# Gravitational bremsstrahlung in plasmas and clusters

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We study the gravitational bremsstrahlung owing to collisions mediated by a $1/r$ potential. We combine classical and first order Born approximation results in order to construct an approximate gravitational "Gaunt factor" for the total emitted energy. We also obtain the cross section with an angular momentum cutoff and hence the cross section for emission via close hyperbolic encounters in a gravitating cluster. These effects are the dominant source of very high frequency gravitational noise in the Solar System. The total gravitational wave power of the Sun is $76 \pm 20$ MW.

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## I. INTRODUCTION

This paper reviews and extends the study of gravitational bremsstrahlung during collisions in a $1/r$ potential. In practice this is Coulomb collisions and gravitational "collisions" (i.e. hyperbolic encounters) where the potential is well approximated as $1/r$. Such processes take place in plasmas such as stellar interiors, and in gravitating clusters such as those of black holes believed to be present in many galactic nuclei, or in the early Universe. However the motivation to study these processes is mainly their innate interest. They involve a combination of quantum theory and dynamic gravitation. For Coulomb collisions in the Sun the resulting gravitational wave amplitude is small and undetectable on Earth using any technology liable to be realiz in the near future, but in principle it contributes to the limits on coherence of matter-wave interferometry owing to gravitational field noise [1–4].

Introductory material is set out in the first two sections below. Section II surveys previous work on gravitational wave (GW) emission during collisions in a $1/r$ potential at low (nonrelativistic) speeds. Section III introduces notation and methods. Section IV obtains the total cross section for the GW energy emission after integrating over impact parameter. This consists in reporting existing work treating classical and quantum (first order Born approximation) limits, and providing approximate formulas for the intermediate regime. Section V considers emission during a single hyperbolic encounter. Section VI presents the cross section obtained if one imposes a cutoff on the angular momentum. This is useful for the case of attractive forces, where it makes sense to separate the collisions into those leading to capture and those where the bodies escape to infinity. Section VII obtains the GW energy emission cross section for close hyperbolic encounters in a gravitating cluster. Section VIII estimates the total GW power of the Sun. Section IX concludes.

## II. HISTORICAL SURVEY

Early work on graviton emission during scattering of fundamental particles was carried out by Ivanenko and Sokolov (1947, 1952) [5,6]. In 1965 Weinberg calculated gravitational bremsstrahlung during Coulomb collisions using quantum field theory, in the limit where the gravitons are "soft," meaning they have negligible impact on the energy-momentum in lines of Feynman diagrams on or near the mass shell [7]. The following year Carmeli confirmed this and also provided a classical calculation, for a repulsive potential, finding the total emitted energy after integration over impact parameters [30]. His clever method of calculation did not require an expression for the emitted energy in each hyperbolic encounter. Boccaletti (1972) extended this method to the Yukawa potential, and estimated emission from neutron stars [29]. Meanwhile Barker *et al.* 1969 gave the Born approximation calculation for graviton emission during collisions in a $1/r$ potential, among other results [8]. Emission from binary stars on Keplerian orbits had also been calculated, pioneered by Peters and Matthews (1963) [9,10].

The above all concern low velocities and Euclidean geometry. Pioneering calculations for the case of a Schwarzschild-Droste metric and arbitrary velocity were provided by Peters (1970) [11]. Since then there has been a very large body of work devoted to post-Newtonian

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corrections to orbits and scattering, with impressive recent progress, see for example the remarkable [12,13] and references therein. In the present survey we will not pursue the high-velocity or non-Newtonian cases. We restrict to $v \ll c$ and $r \gg R_S$ for participating massive objects (with $R_S$ the Schwarzschild radius) and the quadrupole approximation applies.

Gal'Tsov and Grats (1974) carried out Born approximation calculations, giving some further information not included in Barker *et al.* [14]. They subsequently (1983) extended their study towards a more complete kinetic theory of a plasma such as that of the Sun [15].

The first person to have correctly reported the total GW energy emitted during a hyperbolic encounter in a $1/r$ potential, according to classical (not quantum) physics, appears to be Turner (1977), correcting a minor error in a previous calculation by Hansen [16,17]. This work was duly noted in a comprehensive review by Kovacs and Thorne in 1978, who comment: "Such computations are straightforward and simple," but in view of the fact that errors exist in the literature (we will point out some more in the next paragraphs), such computations are clearly not straightforward for ordinary mortals [18].

Dehnen and Ghaboussi 1985 treated a general central potential and report some useful results for that case [19,20]. They apply their methods to the $1/r$ potential as an example and obtain the total scattered energy. Their formula agrees with that of Turner. They did not cite Turner, presumably an indication that they were not aware of his work. (Different authors report the formula in terms of different parameters so the agreement is not self-evident; we shall display both versions in Sec. V.)

Further reviews of astrophysical sources of gravitational waves are provided by [1,2,21]. Whereas Papini and Valluri discuss bremsstrahlung inside stars along with other processes, Cutler and Thorne do not because their review is focused on signals that may be detectable now or in the near future.

Recently a further case has gained interest: the emission from clusters of black holes which may have been produced in the early Universe or in the centers of galaxies [22–26]. The emission is partly from masses in bound orbits, and partly from a background of close hyperbolic encounters. In this work we are concerned with the latter, because it has received less attention in the literature and because it can, in principle, dominate, depending on the parameters of any given cluster. Capozziello *et al.* (2008) calculated the power and total emitted energy per encounter in the case $r \gg R_S$ where the gravitational potential is Newtonian to good approximation. Their results reproduce those of Turner and of Dehnen and Ghaboussi though they cite neither; they cite the review by Kovacs and Thorne which includes Turner but they do not make the comparison. De Vittori *et al.* (2012) follow the method of Capozziello explicitly but their Eq. (6) has a sign error in the last term and their Eq. (8) has the total power too large by a factor of 4. García-Bellido and Nesseris, and also Gröbner *at al.*, point out further mistakes. In view of these discrepancies a new calculation may be useful and we provide one.

The spectrum of the emitted radiation was treated by various authors, with noteworthy contributions from Turner, O'Leary *et al.*, De Vittori *et al.*, García-Bellido and Nesseris and Gröbner *at al.*. (Gröbner *et al.*'s opening statement that De Vittori *et al.* constitutes "the first calculation of the frequency spectrum" understates the contribution of Turner who gave explicit formulas for the cases of eccentricity $e = 1$ and $e \gg 1$ and much of the analysis for general $e$; subsequent authors completed the Fourier integrals for all $e$). Some mistakes in [23] are corrected in [25,27].

The existing studies for electrical plasmas and those for gravitating clusters appear to be unaware of one another although they are often calculating the same things (i.e. emission during scattering in a $1/r$ potential). The present work does the following: (i) bring together the two communities just outlined; (ii) present the work of Galt'sov and Grats afresh; (iii) estimate the case, intermediate between classical and quantum, which is not amenable to classical nor Born approximations, obtaining an approximate "Gaunt factor" for the total emitted power; (iv) obtain an emission cross section by using an angular momentum cutoff; (v) show how the above can be applied to calculate the emission from gravitating clusters and from a stellar plasma.

