Black holes in massive Einstein-dilaton gravity

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In this paper, we focus on massive Einstein-dilaton gravity including the coupling of dilaton scalar field to massive graviton terms, and then derive static and spherically symmetric solutions of dilatonic black holes in four dimensional spacetime. We find that the dilatonic black hole could possess two horizons (event and cosmological), extreme (Nariai) and naked singularity for the suitably fixed parameters. In addition, we investigate thermodynamic properties of these dilatonic black holes, and check the corresponding first law of black hole thermodynamics. Extending to the massive Einstein-dilaton gravity in high dimensions, we further obtain the dilatonic black hole solutions in $(d + 1)$ dimensional spacetime.

I. INTRODUCTION

Despite many successes agreement with observations, the Einstein's general relativity (GR) might be searched for alternatives due to the cosmological constant problem [1], and the origin of acceleration of our universe based on the supernova data [2,3] and cosmic microwave background (CMB) radiation [4,5]. An alternative theory of GR is dilaton gravity, coming from the low energy limit of string theory, in which Einstein's gravity is recovered along with a scalar dilaton field by nonminimal coupling to other fields such as axion and gauge fields [6]. The presence of dilaton field is necessary and plays an essential role in string theory if one couples the gravity to other gauge fields. Until now, many attempts have been made to investigate the dilaton gravity theory. For instance, Refs. [7–13] discovered that the dilaton field changes the causal structure of the black hole and leads to the curvature singularities at finite radii. The dilaton potential can be regarded as the generalization of the cosmological constant, and also change the asymptotic behavior of the solutions. Combined three Liouville-type dilaton potentials, the black hole solutions of dilaton gravity in the background of (A)dS spaces were investigated in Refs. [14,15]. In addition, the scalar-tensor type generalizations of general relativity have been also investigated by containing various kinds of curvature corrections to the usual Einstein-Hilbert Lagrangian coupled to the dilaton scalar field [16–18]. A particular model called the Einstein-dilaton-Gauss-Bonnet (EdGB) gravity was extensively studied in Refs. [19,20]. It found that the scalar dilaton is a secondary hair because the dilaton charge is expressed in terms of the black hole mass. Later, the black holes in various dimensions [21–23]. rotating black holes [24,25], wormholes [26], and rapidly rotating neutron stars [27] were investigated in EdGB gravity.

From the perspective of modern particle physics [28,29], gravity field can be treated as a unique theory of a spin-2 graviton. Massive gravity is a straight forward and natural modification by simply giving a mass to the graviton, dating back to 1939 when Fierz and Pauli [30] constructed a linear theory of massive gravity, which is always plagued with the Boulware-Deser ghost in nonlinear level [31,32]. Notice that the authors of [33] constructed a theory where the Boulware-Deser ghost was eliminated by introducing higher order interaction terms into the Lagrangian. Then, the ghost-free massive theory known as dGRT massive gravity was discussed in Refs. [34,35]. In dGRT massive gravity, a class of (charged) black hole solutions [36–38] and their corresponding thermodynamics [39–42] in asymptotically AdS spacetime were investigated, and the coefficients in the potential associated with the graviton mass were shown to

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play the same roles as as the charge in thermodynamic phase space. Other solutions of black holes were also studied in massive gravity [43–49]. Recently, the so-called Quasi-dilaton massive gravity has been also investigated in Refs. [50–54], which are scalar extended dRGT massive gravity with a shift symmetry. Inspired by these, we would like to extend the study by considering the nonminimal coupling of dilaton field to graviton, and derive analytically solutions of dilatonic black holes in massive dilaton gravity.

The paper is organized as follows. In Sec. II, we will present the static and spherically symmetric black hole solutions in four dimensional massive Einstein-dilaton gravity, and investigate the solution structures of dilatonic black holes. In Sec. III, we will discuss the thermodynamic properties of these black holes. Considering the massive Einstein-dilaton gravity in high dimensional spacetime, we will derive the $(d + 1)$ dimensional solutions of black hole in Sec. IV. Finally, we close the paper with discussions and conclusions in Sec. V.

II. BLACK HOLE IN MASSIVE EINSTEIN-DILATON GRAVITY

The action for massive gravity with a nonminimal coupling of dilaton field φ in four dimensional spacetime is given by

$$
I = \frac{1}{16\pi} \int d^4x \sqrt{-g} \bigg[\mathcal{R} - 2(\nabla\varphi)^2 - V(\varphi) + m_0^2 \sum_{i=1}^{4} c_i e^{-2\beta_i \varphi} \mathcal{U}_i(g, h) \bigg],
\tag{1}
$$

where $\varphi = \varphi(r)$ is the dilaton scalar field, and $V(\varphi)$ is a potential for φ. The last term in the action denotes general form of nonminimal coupling between the scalar field and massive graviton with coupling constants β_i. Here m_0 is the mass of graviton, and c_i are the number of dimensionless coupling coefficients. Moreover, $\mathcal{U}_i$ are symmetric polynomials of the eigenvalues of the 4×4 matrix $K^\mu_{\ \nu} = \sqrt{g^{\mu\alpha} h_{\alpha\nu}}$ in which h is a fixed rank-2 symmetric tensor, satisfying the following recursion relation [34]

$$
\begin{aligned}
\mathcal{U}_1 &= [K] = K^\mu_{\ \mu}, \\
\mathcal{U}_2 &= [K]^2 - [K^2], \\
\mathcal{U}_3 &= [K]^3 - 3[K][K^2] + 2[K^3], \\
\mathcal{U}_4 &= [K]^4 - 6[K^2][K]^2 + 8[K^3][K] + 3[K^2]^2 - 6[K^4].
\end{aligned}
\tag{2}
$$

Varying the action with respect to the field variables $g_{\mu\nu}$ and φ, the equations of motion are obtained as

$$
\begin{aligned}
G_{\mu\nu} &= \mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} \\
&= 2\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}[V + 2(\nabla\varphi)^2]g_{\mu\nu} + m_0^2\chi_{\mu\nu},
\end{aligned}
\tag{3}
$$

$$
\nabla^2\varphi = \frac{1}{4}\left[\frac{\partial V}{\partial\varphi} - m_0^2 \sum_{i=1}^{4} \frac{\partial\tilde{c}_i}{\partial\varphi}\mathcal{U}_i \right],
\tag{4}
$$

where

$$
\tilde{c}_i = c_i e^{-2\beta_i\varphi},
\tag{5}
$$

$$
\begin{aligned}
\chi_{\mu\nu} =& \frac{\tilde{c}_1}{2}(\mathcal{U}_1 g_{\mu\nu} - K_{\mu\nu}) + \frac{\tilde{c}_2}{2}(\mathcal{U}_2 g_{\mu\nu} - 2\mathcal{U}_1 K_{\mu\nu} + 2K_{\mu\nu}^2) \\
&+ \frac{\tilde{c}_3}{2}(\mathcal{U}_3 g_{\mu\nu} - 3\mathcal{U}_2 K_{\mu\nu} + 6\mathcal{U}_1 K_{\mu\nu}^2 - 6K_{\mu\nu}^3) \\
&+ \frac{\tilde{c}_4}{2}(\mathcal{U}_4 g_{\mu\nu} - 4\mathcal{U}_3 K_{\mu\nu} + 12\mathcal{U}_2 K_{\mu\nu}^2 - 24\mathcal{U}_1 K_{\mu\nu}^3 + 24K_{\mu\nu}^4).
\end{aligned}
\tag{6}
$$

Now we introduce the static and spherical symmetry metric ansatz

$$
ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 R^2(r)d\Omega^2,
\tag{7}
$$

in which $f(r)$ and $R(r)$ are functions of r and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element for two dimensional spherical subspace with constant curvature.

Since the fiducial metric $h_{\mu\nu}$ in the action (1) plays the role of a Lagrange multiplier to eliminate the BD ghost [36], one can choose an appropriate form to simplify the calculation. The authors of Ref. [36] pointed out that distinguished from the dynamical physical metric $g_{\mu\nu}$, the reference metric $h_{\mu\nu}$ is usually fixed and assumed to be nondynamical in the massive theory. In this work, we will follow [39,40] by choosing the fiducial metric to be

$$
h_{\mu\nu} = \mathrm{diag}(0, 0, c_0^2, c_0^2 \sin^2\theta),
\tag{8}
$$

where c_0 is a positive parameter and we set $c_0 = 1$ for convenience in whole paper.

From the ansatz (8), the interaction potential in Eq. (2) changes into

$$
\mathcal{U}_1 = \frac{2}{Rr}, \qquad \mathcal{U}_2 = \frac{2}{R^2 r^2}, \qquad \mathcal{U}_3 = \mathcal{U}_4 = 0.
\tag{9}
$$

Then, χ^μ_ν in Eq. (6) becomes

$$
\begin{aligned}
\chi^1_1 &= \chi^2_2 = \frac{c_1 r R e^{-2\beta_1\phi} + c_2 c_0^2 e^{-2\beta_2\phi}}{(rR)^2}, \\
\chi^3_3 &= \chi^4_4 = \frac{c_1 e^{-2\beta_1\phi}}{2rR}
\end{aligned}
\tag{10}
$$

and the corresponding components of equation of motion (3) can be simplified to

$$G^1{}_1 = \frac{1}{(rR)^2}[rR(rR)'f' + 2rR(rR)''f + (rR)'^2f - 1]$$

$$= -\frac{1}{2}V(\varphi) - f\varphi'^2 + m_0^2\chi^1{}_1, \tag{11}$$

$$G^2{}_2 = \frac{1}{(rR)^2}[rR(rR)'f' + (rR)'^2f - 1]$$

$$= -\frac{1}{2}V(\varphi) + f\varphi'^2 + m_0^2\chi^2{}_2, \tag{12}$$

$$G^3{}_3 = G^4_4 = \frac{1}{2rR}[(rR)f'' + 2(rR)'f' + 2(rR)''f]$$

$$= -\frac{1}{2}V(\varphi) - f\varphi'^2 + m_0^2\chi^3{}_3. \tag{13}$$

Here the prime $'$ denotes differentiation with respect to the radial coordinate r.

