

General covariance and dynamics with a Gauss law

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We present a four-dimensional generally covariant gauge theory with local degrees of freedom which leads to the Gauss constraint but lacks *both* the Hamiltonian and spatial diffeomorphism constraints. The canonical theory therefore resembles Yang-Mills theory without the Hamiltonian. We describe its observables, quantization, and some generalizations.

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I. INTRODUCTION

The difficulty of constructing a fully satisfactory theory of quantum gravity gives rise to the need for simpler models, which consist of writing down generally covariant theories that are different from general relativity but simpler to analyze from a quantum-theoretical perspective. These include BF theory [1,2], 2 + 1 gravity [3–6], $U(1)^3$ theory [7,8], the Husain-Kuchar (HK) model [9,10], and many others with symmetry reductions [11] and matter.

The Hamiltonian version of the HK theory lacks the Hamiltonian constraint of general relativity, which encodes the dynamics of general relativity and is the most difficult object to quantize in any canonical approach to quantum gravity. As such, the model furnished the hope for clearly separating the question of dynamics from that of kinematics, and thus determining which quantum-gravitational effects are solely kinematical in nature. To this end, subsequent developments revealed subtle and surprising results, such as the fact that black-hole entropy calculations in loop quantum gravity (LQG) can be performed with generally covariant theories that lack black-hole solutions entirely [12].

In this paper, we describe an $SU(2)$ model that, on the face of it, is a slight variation of the HK model, but turns out to have significantly different features: the canonical theory lacks not only the Hamiltonian, but also the diffeomorphism constraint; the remaining Gauss constraint arises with a source term. This leads to new observables in the classical and quantum theories that resemble “quarks on a string.” Furthermore, the presence of only the Gauss constraint gives considerable freedom in constructing

classical solutions, which include metrics corresponding to almost any three-geometry.

The surprising feature that a generally covariant theory realizes dynamics only as internal gauge provides a large class of exactly solvable classical and quantum models with local degrees of freedom; the canonical theory of the model we present may be viewed as a Yang-Mills theory without its Hamiltonian, with nine local degrees of freedom.

The plan of the paper is as follows. In Sec. II, we introduce the model and derive its canonical theory, and in Sec. III, we explain why the model is diffeomorphism invariant despite having a Gauss law as the only first-class constraint and hence gauge generator of the theory. Sections IV and V are devoted to describing certain classical observables and solutions, respectively, and Sec. VI discusses the quantum theory. Finally, Sec. VII describes possible extensions of the model, such as addition of a Chern-Simons boundary term.

II. ACTION AND CANONICAL THEORY

Let M be a four-dimensional spacetime, and let $\phi^i(x)$ and $A_\alpha^i(x)$ be scalars and connection one-forms in the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ [or any other semisimple [13] Lie group of dimension 3, e.g. $SL(2, \mathbb{R})$, $SO(2, 1)$, etc.], and let ϵ_{ijk} denote the Levi-Civita symbol in $\mathfrak{su}(2)$ (or equivalently, the structure constants of the Lie algebra). The action of the model we propose is

$$\begin{aligned} S[\phi, A] &= \frac{1}{2} \int_M d^4x \tilde{\epsilon}^{\alpha\beta\gamma\delta} \epsilon_{ijk} D_\alpha \phi^i D_\beta \phi^j F_{\gamma\delta}^k \\ &= \frac{1}{2} \int_M d^4x \text{Tr}(D\phi \wedge D\phi \wedge F), \end{aligned} \quad (1)$$

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where D and F denote, respectively, the covariant derivative and the curvature 2-form corresponding to the connection A , i.e.

$$D_\alpha \gamma^i = \partial_\alpha \gamma^i + \epsilon^i_{jk} A_\alpha^j \gamma^k \quad (2)$$

$$F_{\alpha\beta}^i = \partial_{[\alpha} A_{\beta]}^i + \epsilon^i_{jk} A_\alpha^j A_\beta^k, \quad (3)$$

and $\tilde{\epsilon}^{\alpha\beta\gamma\delta}$ is the four-dimensional desensitized Levi-Civita symbol.

This action is manifestly invariant both under spacetime diffeomorphisms and $SU(2)$ transformations. It resembles the HK action

$$S_{\text{HK}}[e, A] = \int d^4x \tilde{\epsilon}^{\alpha\beta\gamma\delta} \epsilon_{ijk} e_\alpha^i e_\beta^j F_{\gamma\delta}^k \quad (4)$$

if one replaces $D_\alpha \phi^i$ with $SU(2)$ triads e_α^i . However, as we see below, the canonical theory of (1) is drastically different.

