Endowing black holes with beyond-Horndeski primary hair: An exact solution framework for scalarizing in every dimension

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This work outlines a straightforward mechanism for endorsing primary hair into Schwarzschild black holes, resulting in a unique modification within the framework of a special scalar-tensor theory, the so-called beyond-Horndeski gravity. The derived solutions are exact, showcase primary hair with a regular scalar field profile everywhere, and continuously connect with the vacuum geometry. Initially devised to introduce primary hair in spherically symmetric solutions within general relativity in any dimension, our investigation explores the conditions under which spherically symmetric black holes in alternative gravitational theories become amenable to the endowment of primary hair through a similar pattern. As a preliminary exploration, we embark on the process of endowing primary hair to the Reissner-Nordström black hole. Subsequently, we extend our analysis to encompass spherically symmetric solutions within Lovelock and cubic quasitopological gravity theories.

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I. INTRODUCTION

Undoubtedly, the present era constitutes a remarkable epoch for the exploration of gravitational phenomena. The affirmation of gravitational waves [1], the first black hole image achieved by the Event Horizon Telescope network [2], and the investigation into the precession orbits of stars revolving around compact massive objects [3] collectively present an unprecedented opportunity to scrutinize gravity on a stage far exceeding the well-established scales of the Solar System [4]. Consequently, a discernible dichotomy emerges: Einstein's theory of general relativity (GR) assumes an unequivocal primacy, having demonstrated confirmation in the realm of strong gravity. Simultaneously, a newfound avenue emerges for the evaluation of alternative gravity theories aspiring to enhance GR and provide solutions to phenomena where the theory proves inadequate. In this context, theories incorporating additional degrees of freedom, particularly scalar fields, emerge as economical and archetypal modifications of GR. Since their inception, these models, encapsulated within the framework of scalar-tensor theories of gravity, have undergone extensive scrutiny, primarily in the realm of cosmology [5], though their applicability extends far beyond. These models find their most comprehensive formulation in Horndeski theory [6], alongside subsequent higher-order modifications such as beyond-Horndeski theories [7,8] and degenerated higher order scalar tensor theories (DHOST) [9–12]. Establishing a robust foundation for the validity of these alternative models necessitates a meticulous examination of their spectrum of black hole solutions. This endeavor not only furnishes a theoretical framework for assessing the consistency of these models but also anticipates potential experimental implications.

It is commonly argued that, following gravitational collapse, a black hole can be adequately described by a specific set of parameters, namely its mass, electromagnetic charges, and angular momentum. This perspective implies that no additional distinctive features of the original matter, such as baryon or lepton numbers, persist after the black hole formation. In certain scenarios, this proposition is substantiated, leading to the formulation of no-hair theorems [13]. Here, the term "hair" is used metaphorically to encompass all characteristics that would render black holes nonbald, indicating quantities not subject to a Gauss law and, consequently, not conserved at infinity. This definition originates from the numerical construction of Einstein-Yang-Mills black holes, where a discrete

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parameter, that represents the number of nodes of the gauge function [14,15], emerges.

Over the decades, numerous endeavors have been undertaken to construct solutions to introduce various types of hairs to black hole configurations. Notably, there has been a particular focus on scalar hair, which involves configurations where a nontrivial scalar field profile coexists within the spacetime, primarily driven by the assumption of the inherent simplicity of scalar fields [16]. However, the incorporation of scalar fields into black hole configurations proves to be challenging, with many solutions exhibiting curvature singularities or divergence of the scalar field profile within the domain of outer communications. As no-hair theorems constitute theory-dependent mathematical statements, the scientific community has continuously subjected them to scrutiny to assess their validity. Early contributions by Chase [17] and subsequently, by Bekenstein [18] defined conditions under which minimally coupled scalar fields cannot dress a black hole spacetime. Similar no-hair theorems were also formulated by Hawking for the Brans-Dicke theory [19], later generalized in the presence of self-interaction for the scalar field [20]. In a more contemporary context, modern scalar-tensor theories, predominantly represented by Horndeski gravity and its higher-order extensions beyond-Horndeski and DHOST theories, have faced constraints concerning the emergence of hairy black hole solutions [21–24]. However, the potency of these theorems is contingent upon their underlying assumptions, and it is, therefore, plausible to circumvent them by precisely relaxing some of their key axioms. A substantial body of literature has emerged, delving into the construction and investigation of black holes with scalar hair. This journey began with the discovery of black holes in the context of conformally coupled scalar theories [25–40] and has progressed to encompass more recent configurations found in complex minimally coupled models [41] and the Horndeski theory as well as higher-order scalar-tensor theories [42–67].

The diversification of hairy black hole solutions has led to a nuanced understanding of the scalar hair concept, culminating in the categorization of two distinct types: primary and secondary. The latter denotes black hole spacetimes characterized by a nontrivial scalar field profile, which, crucially, does not introduce any additional parameter to the geometry. Consequently, the backreaction of the spacetime exhibits no explicit manifestation of the scalar hair, preventing these solutions from forming a continuous connection with vacuum black hole spacetimes. In contrast, primary hair designates black hole spacetimes featuring a nontrivial scalar field configuration that alters the spacetime backreaction by incorporating an additional parameter. Consequently, black holes with primary hair can form a continuous connection with vacuum solutions, that is, with their bald counterpart geometries. While solutions with secondary hair constitute the majority of existing exact solutions documented in the literature, those with primary hair have predominantly been constructed through numerical methods [41,68].

In seeking to investigate potential experimental signatures associated with black holes harboring extra degrees of freedom, from an astrophysical perspective, it has been introduced as an intriguing mechanism for the systematic numerical construction of black holes with scalar hair. This process known as "scalarization" represents a pathway by which a vacuum black hole can develop scalar hair through a tachyonic instability, revealing the emergence of black holes with secondary hair at its culmination. The fundamental properties defining a black hole with scalar hair include the presence of black holes exhibiting a consistently regular scalar field configuration. Additionally, their backreaction is contingent upon the existence of a scalar charge (a continuous parameter governing the manifestation of the scalar field profile that, in this case, depends on the mass parameter of the solution). These black hole solutions with secondary hair have undergone thorough investigation in recent years, encompassing not only spherically symmetric configurations [69–74] but also extending to stationary and axially symmetric ones [75–77]. Furthermore, this exploration has extended to hairs beyond the scalar variety, including vectorial and tensorial natures [78–84].