Based on Eqs. (10)–(12), we obtain

$$\frac{(rR)''}{rR} = -\varphi'^2, \tag{14}$$

which can be rewritten as the following form

$$\frac{d^2}{dr^2}\ln R + \frac{2}{r}\frac{d}{dr}\ln R + \left(\frac{d}{dr}\ln R\right)^2 = -\varphi'^2. \tag{15}$$

In order to derive the dilaton field φ, we assume that $R(r)$ could be an exponential function of $\varphi(r)$, such as

$$R(r) = e^{\alpha\varphi}, \tag{16}$$

where α is a constant. Then, Eq. (15) becomes a simple differential equation for φ

$$\alpha\phi''(r) + (\alpha^2 + 1)\phi'(r)^2 + \frac{2\alpha\phi'(r)}{r} = 0. \tag{17}$$

In fact, the similar assumption (16) has been extensively used to look for the charged dilaton black hole solutions [12,13] in Maxwell-dilaton gravity. By solving the Eq. (17), the dilaton field can be obtained as

$$\varphi(r) = \frac{\alpha}{1 + \alpha^2}\ln\frac{\delta}{r}. \tag{18}$$

Here the integration constant δ is supposedly related to some rescaling properties of solution.

Taking the trace of the gravitational field equation (3), one can get

$$f'' + \frac{4(rR)'f'}{rR} + \frac{2f((rR')^2 + 2rR(rR'' + 3R') + R^2 + (rR)^2\varphi'^2)}{(rR)^2} + 2V(\varphi) - \frac{2}{(rR)^2} - m_0^2\sum_{i=1}^{4}\chi^i_i = 0, \tag{19}$$

Considering the G^3_3 component of gravitational field equation (13) together with the assumption (16) and the solution of dilaton field (18), Eq. (19) can be simplified as

$$\frac{f'}{\alpha^2 + 1} + \frac{(1 - \alpha^2)f}{(\alpha^2 + 1)^2 r} + \frac{r}{4\alpha}\frac{\partial V}{\partial\varphi} + \frac{m_0^2}{\alpha\delta}\left(\beta_2 c_2 c_0^2 e^{\frac{1-\alpha^2-2\alpha\beta_2}{\alpha}\varphi} + \beta_1 c_1 c_0\delta e^{-(\alpha+2\beta_1)\varphi}\right) = 0 \tag{20}$$

On the other hand, we further consider the scalar field equation and substitute the metric ansatz (7) and scalar field (18) including Eqs. (10) and (16) into Eq. (4). Then, the scalar field equation becomes

$$-\frac{f'}{\alpha^2 + 1} + \frac{(\alpha^2 - 1)f}{(\alpha^2 + 1)^2 r} - \frac{rV(\varphi)}{2} + \frac{e^{\frac{1-\alpha^2}{\alpha}\varphi}}{\delta} + c_1 c_0 m_0^2 e^{-(\alpha+2\beta_1)\varphi} + \frac{c_2 c_0^2 m_0^2}{\delta}e^{\frac{1-\alpha^2-2\alpha\beta_2}{\alpha}\varphi} = 0 \tag{21}$$

According to Eqs. (20) and (21), we obtain a first order differential equation for dilaton field potential

$$\frac{\partial V(\varphi)}{\partial\varphi} - 2\alpha V(\varphi) + \frac{4\alpha e^{\frac{2\varphi}{\alpha}}}{\delta^2} + \frac{4m_0^2}{\delta^2}\left(c_1\delta(\alpha + \beta_1)e^{-2\varphi(\beta_1 - \frac{1}{2\alpha})} + c_2(\alpha + \beta_2)e^{-2\varphi(\beta_2 - \frac{1}{\alpha})}\right) = 0 \tag{22}$$

The solution to the differential equation (22) can be written as the generalized form of the Liouville scalar potential

$$V(\varphi) = \begin{cases} 2\gamma_0 e^{2\xi_0\varphi} + 2\gamma_1 e^{2\xi_1\varphi} + 2\gamma_2 e^{2\xi_2\varphi}, & \alpha \neq 1 \\ 2\lambda_0\varphi e^{2\varphi} + 2\lambda_1 e^{2\zeta_1\varphi} + 2\lambda_2 e^{2\zeta_2\varphi}, & \alpha = 1 \end{cases} \tag{23}$$

where the last two terms are associating with coupling between the dilaton field and graviton, and the parameters are determined as following

$$\gamma_0 = \frac{\alpha^2}{(\alpha^2 - 1)\delta^2}, \qquad \gamma_1 = \frac{2\alpha c_1 m_0^2(\alpha + \beta_1)}{\delta(2\alpha^2 + 2\alpha\beta_1 - 1)}, \qquad \gamma_2 = \frac{\alpha c_2 m_0^2(\alpha + \beta_2)}{\delta^2(\alpha^2 + \alpha\beta_2 - 1)}$$

$$\xi_0 = \frac{1}{\alpha}, \qquad \xi_1 = -\beta_1 + \frac{1}{2\alpha}, \qquad \xi_2 = -\beta_2 + \frac{1}{\alpha}, \qquad \zeta_1 = 1/2 - \beta_1, \qquad \zeta_2 = 1 - \beta_2$$

$$\lambda_0 = -\frac{2}{\delta^2}, \qquad \lambda_1 = \frac{2(\beta_1 + 1)c_1 m_0^2}{2\beta_1\delta + \delta}, \qquad \lambda_2 = \frac{(\beta_2 + 1)c_2 m_0^2}{\beta_2\delta^2}. \tag{24}$$

Then, the black hole solution $f(r)$ from Eq. (21) can be written as

$$f(r) = \begin{cases} -mr^{\frac{\alpha^2-1}{\alpha^2+1}} - \frac{(\alpha^2+1)\delta^{-\frac{2\alpha^2}{\alpha^2+1}}r^{\frac{2\alpha^2}{\alpha^2+1}}}{\alpha^2-1} - \frac{(\alpha^2+1)^2c_1m_0^2\delta^{-\frac{\alpha(\alpha+2\beta_1)}{\alpha^2+1}}r^{\frac{2\alpha^2+2\beta_1\alpha+1}{\alpha^2+1}}}{(\alpha^2+2\beta_1\alpha+2)(2\alpha^2+2\beta_1\alpha-1)} \\ \qquad - \frac{(\alpha^2+1)^2c_2m_0^2\delta^{-\frac{2\alpha(\beta_2+\alpha)}{\alpha^2+1}}r^{\frac{2\alpha(\alpha+\beta_2)}{\alpha^2+1}}}{(\alpha^2+2\beta_2\alpha+1)(\alpha^2+\beta_2\alpha-1)}, & \alpha \neq 1 \\ \\ -m + \frac{2r}{\delta}\left(\log\left(\frac{\delta}{r}\right) + 2\right) - \frac{4c_1m_0^2\delta^{-\beta_1-\frac{1}{2}}r^{\beta_1+\frac{3}{2}}}{4\beta_1^2+8\beta_1+3} - \frac{2c_2m_0^2\delta^{-\beta_2-1}r^{\beta_2+1}}{\beta_2^2+\beta_2}, & \alpha = 1 \end{cases} \tag{25}$$

where m is an integration constant related to the mass of the black hole as it will be shown below. Therefore, there exist two branch solutions for the dilatonic black holes in the massive Einstein-dilaton gravity.

Notice that, in the absence of the dilaton field ($\alpha = 0$), the solution $f(r)$ in massive gravity reduces to

$$f(r) = 1 - \frac{m}{r} + \frac{1}{2}c_1 m_0^2 r + c_2 m_0^2, \tag{26}$$

which was presented in Ref. [39]. Here we have set $c_0 = 1$. Obviously, the solution $f(r)$ in Eq. (26) does not describe an asymptotically flat spacetimes unless $m_0 = 0$. For the dilatonic black hole solution $f(r)$ in the dilaton massive gravity, the dominant term of metric function $f(r)$ approaches

$$\lim_{r\to\infty} f(r) = \begin{cases} -\frac{(\alpha^2+1)\delta^{-\frac{2\alpha^2}{\alpha^2+1}}r^{\frac{2\alpha^2}{\alpha^2+1}}}{\alpha^2-1} - \frac{(\alpha^2+1)^2c_1m_0^2\delta^{-\frac{\alpha(\alpha+2\beta_1)}{\alpha^2+1}}r^{\frac{2\alpha^2+2\beta_1\alpha+1}{\alpha^2+1}}}{(\alpha^2+2\beta_1\alpha+2)(2\alpha^2+2\beta_1\alpha-1)} \\ \qquad - \frac{(\alpha^2+1)^2c_2m_0^2\delta^{-\frac{2\alpha(\beta_2+\alpha)}{\alpha^2+1}}r^{\frac{2\alpha(\alpha+\beta_2)}{\alpha^2+1}}}{(\alpha^2+2\beta_2\alpha+1)(\alpha^2+\beta_2\alpha-1)}, & \alpha \neq 1 \\ \\ \frac{2r}{\delta}\left(\log\left(\frac{\delta}{r}\right) + 2\right) - \frac{4c_1m_0^2\delta^{-\beta_1-\frac{1}{2}}r^{\beta_1+\frac{3}{2}}}{4\beta_1^2+8\beta_1+3} - \frac{2c_2m_0^2\delta^{-\beta_2-1}r^{\beta_2+1}}{\beta_2^2+\beta_2}, & \alpha = 1 \end{cases} \tag{27}$$

at the infinity. For example, taking $\alpha = 0.6$, $\beta_1 = 0.5$ and $\beta_2 = 0.3$, we have

$$\lim_{r\to\infty} f(r) = 2.125r^{0.529412} - 1.9527c_1 m_0^2 r^{1.70588} + 2.33771c_2 m_0^2 r^{0.794118}, \tag{28}$$

where we set $\delta = 1$ for simplify. Clearly, the metric function is also not asymptotically flat but asymptotic infinity in general.

For spacetime singularities, we calculate the Ricci and Kretschmann scalars

$$\mathcal{R} = -\frac{4f'(rR)'}{rR} - \frac{f(2((rR)')^2 + 4rR(rR)'')}{(rR)^2} + \frac{2}{(rR)^2} - f''(r), \tag{29}$$

$$\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma} = f\left(\frac{8f'(rR)'(rR)''}{(rR)^2} - \frac{8((rR)')^2}{(rR)^4}\right) + \frac{4f'^2((rR)')^2}{(rR)^2} + \frac{4f^2(((rR)')^4 + 2(rR)^2((rR)'')^2)}{(rR)^4} + \frac{4}{(rR)^4} + f''^2. \tag{30}$$

Note that both of Ricci and Kretschmann scalars are not singular at the horizons, therefore these points are just singularity of coordinate as it should be for a black hole. Considering the leading terms of asymptotical behaviors of metric at the origin, we obtain

$$\lim_{r\to 0}\mathcal{R} \sim mr^{-\frac{\alpha^2+3}{\alpha^2+1}}, \qquad \lim_{r\to 0}\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma} \sim mr^{-\frac{\alpha^2+5}{\alpha^2+1}}, \qquad (31)$$

where Ricci and Kretschmann scalars are divergence at origin $r = 0$, finite for $r > 0$, which suggests the origin $r = 0$ is an essential and physical singularity in the spacetime.

