Contributions of the subprocess $K_0^*(1430) \to K\eta'$ in the charmless three-body B meson decays

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We study the contributions for $K\eta'$ pair originating from the scalar intermediate state $K_0^*(1430)$ in the three-body decays $B \to K\eta' h$ ($h = \pi$, K) within the perturbative QCD approach. The contribution of $K_0^*(1430) \to K\eta'$ is described by the Flatté formula with coupled channels $K\pi$, $K\eta$, and $K\eta'$. The strong coupling constants $g_{K_0^* K\eta^{(\prime)}}$ are extracted from $g_{K_0^* K\pi}$ within flavor SU(3) symmetry. In spite of the strong depression by phase space near the threshold of $K\eta'$, the CP averaged branching fractions for the $B \to K_0^*(1430)h \to K\eta' h$ decays are predicted to be on the order of 10^{-8} to 10^{-5}, which are non-negligible for the corresponding three-body B decays. Since the Kη system is almost decoupled from the even-spin strange mesons under flavor SU(3) symmetry, those quasi-two-body B decays with subprocess $K_0^*(1430) \to K\eta$ shall have quite small branching ratios and are not taken into account in this work. We also estimate that the branching fraction for $K_0^*(1430) \to K\eta'$ is about one fifth of that for $K_0^*(1430) \to K\pi$. The predictions for the relevant decays are expected to be tested by the LHCb and Belle-II experiments in the future.

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I. INTRODUCTION

It is known that the identification of the scalar resonances is experimentally difficult and the underlying inner structure of the scalar mesons remains debated on the theoretical side. The light scalars, especially those with the mass below 1 GeV, are generally suggested as multiquark states, hadronic molecules, or glueballs, etc., besides the conventional quark-antiquark configuration [1–5]. The resonance $K_0^*(1430)$, the least controversial one in the light scalar states, has been observed decaying dominantly to the $K\pi$ pair for more than three decades [6] and can be constructed as a $q\bar{q}$ state with $J^{PC} = 0^{++}$ [7]. In recent years, the other two coupled channels $K\eta$ and $K\eta'$ for $K_0^*(1430)$ have also been measured by different collaborations. For example, in Ref. [8] the *BABAR* collaboration reported the first observation of the $K_0^*(1430) \to K\eta$ decay in the process $\eta_c \to \eta K^+ K^-$ and gave a relative branching fraction ($\mathcal{B}$)

$$R_{K\eta} = \frac{\mathcal{B}(K_0^*(1430) \to K\eta)}{\mathcal{B}(K_0^*(1430) \to K\pi)} = (9.2 \pm 2.5^{+1.0}_{-2.5})\%. \quad (1)$$

Meanwhile, the $K_0^*(1430) \to K\eta'$ decay was firstly observed in the measurement of $\chi_{cJ} \to \eta' K^+ K^-$ by the BESIII collaboration [9]. Very recently, the Dalitz plot analysis for $\eta_c \to \eta' K^+ K^-$ was performed and the ratio $R_{K\eta'} = \frac{\mathcal{B}(K_0^*(1430)\to K\eta')}{\mathcal{B}(K_0^*(1430)\to K\pi)}$ was measured to be quite large, with the value of $(39.7 \pm 6.4 \pm 5.4)\%$ by the *BABAR* collaboration [10].

Since the scalar states decay mainly into two pseudoscalars, three-body B meson decays provide us rich opportunities to study the scalar resonant states. The contribution of $K_0^*(1430)$ in the S-wave $K\pi$ system has been solidly established in the measurements of the charmless three-body B meson decays by the Belle [11–13], *BABAR* [14–19], and LHCb collaborations [20,21], and the corresponding branching fractions and direct CP violations for the quasi-two-body channels $B \to K_0^*(1430)h \to K\pi h$ ($h = \pi, K$) have also been presented via amplitude analysis. The $K_0^*(1430)$ contributions in most of the concerned three-body B meson decays are found to be considerably large. Particularly, the quasi-two-body components with $K_0^*(1430)$ in the decays $B^+ \to K^+\pi^+\pi^-$ [11,13], $B^0 \to K^0\pi^+\pi^-$ [12], $B^0 \to K^+\pi^-\pi^0$ [18], and $B^+ \to K^0\pi^+\pi^0$ [19]

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were shown to be larger than 50% of the corresponding total branching fractions. Theoretical researches were carried out in parallel. As discussed in Refs. [22–24], the factorization methods for quasi-two-body B meson decays are still valid, and a number of related works within different approaches [25–35] based on factorization theorem have been done in order to explain the data or provide theoretical predictions. In addition, the resonant contributions including $K_0^*(1430)$ in $B^- \to K^+K^-\pi^-$ were considered by adopting the light-cone sum rule approach [36], where the matrix element $\langle R, M|\mathcal{O}_i|B\rangle$ as a whole was calculated. But, there are not many discussions on the contributions from subprocesses $K_0^*(1430) \to K\eta^{(\prime)}$ for the charmless three-body B meson decays on both the theoretical and experimental sides.

Within the flavor SU(3) symmetry, the $K\eta$ branching fraction for the strange states like K^* with even spin is expected to be strongly suppressed, but for odd-spin states it will be quite substantial [37–39]. This is supported by the data $\mathcal{B}(K_2^*(1430) \to K\eta) = (0.15^{+0.34}_{-0.10})\%$ and $\mathcal{B}(K_3^*(1780) \to K\eta) = (30 \pm 13)\%$ presented by the *Particle Data Group* (PDG) [7]. Since the $K\eta$ system almost decouples from the even-spin K^* resonances, the relatively large $R_{K\eta}$ for $K_0^*(1430)$ presented by *BABAR* [8] is not in complete agreement with the SU(3) expectation and needs further confirmation. In addition, there is no significant amplitude observed for the $K_0^*(1430) \to K\eta$ decay in the study of $K^-p \to K^-\eta p$ interaction by LASS collaboration [39]. The situation will be reversed for the $K\eta'$ channel and the even-spin K^* states are expected to couple preferentially to $K\eta'$. In the description of $K_0^*(1430)$ contribution with coupled channels, the $K\pi$ and $K\eta'$ systems are included while the term for $K\eta$ is usually ignored in the literature [9,10,40–45]. In this context, we will focus on the contributions from $K_0^*(1430)$ for the $K\eta'$ final state in the three-body decays $B \to K\eta' h$ in this work.