Efforts directed towards the construction of exact black hole solutions featuring primary hair remain limited. This scarcity primarily stems from the intricate nature of the theories within which the search for such hair is conducted, compounded by the challenges posed by the underlying complexities embedded in no-hair theorems. Furthermore, scalarizing processes in the presence of a cosmological constant or within an arbitrary number of dimensions pose considerable difficulties and represent a direction far less explored [85,86]. These constraints significantly limit the applicability of these black holes for exploration, particularly in realms such as black hole thermodynamics or other semiclassical phenomena within the framework of the AdS/CFT conjecture [87–91], just to name a few examples. Consequently, the pursuit of exact black hole solutions with primary hair, or, due to the similarities in the process, exact scalarized black holes[1] becomes an intriguing avenue of investigation.

Recently, analytical black hole solutions featuring primary scalar hair were discovered in [92] within the framework of beyond-Horndeski theories [7] in four dimensions. Beyond-Horndeski theories represent extensions of the

[1]The determination of whether these exact black hole solutions with primary hair exhibit a tachyonic instability is not the primary focus of this comparison. Instead, we emphasize the significance of a black hole possessing a regular scalar field configuration everywhere and well-defined geometry, in this particular case connecting with the vacuum via the vanishing of the scalar charge.

well-known Horndeski theories [6], incorporating higher-order derivatives while avoiding Ostrogradski ghosts. These theories have demonstrated considerable promise in the exploration of compact objects, as evidenced in works such as [64–66]. This promise extends to the construction of scalarized black holes or black holes with primary hair. In particular, the authors of [92] demonstrated the existence of an extension of the Schwarzschild black hole within precise beyond-Horndeski models, specifically when $G_2 \sim X^2$, $G_4 \sim X^2$, and $F_4 \sim cte$ [see the action (1) below]. Remarkably, the resulting spacetime remains described by the Schwarzschild metric, augmented by a term proportional to the scalar hair. Furthermore, it was revealed that the inclusion of this additional term enables the elimination of the central singularity through a specific tuning between the mass and the hair.

In this study, we aim to broaden the findings established in [92] by extending the class of beyond-Horndeski theories capable of accommodating similar black hole solutions endowed with primary scalar hair. Despite the original theory being defined in four dimensions, our investigation will encompass the arbitrary dimensional case. Our focus is on demonstrating that the existence of hairy solutions and extending the Schwarzschild black hole paradigm can be guaranteed for a broader selection of theory functions. Specifically, we will establish that a two-parametric subclass of actions, characterized by functions G_2 and G_4 within the framework of (1), facilitates the emergence of such scalarized solutions. The resulting metric solution comprises a superposition of the Schwarzschild-(A)dS function with an additional component proportional to the scalar hair. We will further expand this pattern by demonstrating that, under specific hypotheses, this two-parametric class of beyond-Horndeski theories can be coupled with other actions of pure gravity (beyond-GR and potentially involving additional dynamical fields). This coupling allows for the extension of purely static black hole solutions to static black holes with primary hair.

This paper is organized as follows: In Sec. II, we elaborate our approach to endowing vacuum black holes with primary scalar hair. In essence, we outline the generic construction of scalarized black holes within the domain of spherically symmetric solutions. We delve into the explicit construction of exact scalarized Schwarzschild black holes within a theory characterized by the form (1). Specifically, we demonstrate that solutions featuring primary hair and smoothly connecting to the Schwarzschild-(A)dS solution may exist for a subset of actions (1) parametrized by G_2 and G_4. The specific scenario where both coupling functions are proportional is examined in detail. Section III introduces a set of conditions under which the framework outlined in Sec. II can be extended to encompass other gravity theories, whether purely geometrical or involving additional matter fields (distinct from the beyond-Horndeski scalar field, ϕ). We illustrate how black holes in alternative theories can be enhanced to exhibit primary hair of the beyond-Horndeski type, as defined by (1). To illustrate this, we demonstrate how the Einstein-Maxwell theory supports black holes with primary hair, thereby explicitly constructing a scalarized version of the Reissner-Nordström black hole. Subsequently, we extend our exploration to the construction of black holes with primary hair in Lovelock and cubic quasitopological gravities. Finally, Sec. IV is dedicated to concluding and proposing several avenues for further exploration and generalization of the framework presented herein. Given the generic nature of our approach, we allocate an Appendix dedicated to the construction of specific black hole configurations. In particular, we delve into the cases of GR and the Einstein-Gauss-Bonnet theory.

II. SCALARIZING THE SCHWARZSCHILD BLACK HOLE

The primary objective of this work is to introduce primary hair onto initially bald black hole solutions, beginning within the framework of general relativity (GR) and subsequently, extending to other geometric theories of gravity. Despite the extensive literature on the construction of black holes with hair, these solutions typically manifest secondary hair, exemplified by stealth black holes or standard black holes lacking a continuous limit with the vacuum (bald) geometry. Various techniques have been employed to construct these solutions, ranging from scalar fields that do not share the same symmetries as the geometry to the utilization of disformal transformations. A prevalent characteristic of most solutions is their emergence within theories that feature shift symmetry. This allows the scalar field equation to be formulated as a current conservation law, facilitating the integration of field equations. Another commonly adopted strategy involves stipulating a constant kinetic term for the scalar field. This considerably simplifies the contribution of the scalar sector within a given scalar-tensor theory, reducing the problem to finding stealth black holes by adjusting the Lagrangian functions. Such solutions are attainable when the scalar field profile exhibits a linear time dependence, a characteristic that, due to the shift invariant nature of the models, does not compromise the stationary nature of the solutions. More challenging is the discovery of solutions with a nonconstant kinetic term, representing black holes with novel back-reactions. In the subsequent sections, we will assimilate various elements from the existing literature and combine them in a manner that facilitates the systematic construction of exact black hole solutions with primary hair. This approach is applicable to a sufficiently general class of theories encapsulated within beyond-Horndeski gravity and generically contain a nontrivial kinetic term for the scalar field.

A. Beyond-Horndeski theory and the scalarization scheme

As our initial aim is to scalarize the Schwarzschild black hole and its higher dimensional extension, the Schwarzschild-Tangherlini black hole, we consider the action of quadratic beyond-Horndeski gravity [7] elevated to an arbitrary dimension, d,

$$S = \int d^d x\sqrt{-g}\big[G_2(X) + G_4(X)R + \big(G_{4,X} + 2XF_4(X)\big)\big((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\big) + 2F_4(X)\big(\Box\phi\phi^{\mu}\phi_{\mu\nu}\phi^{\nu} - \phi_{\mu}\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho}\big)\big]. \qquad (1)$$

For simplicity, we have defined $\phi_\mu = \partial_\mu\phi$ and $\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$, where the coupling functions G_2, G_4, and F_4 depend solely on the kinetic term $X = -\frac{1}{2}\phi_\mu\phi^\mu$. Here, $G_{4,X}$ stands for the derivative of G_4 with respect to X, i.e., $G_{4,X} = \frac{dG_4(X)}{dX}$, and $\epsilon_{\mu\nu\rho\sigma}$ stands for the Levi-Civita tensor. Notice that GR is naturally included by a proper choice of the theory function G_4. Action (1) is invariant under a constant shift of the scalar field $\phi \to \phi + \mathrm{cst}$, a heritage from the Galileon origin of the model [93], and is parity invariant $\phi \to -\phi$ as it is quadratic in the derivatives of the scalar field.

