^\mu)$, for any $\lambda \in \mathbb{R}, x^\mu \in \mathbb{R}^{d-1}$; (b) a polynomial $h_\ell\colon \mathbb{R}^{d-1} \to \mathbb{C}$ is harmonic in $\mathbb{R}^{d-1}$ if $\Box_E h_\ell = 0$, where $\Box_E$ is the Laplacian on $\mathbb{R}^{d-1}$; (c) a spherical harmonic of degree ℓ on $\mathcal{S}^{d-2}$ is a function $Y_\ell\colon \mathcal{S}^{d-2} \to \mathbb{C}$ such that, for some homogeneous and harmonic polynomial in $\mathbb{R}^{d-1}$, h_ℓ, $Y_\ell(\theta) = h_\ell(\theta)$ for all $\theta = (\theta^2, \ldots \theta^{d-1}) \in \mathcal{S}^{d-2}$. Spherical harmonics on the $(d - 2)$-sphere can also be constructed recursively by dimensional reduction, see, e.g., [28], but the approach becomes cumbersome when constructing vector spherical harmonics. For more details on the matter we refer to [27–30], see also [24].

2. Scalar spherical harmonics

In spherical coordinates, the $\mathbb{R}^{d-1}$ line element, ds^2, is related to the $\mathcal{S}^{d-2}$ line element, $(d\Omega^{d-2})^2 = \hat{g}_{ij} d\theta^i d\theta^j$, by $ds^2 = dr^2 + r^2 (d\Omega^{d-2})^2$. Using Eq. (11), the nonvanishing Christoffel symbols associated to $g_{\mu\nu}$ in these coordinates are

$$\Gamma^r{}_{ij} = -r\hat{g}_{ij}, \qquad \Gamma^i{}_{rj} = \frac{1}{r}\delta^i_j, \qquad \Gamma^i{}_{jk} = \hat{\Gamma}^i{}_{jk}, \tag{A1}$$

and the condition for a homogeneous polynomial of degree ℓ, h_ℓ, to be harmonic in $\mathbb{R}^{d-1}$, becomes

$$\Box_E h_\ell = \frac{1}{r^{d-2}} \partial_r (r^{d-2} \partial_r h_\ell) + \frac{1}{r^2} \hat{\Box} h_\ell = 0, \tag{A2}$$

where $\hat{\Box} = \hat{g}^{ij}\hat{\nabla}_i\hat{\nabla}_j$. Since h_ℓ is homogeneous, it follows that $h_\ell(x)|_S = r^\ell h_\ell(\hat{x})|_S = r^\ell Y_\ell(\theta)$, where $x^\mu = r\hat{x}^\mu$, so that, substituting in Eq. (A2), one has

$$\hat{\Box} Y_\ell = -\ell(\ell+d-3)Y_\ell, \quad \ell = 0,1,2,\ldots. \tag{A3}$$

Y_ℓ are called scalar spherical harmonics. Besides being eigenfunctions of $\hat{\Box}$, it can also be shown that they form a complete and orthogonal set on $\mathcal{S}^{d-2}$.

3. Vector spherical harmonics

Vector spherical harmonics can be constructed in the same way as scalar spherical harmonics, only this time one starts with vector functions $V^\ell_\nu\colon \mathbb{R}^{d-1} \to \mathbb{C}^{d-1}$. Using the Helmholtz-Hodge theorem, V^ℓ_ν can be written as, in analogy to Eq. (15),

$$V^\ell_\nu dx^\nu = V^\ell_r dr + (W^\ell_i + \hat{\nabla}_i\sigma^\ell)d\theta^i, \quad \hat{\nabla}^i W^\ell_i = 0, \tag{A4}$$

where W^i_ℓ is a vector on the $(d-2)$-sphere and V^ℓ_r and σ^ℓ are scalars. Expanding $\Box_E V^\ell_\nu$ in spherical coordinates, and assuming that V^ℓ_ν is harmonic, one has

$$\Box_E W^\ell_i = \partial_r^2 W^\ell_i + \frac{d-4}{r}\partial_r W^\ell_i - \frac{d-3}{r^2}W^\ell_i + \frac{1}{r^2}\hat{\Box} W^\ell_i = 0, \tag{A5}$$