The perturbative QCD (PQCD) approach [46–48] based on the k_T factorization has been widely adopted in the investigations of the quasi-two-body B meson decay processes in recent years [34,35,49–67]. Employing the same method, we have studied the whole sixteen quasi-two-body decays with the type of $B \to K_0^*(1430)h \to K\pi h$ in Ref. [34] and found that the PQCD predictions for the relevant decays agree well with the existing experimental results from the *BABAR*, Belle, and LHCb collaborations. Now we will extend the previous work to the calculation of the contributions from the subprocess $K_0^*(1430) \to K\eta'$ in the charmless three-body $B \to K\eta' h$ decays. In Ref. [10], evidence for $K_0^*(1950)$ in the $\eta_c \to \eta' K^+K^-$ decay was also found and the width was measured as $\Gamma_{K_0^*(1950)} = (80 \pm 32 \pm 20)$ MeV, which is quite different from the commonly used value $(201 \pm 34 \pm 79)$ MeV presented by LASS [6]. Due to the lack of the well-defined distribution amplitudes for $K_0^*(1950)$ and the still large uncertainty of its parameters, we shall leave it to the future studies.

The rest of this paper is organized as follows. In Sec. II, we briefly describe the PQCD framework for the quasi-two-body $B \to K_0^*(1430)h \to K\eta' h$ decays. In Sec. III, we provide the numerical results and give necessary discussions. The summary is presented in Sec. IV.

II. FRAMEWORK

In the PQCD approach, the decay amplitude of the hadronic B meson decay is expressed as the convolution of a hard kernel with the distribution amplitudes (or wave functions) for the initial B meson and the final state hadrons. For a quasi-two-body decay, the two-meson distribution amplitudes are introduced to describe the interaction between the meson pair which proceeds by the intermediate state. Then, the factorization formula of the decay amplitude $\mathcal{A}$ for the $B \to K_0^*(1430)h \to K\eta' h$ decay has the form [22,34,50]

$$\mathcal{A} = \phi_B \otimes \mathcal{H} \otimes \phi^{S\text{-wave}}_{K\eta'} \otimes \phi_h, \tag{2}$$

where the hard kernel $\mathcal{H}$ contains only one hard gluon exchange. The symbols ϕ_B, $\phi^{S\text{-wave}}_{K\eta'}$, and ϕ_h represent the distribution amplitudes for the B meson, the $K\eta'$ pair, and the bachelor meson h (π or K) respectively, which absorb the nonperturbative dynamics in the hadronization processes. In this work, the same distribution amplitudes for the heavy B meson, and the light K and π mesons are adopted as those listed in Refs. [34,52] and the references therein.

For the $K\eta'$ system, the scalar form factor $F_0^{K\eta'}(s)$ can be written via the following matrix element [68,69]

$$\begin{aligned}\langle K\eta'|\bar{q}_2 q_1|0\rangle &= C_{K\eta'}\frac{\Delta_{K\pi}}{m_{q_2} - m_{q_1}} F_0^{K\eta'}(s) \\ &= C_{K\eta'} B_0 F_0^{K\eta'}(s),\end{aligned} \tag{3}$$

where the isospin factor $C_{K\eta'} = \frac{2}{\sqrt{3}}$, the mass-squared difference $\Delta_{K\pi} = m_K^2 - m_\pi^2$, the quarks $q_1 = s$ and $q_2 = (u, d)$ for $K = (K^+, K^0)$, and $q_1 = (u, d)$ and $q_2 = s$ for $K = (K^-, \bar{K}^0)$. When there is only the intermediate state $K_0^*(1430)$ for the $K\eta'$ system, one also has [70]

$$\begin{aligned}\langle K\eta'|\bar{q}_2 q_1|0\rangle &= \langle K\eta'|K_0^*\rangle \frac{1}{\mathcal{D}_{K_0^*}} \langle K_0^*|\bar{q}_2 q_1|0\rangle \\ &= \Pi_{K_0^* K\eta'}(s)\langle K_0^*|\bar{q}_2 q_1|0\rangle,\end{aligned} \tag{4}$$

where $\langle K\eta'|K_0^*\rangle$ is the coupling constant for the resonance $K_0^*(1430)$ with $K\eta'$ and $\frac{1}{\mathcal{D}_{K_0^*}}$ stands for the propagator. Then, we have the vertex function related form factor

$$F_0^{K\eta'}(s) = \frac{\Pi_{K_0^* K\eta'}(s)\langle K_0^*|\bar{q}_2 q_1|0\rangle}{C_{K\eta'} B_0} = \frac{g_{K_0^* K\eta'} m_{K_0^*} \bar{f}_{K_0^*}}{C_{K\eta'} B_0 \mathcal{D}_{K_0^*}}, \tag{5}$$

by using the definition of the scalar decay constant $\langle K_0^*|\bar{q}_2 q_1|0\rangle = m_{K_0^*}\bar{f}_{K_0^*}$. The value of $\bar{f}_{K_0^*}$ is related to the vector decay constant $f_{K_0^*}$ through $\bar{f}_{K_0^*} = \frac{m_{K_0^*} f_{K_0^*}}{m_{q_2}(\mu) - m_{q_1}(\mu)}$ and the result $f_{K_0^*(1430)} m^2_{K_0^*(1430)} = (0.0842 \pm 0.0045)\ \mathrm{GeV}^3$ [71] is employed in this work.