In order to comprehend the behavior of the metric function deeply, we would like to give graphical dependence of the function $f(r)$ both for $\alpha \neq 1$ and $\alpha = 1$, and we set $m_0 = \delta = m = 1$ for simplify in following discussions.

A. $\alpha \neq 1$

For $\alpha \neq 1$, Figs. 1 and 2 depict the plots of $f(r)$ versus r, indicating the presence of black hole horizons. Figure 1(a) shows that when $c_1 > 0$ and $c_2 > 0$, there is only one black

hole horizon, and $f(r)$ approaches to $+\infty$ if $r \to \infty$. However, when $c_1 < 0$, there are two horizons: an event horizon and a cosmological horizon, and $f(r)$ approaches to $-\infty$ as $r \to \infty$. As c_1 decreases to a certain value, an extremal black hole known as the Nariai black hole is formed, exhibiting a coincidence of the event and cosmological horizons. A naked singularity may also appear with the decreasing of c_1. Similarly, Fig. 1(b) shows that when $c_2 < 0$ and $c_1 < 0$, there is only one black hole horizon, but when $c_2 > 0$, there are two horizons: an event horizon and a cosmological horizon. It is worth noting that as shown in Eq. (26) for black hole solution in massive gravity, the parameter c_2 does not affect the asymptotic behavior of the solution $f(r)$. However, with the nonminimal coupling to the dilaton field, c_2 plays an important role in the behavior of solution $f(r)$ because the term of c_2 can become the dominant term with suitable parameters α, β_1, and β_2 from Eq. (27). The influences of the parameters α, β_1, and β_2 are also respectively depicted in Fig. 2.

B. $\alpha = 1$

When $\alpha = 1$, there is still the presence of one black hole horizon with or without a cosmological horizon for
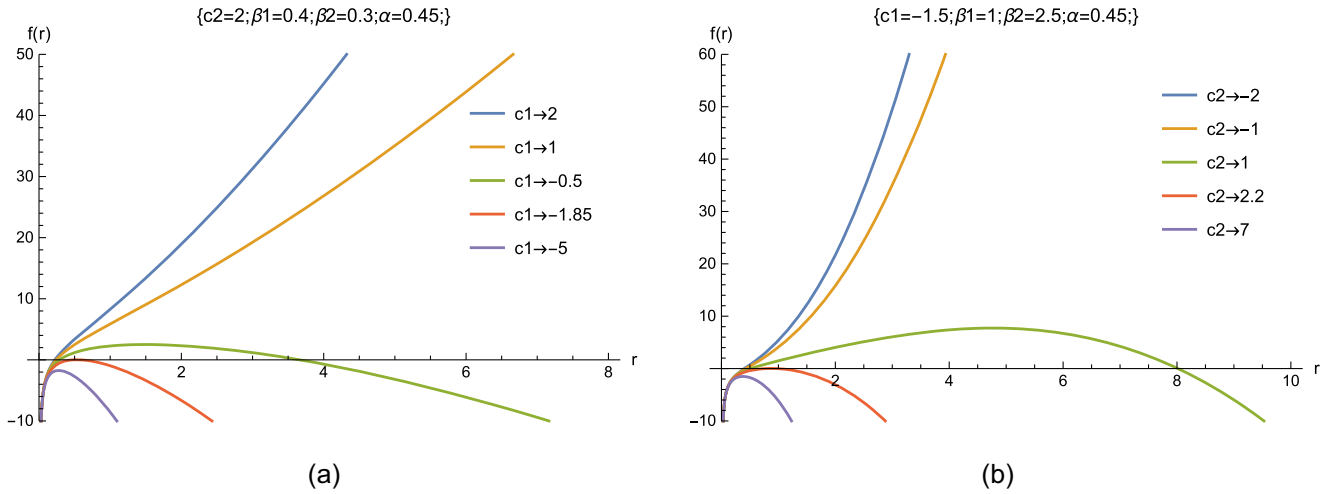

FIG. 1. The function $f(r)$ versus r for different values of c_1 and c_2 in case $\alpha \neq 1$.
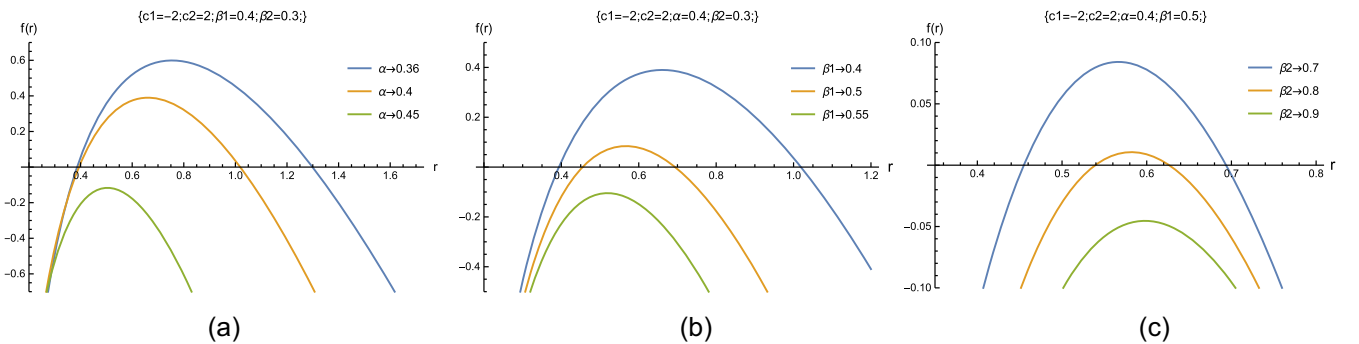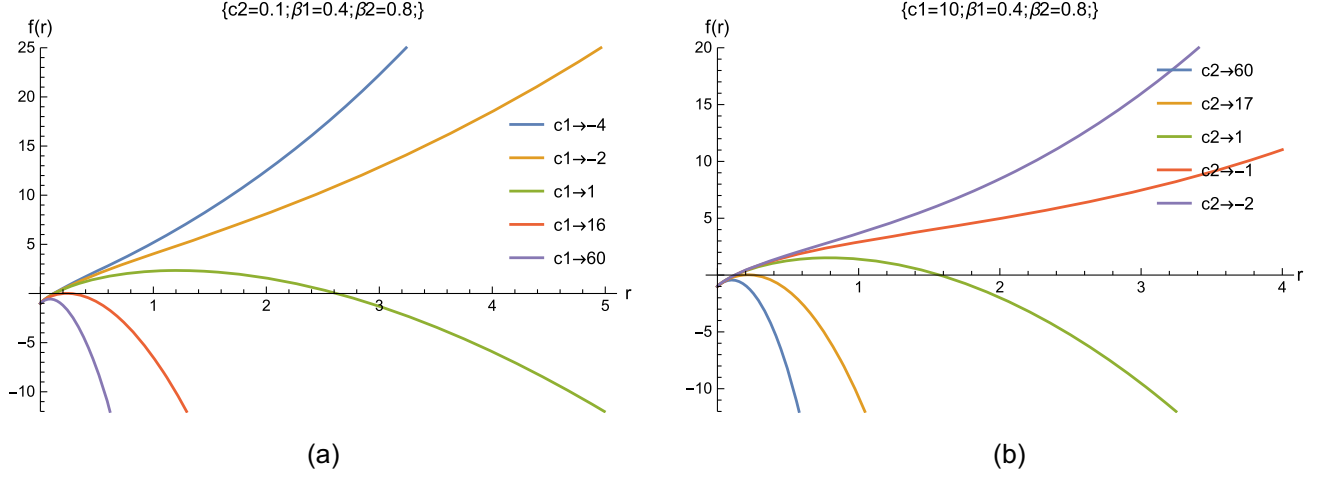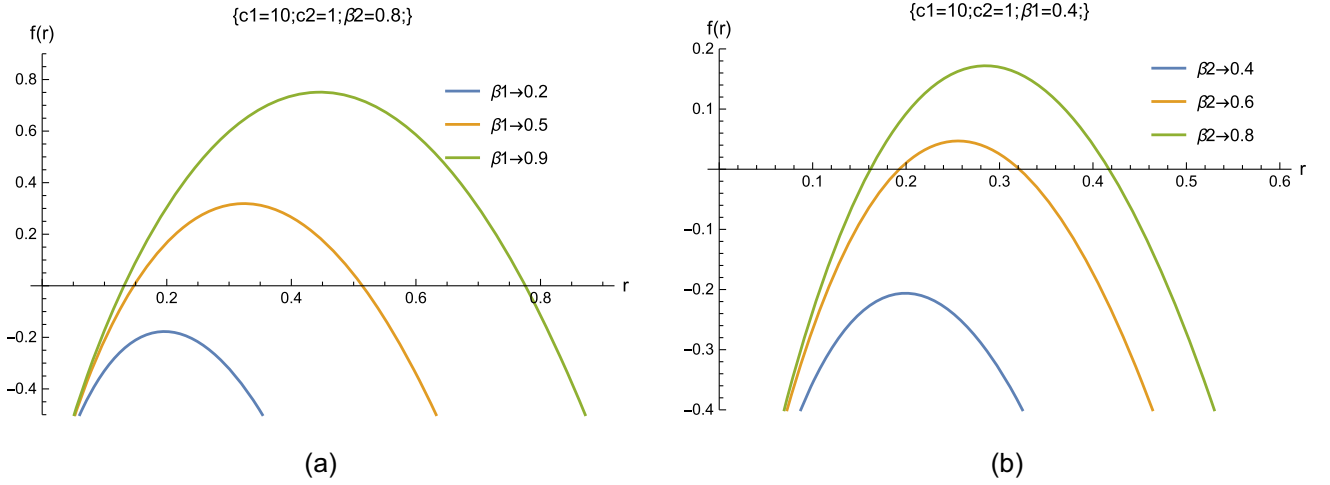

FIG. 2. The function $f(r)$ versus r for different values of α, β_1 and β_2 in the case $\alpha \neq 1$.

