Kinetic theory formulation of the *P*- and *CP*-odd terms in the photon self-energy in a medium

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(Received 20 January 2024; accepted 29 February 2024; published 26 March 2024)

In an optically active medium, such as a plasma that contains a neutrino background, the left-handed and right-handed polarization photon modes acquire different dispersion relations. We study the propagation of photons in such a medium, which is otherwise isotropic, within the framework of the covariant collissionless Boltzmann equation incorporating a term that parametrizes the optical activity. Using the linear response approximation, we obtain the formulas for the components of the photon polarization tensor, expressed in terms of integrals over the momentum distribution function of the background particles. The main result here is the formula for the *P*- and *CP*-breaking component of the photon polarization tensor in terms of the parameter involved in the new term we consider in the Boltzmann equation to describe the effects of optical activity. We discuss the results for some particular cases, such as long-wavelength and nonrelativistic limits, for illustrative purposes. We also discuss the generalizations of the *P*- and *CP*-breaking term we included in the Boltzmann equation. In particular we consider the application to a plasma with a neutrino background and establish contact with calculations of the photon self-energy in those systems in the framework of thermal field theory.

DOI: 10.1103/PhysRevD.109.056022

I. INTRODUCTION AND OUTLINE

It was shown sometime ago, in the context of *thermal field theory* (TFT), that the general expression for the photon self-energy in an isotropic medium, consistent with gauge and Lorentz invariance in four dimensions, may contain a term which signals *P*- and *CP*- symmetry breaking, either in the Lagrangian, or in the background, or both [1,2]. The effect of this term is that the dispersion relation of the photon transverse modes, which would otherwise be degenerate, are split according to the polarization of the propagating mode. A consequence of this is the rotation of the linear polarization of an electromagnetic wave traveling in such media, or birefringence. These effects arise, for example, when photons propagate in a medium that contains a neutrino background [3–5].

On the other hand it has also been shown that the effect mentioned can be described in the context of the classical Maxwell equations in terms of an additional electromagnetic *constant* ζ, besides the standard dielectric (ϵ) and magnetic (μ) constants [6]. A covariant version of this formulation has also been given [7]. Following Ref. [6] we refer to ζ as the *activity constant*, and to this kind of medium as an *optically active medium*.

Our objective in the present work is to revisit the study the propagation of photons in such media, in the framework of the covariant, collisionless, Boltzmann equation, but incorporating a term that gives rise to the optical activity effects mentioned above. This approach, which lies somewhere in the middle between the two approaches mentioned, that is, TFT on one hand, and a purely phenomenological description on the other, could be more suitable than those two approaches in certain situations.

Recently, the optical activity effects on the cosmic microwave photons as they travel through the medium of the cosmic neutrino background have been considered in the context of the cosmic birefringence [8]. Independently of whether or not the photon-neutrino interactions are responsible for the observed effects in this context, or whether another source of the optical activity may be required as suggested in Ref. [8], our work may be useful for further development in this area, and other astrophysical contexts in which optical activity plays a role.

While in the present work we restrict ourselves to the collisionless Boltzman equation, modified to incorporate the optical activity effects, the method we use can be extended to include the effect of collisions in an optically active plasma, for example by adapting the techniques used

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for this purpose in the framework of the Boltzmann equation for an ordinary plasma (see, e.g., Ref. [9] and references therein). For example, in the case of photons propagating in a plasma medium with a neutrino background, it may be important to include also the effects of collisional interactions. While such interactions typically lead to *damping effects*, under the appropriate circumstances can also lead to growth effects. An example of a growth effect was provided in Ref. [4], with regard to the evolution of a magnetic field perturbation in such a medium. This particular phenomena has been considered by Semikoz and Sokoloff [10] as a mechanism for the generation of large-scale magnetic fields in the early Universe as a consequence of the neutrino-plasma interactions. In this sense the present work provides the mechanism for considering the effects of collisions in an optically active medium on a firm ground.

In short, our proposal is to consider the relativistic collisionless Boltzmann equation for a given particle specie "a," of charge q_a,

$$p\cdot\partial^{(x)}f_a = -q_a F^{\mu\nu}p_\nu\partial^{(p)}_\mu f_a, \tag{1.1}$$

modified by adding a term proportional to the dual electromagnetic tensor $\tilde{F}_{\mu\nu}$, as we state precisely below, including the explanation of the various symbols that enter in this equation. The strategy is to use the linear response approach to obtain the expression for the induced current, and thereby the photon polarization tensor, in terms of the particle distribution functions, and study various aspects of the results that could be useful for applications, such as the corresponding dispersion relations and the interpretation in terms of the activity constant. In the present work we restrict ourselves to implement this program for an isotropic system, that is, the momentum distribution functions of all the particle species are isotropic in the rest frame of the system. The generalization of the approach to other cases, for example a two-stream component plasma is straightforward from a conceptual point of view, although of course the details will be different and they can be important in specific physical contexts.

The outline of the rest of the paper is as follows. In Sec. II we present the covariant Boltzmann equation, that includes the P- and CP-breaking term to describe the effects of an optically active medium. For consistency, we discuss some particular features and consequences of the equation, including current conservation and the role of the discrete symmetries of the new term. In Sec. III, we consider the solution of the equation, using the standard linearization method. The expression for the induced current is determined, in terms of integrals of the momentum distribution functions of the particles. There we establish contact with the photon polarization tensor, or equivalently the photon self-energy, in the TFT language, specifically as used in Ref. [1], and the formulas for the components of the polarization tensor are obtained. The results for the longitudinal and transverse components of the photon self-energy are the familiar ones. The formula for the P- and CP-breaking term is the new result here. In Sec. IV we discuss some details of the results obtained, and consider specifically some particular cases (e.g., the long-wavelength and the nonrelativistic limit) that are useful in many applications and can serve as benchmark references for more general situations. We also point out possible generalizations of the P- and CP-breaking term we included in the Boltzmann equation, in particular how it applies in the context of a plasma with a neutrino gas as a background, and establish contact with calculations of the photon self-energy in such backgrounds in the framework of TFT [3]. Possible avenues for extensions and exploration of the present work are mentioned in Sec. V.