Variation of the action with respect to A and ϕ yields the equations of motion:

$$\delta\phi: D \wedge (D\phi \wedge F) = 0, \quad (5)$$

$$\delta A: D \wedge (D\phi \wedge D\phi) + \phi \times (D\phi \wedge F) = 0, \quad (6)$$

where “ \times ” denotes an internal cross product using the structure constants, i.e. $(u \times v)^i = \epsilon^i_{jk} u^j v^k$. Using the Bianchi identity $D_{[\alpha} F_{\beta\gamma]} = 0$ and $D_{[\alpha} D_{\beta]} \lambda = F_{\alpha\beta} \times \lambda$ for any section λ of the $SU(2)$ bundle associated with the spacetime, the preceding equations can be recast as

$$\begin{aligned} \tilde{\epsilon}^{\alpha\beta\gamma\delta} (F_{\alpha\beta} \times \phi) \times F_{\gamma\delta} &= 0, \\ \tilde{\epsilon}^{\alpha\beta\gamma\delta} [(F_{\beta\gamma} \times \phi) \times D_\delta \phi + \phi \times (D_\beta \phi \times F_{\gamma\delta})] &= 0. \end{aligned}$$

Evidently, the first equation is true if and only if F and ϕ are internally parallel, i.e.

$$F_{\alpha\beta} \times \phi = 0. \quad (7)$$

Substituting this equation and its covariant derivative into the second equation trivially satisfies the latter. Thus, any connection is a solution to the theory, provided one chooses a section ϕ parallel to the curvature F in $\mathfrak{su}(2)$! One might say that the equations of motion concern only the “internal space” of the theory; from this perspective, it is not surprising that the canonical theory, as we shall see below, only has an internal Gauss constraint.

To perform canonical decomposition of the action (1), we assume M has the topology $\mathbb{R} \times \Sigma$, where Σ is a compact three-dimensional, differentiable manifold. Then, choosing the coordinates $x^\alpha = (x^0, x^a)$, with $x^0 \in \mathbb{R}$ and $x^a \in \Sigma$, the $3 + 1$ form of the action is

$$\begin{aligned} S &= \int d^4x \epsilon_{ijk} \tilde{\epsilon}^{0abc} \{ (\dot{\phi}^i + A_0^i \epsilon^i_{lm} A_a^l \phi^m) D_a \phi^j F_{bc}^k \\ &\quad + D_a \phi^i D_b \phi^j (\dot{A}_c^k + \epsilon^k_{lm} A_0^l A_c^m) \}, \end{aligned} \quad (8)$$

where the overdot indicates ∂_0 . This identifies the momenta conjugate to ϕ^i and A_a^i :

$$\tilde{E}_i^a := \tilde{\epsilon}^{abc} \epsilon_{ijk} (D_b \phi^j) (D_c \phi^k), \quad (9)$$

$$\tilde{p}_i := \tilde{\epsilon}^{abc} \epsilon_{ijk} (D_a \phi^j) F_{bc}^k, \quad (10)$$

where $\tilde{\epsilon}^{abc} := \tilde{\epsilon}^{0abc}$. The canonical action is then

$$S = \int d^4x [\tilde{E}_k^c \dot{A}_c^k + \tilde{p}_i \dot{\phi}^i + A_0^k (D_c \tilde{E}_k^c + \epsilon_{kl}^i \phi^l \tilde{p}_i)]. \quad (11)$$

There are thus two sets of configuration variables, the $\mathfrak{su}(2)$ -valued connection A and scalar ϕ . Variation with respect to A_0^k gives the constraint

$$\tilde{G}_k = -(D_c \tilde{E}_k^c + \epsilon_{kl}^i \phi^l \tilde{p}_i) \approx 0. \quad (12)$$

This is a Gauss law with a source term; it is readily verified that the constraint algebra is first class:

$$\begin{aligned} \{G(\lambda), G(\mu)\} &= \left\{ \int d^3x \lambda^m D_a \tilde{E}_m^a(x), \int d^3y \gamma^n D_b \tilde{E}_n^b(y) \right\} \\ &\quad + \left\{ \int d^3x \lambda^m \epsilon_{ml}^s \phi^l \tilde{p}_s(x), \int d^3y \gamma^n \epsilon_{ij}^k \phi^j \tilde{p}_k(y) \right\} \\ &= G([\lambda, \mu]). \end{aligned} \quad (13)$$

The Hamiltonian is therefore a linear combination of first-class constraints

$$H = \int d^3x \lambda_k G^k(x), \quad (14)$$

and it is evident that there is neither a Hamiltonian nor a spatial diffeomorphism constraint; the reason for their absence is explored in the next section.

