Higher-order NLO radiative corrections to polarized muon decay spectrum

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Higher-order QED radiative corrections to the muon decay spectrum are evaluated within the QED structure function approach in the next-to-leading-order logarithmic approximation. New analytical results are given in the $\mathcal{O}(\alpha^3 \ln^2(m_\mu^2/m_e^2))$ order. Earlier results in the $\mathcal{O}(\alpha^2 \ln^1(m_\mu^2/m_e^2))$ and $\mathcal{O}(\alpha^3 \ln^3(m_\mu^2/m_e^2))$ orders are partially corrected. Numerical estimates of different contributions are presented.

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I. INTRODUCTION

Studies of muon decay

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu \tag{1}$$

are one of the cornerstones of modern particle physics. This process is almost a pure weak-interaction process with small QED, QCD, and possibly new physics additions. High-precision and high-sensitivity experiments with muons can test small deviations from the Standard Model (SM) predictions, which would be the traces of new physics. Differential distributions in muon decays allow for the study of properties of weak interactions, including even the Dirac or Majorana nature of neutrinos [1,2]. Such experiments as TWIST [3,4], Mu2e [5], and Mu3e [6] require accurate advanced theoretical predictions. The precision of the predictions can be increased by the calculation of higher-order radiative corrections.

QED corrections to the muon lifetime are known from the works [7–13] up to the $\mathcal{O}(\alpha^2)$ order, and the $\mathcal{O}(\alpha^3)$ corrections were also recently calculated [14]. The TWIST experiment required corrections to the muon decay spectrum in at least the $\mathcal{O}(\alpha^2)$ order. In Ref. [15], radiative corrections to the unpolarized muon decay spectrum to the order $\mathcal{O}(\alpha^2 L)$ where $L \equiv \ln(m_\mu^2/m_e^2)$ were found, and in Ref. [16], radiative corrections to the polarized muon decay spectrum in the $\mathcal{O}(\alpha^3 L^3)$ and $\mathcal{O}(\alpha^2 L)$ orders were calculated analytically. Complete two-loop QED corrections to the muon decay spectrum were calculated numerically [17] in a restricted kinematics domain.

Recently, the results for the pure-photonic part of the higher-order QED corrections in the leading and next-to-leading logarithmic approximations were presented in [18].

Our aim here is to calculate $\mathcal{O}(\alpha^3 L^2)$ order corrections to the electron energy spectrum in decays of polarized and unpolarized muons. The paper is organized as follows: In the next section, we describe the application of the QED structure function formalism for the calculation of corrections to the muon decay spectrum. Section III contains numerical results and a discussion about the factorization scale choice. Lengthy formulas with analytic results are shifted to the appendixes.

II. CORRECTIONS TO THE ELECTRON ENERGY SPECTRUM

Analytic calculations of higher-order radiative corrections to the muon decay spectrum, as well as to differential distributions of other processes like Bhabha scattering and electron-positron annihilation, are very difficult because of the presence of several energy scales. Only a few complete results for $\mathcal{O}(\alpha^2)$ QED radiative corrections to differential distributions are known.

On the other hand, the bulk of QED radiative corrections typically come from the terms enhanced by powers of the large logarithms $\ln Q^2/m^2$, where Q^2 is the square of the characteristic energy scale and m is the mass of a light charged lepton—e.g., $L = \ln M_Z^2/m_e^2 \approx 24$ for the process of e^+e^- annihilation into Z bosons.

In the QED structure function approach [19], one can get corrections enhanced by the large logarithms by performing a convolution of a hard scattering cross section and the corresponding parton distribution or fragmentation functions, which are independent of the process. In general, the large logarithm reads

$$L = \ln\frac{\mu_F^2}{\mu_R^2}, \tag{2}$$

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where μ_F is the factorization scale and μ_R is the renormalization scale. If $\mu_F \gg \mu_R$, the corrections proportional to powers of L yield the most significant contributions. In the case of muon decay, we can calculate the electron energy spectrum in the following way [15]:

$$\frac{d\Gamma}{dcdx} = \sum_{j=e,\gamma} \int_x^1 \frac{dz}{z} \frac{d^2\hat{\Gamma}_j}{dcdz}(z, c, \mu_F, \mu_R) D_{ej}\left(\frac{x}{z}, \mu_F, \mu_R\right), \quad (3)$$

where c is the cosine of the angle θ between the muon polarization vector and the electron momentum. Above, z is the energy fraction of the parton j produced in muon decay, x is the energy fraction of the resulting massive electron, $d\hat{\Gamma}_j/(dcdz)$ is the energy and angle distribution of the massless parton j, D_{ej} is the fragmentation function that describes the probability density for the transformation of the massless parton j into the physical electron in the final state, and μ_F is the factorization scale. Here, the standard modified minimal subtraction scheme is used. We can take $\mu_F = m_\mu$ and $\mu_R = m_e$, so the large logarithm is

$$L = \ln \frac{m_\mu^2}{m_e^2} \approx 10.66. \quad (4)$$

The perturbative expansion of the kernel coefficient function $d\hat{\Gamma}_j/(dcdz)$ in powers of α reads

$$\frac{1}{\Gamma_0} \frac{d\hat{\Gamma}_j}{dcdz}(z, \mu_F, \mu_R) = A_j^{(0)}(z) + \frac{\alpha(\mu_F^2)}{2\pi} \hat{A}_j^{(1)}(z) + \left(\frac{\alpha(\mu_F^2)}{2\pi}\right) \hat{A}_j^{(2)}(z) + \cdots, \quad (5)$$

where $\Gamma_0 = G_F^2 m_\mu^5/(192\pi^3)$, $A_j^{(0)}(z) = z^2(3-2z)\delta_{je}$, and $\alpha(\mu_F^2)$ is the renormalized fine structure constant taken at the factorization energy scale.

Here, we use process-independent fragmentation functions, calculated by solving the QED evolution equation

$$D_{ba}\left(x, \frac{\mu_R^2}{\mu_F^2}\right) = \delta(1-x)\delta_{ba} + \sum_{i=e,\bar{e},\gamma} \int_{\mu_R^2}^{\mu_F^2} \frac{d t \alpha(t)}{2\pi t} \int_x^1 \frac{dy}{y} \times D_{ia}\left(y, \frac{\mu_R^2}{t}\right) P_{bi}\left(\frac{x}{y}, t\right). \quad (6)$$

For the initial conditions and other details, see Ref. [20]. Note that here and in what follows, we apply the natural choice of the QED renormalization constant $\mu_R = m_e$.