II. *P*- AND *CP*-BREAKING KINETIC EQUATION

Without further preamble, the equation we consider is,

$$p\cdot\partial^{(x)}f_a = [-q_a F^{\mu\nu} + \gamma_a\tilde{F}^{\mu\nu}]p_\nu\partial^{(p)}_\mu f_a. \tag{2.1}$$

We will discuss some generalizations of this equation in Sec. IV B. In this expression, $f_a(x,p)$ is the number density of the particles specie "a," of charge q_a, in the plasma, expressed as a function of the four-vectors x^μ and p^μ, and normalized as specified below [e.g., see Eq. (2.10)]. Further, $\tilde{F}^{\mu\nu}$ is the dual electromagnetic field tensor,

$$\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}, \tag{2.2}$$

and we are using the shorthand symbols

$$\partial^{(x)}_\mu = \frac{\partial}{\partial x^\mu},$$
$$\partial^{(p)}_\mu = \frac{\partial}{\partial p^\mu}. \tag{2.3}$$

We use the conventions such that $g^{\mu\nu}$ has diagonal elements $(1,-1,-1,-1)$ and $\epsilon^{0123} = +1$.

The parameter γ_a is a phenomenological parameter, which in this approach is unknown. However Eq. (2.1), or its generalizations, as we will discuss, parametrize effectively the results of the calculations of optical activity in some systems, such as those considered in the references already cited (e.g., Refs. [3–5]). In this sense this approach serves as a bridge between those calculations on one hand, and a pure phenomenological description in terms of the electrodynamics equations on the other (e.g., Refs. [1,6,7]).

Before entering in the practical calculations, there are various aspects of these equations that are worth discussing. We consider them below.

A. Current conservation

The electromagnetic current density is

$$j_\mu = \sum_a q_a J_{a\mu}, \tag{2.4}$$

where

$$J_{a\mu} = 2\int \frac{d^4p}{(2\pi)^3}\delta(p^2 - m_a^2)\theta(p\cdot u)p_\mu f_a. \tag{2.5}$$

We have introduced the velocity four-vector of the medium, u^μ, which has components

$$u^\mu = (1, \vec{0}), \tag{2.6}$$

in the medium's own rest frame. Notice that

$$\delta(p^2 - m_a^2)\theta(p\cdot u) = \frac{1}{2E_{ap}}\delta(p^0 - E_{ap}), \tag{2.7}$$

with

$$E_{ap} = \sqrt{|\vec{p}|^2 + m_a^2}, \tag{2.8}$$

since $\theta(p\cdot u) = 0$ for the negative solution $p^0 = -E_{ap}$. Therefore Eq. (2.5) reduces to

$$J_{a\mu} = \int \frac{d^3p}{(2\pi)^3 E_{ap}} p_\mu f_a, \tag{2.9}$$

which is the conventional expression for the current density four-vector of each specie. In particular, its zeroth component is the particle number density of each specie,

$$n_a = J_a^0 = \int \frac{d^3p}{(2\pi)^3} f_a. \tag{2.10}$$

We now consider the divergence of the current density. From Eq. (2.5),

$$\partial^{(x)}\cdot J_a = 2\int \frac{d^4p}{(2\pi)^3}\delta(p^2 - m_a^2)\theta(p\cdot u)(p\cdot\partial^{(x)} f_a), \tag{2.11}$$

and using Eq. (2.1)

$$\partial^{(x)}\cdot J_a = A_a^{\mu\nu} J_{a\mu\nu}, \tag{2.12}$$

where we have defined

$$A_a^{\mu\nu} = -q_a F^{\mu\nu} + \gamma_a \tilde{F}^{\mu\nu}, \tag{2.13}$$

and

$$\begin{aligned} J_{a\mu\nu} &\equiv \int d^4p\,\delta(p^2 - m_a^2)\theta(p\cdot u)p_\nu \partial_\mu^{(p)} f_a \\ &= -\int d^4p\, f_a \partial_\mu^{(p)}[p_\nu \delta(p^2 - m_a^2)\theta(p\cdot u)]. \end{aligned} \tag{2.14}$$

In the second equality in Eq. (2.14) we have integrated by parts, dropping the surface term since f_a vanishes for infinite momentum. Current conservation is a consequence of the fact that, while $A_a^{\mu\nu}$ is antisymmetric,

$$A_a^{\mu\nu} = -A_a^{\nu\mu}, \tag{2.15}$$

$J_{a\mu\nu}$ is, as we show below, symmetric,

$$J_{a\mu\nu} = J_{a\nu\mu}. \tag{2.16}$$

Therefore,

$$A_a^{\mu\nu} J_{a\mu\nu} = 0, \tag{2.17}$$

which together with Eq. (2.12) imply that

$$\partial^{(x)}\cdot J_a = 0. \tag{2.18}$$

The proof of Eq. (2.16) is by straightforward algebra. Taking derivative that appears in the integrand of Eq. (2.14) we obtain,

$$\begin{aligned} \partial_\mu^{(p)}[p_\nu \delta(p^2 - m_a^2)\theta(p\cdot u)] &= g_{\mu\nu}\delta(p^2 - m_a^2)\theta(p\cdot u) \\ &\quad + p_\nu \partial_\mu^{(p)}[\delta(p^2 - m_a^2)\theta(p\cdot u)], \end{aligned} \tag{2.19}$$

and for the second term in Eq. (2.19)

$$\begin{aligned} \partial_\mu^{(p)}[\delta(p^2 - m_a^2)\theta(p\cdot u)] &= p_\mu \delta'(p^2 - m_a^2)\theta(p\cdot u) \\ &\quad + u_\mu \delta(p^2 - m_a^2)\delta(p\cdot u). \end{aligned} \tag{2.20}$$

In the last formula, the product of the two *delta* functions give zero because the two conditions,

$$p^0 = \pm\sqrt{|\vec{p}|^2 + m_a^2}, \tag{2.21}$$

and

$$p^0 u^0 = \vec{p}\cdot\vec{u}, \tag{2.22}$$

cannot be satisfied simultaneously. Therefore we get

$$J_{a\mu\nu} = -\int d^4p f\theta(p\cdot u)[g_{\mu\nu}\delta(p^2-m_a^2) + p_\mu p_\nu \delta'(p^2-m_a^2)], \tag{2.23}$$

which explicitly verifies Eq. (2.16).