At this stage it is important to check whether any constraints have been missed. We now show that this is not the case. Let us first note that there can be no algebraic relations between ϕ^i , A_a^i and E^{ai} , since the former two are independent configuration variables, and the last one is made from the gauge-covariant derivatives of the first (9); similarly the momenta \tilde{E}^{ai} and \tilde{p}^i are algebraically independent since the latter, from (10), is a function of the curvature $F(A)$; the possible nontrivial quadratic combinations of the momenta (with $B^{ai} = \tilde{\epsilon}^{abc} F_{ab}^i$) are

$$\tilde{p}_i \tilde{p}^i = B^{aj} B_j^b D_a \phi^i D_b \phi_i - B^{aj} B_i^b D_a \phi^i D_b \phi_j, \quad (15)$$

$$\tilde{E}^{ai}\tilde{E}_i^b = 2\tilde{\epsilon}^{acd}\tilde{\epsilon}^{bef}D_c\phi^jD_d\phi^kD_e\phi_jD_f\phi_k, \quad (16)$$

$$(\tilde{E}^a \times \tilde{E}^b)_i = \epsilon_i^{jk}\tilde{E}_j^a\tilde{E}_k^b = 2\tilde{E}_j^a\tilde{\epsilon}^{bcd}D_c\phi_iD_d\phi^j, \quad (17)$$

$$(\tilde{p} \times \tilde{E}^a)_i = \epsilon_i^{jk}\tilde{p}_j\tilde{E}_k^a = 2\tilde{p}_j\tilde{\epsilon}^{abc}D_b\phi_iD_c\phi^j; \quad (18)$$

inspecting the right-hand sides of these expressions shows that there can be no quadratic relationships between the momenta. We thus conclude that there are no ‘‘hidden’’ secondary constraints in the model. Another way to establish this via the Dirac procedure for constrained Hamiltonian systems [14,15] is given in the Appendix.

The theory thus has three ϕ^i and nine A_a^i local phase-space configuration degrees of freedom subject to the $SU(2)$ Gauss law. Hence it has a net of nine unconstrained degrees of freedom per space point [15]; in comparison the HK model has three, since it has only the connection A_a^i as the configuration variable, and both the Gauss and diffeomorphism constraints.

Like the HK model, the action (1) has a special vector density

$$\tilde{u}^\alpha = \epsilon^{\alpha\beta\gamma\delta}\epsilon_{ijk}D_\beta\phi^iD_\gamma\phi^jD_\delta\phi^k, \quad (19)$$

which satisfies the ‘‘parallel transport’’ equation $\tilde{u}^\alpha D_\alpha\phi^i = 0$; a degenerate spacetime metric

$$g_{\alpha\beta} = D_\alpha\phi^iD_\beta\phi^j\delta_{ij}; \quad (20)$$

in the canonical theory there is a scalar density of weight 1,

$$\tilde{\epsilon} = \frac{1}{3!}\epsilon_{ijk}\tilde{\epsilon}^{abc}D_a\phi^iD_b\phi^jD_c\phi^k, \quad (21)$$

which may be used to define the inverse triad

$$e^{ai} = \frac{1}{\tilde{\epsilon}}\epsilon^{ijk}\tilde{\epsilon}^{abc}D_b\phi^jD_c\phi^k, \quad (22)$$

which satisfies $e^{ai}e_{aj} = \delta_j^i$ and $e^{ai}e_{bi} = \delta_b^a$. Thus like the HK model, Eq. (1) is a theory of three-geometries with an invertible spatial metric $q_{ab} = e_a^i e_b^i$.

III. THE CASE OF THE MISSING CONSTRAINTS

The action (1) is manifestly diffeomorphism invariant, but the canonical theory contains only a Gauss constraint. Since one expects continuous symmetries of the action to be generated by the first-class constraints, we have a puzzle: how do we understand the absence of the diffeomorphism and Hamiltonian constraints?

By way of preliminaries, let us recall how this question is answered in other known diffeomorphism-invariant theories of connections where the first-class constraints generate only the usual gauge transformations of the gauge fields. Examples include BF theory and its

generalizations [1,2], Chern-Simons theory, and $2+1$ gravity [3]. In these theories $F = 0$ is an equation of motion. As a result the Lie derivative of the field variables with respect to a vector field ζ that generates a diffeomorphism is a gauge transformation on shell:

$$\begin{aligned} \mathcal{L}_\zeta A_a &= \zeta^c \partial_c A_a + A_c \partial_a \zeta^c \\ &= D_a(\zeta^c A_c) + \zeta^c F_{ca}. \end{aligned} \quad (23)$$