In the unpolarized case, the relevant coefficient functions are

$$\hat{A}_{U,e}^{(0)}(z) = f_e^{(0)}(z), \qquad \hat{A}_{U,e}^{(1)}(z) = f_e^{(1)}(z), \quad (7)$$

$$\hat{A}_{U,\gamma}^{(0)}(z) = 0, \qquad \hat{A}_{U,\gamma}^{(1)}(z) = f_\gamma^{(1)}(z). \quad (8)$$

In the polarized case, the corrections include the parts which are dependent on the polarization degree P_μ and the cosine of the angle between the muon spin and electron's momentum $c = \cos\theta$:

$$\hat{A}_{P,e}^{(0)} = f_e^{(0)}(z) + cP_\mu g_e^{(0)}(z), \quad (9)$$

$$\hat{A}_{P,e}^{(1)}(z) = f_e^{(1)}(z) + cP_\mu g_e^{(1)}(z), \quad (10)$$

$$\hat{A}_{P,\gamma}^{(0)}(z) = 0, \quad (11)$$

$$\hat{A}_{P,\gamma}^{(1)}(z) = f_\gamma^{(1)}(z) + cP_\mu g_\gamma^{(1)}(z). \quad (12)$$

The expression for the differential distribution of electrons (averaged over electron spin states) in a polarized muon decay reads [16]

$$\frac{d^2\Gamma}{dzdc} = \Gamma_0(F(z) \pm cP_\mu G(z)), \qquad z = \frac{2m_\mu E_e}{m_\mu^2 + m_e^2},$$
$$z_0 \le z \le 1, \qquad z_0 = \frac{2m_\mu m_e}{m_\mu^2 + m_e^2}, \quad (13)$$

where "+" and "−" correspond to e^+ and e^-, respectively; G_F is the Fermi coupling constant; and E_e and $z = 2E_e/m_\mu$ are the energy and the energy fraction of the electron (or positron), respectively.

The complete expression for the spectrum functions ($H = F$, G) up to the $\mathcal{O}(\alpha^3 L^2)$ order reads

$$\begin{aligned} H(z) &= h_e^{(0)}(z) + \frac{\alpha}{2\pi} h_1 \\ &\quad + \left(\frac{\alpha}{2\pi}\right)^2 \left\{ \left[h_2^{(0,\gamma)} + h_2^{(0,NS)} + h_2^{(0,S)}\right] \frac{L^2}{2} \right. \\ &\quad \left. + \left[h_2^{(1,\gamma)} + h_2^{(1,NS)} + h_2^{(1,S)} + h_2^{(1,int)}\right] L \right\} \\ &\quad + \left(\frac{\alpha}{2\pi}\right)^3 \left\{ \left[h_3^{(0,\gamma)} + h_3^{(0,NS)} + h_3^{(0,S)}\right] \frac{L^3}{6} \right. \\ &\quad \left. + \left[h_3^{(1,\gamma)} + h_3^{(1,NS)} + h_3^{(1,S)} + h_3^{(1,int)}\right] \frac{L^2}{2} \right\} \\ &\equiv h_e^{(0)}(z) + \sum_{i,j} \alpha^i L^j H_{ij}(z), \end{aligned} \quad (14)$$

where the indices γ, NS, S and int correspond to the pure photonic contribution, the nonsinglet fermion pair one, the singlet fermion pair one, and the interference of the singlet and nonsinglet pair corrections. Here and in what follows, we omit the arguments of the functions h_i for convenience with $h \equiv f$ or $h \equiv g$.

To get the contribution of the order $\mathcal{O}(\alpha^3 L^2)$, we have to make convolutions of the fragmentation functions with functions $h_e^i(z)$ and $h_\gamma^i(z)$:

$$\left(h_e^{(0)}(z)+\frac{\alpha}{2\pi}h_e^{(1)}(z)\right)\otimes[D_{ee}]_T + \left(h_\gamma^{(0)}(z)+\frac{\alpha}{2\pi}h_\gamma^{(1)}(z)\right)\otimes[D_{e\gamma}]_T, \tag{15}$$

and we take only the terms proportional to α^3L^2 from the result.

Expressions for the polarized part can be received from Eq. (15) by the substitution $f_i^{(j,r)} \to g_i^{(j,r)}$. The index T in the above equation marks (timelike) fragmentation functions.

We recalculate the $\mathcal{O}(\alpha^2L)$ corrections and find that the term $d_{\gamma e}^{(1)}(x)\otimes P_{e\gamma}^{(0)}$ [see Eq. (A8)] has been missed in the electron fragmentation function used in Ref. [15]. The difference is

$$F_{21}^{\rm new}(z)-F_{21}^{\rm old}(z)=\frac{46z^3}{27}+\frac{151z^2}{6}-19z-\frac{8}{3z}-\frac{281}{54} - \left(\frac{64}{9}+\frac{4}{3z}+18z+\frac{38z^2}{3}\right)\ln z - \left(\frac{5}{3}+4z\right)\ln^2 z. \tag{16}$$

A similar change with respect to the result given in Ref. [16] is found for the function $G_{21}(z)$, and the corrected expression for it is shown in Appendix A.

In our previous work [20], we also corrected a mistake in the result for the $\mathcal{O}(\alpha^3L^3)$ singlet contribution to structure and fragmentation functions obtained in Ref. [21]. As a result, the singlet part of the NLO electron fragmentation function should read

$$[D_{ee}^S]_T=\left(\frac{\alpha}{2\pi}\right)^2 L(P_{ee}^{(1),S}+d_{\gamma e}^{(1)}(x)\otimes P_{e\gamma}^{(0)}) + \left(\frac{\alpha}{2\pi}\right)^2 L^2\frac{1}{2}P_{\gamma e}^{(0)}\otimes P_{e\gamma}^{(0)} + \left(\frac{\alpha}{2\pi}\right)^3 L^3\left(\frac{1}{6}P_{ee}^{(0)}\otimes P_{ee}^{(0)}\otimes P_{ee}^{(0)}+\frac{2}{9}P_{\gamma e}^{(0)}\otimes P_{e\gamma}^{(0)}\right) + \mathcal{O}(\alpha^3L^2). \tag{17}$$