B. Modification of the Boltzmann equation

If the distribution function is treated as a function only of $\vec{p}$ rather than $\vec{p}$ and p^0 separately, in other words we explicitly set

$$f_a^{(s)}(t,\vec{x},\vec{p}) \equiv f_a(p)|_{p^0=E_p}, \tag{2.24}$$

it is well known that the covariant equation given in Eq. (1.1) is equivalent to (see, e.g., Ref. [11])

$$\partial_t f_a^{(s)} + \vec{v}_a\cdot\nabla_x f_a^{(s)} = -\vec{F}_a\cdot\nabla_p f_a^{(s)}, \tag{2.25}$$

with

$$\vec{F}_a = q_a(\vec{E} + \vec{v}_a\times\vec{B}). \tag{2.26}$$

Equation (2.25) is the standard form of the Boltzmann equation for charged particles in an electromagnetic field.

In the case that f_a satisfies Eq. (2.1) the corresponding equation, analogous to Eq. (2.25), is

$$\partial_t f_a^{(s)} + \vec{v}_a\cdot\nabla_x f_a^{(s)} = (-\vec{F}_a + \vec{G}_a)\cdot\nabla_p f_a^{(s)}, \tag{2.27}$$

where $\vec{F}_a$ is given above, and

$$\vec{G}_a = \gamma_a(\vec{B} - \vec{v}_a\times\vec{E}). \tag{2.28}$$

The result given in Eq. (2.27) follows easily from the fact that the elements of $\tilde{F}_{\mu\nu}$ are obtained from $F_{\mu\nu}$ by making the replacement

$$\begin{aligned}\vec{E}&\to\vec{B},\\ \vec{B}&\to-\vec{E},\end{aligned} \tag{2.29}$$

which implies that $\vec{G}_a$ is obtained from $\vec{F}_a$ by making the same replacement. Thus, for example, in the presence of only a magnetic field $\vec{B}$, the equation is

$$\partial_t f_a^{(s)} + \vec{v}_a\cdot\nabla_x f_a^{(s)} = (-q_a\vec{v}_a\times\vec{B} + \gamma_a\vec{B})\cdot\nabla_p f_a^{(s)}. \tag{2.30}$$

A quick observation that follows from Eq. (2.27) is that some discrete space-time symmetries are broken in the system when the $\tilde{F}_{\mu\nu}$ term is present in the Boltzmann equation. This is obvious, for example for parity (P), from the fact that $\vec{E}$ and $\vec{B}$ have opposite phase under a P transformation, and therefore the same holds for $\vec{F}$ and $\vec{G}$.

III. LINEARIZATION OF THE KINETIC EQUATION AND $\pi_{\mu\nu}$

A. Linearization and the induced current

The dispersion relations for the propagating photons are obtained by linearizing the kinetic equation. We put

$$f_a = f_{a0} + f_{a1} + \cdots, \tag{3.1}$$

where f_{a0} is the equilibrium distribution, and f_{a1} is linear in $F^{\mu\nu}$. Substituting Eq. (3.1) in Eq. (2.1), and retaining only terms that are linear in $F^{\mu\nu}$, gives

$$p\cdot\partial^{(x)} f_{a1} = [-q_a F^{\mu\nu} + \gamma_a\tilde{F}^{\mu\nu}]p_\nu\partial_\mu^{(p)} f_{a0}. \tag{3.2}$$

The next step is to consider the momentum space equation corresponding to Eq. (3.2). Denoting the wave vector by k^μ, the momentum space equation is obtained from Eq. (3.2) by making the replacements

$$\begin{aligned}F_{\mu\nu}&\to f_{\mu\nu},\\ \tilde{F}_{\mu\nu}&\to\tilde{f}_{\mu\nu},\\ f_{a1}&\to\hat{f}_{a1},\\ \partial_\mu^{(x)} f_{a1}&\to -ik_\mu\hat{f}_{a1},\end{aligned} \tag{3.3}$$

with the understanding that the functions on the right-hand side are the Fourier transforms of those on the left. In particular,

$$\tilde{f}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}f_{\alpha\beta}, \tag{3.4}$$

in correspondence with Eq. (2.2). Furthermore, remembering that we are considering an isotropic system, f_{a0} is a function only of

$$\mathcal{E}\equiv p\cdot u, \tag{3.5}$$

in which case

$$\partial_\mu^{(p)} f_{a0} = u_\mu f'_{a0}, \tag{3.6}$$

where

$$f'_{a0}\equiv\frac{\partial f_{a0}}{\partial\mathcal{E}}. \tag{3.7}$$

With these substitutions, the momentum space equation corresponding to Eq. (3.2) is,

$$(-ik\cdot p)\hat{f}_{a1} = [-q_a f^{\mu\nu} + \gamma_a\tilde{f}^{\mu\nu}]u_\mu p_\nu f'_{a0}, \tag{3.8}$$

which gives

$$\hat{f}_{a1} = \left(\frac{i}{k\cdot p}\right)[-q_a f^{\mu\nu} + \gamma_a \tilde{f}^{\mu\nu}]u_\mu p_\nu f'_{a0}. \quad (3.9)$$

The induced current is obtained from Eqs. (2.4) and (2.9) using Eqs. (3.1) and (3.9). The terms containing the equilibrium distributions f_{a0} do not contribute. This is most easily seen by going to the medium's own rest frame. Due to the isotropy condition, the vector current density $\vec{j}_a$ is zero, while the total j^0 is zero assuming that the charge density is zero in equilibrium. Thus,

$$j_\mu = i\sum_a q_a[-q_a f^{\lambda\nu} + \gamma_a \tilde{f}^{\lambda\nu}]u_\lambda I^{(1)}_{a\mu\nu}, \quad (3.10)$$

where

$$I^{(1)}_{a\mu\nu} = \int \frac{d^3p}{(2\pi)^3 E_{ap}} \frac{f'_{a0}}{k\cdot p} p_\mu p_\nu. \quad (3.11)$$

The next step is to write Eq. (3.10) in terms of the vector potential rather than the field. We consider separately the two terms in Eq. (3.10).