The strong coupling constant $g_{K_0^* K\pi}$ can be determined from the measured partial width $\Gamma_{K_0^* \to K\pi}$ with the relation [27,30]

$$\Gamma_{K_0^* \to h_1 h_2} = \frac{q_0}{8\pi m^2_{K_0^*}} g^2_{K_0^* h_1 h_2}. \tag{6}$$

The q_0 denotes the value at $s = m^2_{K_0^*}$ for the magnitude of the momentum for the daughter h_1 or h_2 which is defined as

$$q = \frac{1}{2}\sqrt{[s - (m_{h_1} + m_{h_2})^2][s - (m_{h_1} - m_{h_2})^2]/s} \tag{7}$$

in the rest frame of the resonance $K_0^*(1430)$. But in view of the insufficient studies on the partial width for $K_0^*(1430) \to K\eta'$ and the fact that the pole mass of $K_0^*(1430)$ is slightly smaller than the threshold for $K\eta'$ in the *Review of Particle Physics* [7], it is not suitable to calculate the strong coupling constant for $K\eta'$ with the above formulae directly. We treat it under the flavor SU(3) symmetry and extract $g_{K_0^* K\eta'}$, together with $g_{K_0^* K\eta}$, from the relations

$$\frac{g_{K_0^* K\eta}}{g_{K_0^* K\pi}} = \sqrt{\frac{1}{3}}\cos\phi - \sqrt{\frac{2}{3}}\sin\phi = -0.070, \tag{8}$$

$$\frac{g_{K_0^* K\eta'}}{g_{K_0^* K\pi}} = \sqrt{\frac{1}{3}}\sin\phi + \sqrt{\frac{2}{3}}\cos\phi = 0.998, \tag{9}$$

with $g_{K_0^* K\pi}$ obtained from Eq. (6). The mixing angle $\phi = 39.3°$ [72,73] in the quark flavor basis is employed while the physical η and η' are known as the mixtures from $\eta_q = \frac{u\bar{u} + d\bar{d}}{\sqrt{2}}$ and $\eta_s = s\bar{s}$ through the relation

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \eta_q \\ \eta_s \end{pmatrix}. \tag{10}$$

According to the discussions in Refs. [9,10,40–44], we adopt the Flatté parametrization [74] for the denominator of the propagator

$$\mathcal{D}_{K_0^*} = m^2_{K_0^*} - s - i[g^2_{K\pi}\rho_{K\pi}(s) + g^2_{K\eta}\rho_{K\eta}(s) + g^2_{K\eta'}\rho_{K\eta'}(s)], \tag{11}$$

where the phase space factor $\rho_{ab}(s) = \frac{2q}{\sqrt{s}}$. It should be noted that the definition of coupling constants g_{ab} here is different by a factor of $\frac{1}{4\sqrt{\pi}}$ from the $g_{K_0^* ab}$ shown in Eqs. (8) and (9). One can find different values of $g^2_{K\pi}$, $g^2_{K\eta}$, and $g^2_{K\eta'}$ in Refs. [10,40–42,75], which are listed in Table I. We employ $g^2_{K\pi} = (0.412 \pm 0.130)\ \mathrm{GeV}^2$, $g^2_{K\eta} = (0.00204 \pm 0.00064^{+0.00113}_{-0.00089})\ \mathrm{GeV}^2$, and $g^2_{K\eta'} = (0.410 \pm 0.129 \pm 0.001)\ \mathrm{GeV}^2$ in this work through the relation

$$g^2_{K\pi(K\eta^{(\prime)})} = \frac{g^2_{K_0^* K\pi(K\eta^{(\prime)})}}{16\pi}, \tag{12}$$

where the first error comes from the uncertainty of $\Gamma_{K_0^*(1430) \to K\pi}$ with $\Gamma_{K_0^*(1430)} = (270 \pm 80)$ MeV and $\mathcal{B}(K_0^*(1430) \to K\pi) = (93 \pm 10)\%$ [7], and the second error for $g^2_{K\eta}$ and $g^2_{K\eta'}$ arises from the mixing angle $\phi = (39.3 \pm 1.0)°$ [72,73].

The S-wave $K\eta'$ distribution amplitudes along with the subprocess $K_0^*(1430) \to K\eta'$ are defined in the same way as those for the $K\pi$ system [34]

$$\Phi_{K\eta'}(z,s) = \frac{1}{\sqrt{2N_c}}[\not{p}\phi(z,s) + \sqrt{s}\phi^s(z,s) + \sqrt{s}(\not{v}\not{n} - 1)\phi^t(z,s)], \tag{13}$$

with the momentum p for the $K\eta'$ pair, the momentum fraction z for the spectator quark, the squared invariant mass $s = p^2 = m^2_{K\eta'}$ and the dimensionless vectors $v = (0, 1, 0_\mathrm{T})$ and $n = (1, 0, 0_\mathrm{T})$. The twist-2 and twist-3 light-cone distribution amplitudes are parametrized as [76–78]

TABLE I. Comparison of parameters $g^2_{K\pi}$, $g^2_{K\eta}$, and $g^2_{K\eta'}$ fitted in different literature.

$g^2_{K\pi}$ (GeV^2)	$g^2_{K\eta}$ (GeV^2)	$g^2_{K\eta'}$ (GeV^2)	Refs.
$0.458 \pm 0.032 \pm 0.044$	0	$(1.50 \pm 0.24 \pm 0.24)g^2_{K\pi}$	[10]
0.353	0	$1.15g^2_{K\pi}$	[40]
0.284 ± 0.009 (0.299 ± 0.005)	0	0.039 ± 0.053 (0.053 ± 0.016)	[41]
0.284 ± 0.012	0	$(0.62 \pm 0.06)g^2_{K\pi}$	[42]
0.515	0.030	0.671	[75]
0.412 ± 0.130	$0.00204 \pm 0.00064^{+0.00113}_{-0.00089}$	$0.410 \pm 0.129 \pm 0.001$	This work