FIG. 3. The function $f(r)$ versus r for different values of c_1 and c_2 in case $\alpha = 1$.

FIG. 4. The function $f(r)$ versus r for different values of β_1 and β_2 in case $\alpha = 1$.

different asymptotical behaviors, as shown in Fig. 3. The influences of other parameters β_1 and β_2, are separately demonstrated in Figs. 4. Similarly to the case when $\alpha \neq 1$, it is possible to have an extremal black hole, known as the Nariai black hole, by selecting appropriate parameter values. Additionally, changing one of the parameters can lead to the appearance of a naked singularity.

III. THERMODYNAMICS OF DILATONIC BLACK HOLES

Now we plan to investigate the thermodynamics of theses dilatonic black holes in massive Einstein-dilaton gravity. According to the definition of ADM mass [55] and definition of horizon $f(r_h) = 0$, the mass of dilatonic black hole is given by

$$
M = \frac{\delta^{\frac{2\alpha^2}{1+\alpha^2}}m}{2(1+\alpha^2)}
$$

$$
= \begin{cases} \frac{r_h}{2(1-\alpha^2)} - \frac{(\alpha^2+1)c_0c_1m_0^2\delta^{\frac{\alpha(\alpha-2\beta_1)}{\alpha^2+1}}r_h^{\frac{\alpha^2+2\alpha\beta_1+2}{\alpha^2+1}}}{2(\alpha^2+2\alpha\beta_1+2)(2\alpha^2+2\alpha\beta_1-1)} - \frac{(\alpha^2+1)c_0^2c_2m_0^2\delta^{\frac{2\alpha\beta_2}{\alpha^2+1}}r_h^{\frac{\alpha^2+2\alpha\beta_2+1}{\alpha^2+1}}}{2(\alpha^2+\alpha\beta_2-1)(\alpha^2+2\alpha\beta_2+1)}, & \alpha \neq 1 \\ \frac{1}{2}r_h\left(\log\left(\frac{\delta}{r_h}\right)+2\right) - \frac{c_0c_1m_0^2r_h^2\left(\frac{\delta}{r_h}\right)^{2-\beta_1}}{4\beta_1(\beta_1+2)+3} - \frac{c_0^2c_2m_0^2r_h\left(\frac{\delta}{r_h}\right)^{-\beta_2}}{2(\beta_2^2+\beta_2)}, & \alpha = 1 \end{cases}
$$

(32)

The ADM mass as a function of black hole radius are plotted in Figs. 5 and 6 for both cases of $\alpha \neq 1$, and $\alpha = 1$. In each case, there exists a maximum mass M_{max}, and there are two black holes with the same mass, distinguished by their size (a smaller
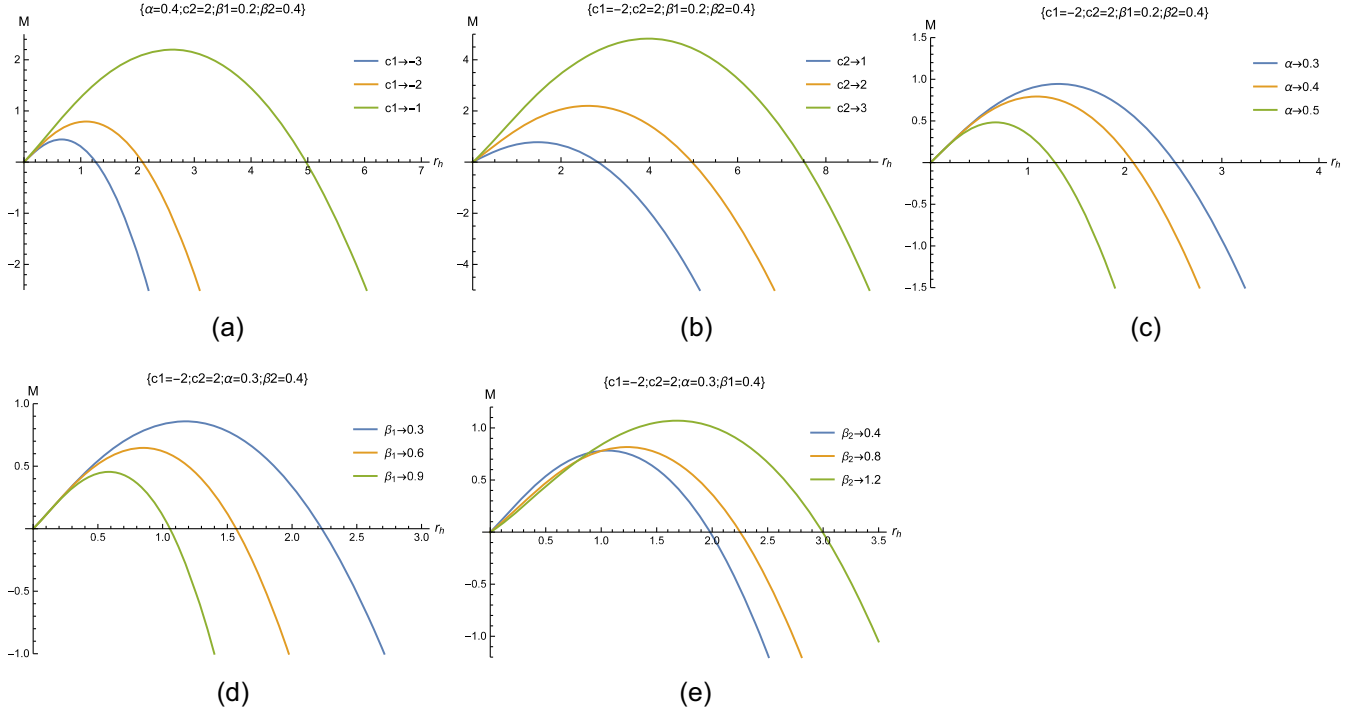
FIG. 5. The mass M of black hole versus r_h in the case $\alpha \neq 1$.

one and a larger one). The ADM mass is always positive, and therefore there is also a maximum radius $r_{\max}$ at which the mass vanishes. It can be observed that for $\alpha \neq 1$, the maximum mass $M_{\max}$ increases with c_1 and c_2 but decreases with α [see Figs. 5(a)–5(c)]. Conversely, $M_{\max}$ decreases with c_1 and c_2 when $\alpha \neq 1$ [see Figs. 6(a) and 6(b)]. The remaining figures demonstrate the effects of the coupling of dilaton field. The maximum mass always decreases with β_1 for both cases of α, as shown in Figs. 5(d) and 6(c), suggesting that the coupling of dilaton field to the first
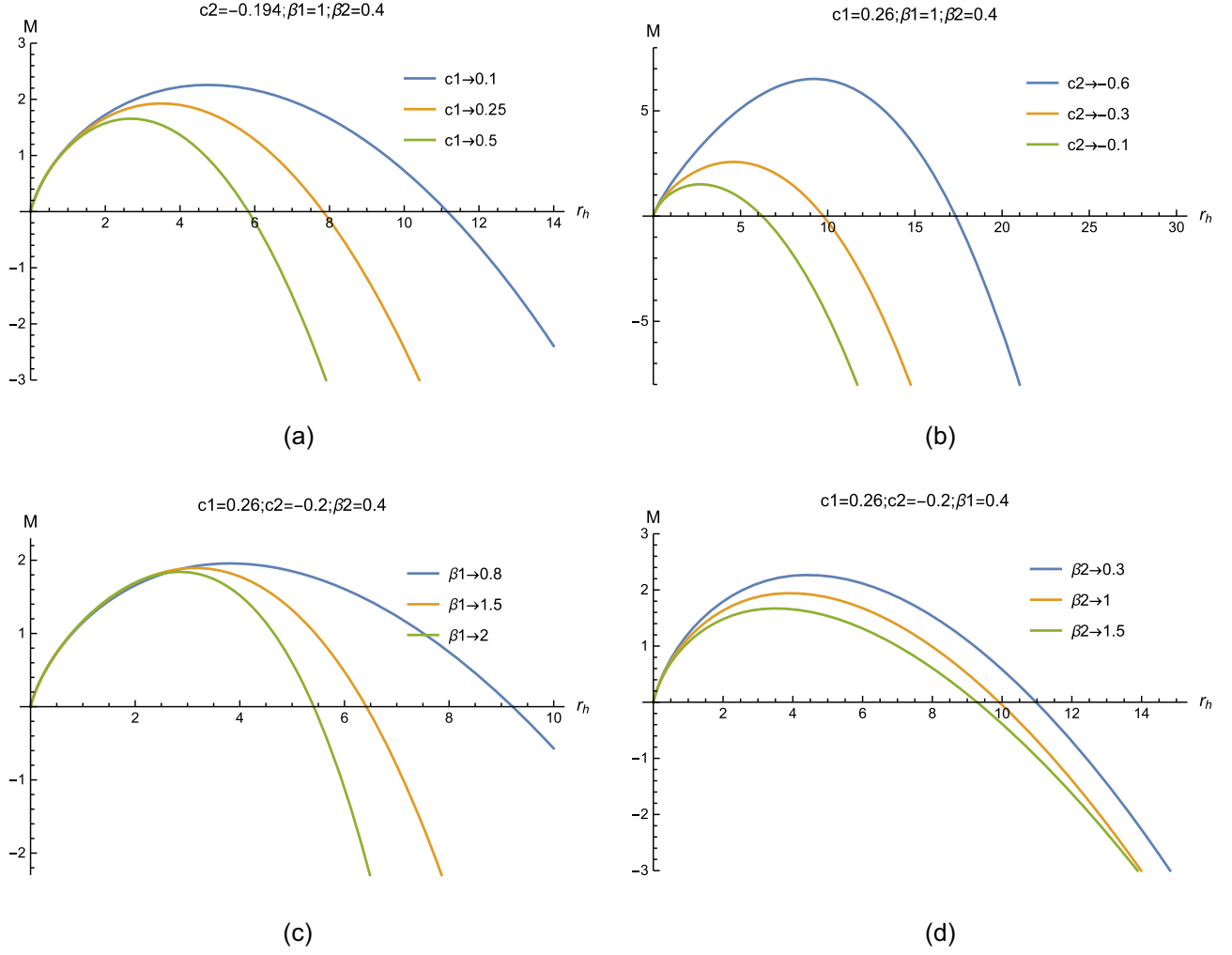
terms of the graviton could always reduced the maximum mass of black holes. However, for β_2, $M_{\max}$ increases when $\alpha \neq 1$ but decreases when $\alpha = 1$ [see Figs. 5(e) and 6(d)].