The functions $f_a^{(i)}$ and $g_a^{(i)}$ read [9]

$$f_e^{(0)}(z)=z^2(3-2z), \tag{18}$$

$$f_e^{(1)}(z)=2z^2(2z-3)(4\zeta(2)-4\mathrm{Li}_2(z)+2\ln z^2 - 3\ln z\ln(1-z)-\ln(1-z)^2) + \left(\frac{5}{3}-2z-13z^2+\frac{34}{3}z^3\right)\ln(1-z) + \left(\frac{5}{3}+4z-2z^2-6z^3\right)\ln z+\frac{5}{6}-\frac{23}{3}z - \frac{3}{2}z^2+\frac{7}{3}z^3, \tag{19}$$

$$f_\gamma^{(0)}(z)=0, \tag{20}$$

$$f_\gamma^{(1)}(z)=\ln z\left(-\frac{10}{3}+\frac{2}{z}+4z\right) + \ln(1-z)\left(-\frac{5}{3}+\frac{1}{z}+2z-2z^2+\frac{2}{3}z^3\right) + \frac{1}{3}-\frac{1}{z}+\frac{35}{12}z-2z^2-\frac{1}{4}z^3, \tag{21}$$

$$g_e^{(0)}(z)=z^2(1-2z), \tag{22}$$

$$g_e^{(1)}(z)=2z^2(1-2z)(\ln(1-z)^2-4\mathrm{Li}_2(1-z) - \ln(z)\ln(1-z)-2\ln(z)^2) + \left(\frac{11}{3}-\frac{4}{3z}-6z-\frac{17}{3}z^2+\frac{34}{3}z^3\right)\ln(1-z) + \left(-\frac{1}{3}-6z^2-6z^3\right)\ln(z)-\frac{7}{6}+3z+\frac{7}{6}z^2+3z^3, \tag{23}$$

$$g_\gamma^{(0)}(z)=0, \tag{24}$$

$$g_\gamma^{(1)}(z)=\left(\frac{1}{3}-\frac{1}{3z}-\frac{2}{3}z^2+\frac{2}{3}z^3\right)\ln(1-z) + \left(\frac{2}{3}-\frac{2}{3z}\right)\ln z-\frac{2}{3}+\frac{2}{3z}+\frac{11}{12}z-\frac{2}{3}z^2-\frac{1}{4}z^3. \tag{25}$$

The relevant fragmentation functions are shown in Appendix B; see Ref. [20] for details of notation and explicit expressions for these functions.

Convolutions were calculated using our own program in FORM [22] and cross-checked with the help of the HPL [23] and MT [24] Wolfram *Mathematica* packages. The results are presented in Appendix B. A part of the results for the unpolarized case were presented in [25]; here we reproduce them for the sake of completeness.

We have calculated separately the parts of F_{21}, F_{22}, F_{43}, F_{44} and G_{21}, G_{22}, G_{43}, G_{44} with pure-photonic contributions in order to compare them with the results of Ref. [18]. Our results completely agreed with the ones from this work in the orders $\mathcal{O}(\alpha^3L^2)$, $\mathcal{O}(\alpha^4L^4)$, and $\mathcal{O}(\alpha^4L^3)$.

III. FACTORIZATION SCALE CHOICE AND NUMERICAL RESULTS

The factorization scale choice allows some arbitrariness. Above and in earlier papers [15,16], the muon mass was taken as the factorization scale. This choice is certainly good for the leading logarithmic approximation, but it can be optimized if one goes beyond it. We suggest choosing the factorization scale as

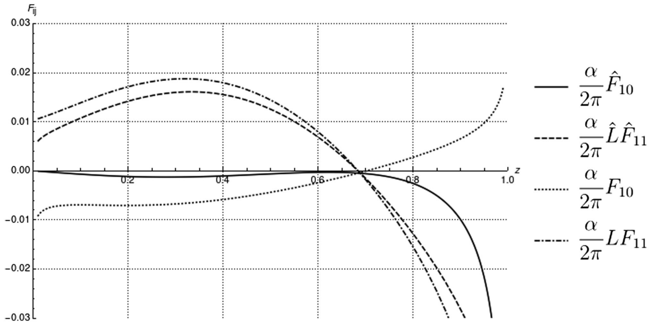

FIG. 1. Contributions in the $\mathcal{O}(\alpha^1 L^0)$ and $\mathcal{O}(\alpha^1 L^1)$ orders for the old and new factorization scales.

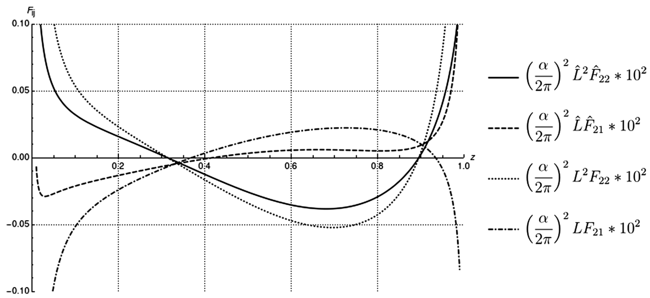

FIG. 2. Contributions in the $\mathcal{O}(\alpha^2 L^2)$ and $\mathcal{O}(\alpha^2 L)$ orders for the old and new factorization scales.

$$\mu_F^2 = m_\mu^2 z(1-z). \tag{26}$$

So, we expand on the powers of the new large logarithm:

$$\hat{L} = L + \Delta L, \qquad \Delta L = \ln z + \ln(1-z). \tag{27}$$

With this choice, NLO contributions are shifted by an additional term

$$\hat{F}_{ab} = F_{ab} - 2\Delta L F_{aa}, \qquad b = a - 1, \tag{28}$$

and the same for the G part. Here, the indices a and b are powers of α and L, respectively; $\hat{F}_{ab}$ is the NLO contribution for the new factorization scale choice, and F_{ab} is for the old one. The LO contributions do not change:

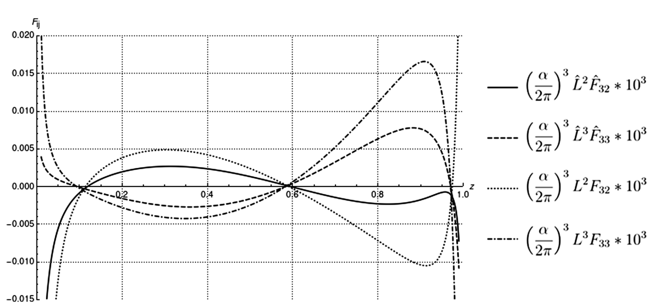

FIG. 3. Contributions in the $\mathcal{O}(\alpha^3 L^3)$ and $\mathcal{O}(\alpha^3 L^2)$ orders for the old and new factorization scales.

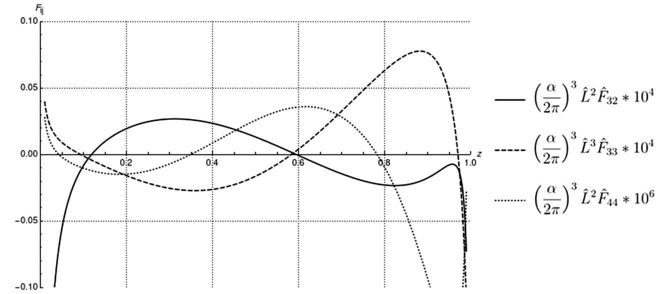

FIG. 4. F part of the corrections of the orders $\mathcal{O}(\alpha^3 L^2, \alpha^3 L^3, \alpha^4 L^4)$ for the new factorization scale.

