Systematic studies of $DDKK$ and $D\bar{D}K\bar{K}$ four-hadron molecules

Ya-Wen Pan,[1] Ming-Zhu Liu,[2,3] Jun-Xu Lu,[1,*] and Li-Sheng Geng[1,4,5,6,†]

[1]*School of Physics, Beihang University, Beijing 102206, China*
[2]*Frontiers Science Center for Rare isotopes, Lanzhou University, Lanzhou 730000, China*
[3]*School of Nuclear Science and Technology, Lanzhou University, Lanzhou 730000, China*
[4]*Beijing Key Laboratory of Advanced Nuclear Materials and Physics, Beihang University, Beijing 102206, China*
[5]*Peng Huanwu Collaborative Center for Research and Education, Beihang University, Beijing 100191, China*
[6]*Southern Center for Nuclear-Science Theory (SCNT), Institute of Modern Physics, Chinese Academy of Sciences, Huizhou 516000, China*

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Assuming that $D_{s0}^*(2317)$ is a DK molecular state with a binding energy of 45 MeV, we investigate the existence of four-hadron molecules, $DDKK$ and $D\bar{D}K\bar{K}$, with the Gaussian expansion method. Their binding energies are 138–155 MeV and 123–163 MeV below the mass thresholds of $DDKK$ and $D\bar{D}K\bar{K}$. The $D\bar{D}K\bar{K}$ state has a decay width of 74–107 MeV due to the complex $K\bar{K}$ interaction. Further theoretical studies of and experimental searches for such four-hadron molecules can help deepen the understanding of the nonperturbative strong interaction in a nontrivial way.

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I. INTRODUCTION

Many studies have been performed on multi-hadron systems. The best-known ones are atomic nuclei and hypernuclei. The deuteron (np) is formed by a proton and a neutron with a binding energy of 2.2 MeV [1], the triton (nnp) is formed by a proton and two neutrons with a binding energy of 8.5 MeV [2], and the α particle ($nnpp$) is formed by two protons and two neutrons with a binding energy of 28.1 MeV [3]. In addition, as an isotope of helium, the ^{3}He nucleus (ppn) is formed by two protons and a neutron. Similar bound states composed of different numbers of antikaons and nucleons have also been studied [4–13]. The $\Lambda(1405)$ can be considered as a quasi-bound $\bar{K}N$ state [14,15]. The $\bar{K}\bar{K}N$ system was found to bind with a binding energy of 10–33 MeV [11]. In Ref. [12], the authors studied multihadron states composed of an antikaon and a varying number of nucleons. They found that the binding energies of $\bar{K}NN$, $\bar{K}NNN$, $\bar{K}NNNN$, and $\bar{K}NNNNN$ are 25–28 MeV, 45–50 MeV, 68–76 MeV, and 70–81 MeV, respectively. In Ref. [13], the studies of the $\bar{K}NNN$ system and $\bar{K}\bar{K}NN$ system in different methods were reviewed. The binding energies of the $\bar{K}NNN$ and $\bar{K}\bar{K}NN$ states are in the range of 17–110 MeV and 31–117 MeV.

The $D_{s0}^*(2317)$ state, discovered by the *BABAR* Collaboration in 2003 [16] and subsequently confirmed by the CLEO Collaboration [17], Belle Collaboration [18], and BESIII Collaboration [19], is a strange-charmed scalar meson and lies about 45 MeV below the DK threshold. It is not easy to interpret the $D_{s0}^*(2317)$ as a conventional $c\bar{s}$ state because its mass is far below the lightest $c\bar{s}$ scalar state predicted in the conventional quark model [20–27]. On the other hand, many studies support that it is a molecular state dynamically generated by the Weinberg-Tomozawa DK interaction [28–38], which is confirmed by many lattice QCD studies [39–43].

The DK bound-state picture naturally inspired studies of DDK, $D\bar{D}K$ [44], and DKK three-body systems [45–48].[1] In Ref. [46], the DDK system forms a three-body molecule with a binding energy of about 70 MeV. In the coupled channel approach [47], the DDK bound state was found to lie about 90 MeV below the DDK threshold. In Ref. [48], it was found that the KK repulsion is of the same order of magnitude as the DK attractive interaction, which prevents the DKK system from binding. Naively, one expects that the $DDKK$ system may bind because, with one more D meson, the extra DK attraction can help bind the four-body

*ljxwohool@buaa.edu.cn
†lisheng.geng@buaa.edu.cn

[1]See, e.g., Ref. [49] for a short review.

system. Indeed, the $DDDK$ system was shown to bind with a binding energy of 74–106 MeV [46]. To obtain a complete picture of multihadron states composed of different numbers of D mesons and kaons, similar to the light nuclear system and nucleon-antikaon system, we study in this work the $DDKK$ and $D\bar{D}K\bar{K}$ systems with the Gaussian expansion method (GEM) [50,51].

This article is organized as follows. In Sec. II, we explain how to use the GEM to solve the four-body Schrödinger equation and construct the four-body wave functions. In Sec. III, we explain how we determine the subsystem potentials involved in the four-body systems. We present and discuss the results in Sec. IV, followed by a summary in Sec. V.