1. $f^{\mu\nu}$ term

For the term with $f^{\mu\nu}$, using

$$f_{\alpha\beta} = -i[k_\alpha A_\beta - k_\beta A_\alpha], \quad (3.12)$$

we have

$$u^\alpha p^\beta f_{\alpha\beta} = -i[(k\cdot u)p_\nu - (k\cdot p)u_\nu]A^\nu, \quad (3.13)$$

which gives

$$f^{\lambda\nu}u_\lambda I^{(1)}_{a\mu\nu} = -i[(k\cdot u)I^{(1)}_{a\mu\nu} - I^{(2)}_a u_\mu u_\nu]A^\nu, \quad (3.14)$$

where $I^{(1)}_{a\mu\nu}$ is given in Eq. (3.11) and

$$I^{(2)}_a = \int \frac{d^3p}{(2\pi)^3 E_{ap}} (p\cdot u) f'_{a0}. \quad (3.15)$$

To arrive at Eq. (3.14) we have used again the fact that we are considering an isotropic system so f_{a0} is a function of $p\cdot u$, and therefore

$$\int \frac{d^3p}{(2\pi)^3 E_{ap}} f'_{a0} p_\mu = I^{(2)}_a u_\mu. \quad (3.16)$$

For the sake of completeness, we mention that the contribution to the current from the term given in Eq. (3.14) is transverse by itself, as can be easily verified explicitly by multiplying by k^μ and using Eq. (3.16).

2. $\tilde{f}^{\mu\nu}$ term

We now derive the relation analogous to Eq. (3.14) for the $\tilde{f}^{\mu\nu}$ term. From Eqs. (3.4) and (3.12), relabeling some of the Lorentz indices, it follows that

$$\tilde{f}^{\lambda\nu}u_\lambda I^{(1)}_{a\mu\nu} = -i\epsilon^{\lambda\nu\alpha\beta}k_\alpha u_\beta A_\nu I^{(1)}_{a\mu\lambda}, \quad (3.17)$$

where $I^{(1)}_{a\mu\nu}$ is defined in Eq. (3.11). Since $I^{(1)}_{a\mu\nu}$ is a symmetric in μ, ν, and depends only on k^μ and u^μ, it is of the form

$$I^{(1)}_{a\mu\nu} = Cg_{\mu\nu} + X_1 u_\mu u_\nu + X_2 k_\mu k_\nu + X_3(k_\mu u_\nu + u_\mu k_\nu). \quad (3.18)$$

Only the C term contributes in Eq. (3.17) and therefore

$$\tilde{f}^{\lambda\nu}u_\lambda I^{(1)}_{a\mu\nu} = -iC_a\epsilon_{\mu\nu\alpha\beta}k^\alpha u^\beta A^\nu. \quad (3.19)$$

C_a can in turn be written in terms of the tensor $R^{\lambda\rho}$ defined in Eq. (A6) as

$$C_a = \frac{1}{2}R^{\lambda\rho}I^{(1)}_{a\lambda\rho}. \quad (3.20)$$

B. $\pi_{\mu\nu}$ and $\pi_{T,L,P}$

Using Eqs. (3.14) and (3.19) in Eq. (3.10), the induced current is then expressed in the form $j_\mu = -\pi_{\mu\nu}A^\nu$ [Eq. (A1)], with

$$\pi_{\mu\nu} = \sum_a \{q_a^2[(k\cdot u)I^{(1)}_{a\mu\nu} - I^{(2)}_a u_\mu u_\nu] - q_a\gamma_a C_a \epsilon_{\mu\nu\alpha\beta}k^\alpha u^\beta\}. \quad (3.21)$$

This expression for $\pi_{\mu\nu}$ can be decomposed in terms of $\pi_{T,L,P}$ as given in Eq. (A2). π_P can be written by inspection, while $\pi_{T,L}$ can be obtained by projecting the term in square brackets in Eq. (3.21), which is symmetric and transverse, with $R^{\mu\nu}$ and $Q^{\mu\nu}$, using Eqs. (A10) and (A9). Thus,

$$\pi_L = -\sum_a q_a^2\left(\frac{k^2 D_a}{\kappa^2}\right),$$
$$\pi_T = \sum_a q_a^2 (k\cdot u)C_a,$$
$$\pi_P = \sum_a iq_a\gamma_a\kappa C_a, \quad (3.22)$$

where C_a has been defined in Eq. (3.20), while

$$D_a = u^\mu u^\nu[(k\cdot u)I^{(1)}_{a\mu\nu} - I^{(2)}_a u_\mu u_\nu]. \quad (3.23)$$

Using Eqs. (3.11) and (3.15), the integral formulas for C_a and D_a can be written in the form

$$C_a = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3 E_{ap}}\left\{p^2 - (p\cdot u)^2 + \frac{1}{\kappa^2}[p\cdot k - (k\cdot u)(p\cdot u)]^2\right\}\frac{f'_{a0}}{k\cdot p},$$

$$D_a = \int \frac{d^3p}{(2\pi)^3 E_{ap}}\{(k\cdot u)(p\cdot u)^2 - (p\cdot u)(k\cdot p)\}\frac{f'_{a0}}{k\cdot p}. \tag{3.24}$$