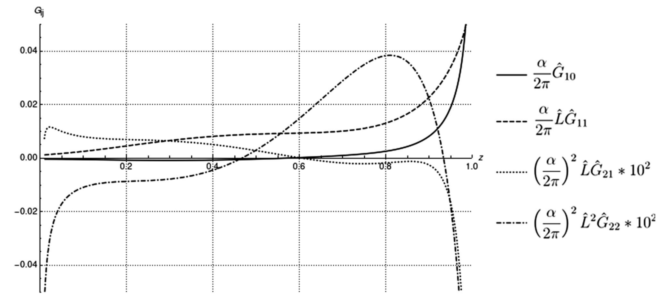

FIG. 5. G part of the corrections of the orders $\mathcal{O}(\alpha, \alpha L, \alpha^2 L, \alpha^2 L^2)$ for the new factorization scale.

$$\hat{F}_{aa} = F_{aa}. \tag{29}$$

The new factorization scale choice increases the difference between the NLO and LO contributions, thus improving the convergence of the expansion in the powers of the large logs. The results for the two factorization scale choices are shown on the plots for $\mathcal{O}(\alpha^1)$ (Fig. 1), $\mathcal{O}(\alpha^2)$ (Fig. 2), and $\mathcal{O}(\alpha^3)$ (Fig. 3).

We present our results for the new factorization scale for F (Fig. 4) and G (Figs. 5, 6) functions. We take corrections of the orders $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha L)$ from [9], and contributions of the order $\mathcal{O}(\alpha)$ are recalculated with the new factorization scale.

We also can look (see Figs. 7–9) at the relative values of the contributions of different orders,

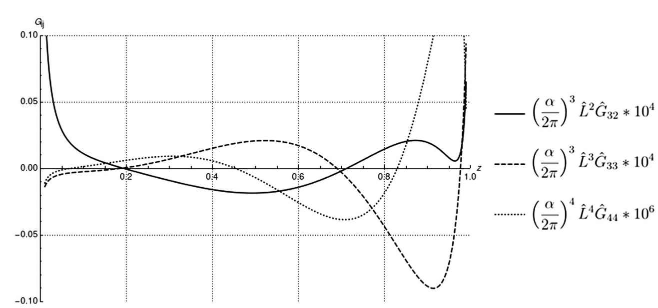

FIG. 6. G part of the corrections of the orders $\mathcal{O}(\alpha^3 L^2, \alpha^3 L^3, \alpha^4 L^4)$ for the new factorization scale.

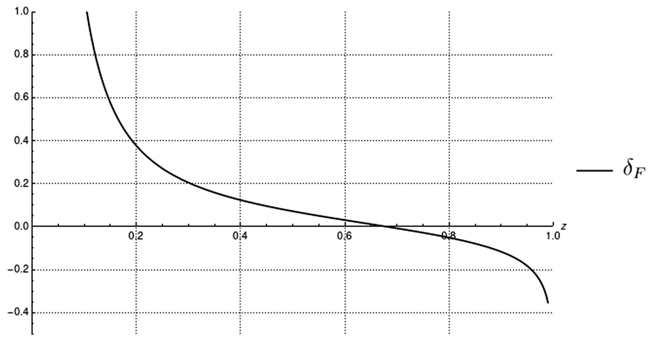

FIG. 7. The full relative correction (31) for the unpolarized case.

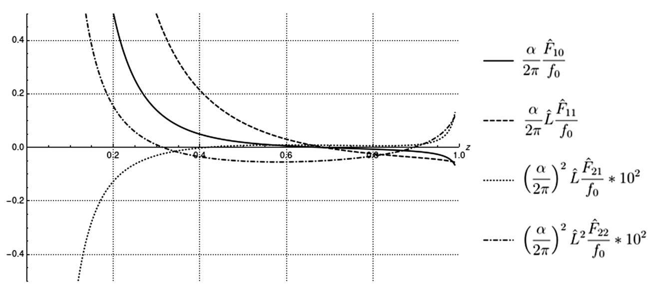

FIG. 8. F-part contributions relative to f_0 in the first two orders.

$$\left(\frac{\alpha}{2\pi}\right)^i \hat{L}^j \frac{\hat{F}_{ij}(z)}{f_0(z)}, \tag{30}$$

and the relative value of the full correction F (in the unpolarized case),

$$\delta_F = \frac{1}{f_0(z)} \sum_{i,j} \left(\frac{\alpha}{2\pi}\right)^i \hat{L}^j \hat{F}_{ij}(z). \tag{31}$$

Here and in what follows, we denote $f_0 \equiv f_e^{(0)}$ and $g_0 \equiv g_e^{(0)}$ for convenience. We cannot show the same picture for the G part because of the zero of the function $g_0(z)$ at $z = 0.5$, but the effect would be of the same order.

One can estimate the theoretical uncertainties related to the arbitrariness in the factorization scale choice in the following way: Let us consider our new factorization scale (26), the conventional factorization scale $\mu_F = m_\mu$, and two variations of the latter, $\bar{\mu}_F = 2m_\mu$ and $\check{\mu}_F = m_\mu/2$. The numerical effect of the scale choice can be seen in Fig. 10, where $\bar{L} \equiv \ln(4m_\mu^2/m_e^2)$ and $\check{L} \equiv \ln[m_\mu^2/(4m_e^2)]$, $\bar{F}_{21} \equiv F_{21} - 2\ln(4)F_{22}$, and $\check{F}_{21} \equiv F_{21} - 2\ln(1/4)F_{22}$ [see Eq. (28)]. Note that the uncertainty in question provides a specific way to estimate unknown higher-order terms. In the given case, those are of the order $\mathcal{O}(\alpha^2 L^0)$. One can see that the standard QCD-like variation of the factorization scale by the factors 1/4 and 4 actually captures the magnitude of the relative $\mathcal{O}(\alpha^2 L^0)$ contribution ($\sim -5 \times 10^{-5}$) known from the numerical estimates presented in [17] (for the most relevant range of z values). Moreover, one can see that the factorization scale proposed above provides the proper sign of the shift—i.e., it leads to absorption of a considerable part of the NNLO corrections into the NLO ones.