II. THEORETICAL FORMALISM

To obtain the binding energies and wave functions of the four-body systems, we solve the four-body Schrödinger equation with the GEM, which has been widely applied to investigate few-body nuclear [50] and hadronic [52,53] systems. The four-body Schrödinger equation reads

$$\left[T + \sum_{1=i<j}^{6} V(r_{ij}) - E\right]\Psi_J^{\text{Total}} = 0, \tag{1}$$

where T is the kinetic-energy operator, $V(r_{ij})$ are the two-body potentials between particle i and particle j, and J is the total angular momentum of the four-body system. The four-body total wave function Ψ_J^{Total} is expressed as a sum of the wave functions of the eighteen rearrangement coordinates [50],

$$\Psi_J^{\text{Total}} = \sum_{c,\alpha} C_{c,\alpha}\Phi^c_{J,\alpha}(\boldsymbol{r}_c, \boldsymbol{R}_c, \boldsymbol{\rho}_c) \quad (c = 1-18), \tag{2}$$

where $C_{c,\alpha}$ are the expansion coefficients of relevant bases, and $c = 1$–18 denote the eighteen Jacobi coordinates of the four-body system. Thanks to the symmetry of the $DDKK$ system, the 18 Jacobi coordinates can be categorized into six groups ($c' = 1$–6), as shown in Fig. 1. α is a collection of parameters $\{l, L, \lambda, L', \Lambda, t, T, I\}$. Here, l, L, and λ are the orbital angular momenta of the coordinates r, R, and ρ in each channel. L' is the coupling of l and L, the total orbital angular momentum Λ is the coupling of L' and λ.t and T are the isospin of the coordinates r and R in each channel, and the total isospin I is the coupling of T and t (or the last particle), which will be discussed later.

Because the spin of D and $K(\bar{K})$ is zero, there is no need to consider the spin wave function. Therefore, the wave function of each channel can be written as

$$\Phi^{c'}_{J,\alpha}(\boldsymbol{r}_{c'}, \boldsymbol{R}_{c'}, \boldsymbol{\rho}_{c'}) = \left[\left[\phi^G_{n_{c'}l_{c'}}(\boldsymbol{r}_{c'})\psi^G_{N_{c'}L_{c'}}(\boldsymbol{R}_{c'})\right]_{L'_{c'}}\chi^G_{\nu_{c'}\lambda_{c'}}(\boldsymbol{\rho}_{c'})\right]_{\Lambda} \otimes H^{c'}_{tT,I}, \tag{3}$$

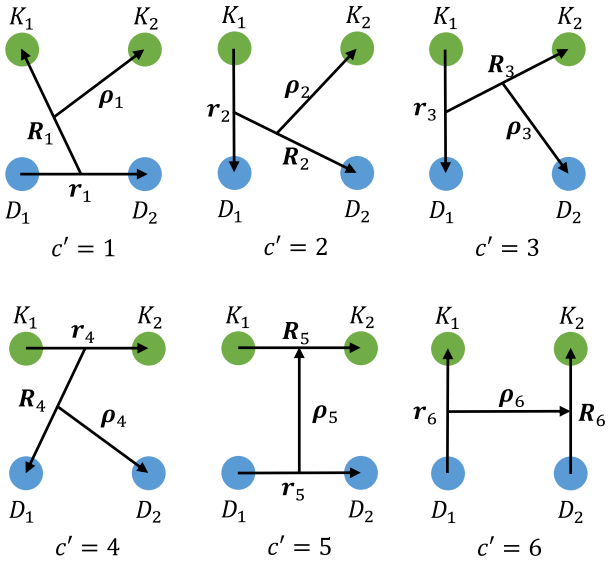

FIG. 1. Six Jacobi coordinates of the $DDKK$ system. Symmetrization is implicit between D mesons and between kaons.

where $\phi^G_{n_{c'}l_{c'}}$, $\psi^G_{N_{c'}L_{c'}}$, and $\chi^G_{\nu_{c'}\lambda_{c'}}$ are the spatial wave functions and $H^{c'}_{tT,I}$ is the total isospin wave function. For the details of the spatial wave functions, one can refer to Ref. [46]. Noting that although in the stochastic variational method of Suzuki *et al.* [54], a single set with nondiagonal Gaussian functions should, in principle, provide the same results, here, we employ different sets of Jacobi variables in the GEM shown in Table I.

As shown in Fig. 1, there are two types of configurations, i.e., K type ($c' = 1$–4) and H type ($c' = 5, 6$). As a result, the total isospin wave function $H^{c'}_{tT,I}$ has two coupling ways, $H^K_{tT,I}$ and $H^H_{tT,I}$, which can be written as

$$\begin{aligned} H^K_{tT,I} &= \left[\left[\left[\eta_{\frac{1}{2}}(i)\eta_{\frac{1}{2}}(j)\right]_t\eta_{\frac{1}{2}}(k)\right]_T\eta_{\frac{1}{2}}(n)\right]_I, \\ H^H_{tT,I} &= \left[\left[\eta_{\frac{1}{2}}(i)\eta_{\frac{1}{2}}(j)\right]_t\left[\eta_{\frac{1}{2}}(k)\eta_{\frac{1}{2}}(n)\right]_T\right]_I, \end{aligned} \tag{4}$$

where $\eta_{\frac{1}{2}}(i)$ is the isospin wave function of particle i. The possible values of t and T are listed in Table I.