Focusing on spherical symmetry, we consider a d-dimensional spacetime configuration of the form,

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{d-2,\kappa},$$
$$\phi(t,r) = qt + \psi(r), \qquad (2)$$

where the $(d-2)$-dimensional base manifold has constant curvature $\kappa = 0, \pm 1$ representing a spherical, hyperbolic, or flat topology, respectively. Here, q is a constant of integration that will also appear in the black hole metric function [92], which will give the constant q the character of primary hair. In addition, it plays a crucial role in the regularity of the scalar field profile. As already mentioned, the action (1) enjoys invariance under a constant translation of the scalar field, and as a consequence, the scalar field equation of motion converts into a conservation law for the scalar Noether current,

$$\mathcal{J}^\mu = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta(\partial_\mu\phi)}, \qquad \nabla_\mu\mathcal{J}^\mu = 0.$$

This property has played a major role in the construction of black hole solutions in Horndeski gravity and its higher order generalizations. Its regularity at the would-be black hole horizon along with a few other assumptions regarding the asymptotic behavior of the would-be solutions and the analyticity of the theory's Lagrangian constitute the cornerstone of no-hair theorems and therefore, almost by transitivity, has paved the road to understand how these theory dependent statements can be circumvented to obtain interesting geometries featuring nontrivial hair.

Along the lines of [94], it is possible to show that for a configuration of the form (2), the independent field equations to be solved reduce to the metric variation equations $\epsilon_{tt} = 0$ and $\epsilon_{rr} = 0$ and vanish the radial Noether current $\mathcal{J}^r = 0$. As a matter of fact, the nondiagonal Einstein equation $\epsilon_{tr} = 0$, sourced by the linear time dependence of the scalar profile, turns out to be proportional to the scalar current $\mathcal{J}^r$, and thus, no flux for the scalar field takes place. As a consequence, the field equations of our theory take the convenient form,

$$\mathcal{J}^r \coloneqq r^2h^2\tilde{G}_{2,X} + (d-2)(d-3)\left(\kappa h^2 - \frac{q^2fh}{2X}\right)\tilde{G}_{4X} + (d-2)q^2h^2\left(\frac{f}{h}\right)' rF_4 - (d-2)\left((d-3)fh^2 + fhh'r - \frac{(d-3)q^2fh}{2X}\right)\mathcal{Z}_X \qquad (3)$$

$$\epsilon_{rr} \coloneqq h^3\bigg[-\frac{(d-2)fh'}{h}r\mathcal{Z} - r^2(a_0 + \tilde{G}_2) - (d-2)(d-3)\kappa(a_1 + \tilde{G}_4) - (d-2)(d-3)f\mathcal{Z} + \frac{(d-2)(d-3)q^2f}{2Xh}\big(\mathcal{Z} + a_1 + \tilde{G}_4\big) - \frac{2(d-2)q^2f}{h}rF_4X'\bigg] - (q^2 - 2hX)\mathcal{J}^r \qquad (4)$$

$$\epsilon_{tt} \coloneqq -\epsilon_{rr} - 2(q^2 - hX)\mathcal{J}^r + 2r^2X'\mathcal{Z}_X\left(\frac{h}{f}\right) - r^2\mathcal{Z}\left(\frac{h}{f}\right)'. \qquad (5)$$

Please note that for convenience, we have rescaled the equations in the following way:

$$\epsilon_{tt} \to 2r^2\epsilon_{tt}, \qquad \epsilon_{rr} \to 2r^2h^3\epsilon_{rr}, \qquad \mathcal{J}^r \to -\frac{r^2h^3}{f\psi'}\mathcal{J}^r.$$

Further, we have voluntarily rewritten the coupling functions G_2 and G_4 as

$$G_2(X) = a_0 + \tilde{G}_2(X), \qquad G_4(X) = a_1 + \tilde{G}_4(X), \qquad \tilde{G}_{4,X} \neq 0, \qquad (6)$$

and hence, the constant a_0 represents an eventual bare cosmological constant, while a_1 corresponds to the standard Einstein-Hilbert term in the action. In addition, it turns out to be advantageous to define the auxiliary function,

$$\mathcal{Z}(X) := 4X^2F_4 + 2X\tilde{G}_{4,X} - (a_1 + \tilde{G}_4). \tag{7}$$

From these independent equations, one can easily visualize the emergence of hairy (scalarized) extensions of the Schwarzschild black hole. Indeed, considering the homogeneous static case $f = h$ in (2), the compatibility of the last Eq. (5) guides us towards two options, imposing either $\mathcal{Z}$ or X to be constant; here, we will consider the first possibility, $\mathcal{Z} = \text{cst}$. The case of constant X, as it is known, naturally leads to the construction of stealth black hole solutions. Hence, for $\mathcal{Z} = \text{cst} = \mathcal{Z}_0$, the radial current equation $\mathcal{J}^r = 0$ given by (3) reduces to the simple expression,

$$r^2\tilde{G}_{2,X} + (d-2)(d-3)\left(\kappa - \frac{q^2}{2X}\right)\tilde{G}_{4,X} = 0, \tag{8}$$

which later will provide the specific radial dependence (at least implicitly) of the kinetic term $X = X(r)$. Next, choosing the constant function $\mathcal{Z}$ to be $\mathcal{Z} = -a_1$, the remaining independent equation $\epsilon_{rr} = 0$ factorizes in the very suitable form,

$$\begin{aligned}&-a_0r^2 + a_1(d-2)\left[rh' + (d-3)h - \kappa(d-3)\right]\\&\quad - 2(d-2)q^2rF_4X' - r^2\tilde{G}_2\\&\quad - (d-2)(d-3)\left(\kappa - \frac{q^2}{2X}\right)\tilde{G}_4 = 0.\end{aligned} \tag{9}$$