To develop thermodynamics of dilatonic black hole, we need to calculate the Hawking temperature of the black hole geometrically associated with the black hole horizon. In terms of the surface gravity κ corresponding to the null killing vector $(\frac{\partial}{\partial t})^a$ at the horizon, the temperature can be written as

$$
T = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \frac{\partial f(r)}{\partial r}\bigg|_{r=r_h}
$$

$$
= \begin{cases}
-\dfrac{\alpha^2+1}{4\pi r_h}\left(\dfrac{\delta^{-\frac{2\alpha^2}{\alpha^2+1}} r_h^{\frac{2\alpha^2}{\alpha^2+1}}}{\alpha^2-1} + \dfrac{c_0 c_1 m_0^2 \delta^{\frac{\alpha(\alpha+2\beta_1)}{\alpha^2+1}} r_h^{\frac{2\alpha^2+2\alpha\beta_1+1}{\alpha^2+1}}}{2\alpha(\alpha+\beta_1)-1} + \dfrac{c_0^2 c_2 m_0^2 \delta^{-\frac{2\alpha(\alpha+\beta_2)}{\alpha^2+1}} r_h^{\frac{2\alpha(\alpha+\beta_2)}{\alpha^2+1}}}{\alpha(\alpha+\beta_2)-1} \right), & \alpha \neq 1 \\[4mm]
\dfrac{\log(\frac{\delta}{r_h})+1}{2\pi\delta} - \dfrac{c_0 c_1 m_0^2 r_h (\frac{\delta}{r_h})^{\frac{1}{2}-\beta_1}}{2\pi(2\beta_1+1)\delta} - \dfrac{c_0^2 c_2 m_0^2 (\frac{\delta}{r_h})^{-\beta_2}}{2\pi\beta_2\delta}, & \alpha = 1
\end{cases}
\tag{33}
$$

The Temperature of black hole as a function of the radius is plotted in Figs. 7 and 8 for different cases: $\alpha \neq 1$ and $\alpha = 1$. For both cases, there is a minimum positive temperature $T_{\min}$ of the black hole, which illustrates there exist two black holes with the same temperature, distinguished by their size (a smaller one and a larger one). Moreover, the minimum temperature ($T_{\min}$) decreases with c_1 and β_1 [see Figs. 7(a), 7(d), 8(a), and 8(c)]. This suggests that the first

term of the graviton, including the effect of coupling to the dilaton field, always reduces the minimum temperature of the black holes. Figure 8(b) demonstrate that the minimum temperature increases with c_2 when $\alpha \neq 1$ but decreases with c_2 when $\alpha = 1$. For $\alpha \neq 1$ [as shown in Figs. 7(c)], the minimum temperature increases with α, indicating that the dilaton field could always enlarge the minimum temperature. In addition, also for both cases, the

FIG. 6.   The mass M of black hole versus r_h in the case $\alpha = 1$.

minimum temperature decreases with β_1 but increases with β_2, illustrating that the coupling of dilaton field to the first term of graviton could always reduce the minimum temperature of black hole but the second term of graviton could always enlarge that of black hole [as shown in Figs. 7(d), 7(e), 8(c), and 8(d)].

The entropy of dilatonic black hole is given by

$$S = \pi r_h^2 R(r_h)^2 = \pi \delta^{\frac{2\alpha^2}{\alpha^2+1}} r_h^{\frac{2}{\alpha^2+1}}. \tag{34}$$

References [37–41] have pointed out that the graviton mass does not significantly affect the form of the entropy, and contributes only as a correction for the horizon radius. In fact, the entropy of these black holes in massive Einstein-dilaton gravity can be also derived from traditional metric ansatz of dilatonic black holes, as shown in the Appendix. Then, we find that the thermodynamic quantities satisfy the first law of black hole thermodynamics

$$dM = TdS. \tag{35}$$

In addition, another important aspect of black hole thermodynamics involves analyzing the thermal stability of black hole solutions. To assess the thermal stability, we compute the heat capacity

$$C = T\frac{\partial S}{\partial T} = T\frac{\partial S/\partial r_h}{\partial T/\partial r_h}, \tag{36}$$

which leads to

$$C = \begin{cases} \dfrac{-8\pi^2(\alpha^2+1)^{-1}T\delta^{\frac{4\alpha^2}{\alpha^2+1}}r^{-\frac{2\alpha^2}{\alpha^2+1}}r_h^{\frac{2}{\alpha^2+1}+1}}{1+\dfrac{\alpha c_0 c_1 m_0^2(\alpha+2\beta_1)\delta^{\frac{\alpha(\alpha-2\beta_1)}{\alpha^2+1}}r^{\frac{2\alpha\beta_1+1}{\alpha^2+1}}}{2\alpha^2+2\alpha\beta_1-1}+\dfrac{c_0^2 c_2 m_0^2(\alpha^2+2\alpha\beta_2-1)\delta^{\frac{2\alpha\beta_2}{\alpha^2+1}}r^{\frac{2\alpha\beta_2}{\alpha^2+1}}}{\alpha^2+\alpha\beta_2-1}}, & \alpha \neq 1 \\[3em] -\dfrac{4\pi^2 T\delta^{\beta_1+\beta_2+\frac{5}{2}}r_h^{3/2}}{c_0 c_1 m_0^2 \delta^{\beta_2+1} r_h^{\beta_1+1}+2c_0^2 c_2 m_0^2 \delta^{\beta_1+\frac{1}{2}}r_h^{\beta_2+\frac{1}{2}}+2\delta^{\beta_1+\beta_2+\frac{1}{2}}\sqrt{r_h}}, & \alpha = 1 \end{cases} \tag{37}$$

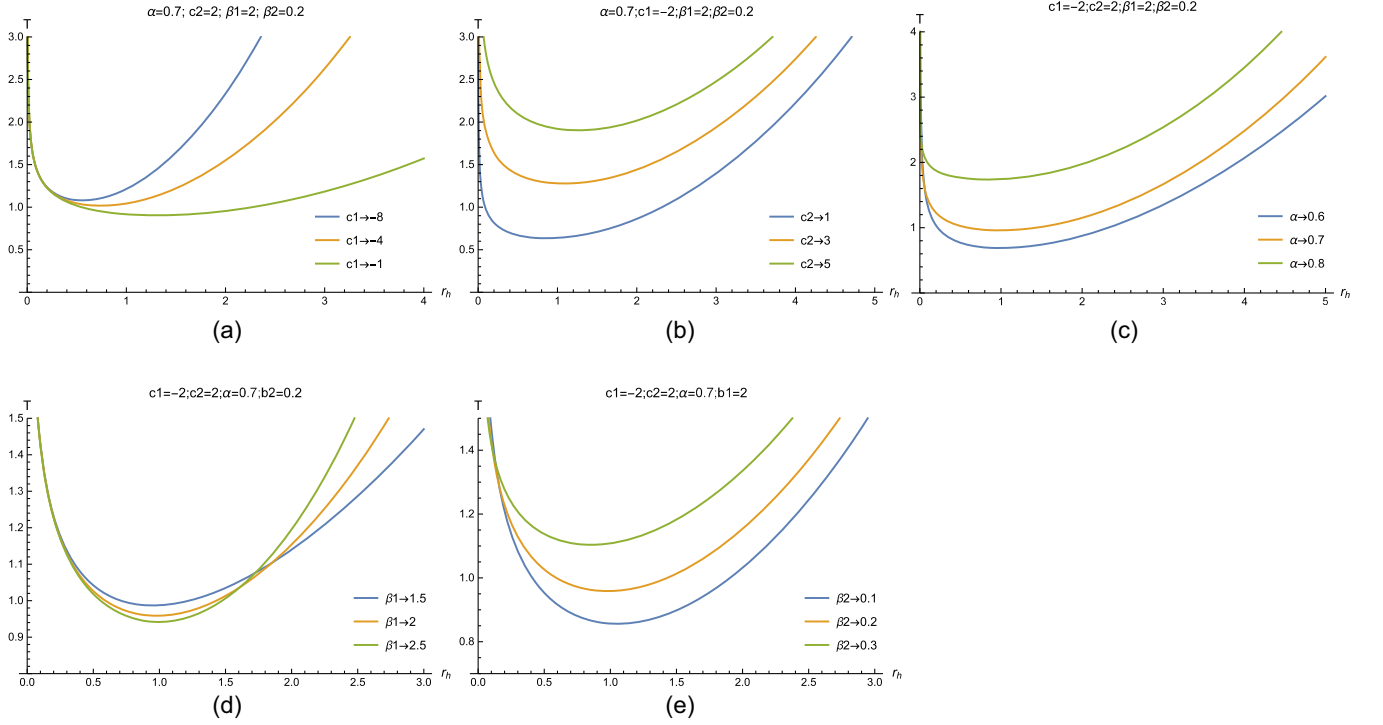From Eq. (36), the nonmonotonic nature of the temperature allows us to conclude that the heat capacity exhibits

FIG. 7. The temperature T versus r_h for $\alpha \neq 1$. The extreme points of each $T - r$ curves $(r_{\text{cri}}, T_{\text{min}})$ are, respectively, (a) (0.555418, 1.07929), (0.737029, 1.01689), (1.29893, 0.90413) for $c_1 = -8, -4, -1$ (b) (0.858517, 0.633958), (1.08183, 1.27758), (1.25738, 1.90305) for $c_2 = 1, 3, 5$ (c) (0.989441, 0.687425), (0.978303, 0.958608), (0.833922, 1.73715) for $\alpha = 0.6, 0.7, 0.8$ (d) (0.948498, 0.986909), (0.978303, 0.958608), (0.992489, 0.941541) for $\beta_1 = 1.5, 2, 2.5$ (e) (1.05142, 0.856112), (0.978303, 0.958608), (0.852074, 1.10364) for $\beta_2 = 0.1, 0.2, 0.3$.
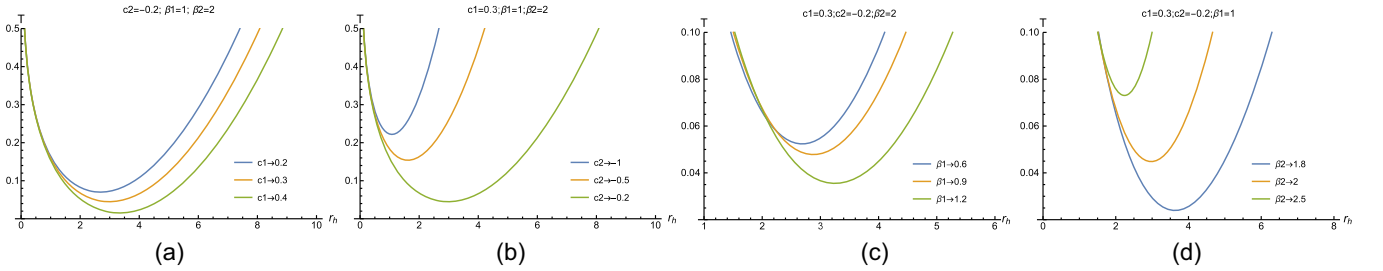