III. TWO-BODY POTENTIALS

A. The DK interaction

In this work, we employ the DK interaction used in Refs. [46,47]. The dominant contribution to the S-wave DK interaction is the Weinberg-Tomozawa (WT) term between a D meson and a kaon, which can be formulated as a standard quantum mechanical potential in the nonrelativistic limit,

$$V_{DK}(q) = -\frac{C_W(I)}{2f_\pi^2}, \tag{5}$$

TABLE I. Four-body isospin space and the Gaussian range parameters for the $I(J^P) = 0(0^+)$ configuration of the $DDKK$ system. Lengths are in units of fm.

c'	Isospin configuration	$n_{\max}$	r_1	$r_{\max}$	$N_{\max}$	R_1	$R_{\max}$	$\nu_{\max}$	ρ_1	$\rho_{\max}$
1	$((DD)_1K)_{1/2}K$	8	0.2	10	6	0.4	10	6	0.6	10
2	$((DK)_{0(1)}D)_{1/2}K$	8	0.1	10	5	0.3	10	6	0.6	10
3	$((DK)_{0(1)}K)_{1/2}D$	8	0.3	10	5	0.6	10	6	0.6	10
4	$((KK)_1D)_{1/2}D$	8	0.1	10	5	0.4	10	6	0.6	10
5	$(DD)_1(KK)_1$	8	0.2	9	6	0.1	10	6	0.2	8
6	$(DK)_{0(1)}(DK)_{0(1)}$	8	0.1	8	5	0.1	10	8	0.2	8

where the pion decay constant $f_\pi \approx 130$ MeV and the strength of the WT term $C_W(I)$ depends on the isospin of the DK system,

$$C_W(0) = 2, \qquad C_W(1) = 0. \tag{6}$$

After Fourier transform and with a local Gaussian regulator, the DK potential in coordinate space reads [46]

$$V_{DK}(r, R_C) = -\frac{C_W(I)}{2f_\pi^2}\frac{e^{-(r/R_C)^2}}{\pi^{3/2}R_C^3}. \tag{7}$$

According to chiral perturbation theory (ChPT) [36], the leading-order S-wave DK interaction in isospin zero is attractive, while the next-to-leading (NLO) order correction is weakly repulsive. The LO DK interaction reads

$$V_{DK}(r, R_C) = C(R_C)\frac{e^{-(r/R_C)^2}}{\pi^{3/2}R_C^3}, \tag{8}$$

where R_C is the cutoff, and $C(R_C)$ is the running coupling constant describing the strength of the LO DK interaction. We should also add a short-range repulsive core in the DK interaction as the NLO correction, whose isospin factor $C_W(I)$ is the same as the LO term. So the total DK interaction can be written as

$$\begin{aligned} V_{DK}(r, R_C) &= C_S\frac{e^{-(r/R_S)^2}}{\pi^{3/2}R_S^3} + C(R_C)\frac{e^{-(r/R_C)^2}}{\pi^{3/2}R_C^3} \\ &= C'_S e^{-(r/R_S)^2} + C'_L e^{-(r/R_C)^2}, \end{aligned} \tag{9}$$

where C'_S, C'_L are coupling constants, and R_S, R_C are cutoffs for the repulsive and attractive potentials. The uncertainty coming from the NLO corrections can be estimated by varying the cutoff R_C within a sensible range. Specifically, we require that $C'_S > |C'_L|$ and $R_S < R_C$ and take $R_S = 0.5$ fm, $R_C = 1.0$, 2.0, 3.0 fm, $C'_S = 0$, 500, 1000, 3000 MeV. For each set of these parameter values, C'_L is determined by reproducing the $D^*_{s0}(2317)$ as a DK bound state with a binding energy of 45 MeV.

B. The DD interaction

Due to a lack of data, we turn to the one-boson exchange (OBE) model [55,56] to describe the $DD(\bar{D})$ interaction. In the OBE model, the potential between two hadrons is generated by exchanging light mesons (i.e., π, σ, ρ, and ω). The OBE model has been successfully employed in accurately describing the nuclear force [55,56] and predicting the existence of heavy hadron molecules [57]. In recent years, it has been widely applied to study newly discovered hadronic molecules [58–70].