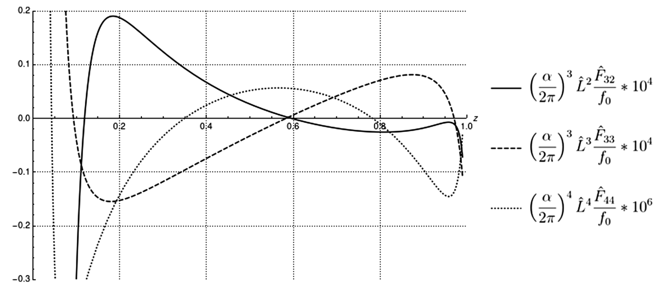

FIG. 9. Higher-order F-part contributions relative to f_0.

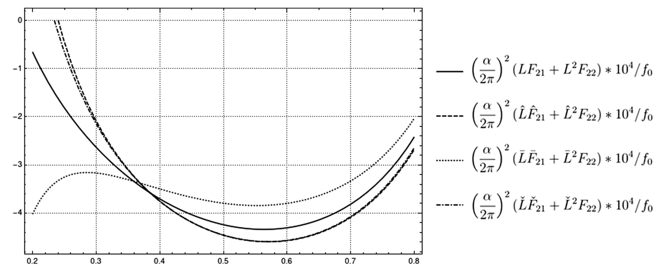

FIG. 10. Relative corrections for different factorization scales.

In accord with the Kinoshita-Lee-Nauenberg theorem [26,27], the terms enhanced by logarithms of the electron mass, (i.e., mass singularities) cancel out in the total decay width (except the ones related to vacuum polarization). The same cancellation happens in the total forward-backward asymmetry of the electron momentum with respect to the muon spin. Note that in both cases, one should take care on events with more than one electron in the final state to avoid double-counting—see a detailed discussion in [16]. So, the presented results are relevant just for differential distributions.

IV. CONCLUSIONS

In this way, we computed radiative corrections to the polarized and nonpolarized muon decay spectrum in the $\mathcal{O}(\alpha^3 L^2)$ and $\mathcal{O}(\alpha^4 L^4)$ orders. A new factorization scale is chosen to improve the convergence of the expansion in the powers of the large logarithm, and thus suppress unknown NNLO effects. With this factorization scale, we have the large logarithm $\hat{L} = \ln\frac{m_\mu^2 z}{m_e^2(1-z)}$, where z is the energy fraction of the electron in the final state.

Two mistakes in earlier results in the $\mathcal{O}(\alpha^2 L)$ and $\mathcal{O}(\alpha^3 L^3)$ orders are corrected. Our results are relevant for high-precision experiments on muon decays. The results can be easily adapted for leptonic modes of tau lepton decays.

ACKNOWLEDGMENTS

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APPENDIX A: SPLITTING AND FRAGMENTATION FUNCTIONS

The next-to-leading-order electron splitting function can be divided into four parts:

$$P_{ee}^{(1,\gamma)} = 2x\,\ln\,x + \ln\,x\,\ln(1-x)\left(-2+\frac{4}{1-x}-2x\right) + \ln^2 x\left(\frac{5}{2}-\frac{4}{1-x}+\frac{5}{2}x\right) + 2 - \frac{4}{1-x}\mathrm{Li}_2(1-x) + 2\mathrm{Li}_2(1-x) - 3x + 2x\mathrm{Li}_2(1-x), \tag{A1}$$

$$P_{ee}^{(1,NS)} = \ln\,x\left(\frac{2}{3}-\frac{4}{3(1-x)}+\frac{2}{3}x\right) - \frac{4}{3} + \frac{4}{3}x, \tag{A2}$$

$$P_{ee}^{(1,S)} = \ln\,x\left(-5-9x-\frac{8}{3}x^2\right) + \ln^2 x(1+x) - 8 - \frac{20}{9x} + 4x + \frac{56}{9}x^2, \tag{A3}$$

$$P_{ee}^{(1,\mathrm{int})} = \ln\,x\left(-5+\frac{3}{1-x}-5x\right) - 7 + \frac{4}{1-x}\mathrm{Li}_2(1-x) - 2\mathrm{Li}_2(1-x) + 8x - 2x\mathrm{Li}_2(1-x). \tag{A4}$$

Note that we have removed the term $\frac{10}{9}P_{ij}^{(0)}$ from the expressions for functions $P_{ij}^{(1,NS)}$ given in Refs. [15,16] because it naturally comes from the running coupling constant and can be kept there, as discussed in [20].

The fragmentation function $[D_{ee}]_T$ can also be divided into four parts:

$$[D_{ee}]_T = [D_{ee}^{\gamma}]_T + [D_{ee}^{S}]_T + [D_{ee}^{NS}]_T + [D_{ee}^{\mathrm{int}}]_T + \mathcal{O}(\alpha^4), \tag{A5}$$

$$[D_{ee}^{\gamma}]_T = \delta(1-x) + \frac{\alpha}{2\pi}d_{ee}^{(1)}(x) + \frac{\alpha}{2\pi}LP_{ee}^{(0)} + \left(\frac{\alpha}{2\pi}\right)^2 L(P_{ee}^{(1,\gamma)} + P_{ee}^{(0)}\otimes d_{ee}^{(1)}) + \left(\frac{\alpha}{2\pi}\right)^2 L^2(P_{ee}^{(0)}\otimes P_{ee}^{(0)}) + \left(\frac{\alpha}{2\pi}\right)^3 L^2\left(\frac{2}{3}P_{ee}^{(1,\gamma)} + P_{ee}^{(0)}\otimes P_{ee}^{(1,\gamma)} + \frac{1}{3}P_{ee}^{(0)}\otimes d_{ee}^{(1)} + \frac{1}{2}P_{ee}^{(0)}\otimes P_{ee}^{(0)}\otimes d_{ee}^{(1)}\right) + \left(\frac{\alpha}{2\pi}\right)^3 L^3\frac{1}{6}P_{ee}^{(0)}\otimes P_{ee}^{(0)}\otimes P_{ee}^{(0)}, \tag{A6}$$

$$[D_{ee}^{NS}]_T = \left(\frac{\alpha}{2\pi}\right)^2 L\left(P_{ee}^{(1),NS} - \frac{10}{9}P_{ee}^{(0)}\right) + \left(\frac{\alpha}{2\pi}\right)^2 L^2\frac{1}{3}P_{ee}^{(0)} + \left(\frac{\alpha}{2\pi}\right)^3 L^2\left(\frac{2}{3}P_{ee}^{(1,NS)} + P_{ee}^{(0)}\otimes P_{ee}^{(1,NS)} - \frac{13}{54}P_{ee}^{(0)} - \frac{10}{9}P_{ee}^{(0)}\otimes P_{ee}^{(0)}\right) + \left(\frac{\alpha}{2\pi}\right)^3 L^3\left(\frac{1}{3}P_{ee}^{(0)}\otimes P_{ee}^{(0)} + \frac{4}{27}P_{ee}^{(0)}\right), \tag{A7}$$