FIG. 8. The temperature T versus r_h for $\alpha = 1$. The extreme points of each $T - r$ curves $(r_{\text{cri}}, T_{\text{min}})$ are, respectively, (a) (2.68273, 0.0700174), (2.97447, 0.0448315), (3.32688, 0.0152299) for $c_1 = 0.2, 0.3, 0.4$ (b) (1.08101, 0.221862), (1.61782, 0.153979), (2.97447, 0.0448315) for $c_2 = -1, -0.5, -0.2$ (c) (2.68804, 0.0523771), (2.88079, 0.047834), (3.24997, 0.0355224) for $\beta_1 = 0.6, 0.9, 1.2$ (d) (3.63175, 0.0239511), (2.97447, 0.0448315), (2.24071, 0.0730591) for $\beta_2 = 1.8, 2, 2.5$.

discontinuities as illustrated in Figs. 9 and 10 for both cases. Figures 9 and 10 show that black holes with relatively large horizon radii $(r_h > r_{\text{cri}})$ are stable thermodynamic systems, while a domain of instability exists for smaller radii below the point of discontinuity $(r_h < r_{\square})$. In other words, the figures clearly illustrate the stable and unstable regions based on the discontinuities observed in the heat capacity as a function of the horizon radius.

To get more information about the phase transition, we can examine the free energy. The Gibbs free energy is $G = M - TS$ reads as

$$G = \begin{cases} \dfrac{r}{4} + \dfrac{\alpha(\alpha^2+1)c_0 c_1 m_0^2 (\alpha+2\beta_1) \delta^{\frac{\alpha(\alpha-2\beta_1)}{\alpha^2+1}} r^{\frac{\alpha^2+2\alpha\beta_1+2}{\alpha^2+1}}}{4(\alpha^2+2\alpha\beta_1+2)(2\alpha^2+2\alpha\beta_1-1)} \\ \quad + \dfrac{(\alpha^2+1)c_2^2 c_2 m_0^2 (\alpha^2+2\alpha\beta_2-1)\delta^{-\frac{2\alpha\beta_2}{\alpha^2+1}} r^{\frac{\alpha^2+2\alpha\beta_2+1}{\alpha^2+1}}}{4(\alpha^2+\alpha\beta_2-1)(\alpha^2+2\alpha\beta_2+1)}, \quad \alpha \neq 1 \\ \dfrac{c_0 c_1 m_0^2 \delta^{\frac{1}{2}-\beta_1} r_h^{\beta_1+\frac{3}{2}}}{4\beta_1+6} + \dfrac{c_0^2 c_2 m_0^2 \delta^{-\beta_2} r_h^{\beta_2+1}}{2\beta_2+2} + \dfrac{r_h}{2}, \quad \alpha = 1. \end{cases} \tag{38}$$

We find that these discontinuities, which correspond to the minima of temperature in the heat capacity curves, indicate the occurrence of Hawking-Page phase transitions.
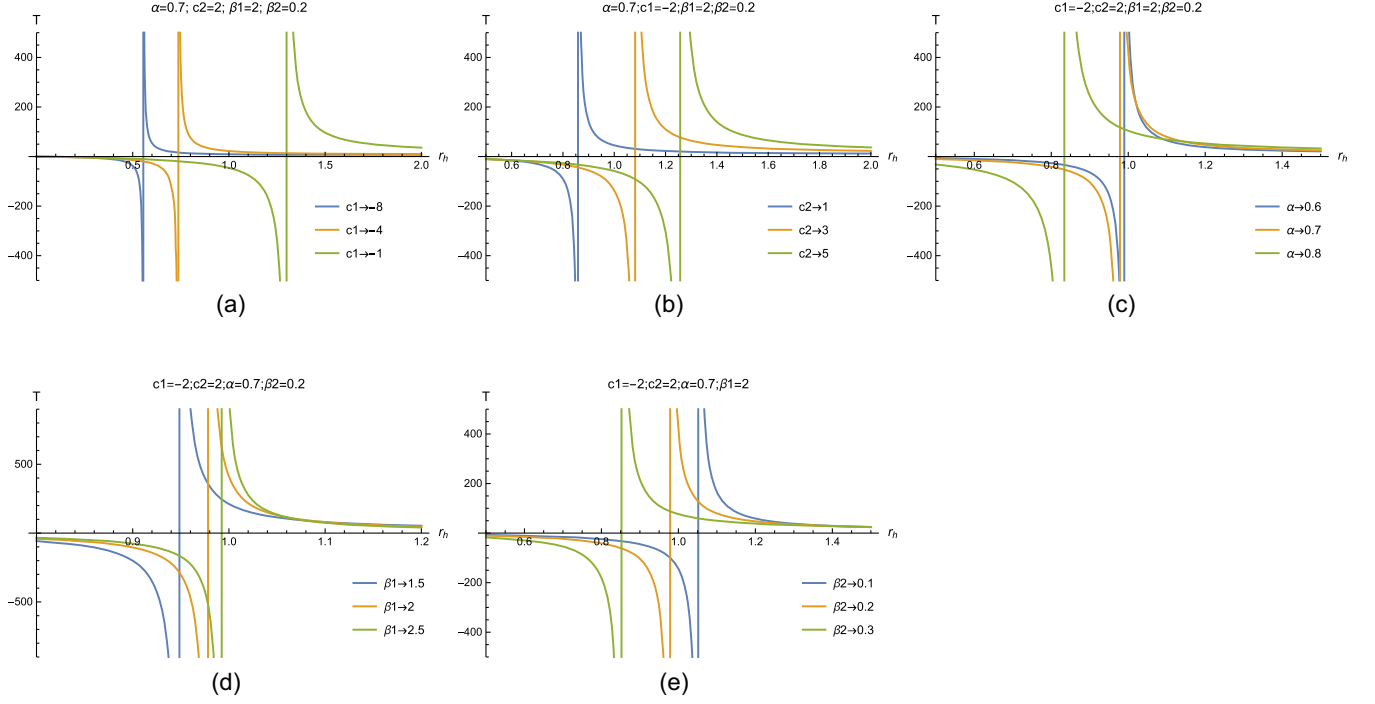
FIG. 9.   The heat capacity C versus r_h for $\alpha \neq 16$.
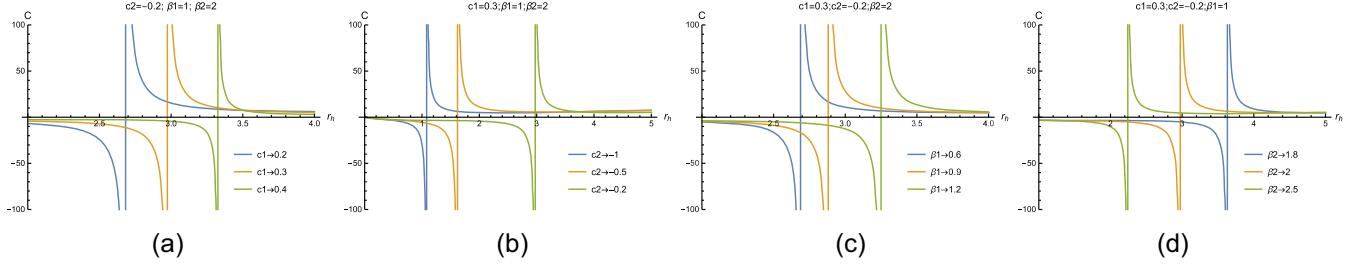
FIG. 10.   The heat capacity C versus r_h for $\alpha = 16$.

Hawking-Page phase transitions were also recently found in dilatonic black holes that are neither asymptotically flat nor AdS as in Ref. [56]. As shown in Figs. 11 and 12, the upper and lower branches correspond to small and large black holes, respectively. A positive Gibbs free energy indicates that the system is in a radiation phase, whereas a Hawking-Page phase transition occurs at the intersection point of the lower branch with $G = 0$. The temperature at this point is known as the Hawking-Page temperature, T_{HP}. The fact that large black holes always have a lower Gibbs free energy compared to small black holes confirms the above arguments regarding their thermal stability. The effects of parameter variations can also be seen from Figs. 11 and 12. Increasing c_1 or β_1 lowers T_{HP}. Similarly, lowering α or β_2 lowers the Hawking-Page temperature T_{HP}. However, T_{HP} increases with c_2 when $\alpha \neq 1$ but decreases with c_2 when $\alpha = 1$.