The $DD(\bar{D})$ potential is generated by exchanging σ, ρ, ω [58],

$$V_{DD(\bar{D})} = V_\sigma(r, \Lambda) + \xi V_\omega(r, \Lambda) + V_\rho(r, \Lambda), \tag{10}$$

where a form factor and a cutoff Λ regularize the contribution of each light meson. ξ equals to $+1$ for the DD system, and -1 for the $D\bar{D}$ system, which are related to each other by the G parity of light mesons. The particular contribution of each meson is written as [58]

$$V_\sigma(r, \Lambda) = -g_\sigma^2 m_\sigma W_C\left(m_\sigma r, \frac{\Lambda}{m_\sigma}\right), \tag{11}$$

$$V_\rho(r, \Lambda) = +\vec{\tau}_1 \cdot \vec{\tau}_2 g_\rho^2 m_\rho W_C\left(m_\rho r, \frac{\Lambda}{m_\rho}\right), \tag{12}$$

$$V_\omega(r, \Lambda) = +g_\omega^2 m_\omega W_C\left(m_\omega r, \frac{\Lambda}{m_\omega}\right), \tag{13}$$

where

$$W_C(x, \lambda) = \frac{e^{-x}}{4\pi x} - \lambda\frac{e^{-\lambda x}}{4\pi\lambda x} - \frac{(\lambda^2 - 1)}{2\lambda}\frac{e^{-x}}{4\pi}. \tag{14}$$

Since we focus on four-body bound states, we only take the real part of the $DD(\bar{D})$ interaction, as argued in Ref. [68]. The masses of the exchanged bosons are $m_\sigma = 0.6$ GeV, $m_\rho = 0.77$ GeV, $m_\omega = 0.78$ GeV, and the couplings are $g_\rho = g_\omega = 2.6$, $g_\sigma = 3.4$. The cutoff Λ is determined by reproducing the mass of the $X(3872)$ as a $D\bar{D}^*$ molecule, yielding $\Lambda = 1.01^{+0.19}_{-0.10}$ GeV [58]. In this work, we set the

cutoff to $\Lambda = 1.0$ GeV and neglect the small uncertainty due to the exploratory nature of the present work.

C. The KK interaction

The kaon-kaon interaction has been widely investigated in studies of $a_0(980)$ [71,72], $KK\bar{K}$ [73], $K\bar{K}N$ [74], $\bar{K}\bar{K}N$ [11], etc. Considering that the typical kinetic energy of the kaon in such systems is small compared to its mass, we adopt the nonrelativistic $KK(K\bar{K})$ potential in the present work. For the S-wave KK effective potential, because of the Bose-Einstein statistics, $I = 0$ is forbidden. Therefore, we only consider the $I = 1$ interaction. Following Refs. [11,73,74], the KK and $K\bar{K}$ potentials can be written as one Gaussian function,

$$V_{KK}(r) = v_0 e^{-(r/b)^2}, \tag{15}$$

where v_0 and b are the strength of the potential and the interaction range, respectively. The interaction ranges of KK and $K\bar{K}$ are assumed to be the same. The strength of the KK interaction is determined by reproducing the lattice QCD scattering length $a_{K^+K^+} = -0.14$ [75]. The strength of the $K\bar{K}$ interaction is determined by fitting to the masses and widths of $f_0(980)$ and $a_0(980)$ [76] assuming that $f_0(980)$ and $a_0(980)$ are quasibound states of $K\bar{K}$ in the $I = 0$ and $I = 1$ channels, respectively. Here, we adopt the parameters for the two cases studied in Refs. [11,73,74]. In case A, $v_0^{KK} = 104$ MeV, $v_0^{K\bar{K}} = -630 - 210i$ MeV, and $b = 0.66$ fm, while in case B, $v_0^{KK} = 313$ MeV, $v_0^{K\bar{K}} = -1155 - 283i$ MeV, and $b = 0.47$ fm.

IV. RESULTS AND DISCUSSIONS

The binding energies, expectation values, and root-mean-square (rms) radii of the $DDKK$ system are given in Table 1. The binding energies are defined with respect to the full-dissociation threshold. Noting that considering only one channel c', we can obtain a sizeable fraction of the binding energy, about 120–130 MeV, about 20 MeV lower than the result from the full GEM. Besides, the single channel results also show that channels $c' = 2, 3, 6$ play a more important role, indicating that the DK cluster dominates the $DDKK$ system. Here, we choose the full GEM to obtain more accurate results to compare with the DDK system. The binding energies as functions of potential parameters are shown in Fig. 2. Clearly, for all parameters studied, the $DDKK$ system is always bound with a binding energy of 138–155 MeV. As the strength of the repulsive core C_S' and the cutoff R_C increases, the binding energy decreases. In addition, as R_C grows, the differences among different C_S' and between case A and case B become smaller. This trend can also be seen in the potentials for different R_C, as shown in Fig. 3. As R_C increases, the total potential becomes flatter, and the differences between case A and case B and among different C_S' decrease, especially for the range of our interest, 1 fm $< r <$ 2 fm, which is responsible for the trend observed above. From the expectation values of the potentials, one concludes that the DK interaction plays a dominant role. The strength of the repulsive KK interaction is compatible with the strength of the attractive DD interaction, which is much smaller than the strength of the attractive DK interaction. Therefore, the differences between case A and case B are minor. The kinetic energy of the four-body system is much smaller