The reflection of this fact in the canonical BF theory arises through the constraints $G^k = D_a E^{ak} = 0$ and $F_{ab}^k = 0$ with the spatial diffeomorphism constraint C_a arising as the linear combination

$$C_a = A_a^k G^k + E^{ak} F_{ab}^k. \quad (24)$$

In the HK model, the Gauss and spatial diffeomorphism constraints are present but the Hamiltonian constraint vanishes identically. Although the latter cannot be written as any linear combination of the former constraints, the Lie-derivative argument sketched above is still available: the theory contains the vector density $\tilde{u}^\alpha \equiv \epsilon^{abcd}e_b^i e_c^j e_d^k \tilde{\epsilon}^{ijk}$, which defines a preferred direction, and satisfies $\tilde{u}^\alpha F_{ab} = 0$ [9]. Using this, one can show, in analogy with the theories discussed above, that spacetime diffeomorphisms are equivalent on shell to the transformations generated by the first-class constraints of the theory, namely the Gauss law and the spatial diffeomorphism constraint. Indeed, given a foliation of spacetime into a family of spacelike hypersurfaces, one can convert \tilde{u} into a vector field u , and then any vector field ζ can be decomposed into a component along u and a component along a leaf Σ of the foliation. Then, up to unimportant multiplicative factors, one can write

$$\begin{aligned} \mathcal{L}_\zeta A_a &= D_a(u^b A_b) + u^b F_{ba} + \mathcal{L}_X A_a \\ \mathcal{L}_\zeta e_a &= (u^b A_b) \times e_a + u^b D_b e_a + \mathcal{L}_X e_a, \end{aligned} \quad (25)$$

where X is a vector field along Σ . The first and last terms in each line are an $SU(2)$ rotation and a spatial diffeomorphism, respectively, and the middle term vanishes by $\tilde{u}^\alpha F_{ab} = 0$, which is a consequence of the equations of motion [9]. Furthermore, one can show that the spatial projections of the equations of motion yield the Gauss and spatial diffeomorphism constraints. Thus although the HK model is not topological with an $F_{ab} = 0$ equation of motion, a related understanding of the identically vanishing Hamiltonian constraint arises there.

We now show that there is an analogous understanding of the vanishing Hamiltonian and diffeomorphism constraints in the canonical theory arising out of the action (1). That is, we can show that all diffeomorphisms are equivalent on shell to the $SU(2)$ gauge transformations generated by the Gauss law (12). To see how this happens let us recall

the special vector density \tilde{u}^α (19) orthogonal to $D_\alpha\phi$; it can be converted into a vector field using (21). Furthermore, in the canonical theory, one has access to the three vector fields e^{ai} (22) which have the property that $e^{ai}D_a\phi^j = \delta^{ij}$. Using this, let us form the spatial vector field

$$X^{ai} = \epsilon^i{}_{jk}\phi^j e^{ak}. \quad (26)$$

Any vector field v^α can be decomposed into a component along $u^\alpha = \tilde{u}^\alpha/\tilde{e}$ and a spatial vector field, which in turn can be written as a linear combination of the X^{ai} . Thus, it suffices to calculate the Lie derivative of the field variables along u^α , and along $w^\alpha = \lambda_i X^{ai}$ for some λ_i . To this end, we first note that

$$\begin{aligned} \mathcal{L}_u\phi^i &= u^\alpha\partial_\alpha\phi^i = u^\alpha D_\alpha\phi - [(u^\alpha A_\alpha) \times \phi]^i \\ &= -[(u^\alpha A_\alpha) \times \phi]^i, \end{aligned} \quad (27)$$

where the last equality follows from $u^\alpha D_\alpha\phi = 0$ —this is evidently an $SU(2)$ gauge transformation with the gauge parameter $u^\alpha A_\alpha^i$. Similarly, for the Lie derivative with respect to w^α , we find

$$\mathcal{L}_w\phi^i = [(\lambda - (A_\alpha w^\alpha)) \times \phi]^i \equiv (\Lambda \times \phi)^i, \quad (28)$$

which is also an $SU(2)$ gauge transformation with the parameter Λ^i . Therefore, by virtue of the special direction u^α , any diffeomorphism of ϕ is equivalent to an $SU(2)$ rotation.

This fact can be used to establish a similar result for A_α^i , provided the equations of motion hold. To see this, observe that if the equations of motion hold, then as shown in the previous section, $F \times \phi = 0$. Now, under a diffeomorphism generated by a vector field v^α , ϕ changes by an $SU(2)$ rotation. Since $F \times \phi = 0$, F must rotate by the same amount as ϕ does (28). Indeed, assuming that ϕ rotates by Λ^i ,

$$\mathcal{L}_v(F_{\alpha\beta} \times \phi) = \mathcal{L}_v F_{\alpha\beta} \times \phi + F_{\alpha\beta} \times (\Lambda \times \phi) = 0:$$

this equation holds if and only if

$$\mathcal{L}_v F_{\alpha\beta} = (\Lambda \times F_{\alpha\beta}), \quad (29)$$

i.e. F , too, rotates by Λ . This constrains the transformation of the connection A : we have, using (23), that

$$\begin{aligned} \mathcal{L}_v F_{\alpha\beta} &= D_{[\alpha} \mathcal{L}_v A_{\beta]} \\ &= (F_{\alpha\beta} \times A_\gamma v^\gamma) - D_{[\alpha} (v^\gamma F_{\beta]\gamma}), \end{aligned} \quad (30)$$

and the last two equations give

$$D_{[\alpha} (v^\gamma F_{\beta]\gamma}) = F_{\alpha\beta} \times (A_\gamma v^\gamma + \Lambda). \quad (31)$$