as well as two coupled equations for the scalars V^ℓ_r and σ^ℓ [27]. Here, we are only interested in the equations for W^ℓ_i. Since V^ℓ_ν is homogeneous, $V^\ell_\mu(x) = r^\ell V^\ell_\mu(\theta)$. This means that $V^\ell_\nu(x)dx^\nu = \frac{\partial x^\mu}{\partial r} r^\ell V^\ell_\mu(\theta)dr + \frac{\partial x^\mu}{\partial \theta^i} r^\ell V^\ell_\mu(\theta)d\theta^i$ and so $W^\ell_i + \hat{\nabla}_i\sigma^\ell = r^{\ell+1}\frac{\partial \hat{x}^\mu}{\partial \theta^i}V^\ell_\mu(\theta)$, where $x^\mu = r\hat{x}^\mu$. Hence, we define $Y_{\ell i}(\theta)$ as the angular dependence of $W^\ell_i(x)$, i.e., $W^\ell_i(x)|_S = r^{\ell+1}Y_{\ell i}(\theta)$, which put in Eq. (A5), it follows that,

$$\hat{\Box} Y_{\ell i} = -[\ell(\ell+d-3)-1]Y_{\ell i}, \quad \ell = 1,2,\ldots. \tag{A6}$$

The vectors $Y_{\ell i}$ are called transverse vector spherical harmonics, as they verify $\hat{\nabla}^i Y_{\ell i} = 0$. One can also construct longitudinal vector spherical harmonics, $Y^L_{\ell i}$, by taking the gradient of scalar fields on $\mathcal{S}^{d-2}$. These are defined as [29]

$$Y^L_{\ell i} \equiv \frac{1}{\sqrt{\ell(\ell+d-3)}}\hat{\nabla}_i Y_\ell, \tag{A7}$$

and have eigenmodes

$$\hat{\Box}\hat{\nabla}_i Y_\ell = -[\ell(\ell+d-3)-(d-3)]\hat{\nabla}_i Y_\ell, \quad \ell = 1,2,\ldots. \tag{A8}$$

4. Properties under rotation and parity transformations

Now, let φ and u^i be, respectively, a scalar field and a vector field on $\mathcal{S}^{d-2}$. The action of the $\mathrm{SO}(d-1)$ Casimir operator, $\hat{J}^2$, on these two entities is $\hat{J}^2\varphi = -\Box_S\varphi$ and $\hat{J}^2 u^i = -(\Box_S - (d-3))u^i$, see [21]. Using Eqs. (A3), (A6), and (A8), one gets the following:

$$\hat{J}^2 Y_\ell = \ell(\ell+d-3)Y_\ell, \tag{A9}$$

$$\hat{J}^2\hat{\nabla}_i Y_\ell = \ell(\ell+d-3)\hat{\nabla}_i Y_\ell, \tag{A10}$$

$$\hat{J}^2 Y_{\ell i} = [\ell(\ell+d-3)+d-4]Y_{\ell i}. \tag{A11}$$

One sees that, for $\ell \geq 1$ and $d > 4$, the Casimir eigenvalues of the scalar spherical harmonics and of the longitudinal vector spherical harmonics are the same. One then expects these modes to mix. On the contrary, the Casimir eigenvalues of the transverse vector spherical harmonics are never equal for $d > 4$ to the eigenvalues of the longitudinal vector spherical harmonics, so that these completely decouple. For $d = 4$, the last are equal and one might expect to have mixed modes. However, in this case, the modes are decoupled due to their different parity eigenvalues. Indeed, under parity transformations $\theta_{i=2} \to \pi - \theta_{i=2}$ and $\theta_{i=3} \to \pi + \theta_{i=3}$, one has $\hat{\mathcal{P}}Y_\ell = (-1)^\ell Y_\ell$, $\hat{\mathcal{P}}\hat{\nabla}_i Y_\ell = (-1)^\ell\hat{\nabla}_i Y_\ell$ and $\hat{\mathcal{P}}Y_{\ell i} = (-1)^{\ell+1}Y_{\ell i}$. Note that a vector, A_{θ_i}, on the $(d-2)$-sphere transforms under parity as $A_{\theta_{i=1}} \to A_{\theta_{i=1}}$ and $A_{\theta_{i\neq 1}} \to -A_{\theta_{i\neq 1}}$.

APPENDIX B: HYPERGEOMETRIC DIFFERENTIAL EQUATION

1. General equation

We study now the solutions of the hypergeometric differential equation, based on [30], see also [21].