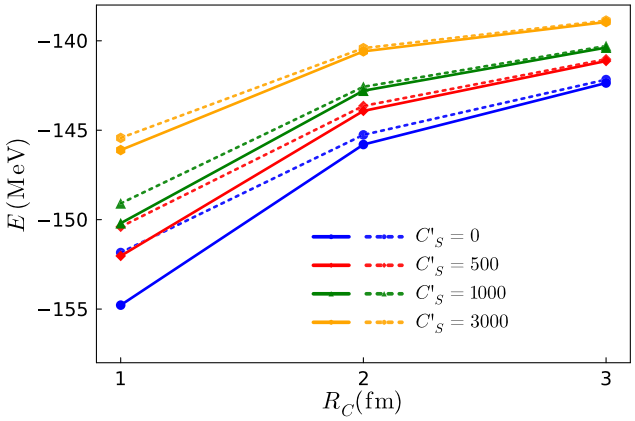

FIG. 2. Binding energies of the $DDKK$ system as functions of the cutoff R_C. The solid lines and dashed lines correspond to case A and case B, respectively. Blue, red, green, and orange lines are for $C_S' = 0$, 500, 1000, 3000 MeV, respectively.

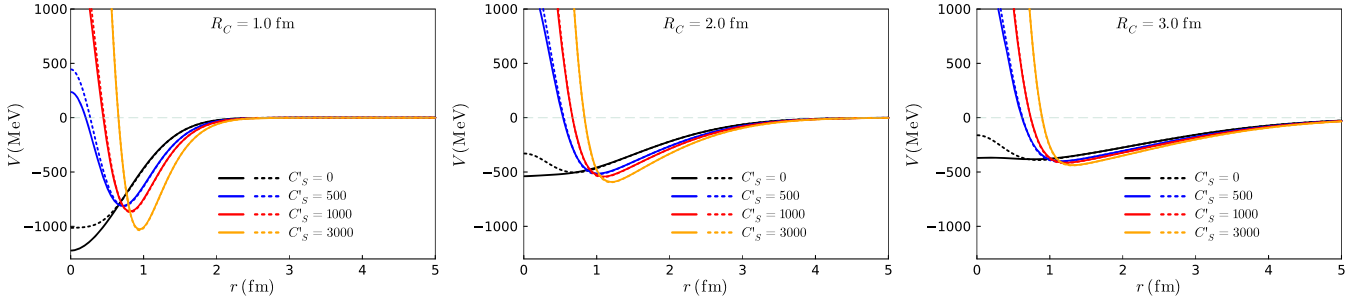

FIG. 3. Total potential of the $DDKK$ system, the solid and dashed lines are for case A and case B. The black, blue, red, and orange lines are for $C_S' = 0$, 500, 1000, 3000 MeV. The left, middle, and right figures are the total potential obtained for $R_C = 1.0$, 2.0, 3.0 fm.

than the kaon mass, which justifies the use of a nonrelativistic potential for the kaon-kaon interaction. The rms radius of the KK subsystem is larger than those of the DK subsystem and the DD subsystem. As R_C and C_S' increase, so do the rms radii, and the rms radii of case B are a bit larger than those of case A. The rms radii of each subsystem are all in the range of 1.0–3.3 fm, consistent with the typical size of a hadronic molecule. Because the basis functions are not orthogonal to each other, we can not get the accurate probability of each channel, which is calculated by $\langle \Psi_J^{c'}|\Psi_J^{c'}\rangle$. However, their relative sizes show that the configurations $(DK)_0(DK)_0$ (38%) and $((DK)_0K)_{1/2}D$ (51%) play a dominant role.

The minimal quark content of the $DDKK$ system is $cc\bar{s}\,\bar{s}$. States containing such quark configurations have been studied in Refs. [77,78]. Reference [78] studied bound and resonant states of double-heavy tetraquarks with strangeness in the chiral quark model. For $cc\bar{s}\,\bar{s}$ with $I(J^P) = 0(0^+)$, only two resonant states with masses of 4176 and 4250 MeV are found. In the quark delocalization color screening model [77], two resonant states with $I(J^P) = 0(0^+)$ are obtained, a molecular resonant state with a mass of around 4256 MeV and a width of around 60 MeV, and a compact resonant state with a mass of around 4308 MeV and a width of around 19 MeV. In our current work, the mass of the $DDKK$ bound state is 4581 MeV, about 300 MeV higher than the states obtained in the quark models, possibly mixing with radially excited states of $cc\bar{s}\,\bar{s}$.

In Ref. [46], the $DDDK$ system is shown to have a binding energy of 91–107 MeV, which is about 50 MeV smaller than the binding energy of the $DDKK$ system. In both systems, the DK interactions are dominant. Intuitively, there are four DK pairs in the $DDKK$ system, one more than in the $DDDK$ system, which can qualitatively explain the difference of binding energy.

Next, we investigate the $D\bar{D}K\bar{K}$ system. There are no identical particles in this system. Therefore, all the 18 Jacobian channels are distinguishable. Compared to the $DDKK$ system, the $D\bar{D}K\bar{K}$ system is unique because the $K\bar{K}$ interaction is complex. To study the $D\bar{D}K\bar{K}$ system, we follow the method of Ref. [74] in which the imaginary

TABLE II. Binding energies, expectation values of the Hamiltonian (potential and kinetic energies), and rms radii of the four-body system $DDKK$. Energies are in units of MeV and radii are in units of fm. The numbers outside and inside the brackets represent cases A and B. The cutoffs R_S and R_C are in units of fm, and the coupling constants C_S' and C_L' are in units of MeV.