## III. NOTATION AND GENERAL APPROACH

For two colliding partners of masses $m_1$, $m_2$ we define the total mass $M = m_1 + m_2$ and the reduced mass $\mu = m_1 m_2/M$. We shall also use the unadorned $m$ (with no subscript) for reduced mass; thus $m \equiv \mu$. A given binary collision is described in the center of mass (COM) frame, such that it consists in a particle of mass $\mu$ moving in a fixed central potential of the form either $V(r) = Z_1 Z_2 e^2/r$ or $V(r) = -G m_1 m_2/r$. It is only necessary to treat one of these two cases since the other is then given by making the replacement $Z_1 Z_2 e^2 \leftrightarrow -G m_1 m_2$. In the following we mostly present the former (Coulomb scattering) since it includes both attractive and repulsive collisions, and also preserves in the notation the distinction between the potential and the role of $G$ in the GW emission process. For a slightly more succinct notation we define $e_1 e_2 \equiv Z_1 Z_2 e^2$. We adopt electromagnetic units such that the Coulomb force between electrons is $e^2/r^2$ and the fine structure constant is $\alpha = e^2/\hbar c$.

For a collision with the masses initially far apart, $v_0$ is the initial velocity and $b$ is the impact parameter. The collision energy is $E = (1/2)\mu v_0^2$ and angular momentum $L = \mu b v_0$.

If a flux $n_2 v$ is incident on a single collision center, then the rate of collisions is $n_2 v \sigma$ where $\sigma$ is the cross section (this defines $\sigma$). If there is a density $n_1$ of collision centers, then the collision rate per unit volume is $n_1 n_2 v\sigma$ if the particle types 1 and 2 are distinct, and it is $(1/2)n_1^2 v\sigma$ if the particle types are not distinct. In this paper we shall write $n_1 n_2 v\sigma$ and expect the reader to understand that in the case of identical particles the factor half must be introduced.

Our discussion is entirely nonrelativistic. This is a good approximation for the core of the Sun, where the Lorentz factor $\gamma \simeq 1.004$ for electrons.

Gravitational bremsstrahlung has some features in common with electromagnetic bremsstrahlung, which has been studied extensively. For the latter, the emitted power per photon solid angle and frequency range is written as a product of an approximate classical expression and a factor $g_{\rm ff}$ called the "free-free *Gaunt factor*" which incorporates quantum and other corrections. Complicated expressions exist for $g_{\rm ff}$ but for many purposes it is useful to have a simpler formula of reasonable accuracy. For the electromagnetic case this has recently been provided by Weinberg [28].

For an approximate classical calculation, one way to proceed is to integrate the emitted power at each moment for a particle moving on the trajectory it would follow if no radiation were emitted. For GW emission this approximation holds very well for particle collisions and we shall adopt it.

Whether in the electromagnetic or GW case, there are two significant energy scales in the collision dynamics: the kinetic energy and the potential energy at a distance of order a de-Broglie wavelength. The former is $(1/2)mv^2$ where $v$ can be taken as the speed at infinity for a repulsive potential, or as the speed at the distance of closest approach for an attractive potential. For low angular momentum the speed and acceleration have very different behaviors for attractive and repulsive cases, leading to different GW emission even though the differential cross section of the collision may be independent of the sign of the potential.

For Coulomb collisions between particles of charges $Z_1 e$, $Z_2 e$ we define the dimensionless parameter $n_{\rm B}$ called the *Born parameter* by Galt'sov and Grats (and called $\xi$ by Weinberg [28]):

$$n_{\rm B} \equiv \frac{|Z_1 Z_2 e^2|}{\hbar v} = |Z_1 Z_2| \alpha \frac{c}{v}. \tag{1}$$

The Born parameter can be read as a statement either about energy or about angular momentum. It is the ratio of the Coulomb energy at $2\lambdabar_{\rm dB}$ to the collision energy. It is also approximately equal to the angular momentum in units of $\hbar$. For a repulsive potential the distance at closest approach is $2n_{\rm B}\lambdabar_{\rm dB}$ according to classical mechanics. The case $n_{\rm B} \lesssim 1$ is the quantum limit; the Born approximation for the scattering holds when $n_{\rm B} \ll 1$. The case $n_{\rm B} \gg 1$ is the classical limit. Thus low temperatures give classical trajectories. The ground state of hydrogen has $n_{\rm B} \approx 1$.

A further relevant energy is that of the emitted photons or gravitons, $h\nu$. We say the photons or gravitons are "soft" when $h\nu \ll (1/2)mv^2$ and "hard" otherwise. The maximum possible emitted photon or graviton energy is equal to the entire kinetic energy $(1/2)mv^2$. More generally if a single photon or graviton is emitted then the initial and final momenta of the scattered particle (e.g. electron) in the COM frame are related by

$$\frac{p_i^2}{2m} - \frac{p_f^2}{2m} = h\nu. \tag{2}$$

The collision process itself has a timescale $\tau \approx r_0/v$ where $r_0$ is the distance of closest approach. Classical mechanics predicts that the emitted spectral power extends up to the angular frequency range near $1/\tau$, but quantum mechanics gives a hard cutoff at $\omega = (1/2)mv^2/\hbar$. The question arises, then, whether the classically "preferred" frequency is available. The condition that $1/\tau$ is less than the cutoff is $2\hbar < mvr_0$, i.e. $n_{\rm B} > 1$.

A further consideration in the gravitational case is whether or not the $1/r$ potential is a good approximation. This can be expressed by the condition $r \gg R_{\rm S}$ already mentioned. We will assume this condition holds, and we show at the end [Eqs. (56)–(58)] that for the orbits under study (namely, noncaptured hyperbolic orbits) this can be subsumed under the condition $v_{\rm max} \ll c$. The reason why this is sufficient, without a further constraint on the impact parameter, is that a constraint is placed implicitly by restricting attention to orbits which do not undergo radiative capture. In other words, slow encounters can only avoid capture by remaining at large distances. The question, whether the methods offer a good approximation in any given case, then becomes a matter of practical astronomy concerning the velocity and position distributions in any given cluster.

### A. Methods of calculation

In the compact source approximation in linearized gravity, the luminosity (i.e. the emitted power) of a source is given by

$$L_{\rm GW} = \frac{G}{5c^5} \langle \dddot{Q}_{ij} \dddot{Q}^{ij} \rangle, \tag{3}$$

where

$$Q^{ij} = \frac{1}{c^2} \int T^{00} \left( x^i x^j - \frac{1}{3}\delta_{ij} x^k x_k \right) {\rm d}^3\mathbf{x} \tag{4}$$

is the quadrupole moment of the mass distribution and the angle bracket indicates an average over a small region of spacetime.

For given collision partners, a collision is parametrized by two quantities: the initial velocity $v_0$ and the impact parameter $b$. We can express the total power generated in some small volume $V$ of a plasma, as a result of collisions between particles of types 1 and 2, as

$$P = V n_1 n_2 \langle v_0 \Sigma \rangle, \tag{5}$$

where $n_1$ and $n_2$ are number densities of two species [$n_1 n_2$ should be replaced by $(1/2)n_1^2$ if the species are identical, as already remarked] and $\Sigma$ is a cross section (to be calculated) with the physical dimensions of energy times area.

We shall obtain $\Sigma$ by calculating the total GW energy emitted during a single collision, integrated over impact parameter $b$. We adopt and compare four methods of calculation, as follows.

(1) Purely classical. We calculate that trajectory in the COM frame. The total emitted energy is $\int L_{\rm GW} {\rm d}t$ per collision, with $\dddot{Q}_{ij}$ obtained from the trajectory. The GW emission cross section is

$$\Sigma = \int_{-\infty}^{\infty} {\rm d}t \int_0^{\infty} 2\pi b {\rm d}b L_{\rm GW}. \tag{6}$$

By exploiting the symmetry of the inward and outward motion this can be written

$$\Sigma = 2\int_{r_0}^{\infty} \frac{{\rm d}r}{|\dot{r}|} \int_0^{b_{\rm max}} 2\pi b {\rm d}b L_{\rm GW}, \tag{7}$$

where $r_0$ is the smallest distance of closest approach and $b_{\rm max}$ is the largest impact parameter whose associated trajectory can reach a given $r$; see Fig. 1 for an elucidation of this.