IV. BLACK HOLE SOLUTIONS IN $(d+1)$-DIMENSIONAL SPACETIME

In this section, we would like to extend to the massive Einstein-dilaton gravity in high dimensional spacetime. The action is given as

$$I = \frac{1}{16\pi} \int d^{(d+1)}x \sqrt{-g} \left[\mathcal{R} - \frac{4}{d-1}(\nabla\varphi)^2 - V(\varphi) \right.$$
$$\left. + m_0^2 \sum_{i=1}^{4} c_i e^{-2\beta_i \varphi} \mathcal{U}_i(g,h) \right], \tag{39}$$

where the last term denotes the general form of the coupling between the scalar field and massive graviton, the $\mathcal{U}_i$ are symmetric polynomials of the eigenvalues of the $(d+1) \times (d+1)$ matrix $K^\mu_\nu = \sqrt{g^{\mu\alpha}h_{\alpha\nu}}$, and satisfy the relation in Eq. (2).
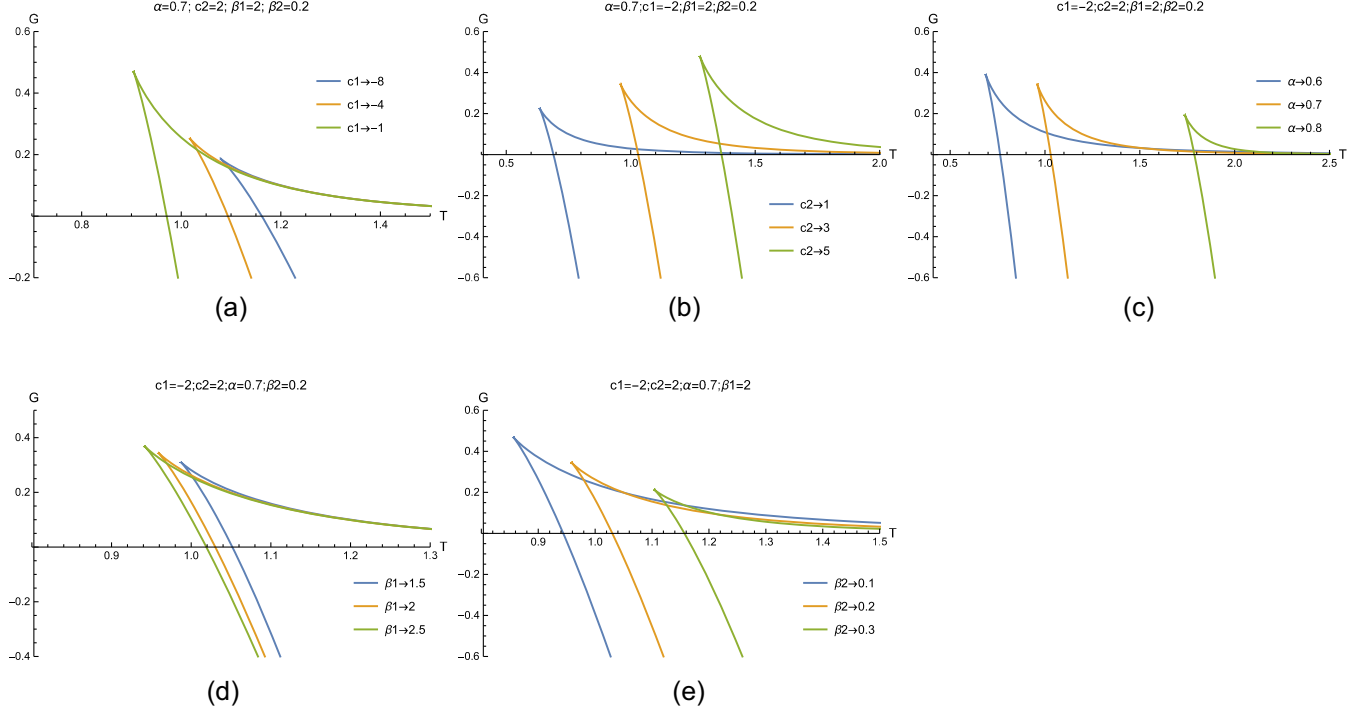
FIG. 11. The Gibbs free energy G versus r_h for $\alpha \neq 16$.
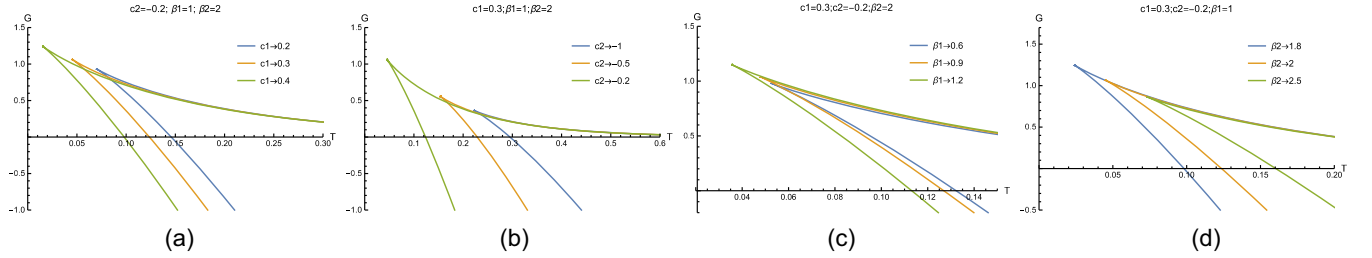
FIG. 12. The Gibbs free energy G versus r_h for $\alpha = 1$.

By varying the action (39) with regard to metric $g_{\mu\nu}$ and φ, the equations of motions can be written as

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \frac{4}{d-1}\partial_\mu\varphi\partial_\nu\varphi - \left(\frac{V}{2} + \frac{2}{d-1}\partial^\rho\varphi\partial_\rho\varphi\right)g_{\mu\nu}$$
$$+ m^2\chi_{\mu\nu}, \qquad (40)$$

$$\nabla^2\varphi = \frac{d-1}{8}\left[\frac{\partial V}{\partial\varphi} - m^2\sum_{i=1}^{4}\frac{\partial\tilde{c}_i}{\partial\varphi}\mathcal{U}_i(g,h)\right], \qquad (41)$$

where the $\tilde{c}_i$ and $\chi_{\mu\nu}$ are defined in Eq. (5).

Now we introduce the metric ansatz for static black hole solution in $(d+1)$-dimensional space-time

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2R^2(r)\sigma_{ij}dx^idx^j, \qquad (42)$$

where $\sigma_{ij}dx^idx^j$ is the line element for $(d-1)$-dimensional spherical Einstein space and volume ω_{d-1}. Considering the following reference metric

$$h_{\mu\nu} = \text{diag}(0, 0, c_0^2\sigma_{ij}), \qquad (43)$$

the interaction potential in Eq. (2) changes into

$$\mathcal{U}_i = \Pi_i \frac{c_0^i}{R^i r^i} \qquad (44)$$

with positive constant c_0 and the notation $\Pi_i = \prod_{j=0}^{i-1}(d-1-j)$.

In order to solve the system of Einstein equation (40), we assume [12,13]

$$R(r) = e^{\frac{2\alpha}{d-1}\varphi}, \qquad (45)$$

Then we can get the dilton field and the Liouville-type dilaton potential $V(\varphi)$ as the following form

$$\varphi(r) = \frac{(d-1)\alpha}{2(1+\alpha^2)} \ln\left(\frac{\delta}{r}\right) \tag{46}$$

$$V(\varphi) = \begin{cases} 2\gamma_0 e^{2\xi_0\varphi} + \sum_{i=1}^{4} 2\gamma_i e^{2\xi_i\varphi}, & \alpha \neq 1 \\ 2\lambda_0 \varphi e^{\frac{4\varphi}{d-1}} + \sum_{i=1}^{4} 2\lambda_i e^{2\zeta_i\varphi}, & \alpha = 1 \end{cases} \tag{47}$$

where the summation term is the general form associating with coupling between the dilaton and massive gravity field, and λ_0, γ_i and $\xi_i (i = 0, 1, \ldots, 4)$ are constants and satisfy

$$\xi_0 = \frac{2}{(d-1)\alpha}, \qquad \xi_i = -\beta_i + \frac{i}{(d-1)\alpha}, \qquad \zeta_i = \frac{i}{d-1} - \beta_i, \qquad \gamma_0 = \frac{\alpha^2(d-2)(d-1)}{2(\alpha^2-1)\delta^2},$$

$$\gamma_i = -\frac{m_0^2 c_0^i c_i \alpha(2\alpha + (d-1)\beta_i)\delta^{-i}\Pi_i}{2(i - 2\alpha^2 + (1-d)\alpha\beta_i)}, \qquad \lambda_0 = -\frac{2(d-2)}{\delta^2}, \qquad \lambda_i = -\frac{m_0^2 c_0^i c_i \Pi_i \delta^{-i}((d-1)\beta_i + 2)}{2((1-d)\beta_i + i - 2)} \tag{48}$$

and then the solutions of Eq. (40) in $(d+1)$-dimensional spacetime can easily be calculated as

$$f(r) = \begin{cases} -mr^{\frac{\alpha^2-d+2}{\alpha^2+1}} - \frac{(\alpha^2+1)^2(d-2)\delta^{\frac{2\alpha^2}{\alpha^2+1}}r^{\frac{2\alpha^2}{\alpha^2+1}}}{(\alpha^2-1)(\alpha^2+d-2)} - \sum_{i=1}^{4} m_0^2 c_0^i c_i A_i r^{\frac{2\alpha^2+\alpha(d-1)\beta_i - i + 2}{\alpha^2+1}}, & \alpha \neq 1 \\ -mr^{\frac{3-d}{2}} - \frac{4(d-2)r((d-1)\log(\frac{r}{\delta})-d-1)}{(d-1)^2\delta} - \sum_{i=1}^{4} m_0^2 c_0^i c_i B_i r^{\frac{(d-1)\beta_i - i + 4}{2}}, & \alpha = 1 \end{cases} \tag{49}$$

with

$$A_i = \frac{i(\alpha^2+1)^2 \delta^{-\frac{\alpha((d-1)\beta_i + \alpha i)}{\alpha^2+1}}\Pi_i}{(d-1)(\alpha^2+\alpha(d-1)\beta_i + d - i)(2\alpha^2+\alpha(d-1)\beta_i - i)},$$

$$B_i = -\frac{4i\Pi_i \delta^{\frac{1}{2}(-d\beta_i+\beta_i - i)}}{(d-1)((d-1)\beta_i - i + 2)((d-1)\beta_i + d - i + 1)} \tag{50}$$

Obviously, the solutions (49) when taking $d = 3$ equals to Eq. (25). According to the definition of ADM mass [55], the mass of dilatonic black hole reads as

$$M = \frac{(d-1)\omega_{d-1}\delta^{\frac{\alpha^2(d-1)}{\alpha^2+1}}m}{16\pi(\alpha^2+1)}, \tag{51}$$

where ω_{d-1} represents the volume of hyper-surface described by $h_{ij}dx^i dx^j$. Therefore, using the definition of horizon $f(r_h) = 0$, the mass of black hole is written in terms of r_h as

$$M = \begin{cases} -\frac{(\alpha^2+1)(d-2)(d-1)\omega_{d-1}\delta^{\frac{\alpha^2(d-3)}{\alpha^2+1}}r_h^{\frac{\alpha^2+d-2}{\alpha^2+1}}}{16\pi(\alpha^2-1)(\alpha^2+d-2)} - \sum_{i=1}^{4} \frac{(d-1)m_0^2 A_i c_0^i c_i \omega_{d-1}\delta^{\frac{\alpha^2(d-1)}{\alpha^2+1}}r_h^{\frac{\alpha^2+\alpha(d-1)\beta_i + d - i}{\alpha^2+1}}}{16\pi(\alpha^2+1)}, & \alpha \neq 1, \\ -\frac{(d-2)\omega_{d-1}\delta^{\frac{d-3}{2}}r_h^{\frac{d-1}{2}}((d-1)\log(\frac{r_h}{\delta})-d-1)}{8\pi(d-1)} - \sum_{i=1}^{4} \frac{im_0^2\omega_{d-1}c_0^i c_i \Pi_i \delta^{\frac{(d-1)(1-\beta_i)-i}{2}}r_h^{\frac{(d-1)\beta_i + d - i + 1}{2}}}{8\pi(i+2-(d-1)\beta_i)(i-2-(d-1)(\beta_i+1))}, & \alpha = 1. \end{cases} \tag{52}$$