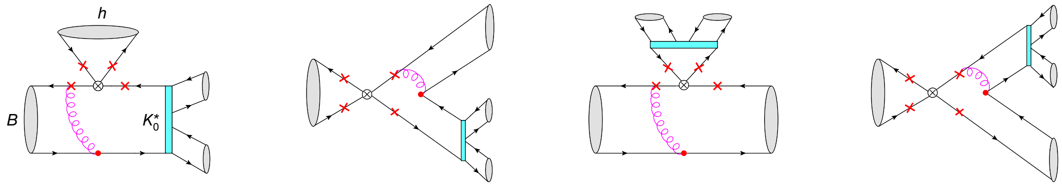

FIG. 1. Typical Feynman diagrams for the quasi-two-body decay processes $B \to K_0^*(1430)h \to K\eta' h$. The symbol $\otimes$ represents the insertion of the four-fermion vertices in the effective theory and $\times$ denotes the possible attachments of hard gluons.

$$\phi(z,s) = \frac{F_{K\eta'}(s)}{2\sqrt{2N_c}}\left\{6z(1-z)\left[a_0(\mu) + \sum_{m=1}^{\infty} a_m(\mu) C_m^{3/2}(2z-1)\right]\right\}, \tag{14}$$

$$\phi^s(z,s) = \frac{F_{K\eta'}(s)}{2\sqrt{2N_c}}, \tag{15}$$

$$\phi^t(z,s) = \frac{F_{K\eta'}(s)}{2\sqrt{2N_c}}(1-2z), \tag{16}$$

where the factor $F_{K\eta'}(s)$ is related to the scalar form factor $F_0^{K\eta'}(s)$ by the equation $F_{K\eta'}(s) = \frac{C_{K\eta'}B_0}{m_{K_0^*}} F_0^{K\eta'}(s)$. The symbols $C_m^{3/2}$ are the Gegenbauer polynomials and the value of a_0 equals to $(m_{(u,d)}(\mu) - m_s(\mu))/\sqrt{s}$ for (K_0^{*+}, K_0^{*0}) and $(m_s(\mu) - m_{(u,d)}(\mu))/\sqrt{s}$ for $(K_0^{*-}, \bar{K}_0^{*0})$. For the Gegenbauer moments a_m, the odd terms with $a_1 = -0.57 \pm 0.13$ and $a_3 = -0.42 \pm 0.22$ at the scale $\mu = 1$ GeV are adopted and the even terms could be neglected according to [76].

The differential branching fraction for the quasi-two-body decays $B \to K_0^*(1430)h \to K\eta' h$ can be written as [7]

$$\frac{d\mathcal{B}}{d\zeta} = \tau_B \frac{q_h q}{64\pi^3 m_B} |\mathcal{A}|^2 \tag{17}$$

with the variable $\zeta = \frac{s}{m_B^2}$ and the B meson mean lifetime τ_B. The magnitude of the momentum for the third meson h besides the $K\eta'$ pair is expressed as

$$q_h = \frac{1}{2}\sqrt{[(m_B^2 - m_h^2)^2 - 2(m_B^2 + m_h^2)s + s^2]/s} \tag{18}$$

in the center-of-mass frame of the resonance $K_0^*(1430)$, where m_h is the mass of the bachelor kaon or pion. The direct CP asymmetry $\mathcal{A}_{CP}$ is defined as

$$\mathcal{A}_{CP} = \frac{\mathcal{B}(\bar{B} \to \bar{f}) - \mathcal{B}(B \to f)}{\mathcal{B}(\bar{B} \to \bar{f}) + \mathcal{B}(B \to f)}. \tag{19}$$

In addition, the decay amplitudes $\mathcal{A}$ in this work have the same expressions as the cases for quasi-two-body decays $B \to K_0^*(1430)h \to K\pi h$ except for the replacement $F_{K\pi}$ to $F_{K\eta'}$ in the two-meson distribution amplitudes. The explicit expressions of the decay amplitudes $\mathcal{A}$, together with the individual amplitudes for the factorizable and nonfactorizable Feynman diagrams as shown in Fig. 1 can be found in the Appendix of Ref. [34].

III. RESULTS AND DISCUSSIONS

In the numerical calculations, the input parameters such as the QCD scale, masses, and decay constants, are summarized as follows (in units of GeV) [7,79]:

$$\begin{aligned} &\Lambda_{\rm QCD}^{f=4} = 0.250, \quad m_{B^\pm} = 5.279, \quad m_{B^0} = 5.280, \quad m_{B_s^0} = 5.367, \\ &m_{\pi^\pm} = 0.140, \quad m_{\pi^0} = 0.135, \quad m_{K^\pm} = 0.494, \quad m_{K^0} = 0.498, \\ &m_\eta = 0.548, \quad m_{\eta'} = 0.958, \quad m_{K_0^*(1430)} = 1.425 \pm 0.050, \\ &f_B = 0.190, \quad f_{B_s} = 0.230, \quad f_\pi = 0.130, \quad f_K = 0.156. \end{aligned} \tag{20}$$

The lifetimes of B mesons are adopted as $\tau_{B^\pm} = 1.638 \times 10^{-12}$ s, $\tau_{B^0} = 1.519 \times 10^{-12}$ s, and $\tau_{B_s} = 1.520 \times 10^{-12}$ s [7]. For the Wolfenstein parameters of the CKM matrix elements, we use the values $A = 0.826^{+0.018}_{-0.015}$, $\lambda = 0.22500 \pm 0.00067$, $\bar{\rho} = 0.159 \pm 0.010$, and $\bar{\eta} = 0.348 \pm 0.010$ provided by PDG [7].

Within the framework of PQCD approach, we calculate the CP averaged branching fractions and the direct CP asymmetries for the concerned quasi-two-body decays $B \to K_0^*(1430)h$ with the subprocess $K_0^*(1430) \to K\eta'$, and list them in Table II. For those PQCD predictions, the first error comes from the uncertainty of shape parameter $\omega_B = 0.40 \pm 0.04$ or $\omega_{B_s} = 0.50 \pm 0.05$ in the $B^{+,0}$ or B_s^0 meson distribution amplitudes; the second error is induced by the uncertainties of the Gegenbauer moments $a_1 = -0.57 \pm 0.13$ and $a_3 = -0.42 \pm 0.22$ in the

TABLE II. PQCD predictions of the CP averaged branching fractions and the direct CP asymmetries for the quasi-two-body $B \to K_0^*(1430)h \to K\eta' h$ decays.