(2) Born approximation. For a quantum treatment in the Born approximation we shall present results obtained by Barker *et al.* and by Gal'tsov and Grats (GG) [8,14].

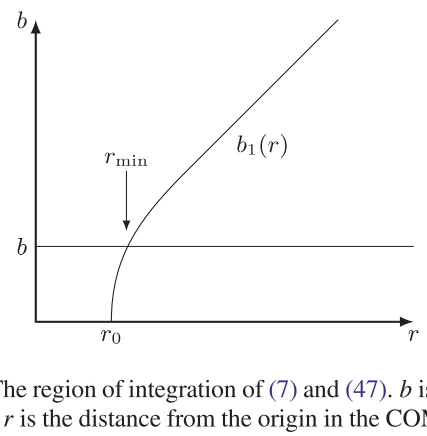


FIG. 1. The region of integration of (7) and (47). $b$ is the impact parameter, $r$ is the distance from the origin in the COM frame. At any given impact parameter $b$, the trajectory does not reach values of $r$ below $r_{\rm min}$ and therefore at any given $r$ it does not reach values of $b$ above $b_1$.

(3) Soft photon theorem. Weinberg has obtained a very general expression for the emission of soft massless particles in any collision. In the nonrelativistic limit his "soft photon theorem" applied to gravitons yields an expression for the power in the radiated spectrum up to frequency $\Lambda/\hbar$:

$$P_{<\Lambda} \simeq V \frac{8G}{5\pi c^5} m^2 v^5 n_1 n_2 \frac{\Lambda}{\hbar} \int \frac{{\rm d}\sigma}{{\rm d}\Omega} \sin^2\theta {\rm d}\Omega, \tag{8}$$

where $\Lambda$ is an energy cutoff which has to be taken low enough so that it is small compared to relevant kinetic energies in the problem, and ${\rm d}\sigma/{\rm d}\Omega$ is the differential cross section for the collision in the absence of radiant emission. The term "soft" here means the graviton's energy momentum is small compared to that of the particle emitting it.

Weinberg's formula does not give the whole emitted power, only the part owing to soft gravitons, and only that part up the frequency cutoff $\Lambda/\hbar$. Therefore we should not expect it to match calculations of the whole power. Nonetheless it offers a useful consistency check. Expressed as a cross section we have

$$\Sigma_{<\Lambda} \simeq \frac{8G}{5\pi c^5} m^2 v^4 \frac{\Lambda}{\hbar} \int \frac{{\rm d}\sigma}{{\rm d}\Omega} \sin^2\theta {\rm d}\Omega. \tag{9}$$

(4) Modified classical. With a view to gaining intuition about the quantum limit, and to obtain formulas which are approximately valid for any initial velocity, we explore the effect of modifying the classical formula (7). This is not a modification to the equation of motion; it is merely a rough method to gain reasonable insight and approximate formulas. The idea is that the quantum behavior can be modeled roughly by using a classical mass distribution with mass density equal to $m|\psi|^2$ where $\psi$ is a wave function in position space, and we suppose this distribution has a peaked (e.g. Gaussian) form with a standard deviation to be discovered and a mean which follows the classical trajectory. We then suppose that, to sufficient approximation, the result of such a model can be estimated by some simple adjustment to the integrand in (7).

One idea, for example, is to replace $r$ in the integrand of (7) with some simple function such as $(r^2 + \Delta^2)^{1/2}$, where $\Delta$ is a parameter to be set so as to reproduce the known behavior in the limits of small and large Born parameter. One would expect this $\Delta$ to be of the order of the de Broglie wavelength. This was explored, and so were other possibilities. In particular, one might leave the integrand unchanged and simply adjust the lower limit of the integral. It was found that this gives a good approximation. This is presented in Secs. IV E, VI.

## IV. TOTAL EMISSION CROSS SECTION

### A. Order-of-magnitude estimate

In order to get some general insight into the results to be discussed, we first present a simple order-of-magnitude estimate of GW radiation during repulsive Coulomb collisions.

From (3) we have

$$L_{\rm GW} \approx \frac{G}{5c^5}\left(\frac{\overline{Mx^2}}{\tau^3}\right)^2 \approx \frac{4G}{5c^5}\left(\frac{E_Q}{\tau}\right)^2, \qquad (10)$$

where $\tau$ is the timescale and $E_Q$ is the part of the kinetic energy associated with nonspherical (i.e. quadrupolar) movements. The timescale of the changing quadrupole moment is $\tau \simeq r_0/v$, where $r_0$ is a characteristic distance scale for a collision at energy $E$ and $v$ is the relative speed of the colliding partners. We take $r_0$ equal to the distance of closest approach in a head-on collision,

$$r_0 = e_1 e_2/E = \frac{2e_1 e_2}{\mu v_0^2}. \qquad (11)$$

The duration of each collision is about $2\tau$ so the emitted energy per collision is $(8G/5c^5)E^2/\tau$. Multiplying this by the collision rate $n_2\sigma v$ and the number density $n_1$, and using $\sigma = 4\pi r_0^2$, we obtain the power per unit volume of the gravitational wave production:

$$\frac{P}{V} \approx n_1 n_2 e_1 e_2 \frac{64\pi G}{5c^5}\frac{E^2}{\mu}. \qquad (12)$$

Equation (12) is compared with the result of a precise calculation in the next section. We there find that it captures correctly the scaling with parameters of the classical result for a repulsive potential, and gets the numerical factor about right.

### B. Classical treatment

We treat the two-body dynamics as a single-body motion of a particle of mass $\mu$ moving in a static potential centered on the origin. Let $D_{ij} \equiv 3Q_{ij}$, then $D_{ik} = \mu(3x_i x_k - x^j x_j \delta_{ik})$ and

$$\dddot{D}_{ik} = 6\mu v_i v_k - 6\frac{dV}{dr}\frac{1}{r}x_i x_k - 2\left[\mu v_j v^j - \frac{dV}{dr}\frac{1}{r}x^j x_j\right]\delta_{ik}.$$

The calculation of $\dddot{D}_{ik}\dddot{D}^{ik}$ is straightforward and the result is given by Boccaletti [29]. For Coulomb collisions one finds

$$L_{\rm GW} = \frac{8G}{15c^5}\frac{(e_1 e_2)^2}{r^4}\left(v^2 + 11v_\perp^2\right), \qquad (13)$$

where $v_\perp^2 = v^2 - \dot{r}^2$ and in this expression $v$, $v_\perp$ and $r$ are all functions of time.

The case of gravitational scattering can be treated by the replacement $e_1 e_2 \to -Gm_1 m_2$.

The potential is

$$V(r) = e_1 e_2/r, \qquad (14)$$

which may be positive or negative, depending on the signs of the charges. In the case of a repulsive force (potential hill), $r_0$ is a positive number equal to the minimum distance attained in a head-on collision. In the case of an attractive force (potential well) $r_0$ has no such interpretation but we retain the formula (11) as a definition, and then $r_0 < 0$.