From here, the following observations are in order: (i) the terms proportional to a_0 and a_1 will vanish identically for a Schwarzschild (A)dS metric function h, (ii) the term involving X' represents a sort of nonhomogeneity, and (iii) the last two terms of the Eq. (9) are a "kind" of first integral with respect to X of the Eq. (8). In fact, using Eq. (8), Eq. (9) can be written as

$$\begin{aligned}&-a_0r^2 + a_1(d-2)\left[rh' + (d-3)h - \kappa(d-3)\right]\\&\quad - 2(d-2)q^2rF_4X' + \tilde{G}_{4,X}(d-2)(d-3)\\&\quad \times\left(\kappa - \frac{q^2}{2X}\right)\left(\frac{\tilde{G}_2}{\tilde{G}_{2,X}} - \frac{\tilde{G}_4}{\tilde{G}_{4,X}}\right) = 0,\end{aligned} \tag{10}$$

an expression that can therefore be satisfied for a metric function h whose homogeneous part is given by the Schwarzschild-(A)dS metric function and whose nonhomogeneity is represented by the terms proportional to F_4 and $\tilde{G}_{4,X}$, and where X is defined implicitly by (8). Unifying all these results, we conclude that the subclass of actions (1) parametrized in terms of $\tilde{G}_2$ and $\tilde{G}_4$ with

$$F_4(X) = \frac{-2X\tilde{G}_{4,X}(X) + \tilde{G}_4(X)}{4X^2}, \tag{11}$$

that is,

$$\begin{aligned}S_{\{\tilde{G}_4(X)\}}[g,\phi] = \int d^dx\sqrt{-g}\Big[&a_0 + a_1R + \tilde{G}_2(X)\\&+ \tilde{G}_4(X)R + \frac{\tilde{G}_4(X)}{2X}\left((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right)\\&+\left(\frac{-2X\tilde{G}_{4,X}(X) + \tilde{G}_4(X)}{2X^2}\right)\\&\times\left(\Box\phi\phi^\mu\phi_{\mu\nu}\phi^\nu - \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\right)\Big]\end{aligned} \tag{12}$$

will admit hairy black hole solutions with primary hair for the ansatz (2), with $f = h$, for a scalar field of the form,

$$\phi(t,r) = qt \pm \int\sqrt{\frac{q^2}{f(r)^2} - \frac{2X(r)}{f(r)}}dr. \tag{13}$$

The metric solution $f = h$ will be the superposition of the Schwarzschild-(A)dS metric function and a nonhomogeneous part, mostly controlled by the primary hair parameter, q,

$$f(r) = \frac{a_0r^2}{a_1(d-1)(d-2)} + \kappa - \frac{2M}{r^{d-3}} + \frac{1}{a_1r^{d-3}}\int\mathcal{H}r^{d-4}dr, \tag{14}$$

where

$$\mathcal{H} = 2q^2rF_4X' - \tilde{G}_{4,X}(d-3)\left(\kappa - \frac{q^2}{2X}\right)\left(\frac{\tilde{G}_2}{\tilde{G}_{2,X}} - \frac{\tilde{G}_4}{\tilde{G}_{4,X}}\right). \tag{15}$$

Notice that since $X \propto q^2$, the absence of the hair in this expression will be consistent only for $\tilde{G}_2 \propto \tilde{G}_4 \propto \sqrt{X}$ [so that F_4 and the last bracket in (15) vanish], reducing the solution to the black hole stealth already found in [92].[2]

It is also interesting to note that the standard falloff M/r^{d-3} can be understood via the Kerr-Schild approach developed in [67]. For the sake of clarity and compactness, here we reproduce briefly the arguments as originally presented for the general case [67]. We start with a seed configuration of the form,

$$ds_0^2 = -h_0(r)dt^2 + \frac{dr^2}{f_0(r)} + r^2d\Omega^2_{d-2,\kappa}, \qquad X = X_0(r), \tag{16}$$

[2]The other possibility will be to chose $\tilde{G}_4 = \text{cst}$, but this is in contradiction with our construction; see Eq. (6).

a solution of the field Eqs. (3)–(5), where h_0 and f_0 are mass-independent functions. Operating with a Kerr-Schild transformation $ds^2 = ds_0^2 + Ma(r) l \otimes l$, where the null geodesic vector field is $l = dt - dr/(\sqrt{h_0 f_0})$, and requiring invariance of the standard kinetic term under the transformation, proves to be equivalent to mapping the original seed functions according to $h_0(r) \to h(r) = h_0(r) - Ma(r)$ and $f_0(r) \to f(r) = f_0(r)(h_0(r) - Ma(r))/h_0(r)$. It is then easy to see that, since X is invariant, these transformations will map the Eqs. (3) and (4) to

$$\frac{\mathcal{J}^r}{h^2} \to \frac{\mathcal{J}^r}{h^2} + M(d-2)\frac{f_0}{h_0}[ra' + a(d-3)]\mathcal{Z}_X,$$
$$\frac{\epsilon_{rr}}{h^3} \to \frac{\epsilon_{rr}}{h^3} + M(d-2)\frac{f_0}{h_0}[ra' + a(d-3)]\mathcal{Z}.$$

Hence, one can conclude that a Kerr-Schild transformation leaving invariant the kinetic term will be a symmetry of the independent equations, provided the Kerr-Schild function $a(r)$ satisfies the equation $ra'(r) + a(r)(d-3) = 0$, that is $a(r) \sim r^{3-d}$.

B. Schwarzschild-like hairy black holes

A very appealing model that allows for explicit analytic expressions is the one characterized by $\tilde{G}_2 = \lambda \tilde{G}_4$, with λ being a constant. As a matter of fact, in this case, the explicit form of the kinetic term is directly identifiable from Eq. (8), yielding

$$X(r) = \frac{(d-2)(d-3)q^2}{2[\lambda r^2 + \kappa(d-2)(d-3)]}. \qquad (17)$$

In addition, the nonhomogenous contribution of the metric $\mathcal{H}$ drastically simplifies, and it is simply given by the beyond-Horndeski function F_4, which from (11) is shown to be determined in terms of $\tilde{G}_4$ only. In consequence, the subclass of actions (12) parametrized in terms of $\tilde{G}_4$, with $\tilde{G}_2(X) = \lambda \tilde{G}_4(X)$, admits a hairy black hole solution with a scalar field,

$$\phi = qt \pm \int \sqrt{\frac{q^2}{f(r)^2}\left(1 - \frac{(d-2)(d-3)f(r)}{\lambda r^2 + \kappa(d-2)(d-3)}\right)}dr, \qquad (18)$$

where the metric reads

$$f(r) = \frac{a_0 r^2}{a_1(d-1)(d-2)} + \kappa - \frac{2M}{r^{d-3}} + \frac{2q^2}{a_1 r^{d-3}} \int r^{d-3} X' \left[\frac{-2X\tilde{G}_{4,X} + \tilde{G}_4}{4X^2}\right] dr. \qquad (19)$$