R_S	R_C	C_S'	C_L'	E	$\langle T\rangle$	$\langle V_{DD}\rangle$	$\langle V_{DK}\rangle$	$\langle V_{KK}\rangle$	r_{DD}	r_{DK}	r_{KK}
0.5	1	0	−320.1	−154.78(−151.84)	239.38(233.70)	−9.07(−8.89)	−396.42(−389.32)	11.33(12.67)	1.03(1.04)	1.20(1.22)	1.54(1.57)
		500	−455.4	−152.03(−150.40)	189.39(187.46)	−7.58(−7.52)	−342.14(−339.24)	8.30(8.90)	1.14(1.14)	1.33(1.34)	1.70(1.72)
		1000	−562.6	−150.20(−149.09)	174.04(173.10)	−6.85(−6.83)	−324.12(−322.40)	6.72(7.04)	1.20(1.21)	1.41(1.42)	1.81(1.83)
		3000	−838.7	−146.11(−145.43)	181.27(180.73)	−6.24(−6.24)	−325.73(−324.63)	4.59(4.91)	1.31(1.31)	1.58(1.59)	2.04(2.05)
0.5	2	0	−149.1	−145.80(−145.26)	113.17(112.63)	−4.95(−4.92)	−258.35(−257.35)	4.32(4.37)	1.43(1.44)	1.69(1.70)	2.15(2.17)
		500	−178.4	−143.92(−143.64)	95.39(95.33)	−4.07(−4.07)	−238.19(−237.83)	2.96(2.93)	1.58(1.58)	1.87(1.88)	2.39(2.40)
		1000	−195.0	−142.79(−142.57)	95.12(95.09)	−3.84(−3.85)	−236.55(−236.26)	2.48(2.45)	1.64(1.64)	1.97(1.97)	2.52(2.53)
		3000	−225.9	−140.59(−140.41)	102.70(102.65)	−3.80(−3.80)	−241.43(−241.17)	1.94(1.91)	1.70(1.70)	2.12(2.12)	2.73(2.74)
0.5	3	0	−107.0	−142.36(−142.17)	73.80(73.71)	−3.30(−3.29)	−215.21(−214.89)	2.34(2.29)	1.75(1.75)	2.07(2.08)	2.64(2.65)
		500	−119.4	−141.13(−141.03)	64.78(64.81)	−2.76(−2.76)	−204.80(−204.67)	1.65(1.59)	1.90(1.90)	2.28(2.28)	2.90(2.90)
		1000	−125.6	−140.38(−140.29)	65.27(65.29)	−2.65(−2.65)	−204.43(−204.31)	1.43(1.38)	1.96(1.96)	2.37(2.37)	3.02(3.03)
		3000	−136.2	−138.95(−138.87)	69.59(69.61)	−2.64(−2.64)	−207.07(−206.96)	1.16(1.12)	2.02(2.02)	2.52(2.52)	3.23(3.24)

TABLE III. Binding energies, expectation values of the Hamiltonian (potential and kinetic energies), and rms radii of the four-body system $D\bar{D}K\bar{K}$. Energies are in units of MeV and radii are in units of fm. The relevant parameter values are $R_S = 0.5$ fm and $C_S' = 0$ MeV.

R_C	C_L'	E	$\langle T\rangle$	$\langle V_{D\bar{D}}\rangle$	$\langle V_{DK+\bar{D}\bar{K}}\rangle$	$\langle V_{\bar{D}K+D\bar{K}}\rangle$	$\langle V_{K\bar{K}}\rangle$	$r_{D\bar{D}}$	$r_{DK/\bar{D}\bar{K}}$	$r_{\bar{D}K/D\bar{K}}$	$r_{K\bar{K}}$
$b = 0.66$ fm $v_0^{K\bar{K}} = -630 - 210i$ MeV											
1	−320.1	$-156.15 - 53.35i$	307.65	−12.68	−228.98	−62.10	−160.04	1.07	1.08	1.16	1.14
2	−149.1	$-133.44 - 42.01i$	186.93	−7.09	−142.76	−44.49	−126.04	1.46	1.51	1.56	1.40
3	−107.1	$-123.60 - 37.42i$	147.15	−4.79	−116.33	−37.39	−112.25	1.81	1.85	1.89	1.57
$b = 0.47$ fm $v_0^{K\bar{K}} = -1155 - 283i$ MeV											
1	−320.1	$-162.95 - 53.17i$	368.87	−13.32	−234.25	−67.22	−217.01	1.04	1.04	1.10	1.04
2	−149.1	$-135.33 - 41.60i$	231.47	−7.24	−144.20	−45.56	−169.80	1.44	1.48	1.53	1.30
3	−107.1	$-124.37 - 37.04i$	186.49	−4.86	−116.99	−37.83	−151.18	1.80	1.83	1.87	1.48