Decay modes	$\mathcal{B}$	$\mathcal{A}_{CP}$
$B^+ \to K_0^*(1430)^0\pi^+ \to K^0\eta'\pi^+$	$5.27^{+2.22+0.54+0.80+0.65}_{-1.43-0.28-0.75-0.87} \times 10^{-6}$	$-0.02^{+0.00+0.00+0.00+0.00}_{-0.00-0.01-0.00-0.00}$
$B^+ \to K_0^*(1430)^+\pi^0 \to K^+\eta'\pi^0$	$2.01^{+0.77+0.23+0.34+0.23}_{-0.50-0.17-0.32-0.31} \times 10^{-6}$	$0.01^{+0.00+0.01+0.00+0.00}_{-0.01-0.01-0.00-0.00}$
$B^+ \to K_0^*(1430)^+\bar{K}^0 \to K^+\eta'\bar{K}^0$	$1.03^{+0.13+0.77+0.17+0.12}_{-0.06-0.40-0.15-0.15} \times 10^{-7}$	$-0.21^{+0.02+0.04+0.06+0.02}_{-0.02-0.02-0.06-0.01}$
$B^+ \to \bar{K}_0^*(1430)^0K^+ \to \bar{K}^0\eta'K^+$	$8.41^{+2.22+2.59+1.29+1.00}_{-1.46-2.16-1.19-1.34} \times 10^{-7}$	$0.35^{+0.06+0.07+0.03+0.01}_{-0.06-0.09-0.02-0.01}$
$B^0 \to K_0^*(1430)^+\pi^- \to K^+\eta'\pi^-$	$4.98^{+2.04+0.35+0.73+0.56}_{-1.31-0.44-0.69-0.79} \times 10^{-6}$	$0.01^{+0.01+0.01+0.00+0.00}_{-0.01-0.02-0.00-0.00}$
$B^0 \to K_0^*(1430)^0\pi^0 \to K^0\eta'\pi^0$	$3.18^{+1.37+0.35+0.41+0.39}_{-0.89-0.14-0.40-0.53} \times 10^{-6}$	$-0.02^{+0.00+0.00+0.00+0.00}_{-0.00-0.01-0.00-0.00}$
$B^0 \to K_0^*(1430)^+K^- \to K^+\eta'K^-$	$1.30^{+0.12+1.06+0.11+0.07}_{-0.09-0.56-0.02-0.13} \times 10^{-8}$	$-0.03^{+0.03+0.26+0.22+0.01}_{-0.00-0.21-0.29-0.01}$
$B^0 \to K_0^*(1430)^-K^+ \to K^-\eta'K^+$	$1.62^{+0.24+1.07+0.06+0.11}_{-0.20-0.71-0.04-0.16} \times 10^{-7}$	$-0.11^{+0.02+0.08+0.01+0.01}_{-0.00-0.05-0.00-0.00}$
$B^0 \to K_0^*(1430)^0\bar{K}^0 \to K^0\eta'\bar{K}^0$	$1.33^{+0.12+1.02+0.18+0.13}_{-0.06-0.62-0.14-0.16} \times 10^{-7}$	…
$B^0 \to \bar{K}_0^*(1430)^0K^0 \to \bar{K}^0\eta'K^0$	$8.32^{+2.15+2.94+1.19+0.96}_{-1.49-2.42-1.11-1.27} \times 10^{-7}$	…
$B_s^0 \to K_0^*(1430)^-\pi^+ \to K^-\eta'\pi^+$	$1.04^{+0.40+0.18+0.00+0.09}_{-0.28-0.17-0.00-0.13} \times 10^{-5}$	$0.18^{+0.03+0.04+0.01+0.00}_{-0.03-0.05-0.01-0.00}$
$B_s^0 \to \bar{K}_0^*(1430)^0\pi^0 \to \bar{K}^0\eta'\pi^0$	$1.17^{+0.41+0.58+0.10+0.10}_{-0.26-0.43-0.09-0.15} \times 10^{-7}$	$0.78^{+0.04+0.00+0.02+0.00}_{-0.07-0.05-0.02-0.00}$
$B_s^0 \to K_0^*(1430)^+K^- \to K^+\eta'K^-$	$5.62^{+1.23+3.02+0.83+0.58}_{-0.65-2.23-0.76-0.79} \times 10^{-6}$	$0.04^{+0.01+0.01+0.01+0.00}_{-0.01-0.02-0.01-0.00}$
$B_s^0 \to K_0^*(1430)^-K^+ \to K^-\eta'K^+$	$6.17^{+1.08+4.33+0.90+0.46}_{-0.73-3.14-0.80-0.71} \times 10^{-6}$	$-0.45^{+0.02+0.09+0.04+0.00}_{-0.02-0.14-0.04-0.00}$
$B_s^0 \to K_0^*(1430)^0\bar{K}^0 \to K^0\eta'\bar{K}^0$	$5.52^{+1.24+2.91+0.86+0.62}_{-0.68-1.53-0.78-0.83} \times 10^{-6}$	…
$B_s^0 \to \bar{K}_0^*(1430)^0K^0 \to \bar{K}^0\eta'K^0$	$4.71^{+0.63+3.71+0.78+0.36}_{-0.43-2.56-0.69-0.55} \times 10^{-6}$	…