$$[D_{ee}^{S}]_T = \left(\frac{\alpha}{2\pi}\right)^2 L(P_{ee}^{(1),S} + d_{\gamma e}^{(1)}(x)\otimes P_{e\gamma}^{(0)}) + \left(\frac{\alpha}{2\pi}\right)^2 L^2\frac{1}{2}P_{\gamma e}^{(0)}\otimes P_{e\gamma}^{(0)} + \left(\frac{\alpha}{2\pi}\right)^3 L^2\left(\frac{1}{2}P_{e\gamma}^{(0)}\otimes P_{\gamma e}^{(1)} + \frac{1}{2}P_{\bar{e}e}^{(0)}\otimes P_{e\bar{e}}^{(1)} + \frac{1}{3}d_{\gamma e}^{(1)}\otimes P_{e\gamma}^{(0)} + \frac{1}{2}d_{\gamma e}^{(1)}\otimes P_{\gamma\gamma}^{(0)}\otimes P_{e\gamma}^{(0)} + \frac{1}{2}P_{\gamma e}^{(0)}\otimes P_{e\gamma}^{(1)} - \frac{10}{9}P_{\gamma e}^{(0)}\otimes P_{e\gamma}^{(0)} + \frac{2}{3}P_{ee}^{(1,S)} + \frac{1}{2}d_{ee}^{(1)}\otimes P_{\gamma e}^{(0)}\otimes P_{e\gamma}^{(0)} + \frac{1}{2}d_{\gamma e}^{(1)}\otimes P_{ee}^{(0)}\otimes P_{e\gamma}^{(0)} + P_{ee}^{(0)}\otimes P_{ee}^{(1,S)}\right) + \left(\frac{\alpha}{2\pi}\right)^3 L^3\left(\frac{1}{6}P_{ee}^{(0)}\otimes P_{ee}^{(0)}\otimes P_{ee}^{(0)} + \frac{2}{9}P_{\gamma e}^{(0)}\otimes P_{e\gamma}^{(0)}\right), \tag{A8}$$

$$[D_{ee}^{\mathrm{int}}]_T = \left(\frac{\alpha}{2\pi}\right)^2 LP_{ee}^{(1,\mathrm{int})} + \left(\frac{\alpha}{2\pi}\right)^3 L^2\left(\frac{2}{3}P_{ee}^{(1,\mathrm{int})} + P_{ee}^{(0)}\otimes P_{ee}^{(1,\mathrm{int})}\right). \tag{A9}$$

The $[D_{e\gamma}]_T$ fragmentation function reads

$$[D_{e\gamma}]_T = \frac{\alpha}{2\pi}d_{e\gamma}^{(1)} + \frac{\alpha}{2\pi}L(P_{e\gamma}^{(0)}) + \left(\frac{\alpha}{2\pi}\right)^2 L\left(P_{e\gamma}^{(1)T} - \frac{10}{9}P_{e\gamma}^{(0)} + P_{ee}^{(0)}\otimes d_{e\gamma}^{(1)}\right) + \left(\frac{\alpha}{2\pi}\right)^2 L^2\left(\frac{1}{3}P_{e\gamma}^{(0)} + \frac{1}{2}P_{\gamma\gamma}^{(0)}\otimes P_{e\gamma}^{(0)} + \frac{1}{2}P_{ee}^{(0)}\otimes P_{e\gamma}^{(0)}\right) + \mathcal{O}(\alpha^3). \tag{A10}$$

APPENDIX B: ANALYTIC RESULTS

Here, we present the analytic formulas for the computed higher-order contributions which appear in Eq. (14). These expressions contain harmonic polylogrithms which are defined in Appendix C below

$$\begin{aligned}
F_{21}(z) = & -\frac{4405}{216} + \frac{2\zeta_2 z^3}{3} - 9\zeta_2 z^2 + \left(8\zeta_2 z^3 - 12\zeta_2 z^2 - \frac{32z^3}{9} - 19z^2 - 13z - \frac{97}{12}\right)\ln(z) + 12\zeta_2 z \\
& + (8z^3 - 12z^2)\left(-\mathrm{Li}_3(z) + \mathrm{Li}_2(z)\ln(z) + \frac{1}{2}\ln(1-z)\ln^2(z) + \zeta_3\right) + \left(-\frac{16z^3}{3} + 6z^2 - 6z\right)\mathrm{Li}_2(1-z) \\
& + (24z^2 - 16z^3)\mathrm{Li}_3(1-z) + (16z^3 - 24z^2)\mathrm{Li}_2(1-z)\ln(1-z) + (8z^3 - 12z^2)\mathrm{Li}_2(1-z)\ln(z) \\
& - 12z^3\zeta_3 - \frac{167z^3}{54} + \left(\frac{16z^3}{3} - 12z\right)\ln^2(1-z) + 18z^2\zeta_3 + \frac{449z^2}{9} + (12z^2 - 8z^3)\ln^3(z) \\
& + \left(-\frac{32z^3}{3} + 11z^2 - 3z - \frac{5}{4}\right)\ln^2(z) + (24z^3 - 36z^2)\ln(1-z)\ln^2(z) + (12z^2 - 8z^3)\ln^2(1-z)\ln(z) \\
& + \left(-\frac{8z^3}{9} + \frac{4z^2}{3} - 16z + \frac{2}{3z} - \frac{8}{3}\right)\ln(1-z) + \left(\frac{8z^3}{3} - 14z^2 + 22z + \frac{20}{3}\right)\ln(1-z)\ln(z) - \frac{1195z}{36} - \frac{3}{z},
\end{aligned} \tag{B1}$$

$$\begin{aligned}
G_{21}(z) = & \left(\frac{2z^3}{3} + 12z + \frac{8}{3z} - 8 - 11z^2\right)\zeta_2 + \left(8\zeta_2 z^3 - 4\zeta_2 z^2 - \frac{44z^3}{9} - \frac{56z^2}{9} - \frac{26z}{3} + \frac{49}{12}\right)\ln(z) + (8z^3 - 4z^2)\big(-\mathrm{Li}_3(z) \\
& + \mathrm{Li}_2(z)\ln(z) + \frac{1}{2}\ln(1-z)\ln^2(z) + \zeta_3\big) + \left(-\frac{16z^3}{3} + 6z^2 - 6z - \frac{8}{3z} + \frac{13}{3}\right)\mathrm{Li}_2(1-z) + (8z^2 - 16z^3)\big(\mathrm{Li}_3(1-z) \\
& - \mathrm{Li}_2(1-z)\ln(1-z)\big) + (8z^3 - 4z^2)\mathrm{Li}_2(1-z)\ln(z) - 12z^3\zeta_3 - \frac{83z^3}{18} + 6z^2\zeta_3 + \frac{1025z^2}{54} + (4z^2 - 8z^3)\ln^3(z) \\
& + \left(\frac{16z^3}{3} + 8z^2 - 12z - \frac{8}{3z} + 8\right)\ln^2(1-z) + \left(-\frac{32z^3}{3} + \frac{43z^2}{3} + \frac{1}{4}\right)\ln^2(z) + (24z^3 - 12z^2)\ln(1-z)\ln^2(z) \\
& + (4z^2 - 8z^3)\ln^2(1-z)\ln(z) + \left(\frac{4z^3}{9} + \frac{98z^2}{9} + \frac{28z}{3} - \frac{10}{9z} - 2\right)\ln(1-z) \\
& + \left(\frac{8z^3}{3} - \frac{86z^2}{3} + 6z - \frac{16}{3}\right)\ln(1-z)\ln(z) - \frac{137z}{12} + \frac{29}{27z} + \frac{415}{72},
\end{aligned} \tag{B2}$$