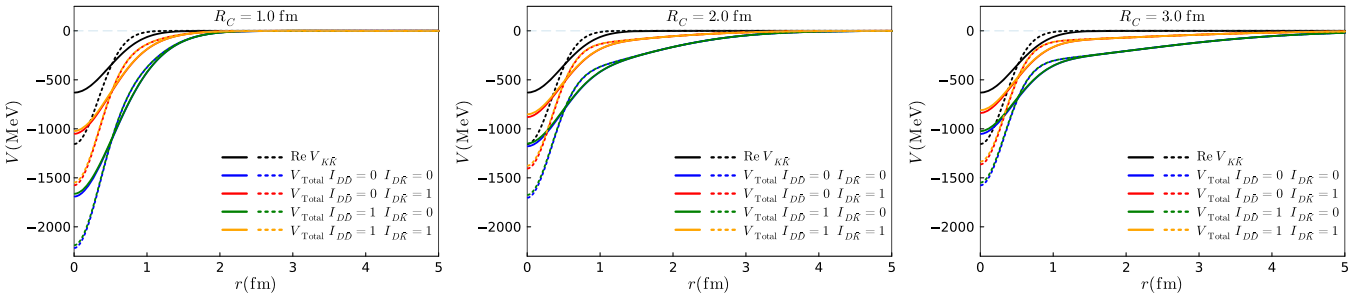

FIG. 4. Total potential of the $D\bar{D}K\bar{K}$ system, the solid and dashed lines are for case A and case B. The black line is for the real part of the $K\bar{K}$ interaction. The left, middle, and right figures are the total potential obtained for $R_C = 1.0$, 2.0, 3.0 fm.

part of the $K\bar{K}$ interaction is treated as a perturbative correction in the $K\bar{K}N$ molecular state. They also discussed the difference between the complex energy obtained in perturbative and nonperturbative treatments for the $K\bar{K}$ two-body system. The conclusion is that the imaginary energies obtained in both methods are similar, while the real energies increase slightly because of the higher-order corrections of the perturbative expansion [74].

We first consider the real part of the Hamiltonian in the GEM calculation. The total wave function Ψ and the binding energy $\mathrm{Re}(E)$ are obtained using the lowest-energy solution. The imaginary part of the energy $\mathrm{Im}(E)$ is estimated by calculating the expectation value of the imaginary part of the Hamiltonian ($\mathrm{Im}(V_{K\bar{K}})$) with the obtained total wave function Ψ,

$$\mathrm{Im}(E) = \langle\Psi|\mathrm{Im}(V_{K\bar{K}})|\Psi\rangle. \tag{16}$$

The complex energy is given as $E = \mathrm{Re}(E) + i\mathrm{Im}(E)$ and the decay width is $\Gamma = -2i\mathrm{Im}(E)$.

According to chiral perturbation theory [36], the leading order $D\bar{K}$ interaction is only half of the leading order DK interaction in isospin zero, while in isospin one, $V^{I=1}_{D\bar{K}} = -V^{I=0}_{D\bar{K}}$. In momentum space, the $I = 0$ $D\bar{K}$ can not form a bound state [36]. However, if we naively reduce by half the DK interactions given in Table II, which is in position space, they can generate a bound state. The reason is that two range parameters R_C and R_S in the form of $\frac{e^{-(r/R_{C(S)})^2}}{\pi^{3/2}R^3_{C(S)}}$ will arise in position space potential due to the Fourier transform with a local Gaussian regulator. While in momentum space, the potential is a constant, meaning that naively reducing the DK potential in position space by half is inappropriate. But if we remove the repulsive core in the DK interaction, i.e., $C'_S = 0$, and take $R_C < 0.78$ fm, one can simultaneously obtain a DK bound state with a binding energy of 45 MeV and an unbound $D\bar{K}$ system. By taking $C'_S = 0$, $R_C = 0.77$ fm, and $b = 0.66$ fm, we can obtain a bound $D\bar{D}K\bar{K}$ state with a binding energy of 166.09 MeV and a width of 117.22 MeV. Besides, we find that the smaller the R_C, the more bound the $D\bar{D}K\bar{K}$ system becomes. Therefore, setting the parameters for the $D\bar{D}K\bar{K}$ system the same as those for the $DDKK$ system, i.e., $R_C = 1$, 2, 3 fm and $b = 0.47$, 0.66 fm, is enough to describe the bound state of the $D\bar{D}K\bar{K}$ system, and can help us to estimate the uncertainty originating from R_C and compare this system with the $DDKK$ system. In addition, we fix $C'_S = 0$ as mentioned above.