From conservation of energy and angular momentum we have

$$v^2 = v_0^2(1 - r_0/r), \qquad v_\perp = v_0 b/r, \qquad (15)$$

where $v_0$ is the initial velocity and $b$ is the impact parameter. Hence

$$\dot{r} = v_0\sqrt{1 - r_0/r - b^2/r^2}. \qquad (16)$$

Using (7) and the above definitions, we have

$$\Sigma = \frac{32\pi G}{15c^5} e_1^2 e_2^2 v_0 \int_{r_{\rm min}}^{\infty} dr \int_0^{b_1} db \frac{(1 - r_0/r) + 11b^2/r^2}{r^4\sqrt{(1 - r_0/r) - b^2/r^2}} b, \qquad (17)$$

where $b_1 = \sqrt{r^2 - rr_0}$. Taking the integration with respect to $b$ first, we have that, for constants $a$, $B$, $C$, $d$,

$$\int \frac{C + db^2}{\sqrt{a - Bb^2}} b db = -\frac{\sqrt{a - Bb^2}}{B}\left[C + \frac{2ad}{3B} + \frac{db^2}{3}\right]. \qquad (18)$$

Therefore

$$\Sigma = \frac{64\pi G}{9c^5}\frac{(e_1 e_2)^2 v_0}{|r_0|}\chi, \qquad (19)$$

where

$$\chi = \frac{5|r_0|}{2}\int_{r_{\rm min}}^{\infty}\frac{1}{r^2}\left(1 - \frac{r_0}{r}\right)^{3/2} dr, \qquad (20)$$

$$= \frac{5}{2}\int_{x_{\rm min}}^{\infty}\frac{1}{x^2}\left(1 \pm \frac{1}{x}\right)^{3/2} dx, \qquad (21)$$

where the plus (minus) sign corresponds to an attractive (repulsive) potential. The lower limit on the integral with respect to $r$ is the smallest $r$ attained in the motion. This is zero for an attractive collision and $r_0$ for a repulsive one. It follows that $x_{\min} = 0$ for an attractive collision and $x_{\min} = 1$ for a repulsive one. Consequently $\chi$ diverges for an attractive collision and one obtains $\chi = 1$ for a repulsive collision. Hence the classical calculation (with no adjustment for quantum effects) yields a divergent result for an attractive collision (owing to infinite acceleration in a head-on collision), and for a repulsive collision yields

$$\Sigma_{\rm r} = \frac{32\pi G}{9c^5} Z_1 Z_2 e^2 m v^3, \tag{22}$$

where we now use $v$ to indicate $v_0$ which makes a comparison with other results more transparent. This is the equation first obtained by Carmeli [[30], Eq. (4.4)]. When substituted into (5) it confirms our rough estimate (12).

### C. Quantum treatment

We now review results of quantum scattering theory for this problem, obtained by previous authors. Both Barker *et al.* and GG treat the Born approximation and give some higher-order results. We shall present the Born approximation and some further observations by GG.

Equation (8) of GG is the same as Eq. (10) of Barker *et al.* after the replacement $(GMm/\hbar c) \to (e^2/\hbar c)$. [This replacement is the one Barker *et al.* point out after their Eq. (15), except that they adopt rationalized electromagnetic units.] In our units, Barker *et al.*, and also GG, find that the contribution to $\Sigma$ of the graviton frequency range $d\omega$, in the case of Coulomb scattering, is

$$d\Sigma = \frac{64 G\hbar}{15c^3}\left(\frac{e_1 e_2}{\hbar c}\right)^2 \left[5x + \frac{3}{2}(1+x^2)\ln\frac{1+x}{1-x}\right]\hbar d\omega, \tag{23}$$

where $x = p'/p$ is the ratio of final to initial momentum of a particle scattering off a fixed potential. For single graviton emission (i.e. Born approximation) we have, by conservation of energy, $\hbar\omega = (p^2 - p'^2)/2m = (1-x^2)p^2/2m$, so $\hbar d\omega = -xp^2/m$. When $\omega$ ranges from 0 to the hard cutoff, $x$ ranges from 1 to 0, so

$$\Sigma = \frac{64G}{15c^5\hbar}(e_1e_2)^2 \frac{p^2}{m}\int_0^1 5x^2 + \frac{3}{2}x(1+x^2)\ln\frac{1+x}{1-x}dx, \\ = (160G/9\hbar c^5)(e_1e_2)^2 m v^2. \tag{24}$$

However one should keep in mind that the Born approximation is only valid when $n_{\rm B} \ll 1$ for both the initial and final momenta. At the hard end of the spectrum $p' \to 0$ so $n_{\rm B} \to \infty$. Therefore the above formula has to be corrected at the hard end. This is the region where $x \to 0$. GG obtain

$$d\Sigma \to \pm \frac{1024\pi G}{15c^5}(e_1e_2)^2 \frac{\tilde{\alpha}c}{v}\frac{d\omega}{(e^{\pm 2\pi\tilde{\alpha}c/xv} - 1)}, \tag{25}$$

where the $+$ sign is for repulsion and the $-$ sign is for attraction, and $\tilde{\alpha} \equiv Z_1 Z_2 \alpha$. Since $xv$ is the final speed, the corrected formula should match the uncorrected one when the final Born parameter $\alpha c/xv \ll 1$, as indeed it does. But at the hard end, $x \to 0$, the spectrum is different in the two cases:

$$d\Sigma \to \frac{1024\pi G}{15c^5}(e_1e_2)^2\frac{\tilde{\alpha}c}{v}d\omega \begin{cases} e^{-2\pi\tilde{\alpha}c/xv} & \text{repulsion} \\ 1 & \text{attraction} \end{cases}. \tag{26}$$

It follows that (24) overestimates the power in the repulsive case and underestimates it in the attractive case; cf. Fig. 2. Note also that $d\Sigma$ scales as $(Z_1Z_2)^3$.

The above Born approximation results apply when $n_{\rm B} \ll 1$. Closed formulas are also available in the other limit, $n_{\rm B} \gg 1$. For repulsion one then regains the classical

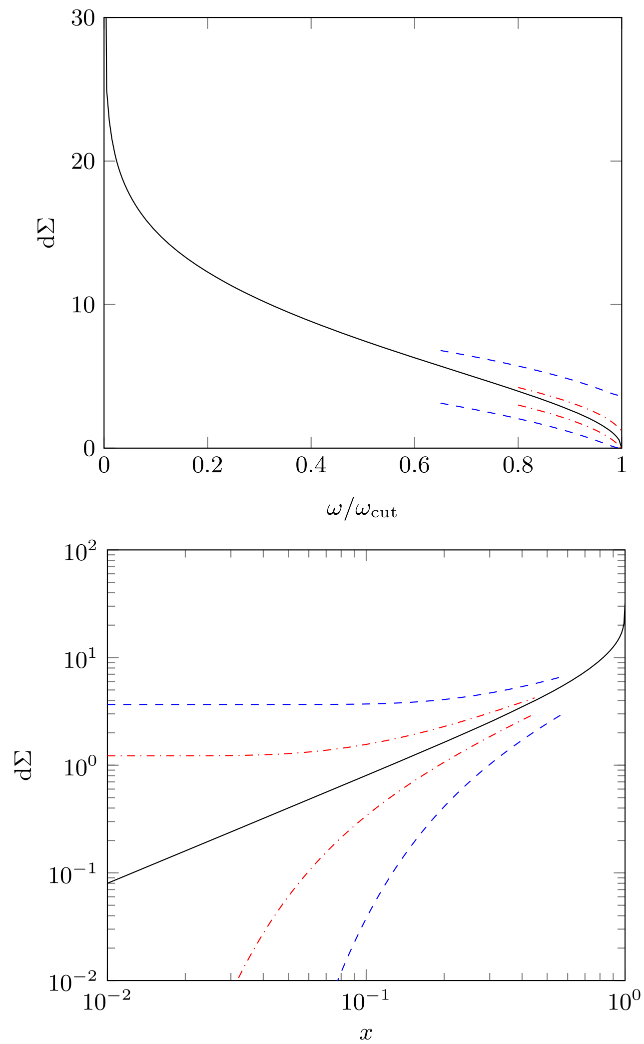


FIG. 2. Spectrum of GW emission in a Coulomb collision in the first order Born approximation for the collision ($n_{\rm B} \ll 1$), as given by (23). The dashed lines show the corrected spectrum near the hard end, Eq. (25). Blue dashed: $v = 0.1c$, red dash-dot: $v = 0.3c$.