A few comments are in order regarding this hairy black hole solution as defined by Eqs. (17)–(19). Firstly, one can recognize that the metric solution is a superposition of the Schwarzschild-(A)dS metric together with a piece proportional to the scalar hair, q. In other words, the scalar hair solution continuously connects to the Schwarzschild-Tangherlini-(A)dS solution, providing a scalarized version of the Schwarzschild-(A)dS black hole in any dimension. Secondly, the integral piece of the metric solution (19) is, of course, a defined modulo which is an integration constant, but since this integral is multiplied by a factor r^{3-d}, this "extra" constant can be absorbed into a redefinition of the mass parameter M. Finally, it is desirable that the metric function behaves asymptotically as the Schwarzschild-(A)dS metric, that is

$$f(r) \sim \frac{a_0 r^2}{a_1(d-1)(d-2)} + \kappa - \frac{2M}{r^{d-3}} + O\left(\frac{1}{r^{d-3}}\right), \qquad (20)$$

namely, neither the (A)dS term, nor the mass falloff are affected by the inhomogeneous contribution in the metric function. Taking into consideration (17) and its derivative, this requirement translates into the condition,

$$\left|\tilde{G}_4 - 2X\tilde{G}_{4,X}\right| \sim \frac{1}{r^\alpha}, \qquad \alpha > 2. \qquad (21)$$

A constraint that particularly affects the use of a $\tilde{G}_4$ function is linear in X, [92]. It is interesting to remark that to have a finite kinetic term everywhere (17), one can simply choose the sign of the coupling λ to be equal to that of the base manifold curvature, $\mathrm{sgn}(\lambda) = \mathrm{sgn}(\kappa)$. Moreover, for a flat base manifold $\kappa = 0$, the kinetic term will express a divergence at the origin $r = 0$, however, hidden behind the would-be event horizon.

III. SCALARIZING THEORIES BEYOND-GR: EINSTEIN-MAXWELL, LOVELOCK, AND CUBIC QUASITOPOLOGICAL

Having established the scheme behind the scalarization of the Schwarzschild-(A)dS black hole, we extend our result to other gravity theories, in particular the cases of Einstein-Maxwell theory, Lovelock gravity, and the so-called cubic quasitopological gravity. To proceed, we start by complementing the two-parametric action (12) with an action depending on the same metric g and a collection of matter fields, denoted by ψ_m, that is different from the original beyond-Horndeski scalar ϕ, yielding[3]

[3]In the eventual case in which the sector defined by $\tilde{\mathcal{L}}_m$ already involves the Einstein-Hilbert piece (respectively, the cosmological constant), we will then consider the action (12) with $a_1 = 0$ (respectively with $a_0 = 0$) in order to avoid a repetition of these terms.

$$
\begin{aligned}
S[g,\phi,\psi_m] = \int d^d x\sqrt{-g}\Big[&a_0 + a_1 R + \tilde{G}_2(X) + \tilde{G}_4(X)R \\
&+ \frac{\tilde{G}_4(X)}{2X}\left((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right) \\
&+ \left(\frac{-2X\tilde{G}_{4,X}(X) + \tilde{G}_4(X)}{2X^2}\right) \\
&\times\left(\Box\phi\phi^\mu\phi_{\mu\nu}\phi^\nu - \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\right)\Big] \\
&+ \int d^d x\sqrt{-g}\tilde{\mathcal{L}}_m(g,\psi_m). \qquad (22)
\end{aligned}
$$

Denoting the field equations coming from the variation of $\tilde{\mathcal{L}}_m$ with the metric as $\tilde{\epsilon}_{\mu\nu}$, we consider the following hypotheses:

(i) The field equations of the Lagrangian $\tilde{\mathcal{L}}_m$ admit a homogeneous static metric solution with purely radial fields of the form,

$$
\begin{aligned}
&ds^2 = -\tilde{f}(r)dt^2 + \frac{dr^2}{\tilde{f}(r)} + r^2 d\Omega^2_{d-2,\kappa}, \\
&\psi_m = \psi_m(r). \qquad (23)
\end{aligned}
$$

(ii) The field equations $\tilde{\epsilon}_{tt}$ and $\tilde{\epsilon}_{rr}$ are a proportional modulo of the field equations associated to the equations defining the other fields, $\psi_m(r)$.

It is easy now to prove that the full action (22) will admit a hairy solution of the form,

$$
\begin{aligned}
&ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{d-2,\kappa}, \\
&\phi(t,r) = qt \pm \int \sqrt{\frac{q^2}{f(r)^2} - \frac{2X(r)}{f(r)}}dr, \\
&\psi_m = \psi_m(r), \qquad (24)
\end{aligned}
$$

with X defined implicitly by (8), and where the metric function f will satisfy the following nonhomogeneous differential equation:

$$
\begin{aligned}
&\tilde{\epsilon}_{rr}\left(r,f,f',f'',\dots,\psi_m,\psi'_m,\psi''_m\cdots\right) - a_0 r^2 \\
&\quad + a_1(d-2)\left[rf' + (d-3)f - \kappa(d-3)\right] \\
&= 2(d-2)q^2 rF_4X' - G_{4,X}(d-2)(d-3) \\
&\quad\times\left(\kappa - \frac{q^2}{2X}\right)\left(\frac{\tilde{G}_2}{\tilde{G}_{2,X}} - \frac{\tilde{G}_4}{\tilde{G}_{4,X}}\right). \qquad (25)
\end{aligned}
$$

In what follows, we will provide three representative examples in addition to a simple counterexample that allows for a deeper understanding of the hypotheses.