distribution amplitudes for the $K\eta'$ system; the third one is due to the chiral masses $m_0^K = (1.6 \pm 0.1)$ GeV and $m_0^\pi = (1.4 \pm 0.1)$ GeV, and the Gegenbauer moment $a_2^h = 0.25 \pm 0.15$ of the bachelor pion or kaon; and the last one is caused by $\Gamma_{K_0^*(1430)} = (270 \pm 80)$ MeV and $\mathcal{B}(K_0^*(1430) \to K\pi) = (93 \pm 10)\%$. The errors coming from the uncertainties of other parameters are small and have been neglected. One interesting thing is that the large uncertainties of the decay width $\Gamma_{K_0^*(1430)}$ and $\mathcal{B}(K_0^*(1430) \to K\pi)$ result in an error around 10% for the branching fractions of the considered quasi-two-body decays. The reason is that the width $\Gamma_{K_0^*(1430)}$ and $\mathcal{B}(K_0^*(1430) \to K\pi)$ are used to calculate the coupling constant $g_{K_0^*K\pi}$ which appears in both the numerator and the denominator of the form factor $F_0^{K\eta'}(s)$; the effects of the variation of $\Gamma_{K_0^*(1430)}$ and $\mathcal{B}(K_0^*(1430) \to K\pi)$ on the branching fractions are partially canceled out.

For the $\eta - \eta'$ mixing angle, we follow the prediction of $\phi = (39.3 \pm 1.0)^\circ$ from Refs. [72,73] in this work. Actually, the value of ϕ has also been investigated in many phenomenological and experimental works [80–91], with results ranging from 38° to 44° approximately. Since the extraction of coupling constants $g_{K\eta^{(\prime)}}$ in this work is related to the mixing angle ϕ, it is necessary to check the ϕ dependence of the PQCD-predicted branching ratios and CP-violating asymmetries for the considered quasi-two-body decays. Taking the decay $B^+ \to K_0^*(1430)^0\pi^+ \to K^0\eta'\pi^+$ as an example, we calculate its branching ratio and direct CP asymmetry with four fixed values of ϕ in the range of 38° to 44°, and list them in Table III. One can find that the PQCD predictions of these two physical observables $\mathcal{B}$ and $\mathcal{A}_{CP}$ are not very sensitive on the variation of the given value of ϕ, and this is because the change of input ϕ in the specified range has a weak effect on the crucial parameter $g_{K_0^*K\eta'}$ extracted from the relation of Eq. (9). The case of $K\eta$ channel in Eq. (8) is quite different, and a considerable error for $g^2_{K\eta}$ caused by $\phi = (39.3 \pm 1.0)^\circ$ can be found in the last line of Table I. When we fix ϕ as 42° and 44°, the values of $g^2_{K\eta}$ are calculated to be 0.00567 GeV^2 and 0.00951 GeV^2, respectively, which are almost three and five times the central value of 0.00204 GeV^2 obtained by employing $\phi = 39.3°$. In this respect, more precise theoretical and experimental studies on the coupling constant for $K_0^*(1430) \to K\eta$ are needed in the future, and it would also be helpful to the determination of mixing angle ϕ. However, the variation of $g^2_{K\eta}$ in the Flatté formula of Eq. (11) is negligible compared to the values of $g^2_{K\pi}$ and $g^2_{K\eta'}$, and does not have much effect on our predictions of the branching ratios and CP-violating asymmetries for the considered quasi-two-body decays. Therefore, the small errors caused by the mixing angle ϕ used in this work are not taken into account in Table II.

TABLE III. The ϕ dependence of the PQCD-predicted branching ratio and direct CP asymmetry for the $B^+ \to K_0^*(1430)^0\pi^+ \to K^0\eta'\pi^+$ decay.

ϕ	38°	40°	42°	44°	39.3°
$\mathcal{B}(10^{-6})$	5.29	5.26	5.22	5.16	5.27
$\mathcal{A}_{CP}$	−0.02	−0.02	−0.02	−0.02	−0.02

In the charmless nonleptonic B meson decays, the direct CP violations arise from the interference between the decay amplitudes for the tree and penguin diagrams. As shown in Table II, there are no direct CP asymmetries for the quasi-two-body decays $B^0_{(s)} \to \bar{K}^*_0(1430)^0 K^0 \to \bar{K}^0 \eta' K^0$ and $B^0_{(s)} \to K^*_0(1430)^0 \bar{K}^0 \to K^0 \eta' \bar{K}^0$ since these decays occur only through the penguin diagrams, while the decays via the $b \to dq\bar{q}$ transition at the quark level generally have sizable direct CP violations due to the effect of parameters in the CKM matrix elements. In this work, the CP averaged branching fractions of the $B \to K^*_0(1430)h \to K\eta' h$ decays are predicted to be on the order of 10^{-8} to 10^{-5} which are non-negligible as expected because of the large coupling constant for $K^*_0(1430) \to K\eta'$. Up to now, there are still not any experimental measurements or theoretical works have been presented for the relevant three-body or quasi-two-body B meson decays, and all of these predictions are expected to be tested by the future experiments from LHCb and Belle-II.

Within the quasi-two-body approximation, the branching ratios of the two-body decay $B \to K^*_0(1430)h$ and the related quasi-two-body decay with cascade decay $K^*_0(1430) \to Kh'$ satisfy the factorization relation

$$\mathcal{B}(B \to K^*_0(1430)h \to Kh'h) \approx \mathcal{B}(B \to K^*_0(1430)h) \times \mathcal{B}(K^*_0(1430) \to Kh'). \tag{21}$$

In Table IV, we list the available experimental data for the branching fractions of the $B \to K^*_0(1430)h$ decays in the *Review of Particle Physics* [7] averaged from the measured results by Belle [12,13], *BABAR* [16,17,19], and LHCb [20]. Considering the fact that the data of those two-body decays are extracted from the measured branching fractions for the related quasi-two-body decays and $\mathcal{B}(K^*_0(1430) \to K\pi) = (93 \pm 10)\%$, together with the PQCD predictions in this work, we obtain that the ratio for the central values of the branching fractions are

$$\frac{\mathcal{B}(B^+ \to K^*_0(1430)^0\pi^+ \to K^0\eta'\pi^+)}{\mathcal{B}(B^+ \to K^*_0(1430)^0\pi^+ \to K\pi\pi^+)} = 14.5\%, \tag{22}$$

$$\frac{\mathcal{B}(B^+ \to K^*_0(1430)^+\pi^0 \to K^+\eta'\pi^0)}{\mathcal{B}(B^+ \to K^*_0(1430)^+\pi^0 \to K\pi\pi^0)} = 18.2\%, \tag{23}$$

TABLE IV. The available experimental measurements for the branching fractions of the $B \to K^*_0(1430)h$ decays from *Review of Particle Physics*.