result (22). For attraction the classical result (with no angular momentum cutoff) diverges; the quantum treatment derived by GG [their Eq. (17)] gives

$$\Sigma_{\rm a} = \frac{8G}{5c^5} 12^{1/3}\Gamma^2(2/3) Z_1 Z_2 e^2 m v^{4/3} (\tilde{\alpha} c)^{5/3}, \quad (27)$$

where the subscript "a" stands for "attractive."

In order to compare the various results, let us define in each case

$$\chi \equiv \Sigma/\Sigma_{\rm r}, \quad (28)$$

where $\Sigma_{\rm r}$ is given by (22). From (24) one obtains

$$\chi_{\rm B} \equiv \frac{\Sigma}{\Sigma_{\rm r}} = \frac{9}{2\pi} n_{\rm B}. \quad (29)$$

Thus quantum effects here act to suppress the power by a factor $9n_{\rm B}/2\pi$ compared to what would be expected classically.

Comparing now attraction and repulsion in the low-velocity limit, we have

$$\chi_{\rm a} \equiv \frac{\Sigma_{\rm a}}{\Sigma_{\rm r}} \simeq 0.6013(\tilde{\alpha} c/v)^{5/3} = 0.6013 n_{\rm B}^{5/3}. \quad (30)$$

The power in the attractive case greatly exceeds that in the repulsive case for low $v$. This is because the relevant speed for the attractive case is not the incident speed but the speed at closest approach. For a classical trajectory at angular momentum $L$, the speed at closest approach is approximately $n_{\rm B} v\hbar/L = \tilde{\alpha} c\hbar/L$ in the limit $n_{\rm B} \gg L/\hbar$. The scaling $v^{4/3}$ exhibited in (27) can be interpreted as the cube of a velocity which makes a compromise (roughly a geometric mean) between $v$ and $n_{\rm B} v$.

The predictions of (22), (29) and (30) are plotted as dashed lines on Fig. 3.

### D. Soft photon theorem

The soft photon theorem has to be applied with caution in the case of Coulomb collisions owing to the divergence of the collision cross section term in (9). That is, the quantity

$$\tilde{\sigma} \equiv \int \frac{{\rm d}\sigma}{{\rm d}\Omega} \sin^2\theta {\rm d}\Omega \quad (31)$$

diverges. Therefore the approximations invoked in the theorem do not hold in the case of the Coulomb potential. The problem is the long-range nature of $1/r$; similar difficulties arising in other scattering problems associated with this potential. In practice in a plasma there will be Debye screening which leads to a Yukawa potential. One then finds that $\tilde{\sigma} \sim v^{-4} \ln v$ in the limit of small screening.

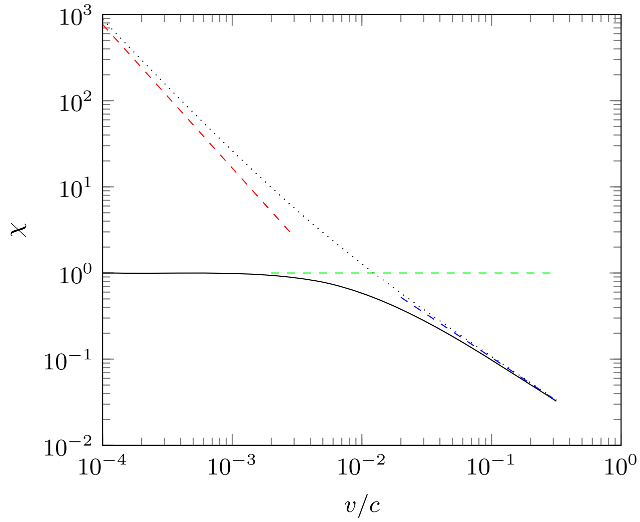


FIG. 3. Predictions for GW radiation in Coulomb collisions. The dashed lines show the limiting cases as described by (22) and (29) (low $v$) and (30) (high $v$). The full (dotted) line shows the predictions of the modified classical method described in Sec. IV E [Eqs. (33), (34)]. The horizontal axis is $\tilde{\alpha}/n_{\rm B}$; this is equal to $v/c$ in the case of electron collisions.

The soft photon/graviton theorem does not give the whole emitted power and one only expects order-of-magnitude agreement with the full $\Sigma$ in general. However by judicious choice of the cutoff $\Lambda$ one may expect to reproduce the full $\Sigma$ to within a factor of 2 for the repulsive case. For Coulomb collisions there are two relevant frequency scales: the inverse of the collision time $|r_0|/v$ (where $v$ is the maximum speed), and the hard cutoff at $K/\hbar$ where $K = (1/2)mv^2$. If we use as $\Lambda$ the smaller of $\hbar v_0/r_0$ and $K$, and take

$$\tilde{\sigma} \simeq 32\pi\alpha^2(\hbar/\mu c)^2(c/v)^4, \quad (32)$$

then the behavior shown in Fig. 3 for repulsive collisions is reproduced by the formula (9) in both limits of low and high $v_0$.

For attractive collisions the soft photon theorem is less successful, but gives a good estimate at high $v_0$ (low Born parameter).

### E. Modified classical

As noted in Sec. III, our modified classical method of calculation is an adjustment of the classical integrals so as to yield a reasonable approximation. We consider the effect of adjusting $x_{\rm min}$ in (21), giving

$$\chi_r(\lambda) = \frac{5}{2}\int_{1+\lambda}^{\infty} \frac{{\rm d}x}{x^2}\left[1 - \frac{1}{x}\right]^{3/2} = 1 - \left[1 + \frac{1}{\lambda}\right]^{-5/2}, \quad (33)$$

$$\chi_a(\lambda) = \frac{5}{2}\int_{\lambda}^{\infty} \frac{{\rm d}x}{x^2}\left[1 + \frac{1}{x}\right]^{3/2} = -1 + \left[1 + \frac{1}{\lambda}\right]^{5/2}, \quad (34)$$

where $\lambda$ is a parameter which one would expect to be of the order of the de Broglie wavelength divided by $|r_0|$.

Defining $\lambda_{\rm dB} \equiv 2\pi\hbar/\mu v_0$, one finds $\lambda_{\rm dB}/|r_0| = \pi/n_{\rm B}$. By setting the parameter value

$$\lambda = 0.5515\pi/n_{\rm B} \tag{35}$$

one finds that (33) reproduces the known results in both classical and quantum limits, and gives reasonable behavior at all $v < c$, see Fig. 3.