This in turn implies

$$v^\gamma F_{\alpha\gamma} = D_\alpha (A_\gamma v^\gamma + \Lambda). \quad (32)$$

Substituting this into (23) gives

$$\mathcal{L}_v A_\alpha^i = -D_\alpha \Lambda^i; \quad (33)$$

this is the sought after result: the Lie derivative of A is an $SU(2)$ gauge rotation. Therefore, provided the equations of motion hold, diffeomorphisms are equivalent to the transformations generated by the Gauss constraint. To complete the analogy with the HK model, one can project Eq. (6) into the spatial surface to obtain the Gauss constraint (12); [Eq. (5) is a scalar and hence not projectable].

From the perspective of constrained Hamiltonian systems, the preceding discussion illustrates the fact that all continuous local symmetries of the action are generated by the first-class constraints; any transformations that are not present in the transformations generated by the first-class constraints should differ from the latter only by equations-of-motion terms or by trivial symmetries [15]. This is what happens here.

IV. OBSERVABLES

Let us recall that an observable is a dynamical function (al) $f(\phi, A, \tilde{E}, \tilde{p})$ that has vanishing Poisson brackets with the constraints. We can define a spatial metric by

$$q_{ab} = \kappa_{ij} D_a \phi^i D_b \phi^j, \quad (34)$$

where κ_{ij} is the Cartan-Killing metric for $SU(2)$. It is readily verified that this is an observable:

$$\{D_a \phi^i(x), G(\lambda)\} = \epsilon^i{}_{mn} \lambda^m D_a \phi^n; \quad (35)$$

hence, $\{q_{ab}, G(\lambda)\} = 0$. Since the metric is gauge invariant, so are all the classical observables that depend on it. Hence, for instance, the area and volume functionals are observables in this model, just as in LQG [16].

Other obvious examples of observables are $SU(2)$ -valued loop variables, which are defined using traces of holonomies around curves in Σ and insertions of configuration and momentum variables along the curves. That is, given a smooth curve $\gamma: [a, b] \rightarrow \Sigma$, we define the Wilson line from a to b :

$$U[\gamma](a, b) := \mathcal{P} \exp\left(\int_a^b dx^\alpha A_\alpha^i \tau_i\right), \quad (36)$$

where \mathcal{P} stands for path ordering and τ_i are the generators of $\mathfrak{su}(2)$. If $a = b$, we get the so-called Wilson loops. Since these are matrices in $SU(2)$, and hence transform under $SU(2)$ via conjugation [i.e. $g \cdot U[\gamma] = g^{-1}(a)U[\gamma]g(b)$], the cyclic invariance of the trace operation entails that

$$T^0[\gamma] := \text{Tr}(U[\gamma](a, a)) \quad (37)$$

is an observable. Similarly, the rest of the familiar T variables, defined via insertions of the momentum \tilde{E}_i^a along Wilson lines, are also observables, e.g.

$$\begin{aligned} T^{a_1 \dots a_n}[\gamma](x_1, \dots, x_n) \\ = \text{Tr}[U[\gamma](a, x_1)\tilde{E}^{a_1}(x_1) \\ \times U[\gamma](x_1, x_2)\tilde{E}^{a_2}(x_2) \dots \tilde{E}^{a_n}U[\gamma](x_n, a)], \end{aligned}$$

where $\tilde{E}^a = \tilde{E}_i^a \tau^i$ and x_1, \dots, x_n are fixed points on the curve γ .

However, unlike in LQG, these are not the only loop observables, since we now have the additional $\mathfrak{su}(2)$ -valued configuration and momentum variables, namely ϕ^i and \tilde{p}_i , that may be inserted along Wilson lines as well. There are numerous such possibilities, ranging from simple end-of-the-curve insertions such as

$$\text{Tr}[\phi(x)U[\gamma](x, y)\phi(y)] \quad (38)$$

to mixed and middle-of-the-curve insertions such as

$$\begin{aligned} \text{Tr}[\phi(x_1)U[\gamma](x_1, x_2)\tilde{E}(x_2)U[\gamma](x_2, x_3) \\ \times \tilde{p}(x_3)U[\gamma](x_3, x_4)\phi(x_4)]. \end{aligned} \quad (39)$$

The former are essentially flux tubes analogous to those found in Yang-Mills theory, except that the nonzero Hamiltonian of the latter can cause these tubes to break to form more such tubes, whereas our model has unbroken tubes floating around in spacetime.