Decay modes	Unit	Data [7]
$B^+ \to K^*_0(1430)^0\pi^+$	(10^{-5})	$3.9^{+0.6}_{-0.5}$
$B^+ \to K^*_0(1430)^+\pi^0$	(10^{-5})	$1.19^{+0.20}_{-0.23}$
$B^+ \to \bar{K}^*_0(1430)^0K^+$	(10^{-7})	3.8 ± 1.3
$B^0 \to K^*_0(1430)^+\pi^-$	(10^{-5})	3.3 ± 0.7

$$\frac{\mathcal{B}(B^0 \to K^*_0(1430)^+\pi^- \to K^+\eta'\pi^-)}{\mathcal{B}(B^0 \to K^*_0(1430)^+\pi^- \to K\pi\pi^-)} = 16.2\%. \tag{24}$$

For $\mathcal{B}(B^+ \to \bar{K}^*_0(1430)^0K^+)$, the experimental value is approximately one order of magnitude less than the theoretical predictions in Refs. [34,92–94] which also cannot be understood in this work. By adopting our previous result $\mathcal{B}(B^+ \to \bar{K}^*_0(1430)^0K^+ \to K^-\pi^+K^+) = (2.86 \pm 0.85) \times 10^{-6}$ [34] and the relation $\mathcal{B}(\bar{K}^*_0(1430)^0 \to K^-\pi^+) = \frac{2}{3}\mathcal{B}(\bar{K}^*_0(1430)^0 \to K\pi)$, the same ratio becomes

$$\frac{\mathcal{B}(B^+ \to \bar{K}^*_0(1430)^0K^+ \to \bar{K}^0\eta'K^+)}{\mathcal{B}(B^+ \to \bar{K}^*_0(1430)^0K^+ \to K\pi K^+)} = 19.6\%. \tag{25}$$

Therefore, we estimate the value of $R_{K\eta'} = \frac{\mathcal{B}(K^*_0(1430)\to K\eta')}{\mathcal{B}(K^*_0(1430)\to K\pi)}$ to be close to 20%, which is about half of *BABAR*'s result $(39.7 \pm 6.4 \pm 5.4)\%$ [10] by using the ratio of measured $\mathcal{B}(\eta_c \to K^-K^*_0(1430)^+ \to K^-K^+\pi^0)$ [8] and $\mathcal{B}(\eta_c \to K^-K^*_0(1430)^+ \to K^-K^+\eta')$ [10] in the corresponding three-body decays, and the relation $\mathcal{B}(K^*_0(1430)^+ \to K^+\pi^0) = \frac{1}{3}\mathcal{B}(K^*_0(1430)^+ \to K\pi)$. Here, the prediction of $R_{K\eta'}$ still has a large margin of error due to the considerable uncertainties for the branching fractions of corresponding quasi-two-body decays by PQCD and experiments. Besides, the validity of the factorization relation in Eq. (21) will also influence our results. In Refs. [31,95], the finite-width effects in three-body B meson decays were discussed in detail and a correction parameter η_R was defined by

$$\eta_R = \frac{\mathcal{B}(B \to Rh_3 \to h_1h_2h_3)_{\Gamma_R\to 0}}{\mathcal{B}(B \to Rh_3 \to h_1h_2h_3)} = \frac{\mathcal{B}(B \to Rh_3) \times \mathcal{B}(R \to h_1h_2)}{\mathcal{B}(B \to Rh_3 \to h_1h_2h_3)} = 1 + \delta \tag{26}$$

where the correction δ was expected to be of order Γ_R/m_R. Since the resonance $K^*_0(1430)$ has a large width, the finite-width effects should be taken into account for the corresponding decays. Numerically, the parameter η_R for $K^*_0(1430)$ within the framework of QCD factorization and the experimental parametrization for the related three-body decay amplitudes were calculated to be 0.83 ± 0.04 and 1.11 ± 0.03, respectively. It indicates that the two-body experimental results for $K^*_0(1430)$ which obtained in the narrow width approximation should be corrected by including the parameter η_R. Meanwhile, the ratio $R_{K\eta'}$ satisfies the relation

$$R_{K\eta'} = \frac{\mathcal{B}(K^*_0(1430) \to K\eta')}{\mathcal{B}(K^*_0(1430) \to K\pi)} = \frac{\eta_{K^*_0K\eta'} \times \mathcal{B}(B \to K^*_0(1430)h \to K\eta'h)}{\eta_{K^*_0K\pi} \times \mathcal{B}(B \to K^*_0(1430)h \to K\pi h)}. \tag{27}$$

If we assume that the independent corrections δ for $\eta_{K_0^* K\eta'}$ by PQCD and $\eta_{K_0^* K\pi}$ by experiments are both in the range of -0.18 to 0.18 based on the $\eta_{K_0^*}^{\mathrm{QCDF(EXPP)}}$ in Refs. [31,95] and the value of $\Gamma_{K_0^*}/m_{K_0^*}$, a maximum error of around 40% will be introduced to the estimation of $R_{K\eta'}$ which can be extracted by the ratio in Eqs. (22)–(24) and the factorization relation in Eq. (21). The similar error for $R_{K\eta'}$ from Eq. (25) should be smaller since both the numerator and the denominator are calculated for the same resonance within the same theoretical framework. Then, we give the estimation of the ratio $R_{K\eta'}$ as $(20 \pm 8)\%$. Still, it tells that the proportion for $K\eta'$ pair in the $K_0^*(1430)$ decay is non-negligible, and it is not in conflict with the presence of a dominant branching fraction for the $K\pi$ decay mode in view of the margin of existing errors in both experimental measurements and theoretical predictions.