The results are listed in Table III. The parameters of the $K\bar{K}$ interaction have tiny effects on the $D\bar{D}K\bar{K}$ system. The binding energies of the $D\bar{D}K\bar{K}$ system range from 123 to 163 MeV, compatible with those of the $DDKK$ system. Although the interaction between D and kaon (antikaon) of the $D\bar{D}K\bar{K}$ system is weaker than that of the $DDKK$ system, the $K\bar{K}$ interaction is strong enough to yield a total potential for the $D\bar{D}K\bar{K}$ system compatible with that for the $DDKK$ system as shown in Fig. 4. From the perspective of expectation values, the dominant interactions in the $D\bar{D}K\bar{K}$ system are the DK interaction and $K\bar{K}$ interaction. As R_C increases and b decreases, the $K\bar{K}$ interaction plays a more important role, which can also be seen in Fig. 4. The rms radii of each subsystem are all in the range of 1.0–2.0 fm. The imaginary part of the expectation values of the total potential $\langle\Psi|\mathrm{Im}(V)|\Psi\rangle$ is about -40 MeV, and the real part $\langle\Psi|\mathrm{Re}(V)|\Psi\rangle$ is about -320 MeV, $|\langle\Psi|\mathrm{Im}(V)|\Psi\rangle| \ll |\langle\Psi|\mathrm{Re}(V)|\Psi\rangle|$, which justifies the perturbative treatment. As for the three-body subsystems of the $D\bar{D}K\bar{K}$ system, $D\bar{D}K$ was found to bind with a binding energy of about 45 MeV [44,79]. For the $DK\bar{K}$ system, a state was found with a mass of about 2833–2855 MeV, made mostly of $Df_0(980)$ [48].

V. SUMMARY AND CONCLUSION

In this work, we studied the four-body systems with $I(J^P) = 0(0^+)$ composed of two D mesons and two kaons (antikaons), $DDKK$ and $D\bar{D}K\bar{K}$. We adopted the OBE model to describe the DD potential, and the cutoff is determined by reproducing the $X(3872)$ as a $D\bar{D}^*$ molecule. We employed the WT term as the LO chiral potential and a repulsive core as the NLO correction in the nonrelativistic limit to describe the S-wave DK interaction. The uncertainties of the DK interaction were estimated by varying the cutoff R_C and coupling constant C'_S. The other

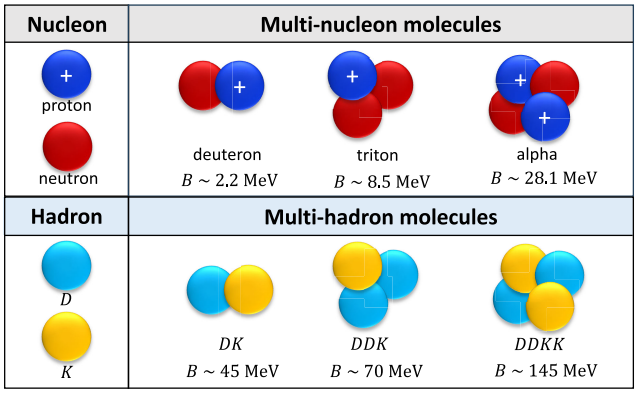

FIG. 5. Analogies between the few-body bound states of nucleons, deuteron, triton, and alpha, and the few-body bound states of D and K.

coupling constant C_L' was determined by reproducing the $D_{s0}^*(2317)$ as a DK bound state. The strength of the $D\bar{K}$ interaction was taken as half of the DK interaction according to chiral perturbation theory. We employed a nonrelativistic form to describe the interaction between two kaons and that between a kaon and an antikaon, which were expressed in the form of one Gaussian function. Two interaction ranges were considered. The strengths of the KK and $K\bar{K}$ interaction were obtained by reproducing the scattering length $a_{K^+K^+} = -0.14$ and fitting to the masses and widths of $f_0(980)$ and $a_0(980)$, respectively.

The four-body Schrödinger equations were solved using the Gaussian expansion method. For the $DDKK$ system, the binding energy is about 138–155 MeV. The dominant contribution is the DK interaction, and the strength of the repulsive KK interaction is compatible with the strength of the attractive DD interaction, which has a tiny impact on the binding energy. As a result, we obtained a complete multihadron picture composed of D mesons and kaons similar to that of nucleons, as shown in Fig. 5. The DK molecule corresponds to deuteron (np), DDK to triton (nnp), and $DDKK$ to the alpha particle ($nnpp$).

Although, the only difference between the $DD\bar{K}\,\bar{K}$ system and the $DDKK$ system is the interaction between the D meson and the antikaon (kaon), the results show that the $DD\bar{K}\,\bar{K}$ system can not bind, which can be understood intuitively due to the unbound two-body subsystems. We note that fully Borromean four-body bound states can exist, although their two-body, as well as three-body subsystems, are unbound, e.g., $\Lambda\Lambda nn$ [80,81]. The $D\bar{D}K\bar{K}$ system is a bit more complicated due to the nonexistence of identical Jacobian coordinates and the imaginary term of the Hamiltonian. We treated the imaginary term perturbatively based on the wave functions obtained with only the real part of the Hamiltonian. The binding energy is 123–163 MeV, and the decay width is 74–107 MeV. The perturbative treatment is justified via $|\langle\Psi|\mathrm{Im}(V)|\Psi\rangle| \ll |\langle\Psi|\mathrm{Re}(V)|\Psi\rangle|$.

Undoubtedly, experimental searches for and further theoretical studies of these four-hadron molecules are essential to verify the molecular nature of the $D_{s0}^*(2317)$ and test our understanding of the DK, $D\bar{K}$, DD, KK, and $K\bar{K}$ interactions. According to Ref. [82], the prompt production rates of these multihadron molecules in e^+e^- colliders might be too small to be realistic. One needs to study other processes, such as heavy-ion collisions, to search for these states.

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