The Proca and electromagnetic field equations can be separated and decoupled to assume the reduced form

$$\partial^2_{r_*} q + \left(\omega^2 - \frac{G}{\sin^2\left(\frac{r_*}{l}\right)} - \frac{H}{\cos^2\left(\frac{r_*}{l}\right)}\right)q = 0, \tag{B1}$$

as it appears in Eqs. (59) and (79), where $r_* = l\arctan(\frac{r}{l})$, $G = \frac{4j(j+d-3)+(d-2)(d-4)}{4l^2}$ for $j \in \mathbb{N}_0$, and H for now is some constant that depends on d, l, μ, and j with units of $\frac{1}{l^2}$. By making the change

$$q(z) = z^\alpha(1-z)^\beta\Theta(z), \tag{B2}$$

with

$$z = \sin^2\left(\frac{r_*}{l}\right), \tag{B3}$$

and recalling $r_* = l \arctan\frac{r}{l}$, one can transform Eq. (B1) into the hypergeometric differential equation

$$z(1-z)\frac{d^2\Theta}{dz^2} + [c - (a+b+1)z]\frac{d\Theta}{dz} - ab\Theta = 0, \tag{B4}$$

with $a = \alpha + \beta + \frac{\omega l}{2}$, $b = \alpha + \beta - \frac{\omega l}{2}$, $c = 2\alpha + \frac{1}{2}$, and $\alpha = \frac{2j+d-2}{4}$, $\beta = \frac{1}{4}(1 + \sqrt{1 + 4Hl^2})$. For d even, the solutions of (B4) are given in terms of the hypergeometric function, ${}_2F_1$, as [30]

$$\begin{aligned} q(z) = {} & Az^{\alpha}(1-z)^{\beta}{}_2F_1[a,b,c;z] \\ & + Bz^{1/2-\alpha}(1-z)^{\beta}{}_2F_1[a-c+1, b-c+1, 2-c; z] \\ & \text{for } d \text{ even}, \end{aligned} \tag{B5}$$

where A and B are constants of integration. For d odd, the indicial roots of Eq. (B4) are separated by an integer, making the solutions presented linearly dependent. In this case, since $c \geq 2$ and it is an integer, $q(z)$ can be written as [30]

$$\begin{aligned} q(z) = {} & Az^{\alpha}(1-z)^{\beta}{}_2F_1[a,b,c;z] + Bz^{\alpha}(1-z)^{\beta} \\ & \times \left[{}_2F_1[a,b,c;z]\ln z + \sum_{i=1}^{\infty} v_i z^i - \sum_{i=1}^{c-1} w_i z^{-i}\right] \\ & \text{for } d \text{ odd}, \end{aligned} \tag{B6}$$

where

$$\begin{aligned} v_i = {} & \frac{(a)_i (b)_i}{(c)_i (i!)}[\Psi(a+i) - \Psi(a) + \Psi(b+i) - \Psi(b) \\ & - \Psi(c+i) + \Psi(c) - \Psi(1+i) + \Psi(1)], \end{aligned} \tag{B7}$$

$$w_i = \frac{(i-1)!(1-c)_i}{(1-a)_i (1-b)_i}, \tag{B8}$$

$(a)_i = \frac{\Gamma(a+i)}{\Gamma(a)}$ if $a > 0$, $(a)_i = (-1)^i(-a-i+1)_i$ if $a < 0$, and the same for $(b)_i$ and $(c)_i$, and $\Psi(a) = \frac{\Gamma'(a)}{\Gamma(a)}$ is the digamma function, see [30].

2. Imposing regularity conditions near $r = 0$, i.e., $z = 0$

To obtain the normal modes, one needs to impose boundary conditions that characterize the system. At $r = 0$ or $z = 0$, one imposes regularity conditions so that the solution does not diverge there.