There is no experimental information about $K_0^*(1430) \to K\eta$ except for the ratio $R_{K\eta}^{\mathrm{exp}} = (9.2 \pm 2.5^{+1.0}_{-2.5})\%$ measured in the Dalitz plot analysis of $\eta_c \to K^+ K^- \eta$ by the *BABAR* collaboration [8]. This observation is not in complete agreement with the SU(3) expectation as mentioned in their own article. According to the definitions in Eqs. (6)–(8) and the relevant parameters, it is easily to get

$$
\begin{aligned}
R_{K\eta} &= \frac{\mathcal{B}(K_0^*(1430) \to K\eta)}{\mathcal{B}(K_0^*(1430) \to K\pi)} = \frac{g^2_{K_0^* K\eta} q_0^{K\eta}}{g^2_{K_0^* K\pi} q_0^{K\pi}} \\
&= \left(\sqrt{\frac{1}{3}}\cos\phi - \sqrt{\frac{2}{3}}\sin\phi\right)^2 \\
&\quad \times \frac{\sqrt{[m^2_{K_0^*} - (m_K + m_\eta)^2][m^2_{K_0^*} - (m_K - m_\eta)^2]}}{\sqrt{[m^2_{K_0^*} - (m_K + m_\pi)^2][m^2_{K_0^*} - (m_K - m_\pi)^2]}} \\
&= 0.39\%. \qquad (28)
\end{aligned}
$$

This value is much less than *BABAR*'s result but agrees with the parametrization for the $K_0^*(1430)$ resonance in Refs. [9,10,40–45] where the $K\eta$ contribution is ignored and the coupling constant $g_{K\eta}$ is set to be zero. Moreover, this result is comparable to the data $\mathcal{B}(K_2^*(1430) \to K\eta) = (0.15^{+0.34}_{-0.10})\%$ by the PDG [7] which supports the suppression of $K\eta$ branching fraction for the even-spin strange mesons under SU(3) with $\eta - \eta'$ mixing. It also indicates that the quasi-two-body B decays with subprocess $K_0^*(1430) \to K\eta$ should have small branching ratios and are more difficult to be observed experimentally. Since the mass of $K_0^*(1430)$ is slightly smaller than $m_K + m_{\eta'}$, it is not proper to calculate the ratio $R_{K\eta'}$ by a formula similar to Eq. (28) directly.

In Fig. 2, we plot the differential branching fraction for the $B^+ \to K_0^*(1430)^0 \pi^+ \to K^0 \eta' \pi^+$ decay with the invariant mass $m_{K\eta'}$ ranging from threshold to 3.5 GeV. For a full resonant state contribution, the main portion of the branching fraction always lies in the region around the pole mass of the intermediate state. But due to the strong suppression of phase space near the threshold, one can find that the peak in the curve of the differential branching fraction for the $B^+ \to K_0^*(1430)^0 \pi^+ \to K^0 \eta' \pi^+$ decay is around 1.55 GeV which cannot be considered as a new resonant contribution. Similar behavior can also be seen in the $\eta' K^\pm$ mass projection of the measured $\eta_c \to \eta' K^+ K^-$ Dalitz plot as shown in Fig. 12(d) of Ref. [10].

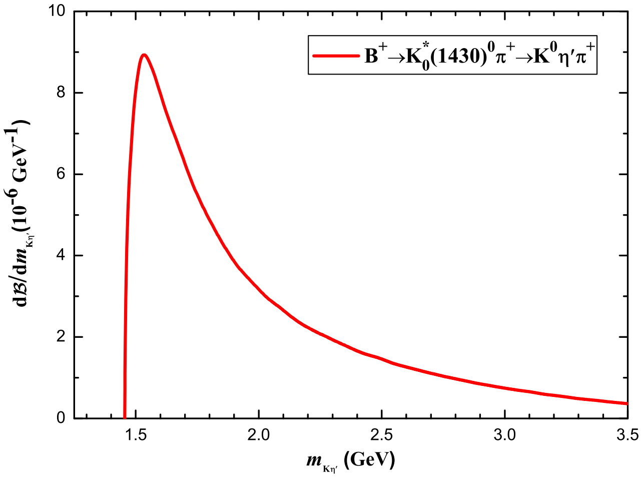

FIG. 2. Differential branching fraction for the $B^+ \to K_0^*(1430)^0 \pi^+ \to K^0 \eta' \pi^+$ decay with the invariant mass $m_{K\eta'}$ ranging from threshold to 3.5 GeV.

IV. SUMMARY

In this work, we studied the contributions from the subprocess $K_0^*(1430) \to K\eta'$ in the charmless three-body $B \to K\eta' h$ decays by employing the PQCD approach. In the description of the $K_0^*(1430)$ contribution, the related form factor was parametrized by the Flatté formula with coupled channels $K\pi$, $K\eta$, and $K\eta'$, and the strong coupling constants $g_{K_0^* K\eta^{(\prime)}}$ were extracted from $g_{K_0^* K\pi}$ under the flavor SU(3) symmetry. The *CP* averaged branching fractions for the $B \to K_0^*(1430) h \to K\eta' h$ decays were predicted to be of the order of 10^{-8} to 10^{-5}, which showed the potential of experimental measurement. In addition, the ratio between $\mathcal{B}(K_0^*(1430) \to K\eta')$ and $\mathcal{B}(K_0^*(1430) \to K\pi)$ was estimated to be about 20% by the comparison of the branching fractions for corresponding quasi-two-body decays. Since the coupling for $K_0^*(1430) \to K\eta$ is strongly suppressed within SU(3) symmetry, the quasi-two-body B decays with subprocess $K_0^*(1430) \to K\eta$ are expected to have small branching ratios and have not been considered in this work. There is no experimental measurement or theoretical analysis for the relevant three-body or quasi-two-body decays, and the PQCD predictions are expected to be tested by the future experiments.

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