For attractive collisions the distance scale where quantum effects must be allowed for is not simply $|r_0|$ but may be considerably smaller. By solving for $r$ the equation $r = h/\mu v$ with $v = v_0(1 + |r_0|/r)^{1/2}$ one finds $r \simeq \pi\lambda_{\rm C}/\alpha$ where $\lambda_{\rm C} = h/\mu c$ is the Compton wavelength. We mention this merely to indicate that the attractive case is less straightforward. We shall choose the parameter $\lambda$ so as to match (30) in the low-velocity limit and (29) in the high-velocity limit. We also have a further piece of information provided by (26), namely that $\chi$ approaches the asymptote from above at small Born parameter in the attractive case. These constraints are achieved by adopting (for example)

$$\lambda = (5.20 + 1.84 n_{\rm B})^{1/3}/n_{\rm B}. \tag{36}$$

The result is shown in Fig. 3.

Equations (33)–(36) together provide a formula for $\chi$ which is approximately valid at all collision speeds $v$. This $\chi$ is the "Gaunt factor" for the total (i.e. integrated over frequency) emission. It allows one to obtain $\Sigma$ by taking $\Sigma_{\rm r}$ given by (22) and multiplying by a correction factor.

The task of calculating gravitational scattering amplitudes and other observables in full is presented in Kosower *at al.* [31]. They offer at once an extensive review, a tutorial and original contributions. There is a procedure to systematize the approximations which allows the classical limit to be taken. Such a calculation remains lengthy. The crude-but-reasonably-accurate method presented here serves partly as a simple formula to apply to study of plasmas and clusters (cf. Sec. VIII), and partly to facilitate the checking of more advanced methods.

## V. POWER AND ENERGY FOR A GIVEN SCATTERING EVENT

So far we have not treated the motion during individual scattering events, because it was convenient to integrate over impact parameter. We now treat individual events of given $b$, $v_0$. We shall present the gravitational (Keplerian), i.e. attractive, case.

The orbit can be described by the parameters $b$, $v_0$ or by a number of other pairs, including $E$, $L$ (energy and angular momentum, both conserved) and $a$, $e$ where $a \equiv GM/v_0^2 = -r_0/2$ and $e$ is the eccentricity defined by

$$e = \sqrt{1 + b^2/a^2}. \tag{37}$$

For a hyperbolic orbit one then finds that the distance of closest approach is

$$r_{\rm min} = a(e-1) = b\sqrt{\frac{e-1}{e+1}} \tag{38}$$

and $e = -1/\cos\phi_0$, where $\phi_0$ is half the total change in azimuthal angle during the encounter (the deflection angle is $2\phi_0 - \pi$).

On a classical model under the adopted assumptions (i.e. motion in a $1/r$ potential), the GW power during the scattering process is given by (13), which, after using the conservation laws (15), gives an expression in terms of $r$ and constants. Turner gives the following formula [Eq. (24) of [17]]:

$$P = \frac{8G^4}{15c^5}\frac{M^3\mu^2(1+e\cos\phi)^4}{[(1+e)r_{\rm min}]^5}\left[e^2\sin^2\phi + 12(1+e\cos\phi)^2\right], \tag{39}$$

where $\phi$ is the azimuthal angle taken from $\phi = 0$ at periastron (the point where $r = r_{\rm min}$). Thus $\phi$ goes from $-\phi_0$ initially to $\phi_0$ finally.

Capozziello *et al.* give [Eq. (21) of [22]]

$$P = \frac{32GL^6\mu^2}{45c^5b^8}f(\phi_0, \psi), \tag{40}$$

where

$$f(\phi_0, \psi) = \frac{\sin^4(\phi_0 - \psi/2)\sin^4(\psi/2)}{\tan^2\phi_0\sin^6\phi_0} \times \left[150 + 72\cos 2\phi_0 + 66\cos 2(\phi_0 - \psi) - 144(\cos(2\phi_0 - \psi) + \cos\psi)\right]. \tag{41}$$

(This formula is quoted incorrectly in [23] where there is a sign error in the last term). Here $\psi \equiv \phi + \phi_0$ (thus $\psi$ goes from 0 initially to $2\phi_0$ finally). If we express $f$ in terms of $\phi$ rather than $\psi$, it simplifies a little:

$$f = \frac{3}{8}\frac{(\cos\phi_0 - \cos\phi)^4}{\tan^2\phi_0\sin^6\phi_0}\left[25 + 12\cos 2\phi_0 - 48\cos\phi_0\cos\phi + 11\cos 2\phi\right]. \tag{42}$$

Equations (13), (39) and (40) give three ways of expressing the same result. They are all equivalent, which one may confirm by employing $r = b\sin\phi_0/(\cos\phi - \cos\phi_0)$ (a standard result of orbital mechanics).

The integral of $P$ over time is conveniently done by converting to an integral over $\phi$. The result was first obtained by Turner:

$$\Delta E = \frac{8G^{7/2}}{15c^5} \frac{M^{1/2} m_1^2 m_2^2}{r_{\rm min}^{7/2}} g(e), \tag{43}$$

with

$$g(e) = \frac{\phi_0 \left(24 + 73e^2 + \frac{37}{4}e^4\right) + \frac{\sqrt{e^2-1}}{12}\left(602 + 673e^2\right)}{(1+e)^{7/2}} \tag{44}$$

(correcting an earlier calculation of Hansen). In order to bring out the comparison with (22), note that

$$\frac{8G^{7/2}}{15c^5} \frac{M^{1/2} m_1^2 m_2^2}{((e+1) r_{\rm min})^{7/2}} = \frac{8G}{15c^5} \frac{GM\mu^2 v_0^3}{b^2 (e^2-1)^{5/2}}. \tag{45}$$

Dehnen and Ghaboussi's result [Eq. (7) of [19]] is

$$\Delta E = \frac{8G(e_1 e_2)^2}{15c^5} \frac{\mu E^2}{L^3} \left[(37 + 366z^2 + 425z^4)\phi_0 + (673/3 + 425z^2) z\right], \tag{46}$$

where $z \equiv -\cot\phi_0 = (e^2-1)^{-1/2}$. This agrees with Turner after one makes the substitution $e_1 e_2 \to -G m_1 m_2$.

The total scattered energy was also obtained by Capozziello *et al.* Their expression is consistent with Turner's if one handles the term $\sqrt{e^2-1}$ correctly. It must be taken positive, which means it is equal to $-\tan\phi_0$ not $\tan\phi_0$ when $e > 1$. Also, [23] gives a result a factor of 4 larger than that of [22]. In view of these issues a further check is useful. We completed the calculation independently and agree with Turner (and therefore also Dehnen and Ghaboussi) and with Capozziello *et al.* as long as the correct sign is taken, as just noted.

## VI. CLASSICAL COLLISIONS WITH ANGULAR MOMENTUM CUTOFF

So far we have surveyed or confirmed existing work, and contributed a small extension in the modified classical method. The remainder of our discussion is mostly new.

Rather than taking the integral (7) over all impact parameters, we now place a lower limit on $b$. This will be useful for two purposes. First, the influence of quantum mechanics on collision cross sections can sometimes be estimated by imposing a low angular momentum cutoff, at a value of order $\hbar$, on a classical collision integral. Second, for attractive collisions the low angular momentum limit has to be considered separately in any case. This is because the approximation that the orbit is almost unaffected by the radiation breaks down.