For even spacetimes, it is seen from Eq. (B5) that for $\alpha > \frac{1}{2}$ one needs to set $B = 0$, considering that $\lim_{z\to 0}{}_2F_1[a,b,c;z] = 1$. For $\alpha = \frac{1}{2}$, which corresponds to the case q_2 with $\ell = 1$ and $d = 4$, the exponent in z vanishes and the solution seems to be finite near $z = 0$. However, this is just an artifact of having removed the origin when separating the field in spherical harmonics, see [21]. Indeed, if one writes the components of the four-dimensional Proca field A_r, A_{θ_2} and A_{θ_3} in terms of q_2 with $\ell = 1$ and the other fields put to zero, one arrives at $A_r = \frac{1}{rf}\sum_{m=-1}^{1} q_{2(0,m)} Y_{(1,m)}$, $A_{\theta_2} = \sum_{m=-1}^{1} q_{2(0,m)}\partial_{\theta_2} Y_{(1,m)}$, and $A_{\theta_3} = -\sum_{m=-1}^{1}\frac{q_{2(0,m)}}{d-3}\partial_{\theta_3} Y_{(1,m)}$, where $q_{2(0,m)}$ is q_2 with $j_2 = 0$, $\ell = 1$ and azimuthal number m. Since near the origin, the spacetime is flat, i.e., $f(r) \simeq 1$, and $q_{2(0,m)} = B_{(1,m)}$ where $B_{(1,1)}$, $B_{(1,0)}$ and $B_{(1,-1)}$ are constants, the behavior of the Proca field near the origin becomes $A_r \simeq \frac{1}{r}(B_{(1,1)}Y_{(1,1)} + B_{(1,0)}Y_{(1,0)} + B_{(1,-1)}Y_{(1,-1)})$, $A_{\theta_2} \simeq B_{(1,1)}\partial_{\theta_2}Y_{(1,1)} + B_{(1,0)}\partial_{\theta_2}Y_{(1,0)} + B_{(1,-1)}\partial_{\theta_2}Y_{(1,-1)}$, $A_{\theta_3} \simeq B_{(1,1)}\partial_{\theta_3}Y_{(1,1)} + B_{(1,0)}\partial_{\theta_3}Y_{(1,0)} + B_{(1,-1)}\partial_{\theta_3}Y_{(1,-1)}$. The components of a vector field transform as $A_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}A_{\mu}$ to give $A_x, A_y, A_z \sim \frac{1}{\sqrt{x^2+y^2+z^2}}$, in Cartesian coordinates. Since the Proca field equations Eq. (6) can be written near the origin in Cartesian coordinates as $\partial^{\nu}\partial_{\nu}A_{\rho} - \mu^2 A_{\rho} = 0$, where $\alpha = \{t, x, y, z\}$, one has $(\partial_x^2 + \partial_y^2 + \partial_z^2)A_{x,y,z} \sim \delta(x)\delta(y)\delta(z)$. Due to this additional delta term, $q_{2(0,m)} = B_{(1,m)}$, with $B_{(1,m)} \neq 0$, cannot be a solution near the origin and one needs to set $B = 0$ in Eq. (B5), even in this special case. Moreover, if one takes the asymptotic expansion of the integrand of the usual energy, $T_{tt}\frac{r^{d-2}}{f}$, near $r = 0$, then one gets divergent terms r^{-1}, r^{-2} and r^{-4} which only vanish if $B = 0$. Thus, in this particular case of $\alpha = \frac{1}{2}$, which corresponds to the case q_2 with $\ell = 1$ and $d = 4$, one has also $B = 0$.

For odd spacetimes, all the terms of Eq. (B6) except the last one vanish in the limit $z \to 0$. For the last term, as $c \geq 2$ and it is an integer, it can be seen that $\sum_{i=1}^{c-1} w_i z^{\alpha-i}$ contains a power r^s with $s \leq -\frac{1}{2}$ always and so the field q would not be square integrable. Thus, one also needs to set $B = 0$ in this case.

3. Imposing Dirichlet boundary conditions at spatial infinity, $r \to \infty$, i.e., $z \to 1$

a. Expansion at spatial infinity for general case

To impose the remaining boundary condition, one uses the transformation law $z \to 1 - z$ of ${}_2F_1$, so that, if $c - a - b = \frac{1}{2} - 2\beta \neq -m'$ with $m' \in \mathbb{N}_0$ one has [30]

$$\begin{aligned} q(z) = {} & Az^{\alpha}(1-z)^{\beta}\left[\frac{\Gamma(c)\Gamma(1-e)}{\Gamma(\bar{a})\Gamma(\bar{b})}{}_2F_1[a,b,e;1-z]\right. \\ & \left. + (1-z)^{\frac{1}{2}-2\beta}\frac{\Gamma(c)\Gamma(e-1)}{\Gamma(a)\Gamma(b)}{}_2F_1[\bar{a},\bar{b},2-e;1-z]\right], \end{aligned} \tag{B9}$$

where $e = 1 - c + a + b$, $\bar{a} = c - a$ and $\bar{b} = c - b$. If $c - a - b = \frac{1}{2} - 2\beta = 0$, one has

$$q(z) = Az^{\alpha}(1-z)^{\beta}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\sum_{i=0}^{\infty}\frac{(a)_i(b)_i}{(i!)^2}(1-z)^i \times[2\Psi(i+1) - \Psi(a+i) - \Psi(b+i) - \log(1-z)]. \quad \text{(B10)}$$