A. Black holes with primary hair in Einstein-Maxwell theory

The simplest case for a theory of the form $\tilde{\mathcal{L}}_m$ satisfying the hypotheses (i) and (ii) is the one of Einstein-Maxwell theory. Indeed, as we know, there exists a simple spherically symmetric solution, the Reissner-Nordström black hole, which is actually found from a set of field equations satisfying that $\tilde{\epsilon}_{tt} \sim \tilde{\epsilon}_{rr}$. Solving (25) and the corresponding Maxwell equations, we obtain

$$
\begin{aligned}
f(r) = &\frac{a_0 r^2}{a_1(d-1)(d-2)} + \kappa - \frac{2M}{r^{d-3}} \\
&+ \frac{2Q^2}{(d-2)(d-3)r^{2(d-3)}} \\
&+ \frac{1}{a_1 r^{d-3}}\int r^{d-4}\Big[2q^2 rF_4X' - \tilde{G}_{4,X}(d-3) \\
&\times\left(\kappa - \frac{q^2}{2X}\right)\left(\frac{\tilde{G}_2}{\tilde{G}_{2,X}} - \frac{\tilde{G}_4}{\tilde{G}_{4,X}}\right)\Big]dr, \qquad (26)
\end{aligned}
$$

$$
A_0(r) = \frac{Q}{(d-3)r^{d-3}}, \qquad (27)
$$

where, again, it is evident how the primary hair is added on top of the bald initial solution. It is interesting to remark that, in this case, as well as in the subsequent cases, stealth black holes are simply found by considering $\tilde{G}_2 \propto \tilde{G}_4 \propto \sqrt{X}$. As noticed in (15) for such a choice of the theory functions, the nonhomogeneous source always vanishes. In this particular subsection, this black hole corresponds to a charged stealth solution defined on top of the Reissner-Nordström metric.

B. Black holes with primary hair in Lovelock gravity

Another appealing example in which the hypothesis (i) and (ii) are fulfilled, is the one of Lovelock gravity [95]. The Lovelock theory represents the natural higher dimensional generalization of Einstein's theory; therefore, it is to be expected that condition (ii) will indeed hold. In addition, it is known that in an arbitrary dimension, and for the complete series representing the whole tower of curvature invariants, up to order $[(d-2)/2]$, a spherically symmetric solution always exists, at least implicitly given by the so-called Wheeler polynomial, of which its most representative explicit case is given by the Boulware-Deser black hole [96], the spherically symmetric solution of the Einstein-Gauss-Bonnet system.

In consequence, considering the Lagrangian of Lovelock gravity of order k (in which the zero and first order terms represent the cosmological constant and Einstein-Hilbert contributions),

$$\tilde{\mathcal{L}}_m(g) = \sum_{k=0}^{[\frac{d-1}{2}]} a_k \frac{(2k)!}{2^k} \delta^{\mu_1}_{[\alpha_1} \delta^{\nu_1}_{\beta_1} \cdots \delta^{\mu_k}_{\alpha_k} \delta^{\nu_k}_{\beta_k]} \prod_{r=1}^{k} R^{\alpha_r \beta_r}{}_{\mu_r \nu_r}, \quad (28)$$

the metric function describing the hairy generalization of the general Lovelock black hole is implicitly given by a root of the following generalized Wheeler polynomial:

$$\sum_{k=0}^{[\frac{d-1}{2}]} \frac{a_k (d-1)!}{(d-2k-1)!} \left(\frac{\kappa - f(r)}{r^2}\right)^k = \frac{2M}{r^{d-1}} - \frac{1}{r^{d-1}} \int r^{d-4} \left[2q^2 r F_4 X' - \tilde{G}_{4,X}(d-3)\left(\kappa - \frac{q^2}{2X}\right)\left(\frac{\tilde{G}_2}{\tilde{G}_{2,X}} - \frac{\tilde{G}_4}{\tilde{G}_{4,X}}\right)\right] dr. \quad (29)$$

From the polynomial (29), explicit cases are easily obtainable, up to the solution of the corresponding algebraic equation. Therefore, solutions like the Boulware-Deser black hole with primary hair or hairy generalizations with even higher corrections in the curvature, such as the cases in which a degenerate vacuum arises, can be straightforwardly studied. In addition, charged solutions follow with ease.

C. Black holes with primary hair in cubic quasitopological gravity

Let us now shift our focus to an alternative yet intriguing higher curvature order gravity, namely, cubic quasitopological gravity. These theories, originally constructed in [97], represent a class of higher curvature order gravity theories that, when assuming spherical symmetry, result in second-order field equations. Notably, they deviate from the general case of Lovelock theories, particularly in any odd dimension. Extensively investigated in the literature [98–105], quasitopological gravities provide a compelling framework for our exploration. In this context, we embark on a complementary approach by augmenting the cubic quasitopological Lagrangian with action (22) for the case of $d = 5$. This extension aims to introduce primary hair to the already identified black holes within this quasitopological model [97],

$$\tilde{\mathcal{L}}_m(g) = a_2 \mathcal{G} + a_3 \left(-\frac{7}{6} R^{\mu\nu}{}_{\lambda\rho} R^{\lambda\sigma}{}_{\nu\tau} R^{\rho\tau}{}_{\mu\sigma} - R^{\lambda\rho}{}_{\mu\nu} R^{\nu\sigma}{}_{\lambda\rho} R^{\mu}{}_{\sigma} - \frac{1}{2} R^{\lambda\rho}{}_{\mu\nu} R^{\mu}{}_{\lambda} R^{\nu}{}_{\rho} + \frac{1}{3} R^{\mu}{}_{\nu} R^{\nu}{}_{\lambda} R^{\lambda}{}_{\mu} - \frac{1}{2} R R^{\mu}{}_{\nu} R^{\nu}{}_{\mu} + \frac{1}{12} R^3\right), \quad (30)$$

where $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$ is the Gauss-Bonnet density. In [97], this cubic theory was shown to have second order traced field equations, and admit black hole solutions that fit our hypotheses. It is then easy to see that for the coupled system, the metric function solution, f, will satisfy the following nonhomogeneous cubic equation:

$$\frac{a_3}{3} \frac{[\kappa - f(r)]^3}{r^2} + a_2[2f(r)(2\kappa - f(r))] + a_1 r^2 (f(r) - \kappa) - \frac{a_0 r^4}{12} = -2M + \int r \left[2q^2 r F_4 X' - \tilde{G}_{4,X}(d-3)\left(\kappa - \frac{q^2}{2X}\right) \times \left(\frac{\tilde{G}_2}{\tilde{G}_{2,X}} - \frac{\tilde{G}_4}{\tilde{G}_{4,X}}\right)\right] dr, \quad (31)$$

and, hence, convert the black hole solution of [97] to a hairy solution of the full theory. A similar construction can also be achieved in any odd dimension $d = 2p - 1$, by considering the general Lagrangian [97],