V. CLASSICAL SOLUTIONS

Since the Hamiltonian constraint, which generates orthogonal transformations of three-dimensional hypersurfaces, vanishes in both the HK model and the model here, both are essentially theories of three-geometry that arise from four-dimensional actions. For the HK model, constraint-free initial data specified on a three-dimensional hypersurface are solutions of the Gauss and spatial diffeomorphism constraints. Since the Hamiltonian constraint vanishes, these data do not evolve. Therefore they are solutions for a three-geometry. This means that constraint-free data for Einstein gravity are a subset of the data for the HK model.

In the model at hand the Gauss constraint contains a source term. Two peculiar features of the model conspire to yield a large class of interesting three-geometries as solutions of the model. First, the three-metric (34) depends solely on the configuration variables ϕ and A . Since these variables are independent, one can set their conjugate momenta \tilde{p}^i and E^{ai} to zero to trivially solve the Gauss constraint, which fortunately is the only constraint to solve.

Second, since the model is valid for arbitrary (semisimple) gauge groups of dimension 3, which include those with Cartan-Killing metrics of mixed signature [e.g. $SO(2, 1)$], one can even construct solutions of 2 + 1 gravity. Thus, by appropriately fixing the form of ϕ and A , while setting \tilde{p} and \tilde{E} to zero, it is possible to construct almost any three-metric, including all those resulting from projecting solutions of the 3 + 1 Einstein equations on three-dimensional hypersurfaces, as well as all the solutions of 2 + 1 gravity. To illustrate these remarks, we explicitly construct the spatial Schwarzschild metric and the Banados-Teitelboim-Zanelli (BTZ) black-hole metric [17].

A. 3d Schwarzschild metric

For $SU(2)$, $\tilde{p} = \tilde{E} = 0$, use spherical coordinates (r, θ, φ) on Σ , and set

$$A = \frac{r}{P(r)}\tau_1 d\theta + \left(\cos\theta\tau_3 - \frac{r}{P(r)}\sin\theta\tau_2 \right) d\varphi, \quad (40)$$

$$\phi = P(r)\tau_3, \quad f(r) = \sqrt{1 - R/r}, \quad (41)$$

$$P(r) = \frac{R}{2} \left[\ln \left| \frac{f(r) + 1}{f(r) - 1} \right| + \frac{2f(r)}{(f(r) + 1)(f(r) - 1)} \right], \quad (42)$$

where τ^i are the Pauli matrices. Then (34) gives the Schwarzschild three-metric

$$ds^2 = f^{-2}(r)dr^2 + r^2 d\Omega^2. \quad (43)$$

B. BTZ black hole

For the gauge group $SO(2, 1)$ the Cartan-Killing metric can be written as $\text{diag}(-1, 1, 1)$. Pick cylindrical coordinates (r, φ, z) on Σ . Then the BTZ metric can be obtained as follows. Let

$$\begin{aligned} A = [P(r)\tau_1 + Q(r)\tau_2]d\varphi + [R(r)\tau_2 + S(r)\tau_3]dz, \\ \phi = T(r)\tau_3, \end{aligned} \quad (44)$$

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad N_\varphi(r) = -\frac{J}{2r^2}, \quad (45)$$

where the last two functions are used to define the BTZ metric. Substituting these into (34) yields the equations

$$T'(r)^2 = N^{-2}(r), \quad (46)$$

$$T^2(r)(P^2(r) - Q^2(r)) = r^2, \quad (47)$$

$$R^2(r)T^2(r) = N^2(r), \quad (48)$$

$$R(r)Q(r)T^2(r) = r^2 N_\varphi(r), \quad (49)$$

which can be solved for the functions P, Q, R, T to obtain A and ϕ : the first equation gives

$$\begin{aligned} T(r) &= \frac{l}{2} \ln \left| \sqrt{h^2(r) + 1} + h(r) \right|, \\ h(r) &= \frac{2r^2 - Ml^2}{\sqrt{J^2 l^2 - M^2 l^4}}, \end{aligned} \quad (50)$$

and the other three are algebraic.

These examples show that almost any desired 3-metric may be constructed by solving for A and ϕ .

VI. QUANTUM THEORY

Since there are now two types of configuration variable, the connection A and the ϕ variables, the connection representation is better termed the $A - \phi$ representation. Physical states are gauge-invariant functionals of ϕ and A —for instance, the loop variables defined above but depending solely on A and ϕ , e.g.

$$\begin{aligned} \Psi(\phi, A) &= \text{Tr}(\phi(x_1)U[\gamma](x_1, x_2)\phi(x_2) \\ &\quad \times U[\gamma](x_2, x_3)\phi(x_3)). \end{aligned} \quad (51)$$

The inner product on these states is well defined, with there being now an additional integration over the ϕ :

$$\langle \Psi_1 | \Psi_2 \rangle = \int d\mu(\phi) \int d\mu(A) \Psi_1^*(\phi, A) \Psi_2(\phi, A), \quad (52)$$

where $\mu(A)$ is a measure on the space of connections (modulo gauge transformations), such as the Ashtekar-Lewandowski measure, and $\mu(\phi)$ is some suitable measure on the space of the ϕ fields (say, a Gaussian measure).