We consider the evaluation of the integrals for C_a and D_a in the medium's rest frame, and thus we set

$$u^\mu = (1, \vec{0}). \tag{3.25}$$

We decompose k^μ in the form

$$k^\mu = (\omega, \vec{\kappa}), \tag{3.26}$$

and in the integrands we will write

$$p^\mu = (E_{ap}, \vec{p}), \tag{3.27}$$

with the understanding that we are working in the rest frame of the medium. In particular f_{a0} is a function only of E_{ap}, and

$$f'_{a0} = \frac{\partial f_{a0}}{\partial E_{ap}}. \tag{3.28}$$

Thus, for example, the numerator of the integrand for D_a in Eq. (3.24)

$$(k\cdot u)(p\cdot u)^2 - (p\cdot u)(k\cdot p) = \omega E_{ap}^2 - E_{ap}(\omega E_{ap} - \vec{\kappa}\cdot\vec{p}) = E_{ap}\vec{\kappa}\cdot\vec{p}, \tag{3.29}$$

while the denominator can be written as

$$E_{ap}(\omega - \vec{\kappa}\cdot\vec{v}_a), \tag{3.30}$$

with

$$\vec{v}_a = \frac{\vec{p}}{E_{ap}}. \tag{3.31}$$

Substituting these in the expression for D_a in Eq. (3.24),

$$D_a = \int \frac{d^3p}{(2\pi)^3}\left[\frac{\vec{\kappa}\cdot\vec{v}_a}{\omega - \vec{\kappa}\cdot\vec{v}_a}\right] f'_{a0}, \tag{3.32}$$

and similarly,

$$C_a = -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{\vec{v}^2_{a\perp}}{\omega - \vec{\kappa}\cdot\vec{v}_a} f'_{a0}, \tag{3.33}$$

where

$$\vec{v}_{a\perp} = \vec{v}_a - \frac{1}{\kappa^2}(\vec{\kappa}\cdot\vec{v}_a)\vec{\kappa}. \tag{3.34}$$

Although we do not indicate it explicitly, it is understood that in these formulas for C_a and D_a, as well as in the formulas for $\pi_{L,T,P}$ given below, the singularity of the integrand at $\omega = \vec{\kappa}\cdot\vec{v}_a$ is to be handled by making the replacement

$$\omega \to \omega + i0^+, \tag{3.35}$$

as usual.

From Eq. (3.22) we then obtain

$$\pi_L = -\frac{k^2}{\kappa^2}\sum_a q_a^2 \int \frac{d^3p}{(2\pi)^3}\left[\frac{\vec{\kappa}\cdot\vec{v}_a}{\omega - \vec{\kappa}\cdot\vec{v}_a}\right] f'_{a0},$$

$$\pi_T = -\frac{\omega}{2}\sum_a q_a^2 \int \frac{d^3p}{(2\pi)^3}\frac{\vec{v}^2_{a\perp}}{\omega - \vec{\kappa}\cdot\vec{v}_a} f'_{a0},$$

$$\pi_P = -\frac{i\kappa}{2}\sum_a q_a\gamma_a \int \frac{d^3p}{(2\pi)^3}\frac{\vec{v}^2_{a\perp}}{\omega - \vec{\kappa}\cdot\vec{v}_a} f'_{a0}. \tag{3.36}$$

The corresponding expressions for $\epsilon_{\ell,t}$ obtained from Eq. (A23) reproduce the standard classic results, e.g.,

$$\epsilon_\ell - 1 = \frac{1}{\kappa^2}\sum_a q_a^2 \int \frac{d^3p}{(2\pi)^3}\left[\frac{\vec{\kappa}\cdot\vec{v}_a}{\omega - \vec{\kappa}\cdot\vec{v}_a}\right] f'_{a0}, \tag{3.37}$$

and similarly for ϵ_t. On the other hand, the formula for π_P, and the corresponding formula for ϵ_p obtained from Eq. (A23), are new.

IV. DISCUSSION

A. Dispersion relations in the long-wavelength limit

The longitudinal dispersion relation is not affected by the γ_a terms, as already indicated. Therefore, we focus on the transverse ones (i.e., polarizations perpendicular to $\vec{\kappa}$) which involve both π_T and π_P. Moreover, we consider specifically the long-wavelength limit,

$$\omega \gg \kappa v_a, \tag{4.1}$$

which is a particularly useful and representative of more general situations.

Using the fact that we are considering the case that the distribution functions f_{a0} are isotropic, by straightforward manipulation of the integrand,

$$C_a(\omega,\kappa \to 0) = \frac{\Omega_a^2}{\omega}, \tag{4.2}$$

where

$$\Omega_a^2 = \int \frac{d^3p}{(2\pi)^3 E_{ap}} \left(1 - \frac{v_a^2}{3}\right) f_{a0}. \tag{4.3}$$

For reference, recall that the plasma frequency ($\omega_{pl,a}$) of each specie is given by

$$\omega_{pl,a}^2 = q_a^2 \Omega_a^2. \tag{4.4}$$

For example, in the nonrelativistic limit,

$$\Omega_a^2 = \frac{n_{a0}}{m_a}, \tag{4.5}$$

where n_{a0} is the equilibrium particle number density of the specie [see Eq. (2.10)]. From Eq. (3.22) we then obtain in the long-wavelength limit

$$\pi_P(\omega,\kappa \to 0) = \frac{i\kappa\gamma_P}{\omega}, \tag{4.6}$$

and the well-known result

$$\pi_T(\omega,\kappa \to 0) = \Omega_0^2, \tag{4.7}$$

where

$$\Omega_0^2 = \sum_a q_a^2 \Omega_a^2, \tag{4.8}$$

and

$$\gamma_P = \sum_a q_a \gamma_a \Omega_a^2. \tag{4.9}$$

As reviewed in Appendix, the transverse dispersion relations are determined as the solutions of

$$k^2 - (\pi_T + \lambda\pi_P) = 0, \qquad (\lambda = \pm). \tag{4.10}$$

Substituting in this equation the results for π_P and π_T given in Eqs. (4.6) and (4.7), respectively, in the long-wavelength limit the equation becomes

$$\omega^2 - (\kappa^2 + \Omega_0^2) - \frac{i\lambda\kappa\gamma_P}{\omega} = 0. \tag{4.11}$$