$$\begin{aligned}
F_{32} = & \frac{53623}{1296} + \frac{1}{108z} + \frac{1201z^3}{162} - \frac{2131z^2}{72} - \frac{49z}{2} + (8z^3 - 4z^2 - 12z)\ln^3(1-z) + \left(\frac{92z^3}{9} - \frac{41z^2}{3} - \frac{7z}{3} - \frac{35}{36}\right)\ln^3(z) \\
& + \left[\frac{142z^3}{9} + \frac{152z^2}{3} + \frac{161z}{12} + (4z^3 - 6z^2)\ln^2(1-z) + \zeta_2(60z^2 - 40z^3) + \left(-\frac{56z^3}{3} + 58z^2 + 44z + \frac{125}{6}\right)\ln(1-z)\right. \\
& \left. + \frac{37}{8}\right]\ln^2(z) + (4z^3 - 6z^2)\mathrm{Li}_2(1-z)^2 + \zeta_2\left(-6z^2 + 16z + \frac{139}{18}\right) + (2z^3 - 3z^2)\{-20\zeta_4 - 18\zeta_2^2 + 52(\mathrm{H}(3,0,z) \\
& + \mathrm{H}(2,0,0,z) + \mathrm{H}(2,1,0,z)) + 40(\mathrm{H}(1,2,0,z) + \mathrm{H}(1,1,1,0,z)) + 32\mathrm{H}(0,0,0,0,z) + 28\mathrm{H}(1,0,0,0,z) \\
& + 48\mathrm{H}(1,1,0,0,z)\} + \left(\frac{136z^3}{9} + \frac{185z^2}{3} - \frac{247z}{3} + \zeta_2(60z^2 - 40z^3) - \frac{160}{3} - \frac{6}{z}\right)\mathrm{Li}_2(1-z) \\
& + \ln^2(1-z)\left(-\frac{62z^3}{9} + \frac{37z^2}{6} - \frac{62z}{3} + \zeta_2(40z^3 - 60z^2) + (16z^3 - 24z^2)\mathrm{Li}_2(1-z) - \frac{121}{18} + \frac{1}{z}\right) \\
& + \left(32z^3 - 40z^2 + 4z - \frac{10}{3}\right)\mathrm{Li}_3(1-z) + \left(\frac{64z^3}{3} + 72z^2 + 97z + \frac{140}{3}\right)\mathrm{Li}_3(z) + (72z^2 - 48z^3)\mathrm{Li}_4(1-z) \\
& + (120z^3 - 180z^2)\mathrm{Li}_4(z) + (120z^2 - 80z^3)\mathrm{S}_{2,2}(z) + \ln(z)\left[\frac{283z^3}{27} - \frac{799z^2}{12} + \frac{539z}{36} + (12z^2 - 8z^3)\ln^3(1-z)\right.
\end{aligned}$$

$$+\left(-\frac{16z^3}{3}-8z^2+36z+10\right)\ln^2(1-z)+\zeta_2\left(-\frac{100z^3}{3}-32z^2-125z-\frac{160}{3}\right)+\left(\frac{8z^3}{3}+22z^2+74z+\frac{185}{6}\right)$$
$$\times\mathrm{Li}_2(1-z)+\ln(1-z)\left(-\frac{4z^3}{3}-11z^2-\frac{z}{2}+\zeta_2(16z^3-24z^2)+(8z^3-12z^2)\mathrm{Li}_2(1-z)-\frac{57}{4}+\frac{8}{3z}\right)$$
$$+(48z^2-32z^3)\zeta_3+\frac{1261}{108}\Bigg]+\ln(1-z)\Bigg[-\frac{155z^3}{27}+\frac{2221z^2}{36}-\frac{677z}{36}+\zeta_2\left(-\frac{20z^3}{3}-14z^2+36z\right)$$
$$+\left(-32z^3+40z^2-4z+\frac{10}{3}\right)\mathrm{Li}_2(1-z)+(48z^2-32z^3)\mathrm{Li}_3(1-z)+(144z^2-96z^3)\mathrm{Li}_3(z)$$
$$+(88z^3-132z^2)\zeta_3-\frac{6281}{108}-\frac{32}{9z}\Bigg]+\left(\frac{8z^3}{3}-92z^2-109z-\frac{125}{3}\right)\zeta_3, \tag{B3}$$