By calculating the Hawking temperature and entropy of black hole as

$$T = \frac{1}{4\pi}\frac{\partial f(r)}{\partial r}\Big|_{r=r_h} = \begin{cases} -\frac{(\alpha^2+1)(d-2)\delta^{-\frac{2\alpha^2}{\alpha^2+1}}r_h^{\frac{\alpha^2-1}{\alpha^2+1}}}{4\pi(\alpha^2-1)} - \sum_{i=1}^{4} \frac{m_0^2 A_i c_0^i c_i(\alpha^2+\alpha(d-1)\beta_i + d - i)r_h^{\frac{\alpha^2+\alpha(d-1)\beta_i - i + 1}{\alpha^2+1}}}{4\pi(\alpha^2+1)}, & \alpha \neq 1, \\ -\frac{(d-2)k(\log(\frac{r_h}{\delta})-1)}{2\pi\delta} + \sum_{i=1}^{4} \frac{im_0^2 c_0^i c_i \Pi_i \delta^{\frac{(1-d)\beta_i - i}{2}}r_h^{\frac{(d-1)\beta_i - i + 2}{2}}}{2\pi(d-1)((1-d)\beta_i + i - 2)}, & \alpha = 1. \end{cases} \tag{53}$$

$$S = \frac{1}{4}\omega_{d-1}\delta^{\frac{\alpha^2(d-1)}{\alpha^2+1}} r_h^{\frac{d-1}{\alpha^2+1}}. \tag{54}$$

It is easily checked that the first law of black hole thermodynamics still holds for dilatonic black hole in high dimensional spacetime

$$dM = TdS \tag{55}$$

V. CONCLUSIONS AND DISCUSSIONS

Considering the nonminimal coupling between graviton and dilaton field, we discussed the massive Einstein-dilaton gravity. According to the gravitational field and dilaton field equations by varying the action, we obtained the static spherically symmetric solutions of dilatonic black hole for $\alpha \neq 1$ and $\alpha = 1$ cases in four dimensional spacetime. Here the dilaton potential $V(\varphi)$ takes a so-called Liouville-type form both for $\alpha \neq 1$ and $\alpha = 1$, and the last two terms of potential are associated with the graviton terms.

Later, we have analyzed the singularity of the solution, we give the Ricci and Kretschmann scalars, which suggest that the horizons of black holes are just singularity of coordinate as it should be. What is more, both of the scalars have the same the asymptotic behavior at the origin, and one can obtain that there is a point of essential located at the origin, and the asymptotic behavior of the solutions is not asymptotically flat but infinity in general when $r \to \infty$. We also show that the black hole solutions can provide one horizon, two horizons (event and cosmological), extreme (Nariai) and naked singularity black holes for the suitably fixed parameters. With dilaton field, the parameter c_2 affects the behavior of the metric function.

We further studied the mass, temperature and entropy of these dilatonic black holes and checked the first law of black hole thermodynamics. The analysis of the mass suggest that both for $\alpha \neq 1$ and $\alpha = 1$, there could exists a maximum of the mass function of black hole horizon. For the temperature of these black holes, and found that it has a minimum positive value $T_{\min}$, distinguishing two sizes of black holes (a smaller one and a larger one) with the same temperature. Moreover, the minimal points of the temperature function correspond to the discontinuous points of the heat capacity, the domain of smaller radii $r_h < r_{cri}$ of black holes lies instable and black holes with relatively large horizon radii $r_h > r_{cri}$ demonstrate stability, which implies the occurrence of Hawking-Page phase transitions in these thermodynamic systems. By investigating the Gibbs free energy, we have found the fact that large black holes always have a lower Gibbs free energy compared to small black holes confirming the arguments of their thermal stability.

Finally, we generalized these discussions to the high dimensional spacetime, and got the $(d + 1)$ dimensional solution of dilatonic black hole both for $\alpha \neq 1$ and $\alpha = 1$ in massive Einstein-dilaton gravity. The corresponding thermodynamic quantities of black holes were also calculated, and we find the first law of black hole thermodynamics still maintains.

APPENDIX: NEW FORM SOLUTION

In this Appendix, we will present the static and spherically symmetric dilatonic black hole solutions with the traditional metric ansatz form in four dimensional spacetime.

We assume the new metric ansatz with

$$ds^2 = -N^2(\tilde{r})f(\tilde{r})dt^2 + f^{-1}(\tilde{r})d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{A1}$$

where $f(\tilde{r})$ and $N(\tilde{r})$ are functions of $\tilde{r}$. Choosing the same reference metric in Eq. (8), the interaction potential in Eq. (2) changes into

$$\mathcal{U}_1 = \frac{2c_0}{\tilde{r}}, \qquad \mathcal{U}_2 = \frac{2c_0^2}{\tilde{r}^2}, \qquad \mathcal{U}_3 = \mathcal{U}_4 = 0, \tag{A2}$$

and χ_ν^μ from Eq. (5) read as

$$\chi^1{}_1 = \chi^2{}_2 = \frac{c_0c_1\tilde{r}e^{-2\beta_1\phi} + c_2c_0^2e^{-2\beta_2\phi}}{\tilde{r}^2},$$

$$\chi^3{}_3 = \chi^4{}_4 = \frac{c_1c_0e^{-2\beta_1\phi}}{2\tilde{r}}. \tag{A3}$$

Then, the corresponding components of the Einstein equation Eq. (3) can be simplified to

$$\tilde{G}^1{}_1 = \frac{-1+f(\tilde{r})+\tilde{r}f'(\tilde{r})}{\tilde{r}^2} = -\frac{1}{2}V(\varphi)-f\varphi'^2+m_0^2\chi^1{}_1, \tag{A4}$$

$$\tilde{G}^2{}_2 = \frac{N'(\tilde{r})[-1+f(\tilde{r})+\tilde{r}f'(\tilde{r})]+2\tilde{r}f(\tilde{r})N'(\tilde{r})}{\tilde{r}^2N(\tilde{r})}$$
$$= -\frac{1}{2}V(\varphi)+f\varphi'^2+m_0^2\chi^2{}_2, \tag{A5}$$

$$\tilde{G}^3{}_3 = \frac{1}{2\tilde{r}N(\tilde{r})}\{3\tilde{r}f'(\tilde{r})N'(\tilde{r})+N(\tilde{r})[2f'(\tilde{r})+\tilde{r}f''(\tilde{r})]$$
$$+ 2f(\tilde{r})[N'(\tilde{r})+\tilde{r}N''(\tilde{r})]\}$$
$$= -\frac{1}{2}V(\varphi)-f\varphi'^2+m_0^2\chi^3{}_3, \tag{A6}$$

where the prime $'$ denotes differentiation with respect to the radial coordinate $\tilde{r}$.

Considering the different forms between new line element (A1) and traditional one (7), we assume the relationship of transformation of the two solutions

$$\tilde{r} = rR(r) = \delta^{\frac{\alpha^2}{\alpha^2+1}} r^{\frac{1}{\alpha^2+1}}, \quad \text{or} \quad r = \tilde{r}^{1+\alpha^2} \delta^{-\alpha^2}. \quad (A7)$$

where $R(r)$ is determined by Eqs. (16) and (18). Based on Eqs. (A3)–(A5), we obtain

$$\frac{N'(\tilde{r})}{\tilde{r}N(\tilde{r})} = \varphi'(\tilde{r})^2. \quad (A8)$$

Therefore, the dilaton field can be written as

$$\varphi(\tilde{r}) = \alpha \ln(\delta/\tilde{r}) \quad (A9)$$

and the function $N(\tilde{r})$ reads as

$$N(\tilde{r}) = (1+\alpha^2)\tilde{r}^{\alpha^2}\delta^{-\alpha^2} \quad (A10)$$

According to the similar calculations, we found the differential equation of dilaton potential $V(\varphi(\tilde{r}))$ is same as Eq. (22) of $V(\varphi(r))$. So the total expansion of $V(\varphi)$ remains consistent with Eq. (23) in the previous text. Finally, we can obtain the solution of metric function $f(\tilde{r})$ as

$$f(\tilde{r}) = \begin{cases} -m\tilde{r}^{-\alpha^2-1} + \frac{1}{1-\alpha^4} - \frac{c_0 c_1 m_0^2 \tilde{r}(\frac{\delta}{\tilde{r}})^{-2\alpha\beta_1}}{(\alpha^2+2\alpha\beta_1+2)(2\alpha^2+2\alpha\beta_1-1)} - \frac{c_0^2 c_2 m_0^2 (\frac{\delta}{\tilde{r}})^{-2\alpha\beta_2}}{(\alpha^2+\alpha\beta_2-1)(\alpha^2+2\alpha\beta_2+1)}, & \alpha \neq 1 \\ 1 + \log(\delta/\tilde{r}) - \frac{m}{\tilde{r}^2} - \frac{c_0 c_1 m_0^2 \tilde{r}(\frac{\delta}{\tilde{r}})^{-2\beta_1}}{4\beta_1^2+8\beta_1+3} - \frac{c_0^2 c_2 m_0^2 \delta^{-2\beta_2} \tilde{r}^{2\beta_2}}{2\beta_2^2+2\beta_2}, & \alpha = 1 \end{cases} \quad (A11)$$

As shown in Refs. [37–41], the graviton mass does not significantly affect the form of the entropy. According to the well-known entropy-area law, the entropy as a pure geometrical quantity corresponding to the radius $\tilde{r}_h$ of black hole can be obtained as

$$S = \pi\tilde{r}^2 = \pi\delta^{\frac{2\alpha^2}{\alpha^2+1}} r_h^{\frac{2}{\alpha^2+1}}. \quad (A12)$$

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