In the limit $\gamma_P \to 0$ we obtain the standard transverse solutions,

$$\omega = \omega_T(\kappa) \equiv \sqrt{\kappa^2 + \Omega_0^2}, \tag{4.12}$$

for either polarization. In the more general situation, assuming

$$|\gamma_P| \ll \frac{2\omega_T^3(\kappa)}{\kappa}, \tag{4.13}$$

the solutions are

$$\omega(\kappa) = \omega_T(\kappa) + i\frac{\lambda\kappa\gamma_P}{2\omega_T^2(\kappa)}. \tag{4.14}$$

The dispersion relations are such that one polarization mode is damped (absorption by the medium) while the other one grows (emission by the medium). Which is one or the other depends on the sign of γ_P, which in turn depends on the relative signs and values of the $q_a\gamma_a$ terms in Eq. (4.9).

B. Generalization

Equation (2.1) is probably the simplest equation of the kind we are discussing, but there are some possible generalizations. Here we mention some of them. We will write them in the generic form

$$p \cdot \partial^{(x)} f_a = [-q_a F^{\mu\nu} + \Gamma_a^{\mu\nu}] p_\nu \partial_\mu^{(p)} f_a. \tag{4.15}$$

The requirement is that $\Gamma^{\mu\nu}$ must contain, in some form, the dual tensor $\tilde{F}^{\alpha\beta}$, and that it is antisymmetric in μ, ν so that the proof of current conservation given in Sec. II A applies in this case as well, with the identification of $A^{\mu\nu} = -q_a F^{\mu\nu} + \Gamma^{\mu\nu}$ in place of Eq. (2.13).

One possibility is

$$\Gamma_a^{\mu\nu} = \gamma_a^{(1)} (u \cdot \partial^{(x)}) \tilde{F}^{\mu\nu}, \tag{4.16}$$

where $\gamma_a^{(1)}$ is a constant parameter. The steps to arrive at Eq. (3.9) for $\hat{f}_{a1}$ apply also in this case, with the identification

$$\gamma_a = -i(k \cdot u)\gamma_a^{(1)}. \tag{4.17}$$

Thus for example, from Eqs. (4.6) and (4.9), we can see that in the long wavelength limit this gives a contribution to π_P of the form

$$\kappa q_a \gamma_a^{(1)} \Omega_a^2. \tag{4.18}$$

The main qualitative difference, relative to the case in which γ_a is independent of ω, is that in the present case the corresponding contribution to π_P is real, which in turn produces a real term in the dispersion relations of opposite sign for the ($\pm$) polarizations.

More generally, we can consider the equation

$$\Gamma_a^{\mu\nu} = \int d^4x' \Gamma_a(x-x')\tilde{F}^{\mu\nu}(x'). \tag{4.19}$$

The momentum space equation in the linear approximation for $\hat{f}_{a1}$ is again Eq. (3.9), but with γ_a being the Fourier transform of $\Gamma_a(x-x')$, that is, γ_a defined by writing

$$\Gamma_a(x-x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')}\gamma_a(k). \tag{4.20}$$

The fact that $\Gamma_a(x-x')$ is real implies that

$$\gamma_a^*(k) = \gamma_a(k)|_{k\to -k}. \tag{4.21}$$

Being a scalar, γ_a is a function of the scalar variables ω and κ defined in Eq. (A3), a fact that we indicate by writing it as $\gamma_a(\omega,\kappa)$ when needed. In particular, Eq. (4.21) actually implies the condition

$$\gamma_a^*(\omega,\kappa) = \gamma_a(-\omega,\kappa). \tag{4.22}$$

This case includes the original Eq. (2.1) (constant γ_a) as well as Eq. (4.16) as special cases, and of course Eq. (4.17) is consistent with Eq. (4.22), as it should be.

C. Neutrino background

An example case in which γ_a has the form given in Eq. (4.17) is afforded by an electron plasma with a neutrino background. The calculation of π_P in that case gives [3]

$$\pi_P(\omega,\kappa\to 0) = \frac{\sqrt{2}G_F\alpha}{3\pi}\left(\frac{\omega_{pl,e}^2}{m_e^2}\right)(n_{\nu_e} - n_{\bar{\nu}_e})\kappa, \tag{4.23}$$

where n_{ν_e} and $n_{\bar{\nu}_e}$ stand for the number densities of the electron neutrinos and antineutrinos, and $\omega_{pl,e}$ is the electron plasma frequency. For simplicity let us consider the nonrelativistic limit, so that only the electrons (no positrons) are present, in which case [i.e., Eqs. (4.4) and (4.5)]

$$\omega_{pl,e}^2 = q_e^2\Omega_e^2 = \frac{q_e^2 n_{e0}}{m_e}, \tag{4.24}$$

where q_e is the electron charge and n_{e0} is the equilibrium electron number density. On the other hand, for this case that we are considering, in the framework of the kinetic equation

$$\pi_P = \frac{i\kappa q_e\gamma_e\Omega_e^2}{\omega}. \tag{4.25}$$

Therefore, in the framework of the kinetic equation, the effects of the neutrino background can be parametrized in terms of a γ_e parameter for the electron of the form

$$\gamma_e = -i\omega\gamma_e^{(1)}, \tag{4.26}$$

with

$$\gamma_e^{(1)} = q_e\frac{\sqrt{2}G_F\alpha}{3\pi m_e^2}(n_{\nu_e} - n_{\bar{\nu}_e}). \tag{4.27}$$

The main lesson here is that the kinetic approach allows us to parametrize the effects produced by the π_P term in the photon polarization tensor in terms of the parameter γ_a. In this framework, γ_a is a phenomenological parameter that must be determined by other means, e.g., thermal field theory in the case of an electron plasma with neutrino background, as we have seen. Nevertheless, the kinetic approach allows us to study further the consequences of the presence of the π_P term, such as the effects of external fields [10], streaming neutrino background [5] or collisional plasmas [9], among others.