In place of Eq. (7) we introduce

$$\Sigma(L, v_0) \equiv 2 \int_{r_{\rm min}}^{\infty} \frac{{\rm d}r}{|\dot{r}|} \int_{L/mv_0}^{b_1} 2\pi b {\rm d}b L_{\rm GW}, \tag{47}$$

where $L$ is the cutoff and the notation on the left-hand side is to indicate explicitly that the result is a function of the cutoff angular momentum $L$ as well as $v_0$. Then in (17) we replace the lower limit 0 on the $b$ integral by $b_0 = L/mv_0$, and $r_{\rm min}$ is given by (38) (and by (49)). After using (18) we obtain

$$\Sigma(L, v_0) = \frac{64\pi G}{9c^5} \frac{(e_1 e_2)^2 v_0}{|r_0|} \chi(L, v_0), \tag{48}$$

where

$$\chi = \frac{|r_0|}{10} \int_{r_{\rm min}}^{\infty} \frac{{\rm d}r}{r^2} \left[25\left(1 - \frac{r_0}{r}\right) + 11\frac{b_0^2}{r^2}\right] \left[1 - \frac{r_0}{r} - \frac{b_0^2}{r^2}\right]^{1/2}.$$

The lower limit on this integral is the smallest $r$ attained in the motion when the impact parameter is $b_0$. This is

$$r_{\rm min} = \frac{1}{2}\left(r_0 + \sqrt{r_0^2 + 4b_0^2}\right), \tag{49}$$

where the positive square root should be taken. (For $L = 0$ this gives $r_{\rm min} = r_0$ for a repulsive collision and $r_{\rm min} = 0$ for an attractive collision.) One finds

$$\chi(L, v_0) = \frac{1}{80|y^5|} \left[6(1+y^2)(85 + 37y^2)\left(\frac{\pi}{2} - \cot^{-1} y\right) - 510y - 562y^3\right], \tag{50}$$

where

$$y \equiv \frac{L v_0}{e_1 e_2} = \pm\sqrt{e^2-1}, \qquad |y| = \frac{L/\hbar}{n_{\rm B}}, \tag{51}$$

where the negative square root is taken for the attractive case. $\chi(L, v_0)$ is plotted as a function of $y$ in Fig. 4. It is remarkable that this $\chi$ is a function of eccentricity alone.

One finds

$$\chi(L, v_0) \to \begin{cases} 1 & y \ll 1, y > 0 \\ (51\pi/8) y^{-5} & |y| \ll 1, y < 0. \\ (111\pi/80) y^{-1} & |y| \gg 1 \end{cases} \tag{52}$$

Positive $y$ means the potential is repulsive. At small $y$ the result is then independent of $L$ and reproduces the classical calculation without any angular momentum cutoff. This is because at small initial velocities the particles do not approach closely in a repulsive potential. At large $y$ the

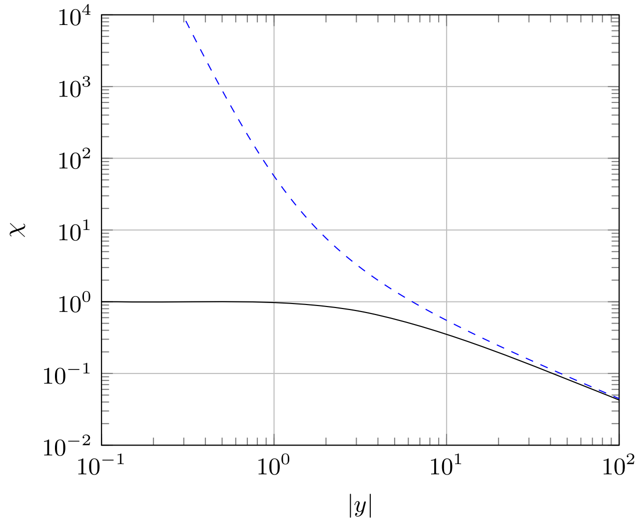


FIG. 4. $\chi(L, v_0)$ given by (50) for attractive (upper line, dashed) and repulsive (lower line, full) collisions.

result exactly reproduces the first order Born approximation (24) in the limit if we take

$$L = \frac{37\pi^2}{120}\hbar \simeq 3.043\hbar. \tag{53}$$

It follows that $\Sigma(3.04\hbar, v_0)$ can be taken as a reasonable approximation to the exact result (i.e. a quantum scattering calculation to all orders) for GW scattering during Coulomb collisions on a repulsive potential, for any collision energy in the nonrelativistic limit. In other words, *for repulsive Coulomb collisions the complete quantum scattering prediction (summed over all orders or Feynman diagrams) closely matches a classical prediction in which low angular momentum states do not contribute at all*. The phrase "closely matches" here signifies exact agreement in the limits of large or small $n_B$, and agreement at some unknown accuracy of order 10% in the case $n_B \sim 1$.

For an attractive potential $\Sigma(3.04\hbar, v_0)$ produces the correct cross section at high $|y|$ but not at low $|y|$. In other words, for an attractive Coulomb collision it is not sufficient merely to place a lower bound on the angular momentum in order to approximate the quantum physics of a collision at low energy.

## VII. GRAVITATING CLUSTERS

In Sec. IV we discussed the total emission cross section, integrating over all impact parameters. For emission from a plasma this is a useful quantity, but for gravitational scattering in general it is not because the approximations break down at low $L$ in an attractive potential. Various situations can arise. Astrophysical bodies are generally not pointlike and can crash into each other or otherwise merge. Also, even on a pointlike model there can be radiative capture. This happens when

$$\Delta E > \frac{1}{2}\mu v_0^2. \tag{54}$$

That is, the emitted energy is larger than the initial energy in the binary system, with the result that an initially unbound pair finishes in a bound state. In a bound state the pair subsequently follows an almost periodic, almost elliptical orbit, gradually losing more energy until the bodies coalesce.

In order to treat a gravitating cluster, we separate the scattering events into those where the bodies emerge to infinity, and those where there is gravitational capture owing to the gravitational radiation. We will employ the condition (54) to separate the two cases, which is valid at a low density of pairs but not at higher density where three-body effects tend to reduce the capture rate [32].

Using (43) on the left-hand side of (54) we find that the limiting case (where $\Delta E = E$) is given by

$$e - 1 = \left(\frac{16}{15}\frac{\mu}{M}\frac{v_0^5}{c^5}g(e)\right)^{2/7}. \tag{55}$$

This method of calculation is approximate since for such collisions the outgoing value of $e$ will not be equal to the initial value, but it gives a reasonable estimate. Equation (55) has $g(e)$ on the right-hand side so it is an implicit equation for $e$ with no analytical solution. But we observe that for $v_0 \ll c$ one has $e - 1 \ll 1$ as one would expect: $e = 1$ is the parabolic orbit where $E = 0$. In this case we can use $g(1)$ on the right-hand side, obtaining

$$e - 1 \simeq \left(\frac{85\pi}{6\sqrt{2}}\frac{\mu}{M}\frac{v_0^5}{c^5}\right)^{2/7}. \tag{56}$$

This agrees with Eq. (17) of [26]. Noncaptured orbits have $e - 1$ larger than this. We should now note two consistency checks. For the Newtonian potential to be valid we require $r_{\min} \gg R_S = 2GM/c^2$ (the Schwarzschild radius). This yields the condition

$$e - 1 \gg 2v_0^2/c^2. \tag{57}$$

This is comfortably satisfied by (56) for $v_0 \ll c$. Also for nonrelativistic mechanics we require $v_{\max} \ll c$. Conservation of angular momentum gives $r_{\min} v_{\max} = b v_0$ and one obtains

$$\frac{e - 1}{e + 1} \gg \frac{v_0^2}{c^2}. \tag{58}$$

Since $e + 1 > 2$ this is a stronger condition than the previous one, but still comfortably satisfied.