If $c - a - b = \frac{1}{2} - 2\beta = -m'$ with $m' \in \mathbb{N}$, one has

$$q(z) = Az^{\alpha}(1-z)^{\beta}\left(\frac{\Gamma(m')\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{i=0}^{m'-1}v'_i(1-z)^{i-m'} - (-1)^{m'}\sum_{i=0}^{\infty}\frac{\Gamma(c)(1-z)^i}{\Gamma(a-m')\Gamma(b-m')}[w'_i\ln(1-z)+t_i]\right), \quad \text{(B11)}$$

with the coefficients defined by

$$v'_i = \frac{(a-m')_i(b-m')_i}{i!(1-m')_i}, \quad \text{(B12)}$$

$$w'_i = \frac{(a)_i(b)_i}{i!(i+m')!}, \quad \text{(B13)}$$

$$t_i = w'_i[\Psi(a+i)+\Psi(b+i)-\Psi(i+1)-\Psi(i+m'+1)]. \quad \text{(B14)}$$

b. Proca field case

In the Proca field case, we have three fields q_1, q_2 and q_3, where the value of j that appears in Eq. (B1) for each field is $j_1 = \ell + 1$, $j_2 = \ell - 1$ and $j_3 = \ell$. The constant H in this case is $H = \frac{(d-2)(d-4)+4\mu^2l^2}{4l^2}$, which leads to the expression of β in Eq. (B1) to become $\beta = \frac{1}{4}(1 + \sqrt{(d-3)^2 + 4\mu^2l^2})$. Moreover, we have $c - a - b \neq -m'$ with $m' \in \mathbb{N}_0$. Therefore the correct expansion at spatial infinity of the Proca field is described by Eq. (B9). Since $\beta > \frac{1}{2}$, the first term of Eq. (B9) vanishes as the gamma function in the numerator is finite. The remaining term must be zero to satisfy the Dirichlet boundary condition, which only occurs if either a or b are nonpositive integers. By requiring that $\omega > 0$, the Dirichlet boundary condition leads to $b = -n$, with $n \in \mathbb{N}_0$, and so the normal mode eigenfrequencies $\omega_k l$ for each field q_k are

$$\omega_k L = 2n + j_k + \frac{d-1}{2} + \frac{1}{2}\sqrt{(d-3)^2 + 4\mu^2L^2}, \quad \text{(B15)}$$

where the normal mode eigenfunctions are given by

$$q_k(r) = A_k\left(\frac{r}{l}\right)^{\frac{2j_k+d-2}{2}}\left(1+\frac{r^2}{l^2}\right)^{n-\frac{\omega_k l}{2}} \times {}_2F_1\left[-n+\omega_k l, -n, j_k + \frac{d-1}{2}; \frac{r^2/l^2}{1+r^2/l^2}\right]. \quad \text{(B16)}$$

c. Maxwell electromagnetic field case

Scalar-type perturbation:

The scalar-type perturbations for the Maxwell field are described by the function q_{12}, where the value of j that appears in Eq. (B1) is $j = \ell$. The constant H in this case is $H = \frac{(d-4)(d-6)}{4l^2}$, which leads to the expression of β in Eq. (B1) to become $\beta = \frac{1}{4}(1 + |d-5|)$.