V. CONCLUSIONS AND OUTLOOK

In this work we have proposed a method to study the propagation of photons in an optically active isotropic medium, based on the covariant collisionless Boltzmann equation. As shown in Sec. II, the covariant Boltzmann equation can be modified by adding a term that gives rise to the optical activity effects, in a way that is consistent with the general requirements of current conservation and symmetry considerations. In Sec. III, using the linear response method, we obtained an expression for the induced current, expressed in terms of integrals over the momentum distribution function of the background particles. There we established contact with the photon polarization tensor, or equivalently the photon self-energy, in the TFT language, specifically as used in Ref. [1], and the formulas for the components of the polarization tensor were obtained. The results for the longitudinal and transverse components of the photon self-energy, $\pi_{L,T}$ respectively, are the familiar ones. The new result here is the formula for the P- and CP-breaking component π_P due to the new term we considered in the Boltzmann equation to describe the effects of optical activity. In Sec. IV we discussed some details of the results obtained, and considered specifically some particular cases (e.g., the long-wavelength and the nonrelativistic limit) that are useful in practical applications and representative of more general situations. To emphasize the usefulness of the method, we pointed out how the P- and CP-breaking term we included in the Boltzmann equation can be generalized, in particular how it applies to a plasma with a neutrino gas as a background, and established contact with calculations of the photon self-energy in such contexts in the framework of TFT [3]. The strength and advantages of the method here presented comes from its semiclassical standpoint, which in many circumstances is more suitable than the thermal field theory approach for incorporating other potentially important effects such as collisions, external fields, stream backgrounds and multicomponent plasmas.

APPENDIX: NOTATION AND CONVENTIONS

We use the notation and conventions used in Ref. [1], which we briefly review here for convenience. The momentum of the propagating photon is denoted by k^μ, and u^μ is the velocity four-vector of the medium, already introduced in Eq. (2.5).

1. Photon polarization tensor

In the context of TFT, the photon self-energy, $\pi_{\mu\nu}$ gives rise to a contribution to the effective Lagrangian of the photon and the corresponding field equation that leads to identify

$$j_\mu = -\pi_{\mu\nu}A^\nu, \tag{A1}$$

as the induced current in the presence in the external field, and whence $\pi_{\mu\nu}$ as the polarization tensor. As discussed in that reference, the most general form of $\pi_{\mu\nu}$ in an isotropic medium is

$$\pi_{\mu\nu}(k,u) = \pi_T(\omega,\kappa)R_{\mu\nu}(k,u) + \pi_L(\omega,\kappa)Q_{\mu\nu}(k,u) + \pi_P(\omega,\kappa)P_{\mu\nu}(k,u), \tag{A2}$$

where ω and κ are the scalar variables

$$\omega = k\cdot u, \qquad \kappa = (\omega^2 - k^2)^{\frac{1}{2}}, \tag{A3}$$

which have the interpretation of the energy and the magnitude of the momentum of the photon, in the rest frame of the medium. The tensors R, Q, P are defined as follows. First, the component of u^μ transverse to k^μ is

$$\tilde{u}_\mu = \tilde{g}_{\mu\nu}u^\nu, \tag{A4}$$

where

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}. \tag{A5}$$

Then,

$$\begin{aligned} Q_{\mu\nu} &= \frac{\tilde{u}_\mu\tilde{u}_\nu}{\tilde{u}^2}, \\ R_{\mu\nu} &= \tilde{g}_{\mu\nu} - Q_{\mu\nu}, \\ P_{\mu\nu} &= \frac{i}{\kappa}\epsilon_{\mu\nu\alpha\beta}k^\alpha u^\beta. \end{aligned} \tag{A6}$$

It is useful to remember that all three tensors are transverse to k^μ, that is

$$k^\mu T_{\mu\nu} = 0 = k^\nu T_{\mu\nu}; \qquad (T = R,Q,P), \tag{A7}$$

and also that R and P are transverse to u^μ as well,

$$u^\mu T_{\mu\nu} = 0 = u^\nu T_{\mu\nu}; \qquad (T = R,P). \tag{A8}$$

They satisfy various product relations, among them

$$R^{\mu\nu}R_{\mu\nu} = R^\mu{}_\mu = 2, \quad Q^{\mu\nu}Q_{\mu\nu} = Q^\mu{}_\mu = 1, \quad P^{\mu\nu}P_{\mu\nu} = -2, \tag{A9}$$

and

$$\begin{aligned} &R^{\mu\lambda}Q_{\lambda\nu} = 0, \qquad Q^{\mu\lambda}P_{\lambda\nu} = 0, \qquad P^{\mu\lambda}R_{\lambda\nu} = P^\mu{}_\nu, \\ &P^{\mu\lambda}P_{\lambda\nu} = R^\mu{}_\nu. \end{aligned} \tag{A10}$$

In the rest frame of the medium, the components of $R_{\mu\nu}$ and $P_{\mu\nu}$ are,

$$\begin{aligned} &R_{00} = R_{0i} = R_{i0} = 0, \qquad R_{ij} = \delta_{ij} + \frac{\kappa_i\kappa_j}{\kappa^2}, \\ &P_{00} = P_{0i} = P_{i0} = 0, \qquad P_{ij} = \frac{i}{\kappa}\epsilon_{ijk}\kappa^k. \end{aligned} \tag{A11}$$

2. Dispersion relations

The equation $\partial^\mu F_{\mu\nu} = j_\nu$, in momentum space becomes

$$[(k^2 - \pi_T)R_{\mu\nu} + (k^2 - \pi_L)Q_{\mu\nu} - \pi_P P_{\mu\nu}]A^\nu = 0, \tag{A12}$$

which determines the dispersion relations and polarization vectors of the propagating modes. To discuss them we recall the definition of the transverse vectors $e^\mu_{1,2}$, which in the rest frame of the medium have components

$$e^\mu_{1,2} = (0, \vec{e}_{1,2}), \tag{A13}$$

where $\vec{e}_{1,2}$ are unit vectors with

$$\vec{e}_{1,2}\cdot\hat{\kappa} = 0, \qquad \vec{e}_2 = \hat{\kappa}\times\vec{e}_1. \tag{A14}$$