$$\mathcal{L}_p = \frac{1}{2^p}\left(\frac{1}{d-2p+1}\right) \delta^{\mu_1\nu_1 \ldots \mu_p\nu_p}_{\lambda_1\rho_1 \ldots \lambda_p\rho_p} \left(C^{\lambda_1\rho_1}_{\mu_1\nu_1} \ldots C^{\lambda_p\rho_p}_{\mu_p\nu_p} - R^{\lambda_1\rho_1}_{\mu_1\nu_1} \ldots R^{\lambda_p\rho_p}_{\mu_p\nu_p}\right) - \gamma_p C^{\mu_1\nu_1}_{\mu_p\nu_p} C^{\mu_1\nu_1}_{\mu_2\nu_2} \ldots C^{\mu_{p-1}\nu_{p-1}}_{\mu_p\nu_p}, \quad (32)$$

where

$$\gamma_p = \frac{(d-4)!}{(d-2p+1)!} \frac{[p(p-2)D(d-3) + p(p+1)(d-3) + (d-2p)(d-2p-1)]}{[(d-3)^{p-1}(d-2)^{p-1} + 2^{p-1} - 2(3-D)^{p-1}]}, \quad (33)$$

and where $C_{\mu\nu\lambda\rho}$ denotes the Weyl tensor. As before, we set a_0 and a_1 to zero in (22). Then, the metric solution $f(r)$ satisfies the following polynomial equation, similar to the pure Lovelock case:

$$\sum_{k=0}^{p} (-1)^k \binom{p}{k} \bar{a}_k \left(\frac{\kappa - f(r)}{r^2}\right)^k = \frac{2M}{r^{d-1}} - \frac{1}{r^{d-1}} \int r^{d-4} \left[2q^2 r F_4 X' - \tilde{G}_{4,X}(d-3)\left(\kappa - \frac{q^2}{2X}\right)\left(\frac{\tilde{G}_2}{\tilde{G}_{2,X}} - \frac{\tilde{G}_4}{\tilde{G}_{4,X}}\right)\right] dr. \quad (34)$$

Note that the dimensional factor of the term coming from $\mathcal{L}_p$ has a slightly different form than those of the Lovelock terms. As we have already mentioned, in odd dimensions, these theories do not coincide. Hence, we have absorbed all factors into their corresponding coupling constants and rescaled them to obtain a more convenient form. This form allows for another simplification when the rescaled coupling constants are such that

$$\bar{a}_0 = \frac{\bar{a}_{p-1}^{p}}{\bar{a}_p^{p-1}}, \qquad \bar{a}_1 = \frac{\bar{a}_{p-1}^{p-1}}{\bar{a}_p^{p-2}},$$
$$\bar{a}_2 = \frac{\bar{a}_{p-1}^{p-2}}{\bar{a}_p^{p-3}}, \ldots, \bar{a}_{p-2} = \frac{\bar{a}_{p-1}^{2}}{\bar{a}_p}. \tag{35}$$

In this case, the polynomial equation of order p reads

$$\frac{\{r^2\bar{a}_{p-1} - \bar{a}_p[\kappa - f(r)]\}^p}{r^{2p}\bar{\alpha}_p^{p-1}} = \frac{2M}{r^{d-1}} - \frac{1}{r^{d-1}}\int r^{d-4}\Bigg[2q^2 rF_4X' - \tilde{G}_{4,X}(d-3)\left(\kappa - \frac{q^2}{2X}\right) \times \left(\frac{\tilde{G}_2}{\tilde{G}_{2,X}} - \frac{\tilde{G}_4}{\tilde{G}_{4,X}}\right)\Bigg]dr. \tag{36}$$

D. A counterexample with the vacuum conformal gravity solution

To appreciate our working hypotheses, we shall provide a counterexample, the case of conformal gravity [106]. There exists an elementary example of a static solution in the homogeneous form (23) for which the field equations $\tilde{\epsilon}_{tt}$ and $\tilde{\epsilon}_{tt}$ are not proportional. Consider the action (22) in $d = 4$ with $a_0 = a_1 = 0$ and

$$\tilde{\mathcal{L}}_m(g) = C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa},$$

where, as before, $C_{\lambda\mu\nu\kappa}$ is the Weyl tensor. Indeed, one can see that in this case, the hypothesis (ii) is broken since

$$\frac{\tilde{\epsilon}_{tt}}{f^2(r)} + \tilde{\epsilon}_{rr} = -\frac{8}{3}\frac{f'''(r)}{r} - \frac{2}{3}f''''(r).$$

Note that, in the vacuum case, the metric solution [106] is such that the right-hand side vanishes identically, and hence, there is no consequence of $\tilde{\epsilon}_{tt}$ not being proportional to $\tilde{\epsilon}_{tt}$. However, in general, in our construction, the non-homogeneity acquired from the primary hair will not be such that the right-hand side of the previous equation vanishes.

IV. FURTHER COMMENTS

In this work, we have explored a scalar-tensor theory within the so-called beyond-Horndeski gravity, a theory that accommodates higher-order terms while preserving the propagation of healthy degrees of freedom [7]. We have deliberately confined our investigation to a scalar theory invariant under a constant shift of the scalar field. In this particular scenario, the presence of a conserved Noether current proves to be an important tool in the search for analytical solutions. Remarkably, the action (1) has demonstrated noteworthy potential in the construction of exact black hole solutions, as evidenced by the recent work of [92]. The solutions presented in this reference have been found for specific coupling functions G_2, G_4, and F_4, and correspond to black holes characterized by primary scalar hair. Notably, these scalarized black holes maintain a continuous connection with the Schwarzschild solution.

In this investigation, we have extended the framework of [92] in two key directions. Firstly, we considered its arbitrary dimensional extension, and secondly, we explored the circumstances under which the beyond-Horndeski model, as considered herein, can be coupled to gravity theories beyond-GR to yield hairy black holes with primary hair. Concerning the former, we have identified the most comprehensive class of actions (1) that facilitates the emergence of black holes with primary scalar hair that are continuously connecting to the Schwarzschild-Tangherlini-(A)dS solution. In relation to the latter point, we have precisely identified the possible alternative gravitational theories (with possibly extra dynamical fields different from the scalar field) that can be coupled with our beyond-Horndeski model to generate scalarized black holes. Subsequently, we have demonstrated how the Reissner-Nordström black hole of Einstein-Maxwell, as well as the static black hole configurations within general Lovelock and cubic quasitopological gravities, can be promoted to configurations featuring primary hair. As in [92], the hair of our solutions is given by the integration constant q. This constant, which accompanies the linear time dependence of the scalar field, ensures the regularity of the scalar field throughout the spacetime. It is essential to note that, although all our solutions exhibit a continuous limit with their bald counterparts, they do not necessarily represent a linear superposition.