For even dimensions, the expansion at spatial infinity of q_{12} is described by Eq. (B9), since $c - a - b \neq -m'$, with $m' \in \mathbb{N}_0$. We distinguish two cases here, $d = 4$ and $d \geq 6$. At spatial infinity for $d = 4$, since $\beta = \frac{1}{2}$, the first term in Eq. (B9) vanishes while the second term is finite. We still impose the Dirichlet boundary conditions and the first term vanishes if $b = -n$, with $n \in \mathbb{N}_0$ and $\omega l > 0$. Therefore, the normal mode frequencies are

$$\omega_{12} = 2n + \ell + 2, \quad \text{for } d = 4. \quad \text{(B17)}$$

For $d \geq 6$ and even, at spatial infinity, since $\beta \geq \frac{1}{2}$, the first term in Eq. (B9) vanishes while the second term seems to diverge for $d > 6$ and seems to assume a finite value for $d = 6$. Despite the $d = 6$ case, we impose the Dirichlet boundary conditions and the second term vanishes if $b = -n$, with $n \in \mathbb{N}_0$ and $\omega l > 0$. Therefore, the normal mode frequencies are

$$\omega_{12} = 2n + \ell + d - 3, \quad \text{for } d \geq 6. \quad \text{(B18)}$$

For odd dimensions, we split the analysis for $d = 5$ and $d \geq 7$. For $d = 5$, the expansion at spatial infinity of q_{12} is described by Eq. (B10), since $c - a - b = 0$. Therefore, at spatial infinity, all the terms vanish in Eq. (B10). The Dirichlet boundary conditions, which impose the field to vanish at $r \to +\infty$, makes the eigenvalue problem ill-defined. One can instead impose a one parameter family boundary condition. For example, one can choose a boundary condition such that the logarithmic term vanishes called Dirichlet-Neumann, which is satisfied if $b = -n$, with $n \in \mathbb{N}_0$, and so the frequency is given by

$$\omega_{12} = 2n + \ell + 2, \quad \text{for } d = 5. \quad \text{(B19)}$$

For $d \geq 7$, the expansion at spatial infinity of q_{12} is described by Eq. (B11), with $\beta > \frac{1}{2}$ and $c - a - b = -m'$

with $m' \in \mathbb{N}$. In this case, the second term vanishes while the first term seems to diverge. We impose the Dirichlet boundary conditions and so the first term vanishes if $b = -n$ with $n \in \mathbb{N}_0$. The normal mode eigenfrequencies are then also described by Eq. (B18).

The eigenfunctions for all the even and odd dimension cases are

$$q_{12}(r) = A_{12}\left(\frac{r}{l}\right)^{\frac{2\ell+d-2}{2}}\left(1+\frac{r^2}{l^2}\right)^{n-\frac{\omega_{12}l}{2}} \times {}_2F_1\left[-n+\omega_{12}l, -n, \ell+\frac{d-1}{2}; \frac{\frac{r^2}{l^2}}{1+\frac{r^2}{l^2}}\right]. \tag{B20}$$

Vector-type perturbation:

The vector-type perturbations are described by the function q_3, where the value of j that appears in Eq. (B1) is $j = \ell$. The constant H in this case is $H = \frac{(d-2)(d-4)}{4l^2}$, which leads to the expression of β in Eq. (B1) to become $\beta = \frac{1}{4}(d-2)$.

For even dimensions, the expansion at spatial infinity of q_3 is described by Eq. (B9), since $\beta \geq \frac{1}{2}$ and $c - a - b \neq -m'$, with $m' \in \mathbb{N}_0$. At spatial infinity, it turns out that the first term in Eq. (B9) vanishes while the second term seems to diverge for $d > 4$ and assumes a finite value for $d = 4$. Despite the $d = 4$ case, we impose Dirichlet boundary conditions and the second term only vanishes if $b = -n$, with $n \in \mathbb{N}_0$. Therefore, the normal mode eigenfrequencies are given by

$$\omega_3 l = 2n + \ell + d - 2. \tag{B21}$$

For odd dimensions, the expansion at spatial infinity of q_3 is described by Eq. (B11), since $\beta > \frac{1}{2}$ and $c - a - b = -m'$, with $m' \in \mathbb{N}$. At spatial infinity, the first term in Eq. (B11) seems to diverge. Imposing the Dirichlet boundary conditions, these terms vanish if again $b = -n$ with $n \in \mathbb{N}_0$. The normal mode eigenfrequencies for odd dimensions have then the same expression as Eq. (B21).

The eigenfunctions for all the even and odd dimension cases are

$$q_3 = A_3\left(\frac{r}{l}\right)^{\frac{2\ell+d-2}{2}}\left(1+\frac{r^2}{l^2}\right)^{n-\frac{\omega_3 l}{2}} \times {}_2F_1\left[-n+\omega_3 l, -n, \ell+\frac{d-1}{2}; \frac{\frac{r^2}{l^2}}{1+\frac{r^2}{l^2}}\right]. \tag{B22}$$

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