A. Spin-network states

As in LQG, quantum states in the model can be realized as spin networks, with one addition. As usual, $SU(2)$ representations label the edges of embedded graphs, corresponding to holonomies of the connection along those edges, i.e. functionals of the form $\Psi(A)$, and these representations are sewn by intertwiners associated with the vertices of the graph. However, here we can also have physical functionals that involve insertions of ϕ at the end of Wilson lines. In a spin network, such states correspond to the ϕ variables sitting at selected vertices of the underlying graph, along with the intertwiners. Thus in general, a typical spin-network state corresponding to a graph Γ is

$$|\Gamma; j_1, \dots, j_n; I_1, \dots, I_n; \phi_1, \dots, \phi_k\rangle, \quad k \leq n, \quad (53)$$

with the obvious inner product. Such states are physical states of the theory, and the area and volume operators of LQG are diagonal physical observables. The quantum tetrahedra are physical states of the theory, and as there

is no spatial diffeomorphism constraint, each graph representing a tetrahedron is a distinct physical state, unlike in the HK model.

VII. DISCUSSION

Our main observation is the theory defined by (1), and its unusual canonical version with missing Hamiltonian and diffeomorphism constraints—it is a theory of three-geometries with local degrees of freedom that is exactly solvable classically and quantum mechanically.

A natural extension of the model arises by adding a Chern-Simons boundary term

$$\begin{aligned} S &= \frac{1}{2} \int_M d^4x \text{Tr}(D\phi \wedge D\phi \wedge F) \\ &\quad + \gamma \int_{\partial M} d^3x \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \end{aligned} \quad (54)$$

Varying the action now yields

$$\begin{aligned} \delta S &= - \int_M d^4x [D \wedge (D\phi \wedge F)]_i \delta\phi^i + \gamma \int_{\partial M} d^3x \tilde{p}_i \delta\phi^i \\ &\quad - \int_M d^4x [D \wedge (D\phi \wedge D\phi) + (\phi \times (D\phi \wedge F))]_k^\alpha \delta A_\alpha^k \\ &\quad + \int_{\partial M} d^3x [\gamma \tilde{e}^{abc} F_{iab} + \tilde{E}_k^c] \delta A_c^k. \end{aligned}$$

Thus, if we insist that the variations in ϕ and A on the boundary ∂M can be nonzero, the conjugate momenta must be constrained on the boundary to ensure a well-defined variational principle:

$$\tilde{E}_k^c|_{\partial M} = -\gamma \tilde{e}^{abc} F_{iab}, \quad \tilde{p}_i|_{\partial M} = 0. \quad (55)$$

The first of these is a condition on conjugate momenta that also arises with a topological bulk theory [18], and in the context of black-hole entropy in LQG for an inner boundary [12,19], where quantization of this condition in the spin-network basis with the area operator provides an area-entropy relation. A similar calculation is possible in the present theory.

Other possible variants of the action include the addition of a bulk topological term $\int d^4x \text{Tr}(F \wedge F)$, more than one connection [20], as well as boundaries with multiple components for identifying corner observables; see, e.g. [21].

As the model and its variants are generally covariant theories with local degrees of freedom, they provide a useful testing ground for other quantization methods, such as those involving spinfoams [22,23] and group-field theory (GFT) [24]. For the former, it is possible to write the action as a BF theory with a ‘‘simplicity’’ constraint term $\Lambda(B - D\phi \wedge D\phi)$; for the latter, the fact that our model has no constraints other than the Gauss law makes it a natural candidate to define directly on a group manifold, in

particular for exploring the question of the manifestation of the Hamiltonian and diffeomorphism constraints in GFT models.

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APPENDIX: DIRAC ANALYSIS

For completeness we present the Dirac analysis [14,15] of the action (1). This ensures consistency with the analysis of Sec. II. The definitions of the momenta (9) and (10) have no explicit dependence on the time derivatives of A_a^i and ϕ_i and so cannot be inverted to express the velocities as

functions of the momenta and configuration variables. We thus have the following primary constraints:

$$\tilde{E}_i^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0^i} \approx 0, \quad (\text{A1})$$

$$\tilde{\psi}_i := \tilde{p}_i - \tilde{\epsilon}^{abc} \epsilon_{ijk} D_a \phi^j F_{bc}^k \approx 0, \quad (\text{A2})$$

$$\tilde{\chi}_i^a := \tilde{E}_i^a - \tilde{\epsilon}^{abc} \epsilon_{ijk} D_b \phi^j D_c \phi^k \approx 0. \quad (\text{A3})$$

Since Eq. (11) is in the form $\int d^3x (p\dot{q} - H)$, the total Hamiltonian *à la* Dirac is

$$H_T = \int d^3x (A_0^k \tilde{G}_k + \lambda^i \tilde{E}_i^0 + \mu^i \tilde{\psi}_i + \rho_a^i \tilde{\chi}_i^a), \quad (\text{A4})$$

where $\lambda_i, \mu^i, \rho_a^i$ are arbitrary functions of spacetime coordinates.