In the broader context of scalar-tensor theories, such as Horndeski, beyond-Horndeski, or DHOST theories [9–11], it is known that certain specific sectors exhibit black hole solutions that are distinctly disconnected from the Schwarzschild solution. In light of this observation, it would be appealing to categorize all these scalar-tensor theories according to whether or not they support solutions that are continuously connecting to vacuum solutions of general relativity. To pursue this task, insights coming from the Kerr-Schild framework developed in [67] could be useful. In this reference, shift-invariant DHOST theories invariant under a Kerr-Schild transformation that preserves the invariance of the kinetic term were identified (with a scalar field possibly linear in the time coordinate). Particularly, those theories featuring a standard falloff mass term in their static black hole solutions emerge as potential candidates for accommodating hairy black holes that are continuously connected with the Schwarzschild solution. Proceeding along this path, we recognize that Kerr's solution can be obtained through a Kerr-Schild transformation employing a null, geodesic, and shear-free congruence. The seed metric, representing the solution with zero mass, corresponds to the flat metric expressed in ellipsoidal coordinates. However, in our scenario, due to the inclusion of the scalar field, particularly the parameter q, the metric solution with zero mass deviates from a flat metric. This departure signifies a substantial alteration from Kerr's metric. Thus, it becomes intriguing to delve into the properties of the null, geodesic, and shear-free congruences of this hairy seed configuration, encompassing the scalar field and metric solutions with $M = 0$. Once these characteristics are established, we can initiate a Kerr-Schild transformation from the seed configuration to generate rotating versions of our solution, thereby discerning the potential astrophysical ramifications of these hairy solutions. Since the seed configuration diverges from Kerr's metric, the resulting metric will not resemble Kerr's, presenting an opportunity for leveraging this metric in current and future gravity experiments.

Finally, the availability of exact black hole spacetimes with primary hair opens avenues for several analyses that can be done analytically. Notably, it becomes pertinent to conduct in-depth investigations into the thermodynamic properties of these spacetimes, compute the corresponding black hole charges, establish methodologies for regularizing a given theory action [107], and substantiate the validity of the first law. Then several applications of these solutions can be performed along the lines of the AdS/CFT conjecture or black hole chemistry [108]. Additionally, a comprehensive analysis of the causal structure, geodesics, and algebraic classification promises a more nuanced understanding of the interplay between vacuum and hairy black holes with primary hair. We anticipate presenting findings on these aspects in the near future.

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APPENDIX: CONCRETE EXAMPLES

For the sake of concreteness, we will give some explicit cases. We start by considering (12)

$$\tilde{G}_2(X) = -\frac{\lambda(d-3)(d-2)}{2n-1} X^n, \tag{A1}$$

$$\tilde{G}_4(X) = -\frac{\lambda}{2n-1} X^n, \tag{A2}$$

$$F_4(X) = \frac{1}{4}\lambda X^{n-2}, \tag{A3}$$

where evidently $\tilde{G}_2 \propto \tilde{G}_4$. Recall that the case $n = \frac{1}{2}$ deserves particular attention as already noticed in [92] (see below). Further, for convenience, we have fixed the constants $a_0 = 0$ and $a_1 = 1$. This provides a kinetic term X of the form,

$$X = \frac{q^2}{2(\kappa + r^2)}, \tag{A4}$$

and the solution of the metric function for the case of GR then reads

$$\begin{aligned} f(r) &= \kappa - \frac{2M}{r^{d-3}} - \frac{2\lambda}{r^{d-3}} \int r^{d-2} X^n dr \\ &= \kappa - \frac{2M}{r^{d-3}} + \frac{\lambda q^2}{2(n-1)} \\ &\quad \times \left[X^{n-1} - \frac{(d-3)}{r^{d-3}} \int r^{d-4} X^{n-1} dr \right] \end{aligned} \tag{A5}$$

$$\begin{aligned} &= \kappa - \frac{2M}{r^{d-3}} - \frac{2^{1-n}\lambda\kappa^{-n}q^{2n}}{d-1} r^2 {}_2F_1 \\ &\quad \times \left(\frac{d-1}{2}, n; \frac{d+1}{2}; -\frac{r^2}{\kappa} \right), \end{aligned} \tag{A6}$$

where ${}_2F_1$ is the Gaussian hypergeometric function. In particular, with n being an integer, this takes an even more simple form; for example, for $n = 3$ and in four dimensions, we obtain

$$f(r) = \kappa - \frac{2M}{r} - \frac{\lambda q^6}{32\kappa} \left[\frac{(r^2 - \kappa)}{(r^2 + \kappa)^2} + \frac{1}{r\kappa^{1/2}} \arctan\left(\frac{r}{\sqrt{\kappa}} \right) \right]. \tag{A7}$$

We can even extend this to include the Gauss-Bonnet term in the action in the same line as we did in the general Lovelock case. This yields a general form of the metric function,

$$f = \kappa + \frac{r^2}{2\alpha(d-4)(d-3)}\left(1 \pm \sqrt{1 + \frac{8\alpha M(d-4)(d-3)}{r^{d-1}} + \frac{8\lambda\alpha(d-4)(d-3)}{r^{d-1}}\int r^{d-2}X^n dr}\right), \tag{A8}$$

where α is the coupling constant of the Gauss-Bonnet term. When carefully taking the limit $\alpha \to 0$ (in the case of the lower sign in front of the square root), this reduces to the Schwarzschild-like case from before. For example, let us give the specific form of the metric function in five dimensions for $n = 3$,

$$f = \kappa + \frac{r^2}{4\alpha}\left(1 \pm \sqrt{1 + \frac{16\alpha}{r^4}\left[M - \lambda\frac{q^6(\kappa + 2r^2)}{32(\kappa + r^2)^2}\right]}\right). \tag{A9}$$

To conclude, as already mentioned for $\tilde{G}_4(X) \propto \sqrt{X}$, the coupling function F_4, as defined by (11), vanishes identically, and consequently, the nonhomogeneity disappears. Hence, in the case of Einstein-Gauss-Bonnet gravity and for

$$\tilde{G}_2(X) = -\lambda(d-3)(d-2)\sqrt{X}, \qquad \tilde{G}_4(X) = -\lambda\sqrt{X}, \tag{A10}$$

we end up with a stealth defined on the Boulware-Deser black hole solution similar to the one found in [59]

$$f(r) = \kappa + \frac{r^2}{2\alpha(d-3)(d-4)} \times \left(1 \pm \sqrt{1 + \frac{8\alpha M(d-3)(d-4)}{r^{d-1}}}\right). \tag{A11}$$

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