In covariant form, they satisfy

$$R_{\mu\nu}e^\nu_a = e_{a\mu}, \qquad Q_{\mu\nu}e^\nu_a = 0, \qquad (a = 1,2), \tag{A15}$$

and

$$e^\mu_2 = -iP^{\mu\nu}e_{1\nu}, \qquad e^\mu_1 = iP^{\mu\nu}e_{2\nu}. \tag{A16}$$

In addition it is useful to introduce

$$e^\mu_3 = \frac{\tilde{u}^\mu}{\sqrt{-\tilde{u}^2}}, \tag{A17}$$

which together with $e^{\mu}_{1,2}$ form a basis in the subspace orthogonal to k^{μ}.

From the fact that $R^{\mu\nu}$ and $P^{\mu\nu}$ acting on $\tilde{u}_{\nu}$ give zero, it follows that $A^{\mu} \sim e^{\mu}_{3}$ is a solution of Eq. (A12) provided

$$k^2 - \pi_L = 0, \tag{A18}$$

which is the equation for the dispersion relation $\omega_L(\kappa)$ for the longitudinal mode. Since the presence of the γ_a term does not affect π_L, the dispersion relation for the longitudinal mode is not affected.

In the absence of the π_P term, Eq. (A15) implies that $A^{\mu} \sim e^{\mu}_{1,2}$, or any combination of them, give a solution of Eq. (A12) if

$$k^2 - \pi_T = 0, \tag{A19}$$

which gives the dispersion relation $\omega_T(\kappa)$, The transverse modes, corresponding to the polarization vectors $e^{\mu}_{1,2}$ are therefore degenerate, with the same dispersion relation $\omega_T(\kappa)$.

As a consequence of the relations in Eq. (A16), neither $A^{\mu} \sim e^{\mu}_{1}$ nor $A^{\mu} \sim e^{\mu}_{2}$ are separately solutions of the equation. At this point it is useful to introduce the circular polarization vectors

$$e^{(\pm)\mu} = \frac{1}{\sqrt{2}}(e^{\mu}_{1} \pm i e^{\mu}_{2}), \tag{A20}$$

which satisfy

$$P^{\mu\nu} e^{(\lambda)}_{\nu} = \lambda e^{(\lambda)\mu}, \quad \lambda = \pm, \tag{A21}$$

in addition to identities analogous to Eq. (A15). It then follows that $A^{\mu} \sim e^{(\pm)\mu}$ are each a solution of the equation, with the corresponding dispersion relation being the solution of

$$k^2 - (\pi_T + \lambda \pi_P) = 0. \tag{A22}$$

3. Dielectric tensor

An equivalent way to express the presence of the π_P term in the photon self-energy is in terms of the components of the dielectric tensor, which are given by [12]

$$1 - \epsilon_t = \pi_T/\omega^2, \quad 1 - \epsilon_{\ell} = \pi_L/k^2, \quad \epsilon_p = \pi_P/\omega^2. \tag{A23}$$

The interpretation is that, in the rest frame of the medium, the induced current vector is given by

$$\vec{j} = i\omega[(1 - \epsilon_{\ell})\vec{E}_{\ell} + (1 - \epsilon_t)\vec{E}_t - i\epsilon_p \hat{\kappa} \times \vec{E}], \tag{A24}$$

where we are writing, in that frame,

$$k^{\mu} = (\omega, \vec{\kappa}), \tag{A25}$$

while $\vec{E}_{\ell}$ and $\vec{E}_t$ denote the components of the electric field parallel and transverse to $\vec{\kappa}$, respectively. The ϵ_p term breaks the degeneracy between the two transverse polarization states of the propagating photon.

[1] J. F. Nieves and Palash B. Pal, *P*- and *CP*-odd terms in the photon self-energy within a medium, Phys. Rev. D **39**, 652 (1989); **40**, 2148 (1989).

[2] Other considerations and conditions, in particular related to time-reversal (T) and CPT transformations were discussed further in, J. F. Nieves and Palash B. Pal, Propagation of gauge fields within a medium, Phys. Rev. D **40**, 1350 (1989).

[3] Subhendra Mohanty, J. F. Nieves, and P. B. Pal, Optical activity of a neutrino gas, Phys. Rev. D **58**, 093007 (1998).

[4] J. F. Nieves and S. Sahu, Electromagnetic effects of neutrinos in an electron gas, Phys. Rev. D **71**, 073006 (2005).

[5] M. Petropavlova and A. Smetana, Towards interferometry of neutrino electromagnetism, Phys. Rev. D **106**, 053003 (2022).

[6] J. F. Nieves and Palash B. Pal, Third electromagnetic constant of an isotropic medium, Am. J. Phys. **62**, 207 (1994).

[7] P. B. Pal, Covariant formulation of electrodynamics in isotropic media, Eur. J. Phys. **43**, 015204 (2021).

[8] Y. Nakai, R. Namba, I. Obata, Y. Qiu, and R. Saito, Can we explain cosmic birefringence without a new light field beyond standard model?, J. High Energy Phys. 01 (2024) 057.

[9] M. Formanek, C. Grayson, J. Rafelski, and B. Müller, Current-conserving relativistic linear response for collisional plasmas, Ann. Phys. (Amsterdam) **434**, 168605 (2021).

[10] V. B. Semikoz and D. D. Sokoloff, Large-scale magnetic field generation by α effect driven by collective neutrino-plasma interaction, Phys. Rev. Lett. **92**, 131301 (2004).

[11] Donald B. Melrose, *Quantum Plasmadynamics: Unmagnetized Plasmas*, Lecture Notes in Physics Vol. 735 (Springer, New York 2008), pp. 98–100.

[12] See, for example, Ref. [1].