We have in (56) an expression for the minimum eccentricity, at any given $v_0$, for noncaptured orbits. Since $e - 1 \ll 1$ we can use $y \equiv -\sqrt{e^2 - 1} \simeq -\sqrt{2}(e - 1)^{1/2}$, and since this is small we can use the small $|y|$ limit of Eq. (50), giving

$$\chi(y) \simeq \frac{51\pi}{32\sqrt{2}}\left(\frac{6\sqrt{2}}{85\pi}\frac{M}{\mu}\frac{c^5}{v_0^5}\right)^{5/7}. \quad (59)$$

Hence the total cross section for emission of gravitational wave energy during hyperbolic (i.e. noncaptured) encounters, in a low-density, low-velocity gravitating cluster is

$$\Sigma = \frac{\pi}{5}\left(\frac{340\pi}{3\sqrt{2}}\right)^{2/7}\frac{GM}{c^2}Gm_1m_2\left(\frac{\mu}{M}\right)^{2/7}\left(\frac{c}{v}\right)^{4/7}. \quad (60)$$

As an example, consider information furnished by O'Leary *et al.* They remark, "20,000 BHs are expected to have segregated into the inner ~1 pc of the Milky Way" [26]. The number density distributions in their Fig. 1 give $n \simeq n_0(r_0/r)^2$ for $r_0 < r < 0.3$ pc, where $r$ is the distance from the center of the Galaxy, $n_0 \simeq 10^{10}$ pc$^{-3}$ and $r_0 = 3 \times 10^{-4}$ pc. They propose black holes in the mass range 5 to $15M_\odot$ and encounters with initial relative speeds of order $v \sim 1000$ km/s. Putting these values into (60) and (5) we obtain a total power from close hyperbolic encounters of black holes in the Galactic Center of order $10^{25}$ watt after averaging over times long enough for many encounters.

## VIII. THE GRAVITATIONAL RADIATION OF THE SUN

Consider now a plasma in thermal equilibrium at the density and temperature of the core of the Sun—cf. Table I. The thermal energy $k_B T_{\rm core} \simeq 1.35$ keV is about twice the Fermi energy of the electrons, and therefore the electron gas is nondegenerate to reasonable approximation. Each electron or proton has a kinetic energy of order $k_B T$ and the rms energy is approximately $E_Q \simeq 2k_B T$.

Gravitational bremsstrahlung in the Sun arises mainly from collisions among electrons, protons and $^4$He nuclei. We shall present the result of integrating the emission over the Sun, treating the collisions as Coulomb collisions. This ignores the effect of Debye screening and therefore cannot be taken as an accurate value for the actual situation. But the Debye screening is not expected to change the overall result by as much as an order of magnitude. Therefore a calculation using the unscreened potential is a useful indicator, and also serves to establish which regime of behavior (low or high Born parameter, attractive or repulsive collisions) dominates.

TABLE I. Some properties of the solar core. pm = picometre. $\lambda_{\rm th}$ is defined such that $n\lambda_{\rm th}^3$ is the onset of degeneracy. $\lambda\!\!\!^{-}_{\rm dB}$ is the distance over which a de Broglie wave acquires a phase of one radian, for a particle of energy $E = (3/2)k_B T_{\rm core}$.

| | | |
|---|---|---|
| $T_{\rm core}$ | $1.57 \times 10^7$ K | |
| $(3/2)k_B T_{\rm core}$ | 2.03 keV | |
| Coulomb distance $r_0$ | 0.7 pm | |
| Plasma wavelength $\lambda\!\!\!^{-}$ | 640 pm | |
| Debye (screening) length $\lambda_D$ | 23 pm | |
| | Electrons | Protons |
| Mean separation | 25 pm | 32 pm |
| $\lambda_{\rm th} = \hbar\sqrt{2\pi/mk_B T}$ | 18.8 pm | 0.43 pm |
| $\lambda\!\!\!^{-}_{\rm dB} = \hbar/\sqrt{2mE}$ | 4.3 pm | 0.10 pm |

TABLE II. Total GW power, in megawatts (MW), from the main types of Coulomb collision in the Sun.

| | e | p | $He^{++}$ |
|---|---|---|---|
| e | 26 | | |
| p | 29 | 0.096 | |
| $He^{++}$ | 21 | 0.048 | 0.004 |
| Total | 76 | 0.14 | 0.004 |

In the solar core we have $|n_B| \simeq 0.06$ for collisions involving electrons. It was remarked by GG that the emission is therefore substantially reduced below the value predicted by the classical calculation (22) (we find one order of magnitude below, not two as suggested by GG). We observe also that it is important to include the attractive (ep and eHe) collisions as well as the repulsive ones.

The total power is obtained by adding the contributions from the various types of collision, integrated over the temperature and density distribution of the Sun. In order to perform such an integral, we adopted the distributions given by the Standard Solar Model [33,34]. The result of the numerical integration is indicated in Table II. We find that the total power is 76 MW (in the absence of Debye screening). This is the first time this power has been calculated with better than order of magnitude accuracy. (The previous best estimate was that of GG who estimated the order of magnitude as 10 MW). It follows that the GW power of the Sun is $76 \pm 20$ MW, where the uncertainty is mostly owing to the as-yet-uncalculated impact of Debye screening.

It is noteworthy that ee, ep and eHe collisions make almost equal contributions. If it were not for the quantum effects, it would not be so. For if we simply set $\chi = 1$ for all the processes, then one finds the ee collisions dominate owing to their smaller reduced mass, leading to higher velocities. The value $\chi = 1$ also leads to a total power 10 times larger, indicating that the quantum effects are important for the conditions of the Sun. Note also that the increased emission for attractive, as compared with repulsive, collisions also raises the contribution of ep and eHe collisions a little, compared with ee.

From the above one may deduce that there is gravitational noise in the Sun with an rms strain amplitude of order $10^{-41}$ at $10^{18}$ Hz owing to Coulomb collisions. This is the dominant source of gravitational noise in the Solar System at this frequency. The energy density of this radiation arriving at Earth is of order $10^{-24}$ Wm$^{-3}$. This is similar to the energy density of relic gravitational waves in the frequency band up to GHz thought to be present owing to early Universe processes [4,35,36]. Owing to their lower frequency, the latter will have larger observable effects.

## IX. CONCLUSION

In conclusion, we have achieved the five aims set out at the end of Sec. II. We have reviewed studies of gravitational bremsstrahlung during Coulomb collisions and presented a formula, based on semiclassical physical reasoning, which is able to reproduce, approximately, the predictions of a full (i.e. quantum) treatment of the total emitted power at any value of the Born parameter, in the nonrelativistic limit. Equations (33)–(36) allow one to calculate the energy cross section with high accuracy in certain limits and with ∼10% accuracy in general. One can thus obtain the power averaged over many collisions in a homogeneous fluid. As an example, we have applied these equations to a treatment of the Sun, obtaining the total emitted power in the approximation where Debye screening is neglected.

Equation (50) [combined with (22)] gives the cross section for gravitational wave emission in the classical (high Born parameter) limit for collisions at a given initial velocity after integrating over impact parameters above a lower limit set by a given angular momentum. This has not previously been calculated. We have used it to obtain, in Eq. (60), the total cross section for emission of GW energy during close hyperbolic encounters where capture does not occur. This can be used to calculate, for example, the time-averaged emission from galactic nuclei by this process.

It has recently been suggested that black hole collisions in the early Universe made a non-negligible contribution to the stochastic gravitational background in the present. One may ask whether Coulomb collisions in the very early Universe made a further non-negligible contribution. I have attempted an estimate of this (unpublished); the estimate suggests that the contribution is negligible but it would be interesting nonetheless to look into this more fully.


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