To ensure preservation of these primary constraints under evolution with H_T , we first note the following algebra:

$$\left\{ \tilde{E}^0(x), \int d^3y A_0(y) \cdot \tilde{G} \right\} = -\tilde{G}(x), \quad (\text{A5})$$

$$\left\{ \tilde{\psi}(x), \int d^3y A_0(y) \cdot \tilde{G}(y) \right\} = -A_0(x) \times \tilde{\psi}(x), \quad (\text{A6})$$

$$\left\{ \tilde{\chi}^a(x), \int d^3y A_0(y) \cdot \tilde{G}(y) \right\} = -A_0(x) \times \tilde{\chi}^a(x), \quad (\text{A7})$$

$$\left\{ \tilde{\psi}(x), \int d^3y \mu(y) \cdot \tilde{\psi}(y) \right\} = \mu(x) \times D_a \tilde{B}^a(x), \quad (\text{A8})$$

$$\left\{ \tilde{\psi}(x), \int d^3y \rho_a(y) \cdot \tilde{\chi}^a(y) \right\} = (\phi(x) \times \rho_a(x)) \times \tilde{B}^a(x), \quad (\text{A9})$$

$$\left\{ \tilde{\chi}^a(x), \int d^3y \mu(y) \cdot \tilde{\psi}(y) \right\} = (\mu(x) \times \tilde{B}^a(x)) \times \phi(x) + \mu(x) \times (\tilde{B}^a(x) \times \phi(x)), \quad (\text{A10})$$

$$\left\{ \tilde{\chi}^a(x), \int d^3y \rho_b(y) \cdot \tilde{\chi}^b(y) \right\} = 2\tilde{\epsilon}^{abc} [\phi(x) \times (\rho_b(x) \times D_c \phi(x))] + 2\tilde{\epsilon}^{abc} [D_c \phi(x) \times (\rho_b(x) \times \phi(x))], \quad (\text{A11})$$

where as before $\tilde{B}^{ai} = \tilde{\epsilon}^{abc} F_{bc}^i$. The first three equations are all weakly zero, while the fourth one is identically zero owing to the Bianchi identity; hence we obtain

$$\{\tilde{E}_0(x), H_T\} = -\tilde{G}(x) \approx 0, \quad (\text{A12})$$

$$\{\tilde{\psi}(x), H_T\} \approx (\phi \times \rho_a) \times \tilde{B}^a \approx 0, \quad (\text{A13})$$

$$\{\tilde{\chi}^a(x), H_T\} \approx [(\mu \times \tilde{B}^a) \times \phi] + [\mu \times (\tilde{B}^a \times \phi)] + 2\tilde{\epsilon}^{abc} [\phi \times (\rho_b \times D_c \phi)] + 2\tilde{\epsilon}^{abc} [D_c \phi \times (\rho_b \times \phi)] \approx 0. \quad (\text{A14})$$

Equation (A12) is a secondary constraint as its right-hand side does not involve a Lagrange multiplier; the other two Eqs. (A13) and (A14) are not secondary constraints, but merely consistency conditions on the Lagrange multipliers μ^i and ρ_i^j (in line with Dirac's method [14,15]). Thus the last two conditions, when set to zero, fix μ^i and ρ_i^j as functions of the phase-space variables. Finally, by virtue of (A5)–(A7), $\{G_i(x), H_T\} \approx 0$. Therefore, the constraint analysis ends, and we conclude that we have four constraints in total: the three primary constraints (A1)–(A3) and one secondary constraint (A12).

Next, we classify the four constraints into first class or second class. It is immediate that (A1) and the Gauss law are first class, whereas (A2) and (A3) are second class. These second-class constraints are solved strongly, which is

equivalent to defining the phase-space variables \tilde{p}_i and \tilde{E}_i^a evident in (A2) and (A3). Lastly, the innocuous primary constraint (A1) can be ignored, for its sole purpose is to provide a full canonical gauge generator that correctly rotates the full spacetime field A_α rather than just its spatial counterpart A_a [25]. Hence we get a canonical theory defined by the momenta (9) and (10) subject to evolution under the Gauss law, as obtained in Sec. II. It is worth emphasizing that the heuristic considerations in Sec. II pertaining to the absence of any nontrivial algebraic relations among the momenta \tilde{p}_i and \tilde{E}_i^a and the configuration variables ϕ^i and A_a^i are systematically reflected in the absence of any secondary second-class constraints in Dirac's procedure: the nontrivial relations (A13) and (A14) are not secondary constraints, but rather consistency conditions on the Lagrange multipliers.

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