$$G_{32}(z)=\frac{11329}{1296}-\frac{131}{108z}-\frac{55z^2}{24}-\frac{421z}{54}+\frac{1273z^3}{162}+(2z-1)z^2(52\mathrm{H}(3,0,z)+40\mathrm{H}(1,2,0,z))$$
$$+\left(8z^3+\frac{20z^2}{3}-12z+8-\frac{8}{3z}\right)\ln^3(1-z)+\left(\frac{92z^3}{9}-\frac{89z^2}{9}+\frac{7}{36}\right)\ln^3(z)$$
$$+\left(\frac{148z^3}{9}+11z^2+\frac{49z}{12}+\zeta_2(20z^2-40z^3)+\left(-\frac{64z^3}{3}+48z^2-6z+6-\frac{4}{3z}\right)\ln(1-z)-\frac{65}{24}\right)\ln^2(z)$$
$$+(4z^3-2z^2)\mathrm{Li}_2(z)^2+\zeta_2\left(-\frac{122z^3}{9}+\frac{40z^2}{9}-30z+\frac{115}{18}+\frac{26}{9z}\right)+\zeta_4(40z^3-20z^2)$$
$$+(104z^3-52z^2)\mathrm{H}(2,0,0,z)+(104z^3-52z^2)\mathrm{H}(2,1,0,z)+(64z^3-32z^2)\mathrm{H}(0,0,0,0,z)$$
$$+(56z^3-28z^2)\mathrm{H}(1,0,0,0,z)+(96z^3-48z^2)\mathrm{H}(1,1,0,0,z)+(80z^3-40z^2)\mathrm{H}(1,1,1,0,z)$$
$$+\left(26z^3-13z^2+\zeta_2(16z^2-32z^3)\right)\mathrm{Li}_2(1-z)+\left(\frac{110z^3}{9}-\frac{112z^2}{9}+16z-\frac{5}{3}-\frac{2}{z}\right)\mathrm{Li}_2(z)$$
$$+\ln^2(1-z)\left[-\frac{50z^3}{9}+\frac{401z^2}{18}+16z+(56z^3-28z^2)\mathrm{Li}_2(1-z)+(40z^3-20z^2)\mathrm{Li}_2(z)-\frac{79}{18}-\frac{11}{9z}\right]$$
$$+\left(32z^3-16z^2+12z-\frac{22}{3}+\frac{8}{z}\right)\mathrm{Li}_3(1-z)+\left(\frac{64z^3}{3}+\frac{56z^2}{3}-15z+\frac{2}{3}+\frac{8}{3z}\right)\mathrm{Li}_3(z)$$
$$+(24z^2-48z^3)\mathrm{Li}_4(1-z)+(120z^3-60z^2)\mathrm{Li}_4(z)+(40z^2-80z^3)\mathrm{S}_{2,2}(z)$$
$$+\ln(z)\Bigg[\frac{319z^3}{27}-\frac{4205z^2}{108}-\frac{25z}{12}+(32z^3-16z^2)\ln^3(1-z)+\left(-\frac{16z^3}{3}-40z^2+12z-10\right)$$
$$\times\ln^2(1-z)+\zeta_2\left(-\frac{92z^3}{3}+\frac{94z^2}{3}+3z+\frac{5}{2}\right)+\left(\frac{74z^3}{9}-\frac{253z^2}{9}-\frac{41z}{6}+\zeta_2(24z^3-12z^2)+\frac{5}{4}-\frac{2}{z}\right)\ln(1-z)$$
$$+\left(-\frac{8z^3}{3}+\frac{2z^2}{3}+\frac{37}{6}-\frac{8}{3z}\right)\mathrm{Li}_2(z)+(16z^2-32z^3)\zeta_3+\frac{136}{27}\Bigg]$$
$$+\ln(1-z)\Bigg[-\frac{191z^3}{27}+\frac{2741z^2}{108}+\frac{19z}{12}+\zeta_2\left(-\frac{20z^3}{3}-\frac{86z^2}{3}+36z-24+\frac{8}{z}\right)-\left(32z^3-16z^2+12z-\frac{22}{3}+\frac{8}{z}\right)$$
$$\times\mathrm{Li}_2(1-z)+(16z^2-32z^3)\mathrm{Li}_3(1-z)+(48z^2-96z^3)\mathrm{Li}_3(z)+(88z^3-44z^2)\zeta_3+\frac{1}{108}+\frac{14}{3z}\Bigg]$$
$$+\left(\frac{8z^3}{3}-\frac{76z^2}{3}-9z+\frac{43}{3}-\frac{8}{z}\right)\zeta_3, \tag{B4}$$

$$
\begin{aligned}
F_{33}(z) = &-\frac{619}{1296}+\frac{20z}{9}+\frac{4}{27z}+\ln(z)\Big[\zeta_2(12z^2-8z^3)+\left(\frac{16z^3}{3}-8z^2\right)\mathrm{Li}_2(z)+\frac{32z^3}{27}+\frac{52z^2}{9}+(8z^3-12z^2)\\
&\times\ln^2(1-z)+\left(-\frac{8z^3}{3}+4z^2-4z-\frac{5}{3}\right)\ln(1-z)-\frac{67z}{36}-\frac{41}{108}\Big]\\
&+\ln(1-z)\Big[\zeta_2(8z^3-12z^2)+(8z^3-12z^2)\mathrm{Li}_2(1-z)-\frac{32z^3}{27}-\frac{44z^2}{9}+\frac{15z}{2}+\frac{4}{9z}+\frac{289}{108}\Big]\\
&+\zeta_2\left(-\frac{16z^3}{3}+\frac{8z^2}{3}+\frac{2z}{3}+\frac{5}{18}\right)+\left(\frac{8z^3}{3}+\frac{4z^2}{3}-\frac{14z}{3}-\frac{35}{18}\right)\mathrm{Li}_2(z)+(12z^2-8z^3)\mathrm{Li}_3(1-z)\\
&+\left(8z^2-\frac{16z^3}{3}\right)\mathrm{Li}_3(z)+\frac{16z^3}{81}-\frac{5z^2}{9}+\left(4z^2-\frac{8z^3}{3}\right)\ln^3(1-z)+\left(\frac{4z^3}{9}-\frac{2z^2}{3}\right)\ln^3(z)\\
&+\left(\frac{8z^3}{3}-4z^2+4z+\frac{5}{3}\right)\ln^2(1-z)+\left(\frac{4z^3}{3}-\frac{2z^2}{3}+(2z^2-\frac{4z^3}{3})\ln(1-z)-\frac{z}{2}-\frac{5}{24}\right)\ln^2(z),
\end{aligned}
\tag{B5}
$$

$$
\begin{aligned}
G_{33}(z) = &-\frac{49}{1296}-\frac{115z}{108}-\frac{4}{81z}+\ln(z)\Big[\zeta_2\left(\frac{4z^2}{3}-\frac{8z^3}{3}\right)+\left(\frac{8z^2}{3}-\frac{16z^3}{3}\right)\mathrm{Li}_2(1-z)+\frac{32z^3}{27}+\frac{68z^2}{27}+(8z^3-4z^2)\\
&\times\ln^2(1-z)+\left(-\frac{16z^3}{3}+\frac{56z^2}{9}-\frac{1}{18}\right)\ln(1-z)+\frac{z}{12}+\frac{1}{108}\Big]\\
&+\ln(1-z)\left(\zeta_2(8z^3-4z^2)+(8z^3-4z^2)\mathrm{Li}_2(1-z)-\frac{32z^3}{27}-\frac{20z^2}{9}-\frac{z}{2}-\frac{4}{27z}-\frac{53}{108}\right)\\
&+\zeta_2\left(-\frac{8z^3}{3}+\frac{20z^2}{3}+\frac{1}{3}\right)+\left(-\frac{8z^3}{3}-\frac{4z^2}{9}-\frac{7}{18}\right)\mathrm{Li}_2(1-z)+(4z^2-8z^3)\mathrm{Li}_3(1-z)+\left(\frac{8z^2}{3}-\frac{16z^3}{3}\right)\\
&\times\mathrm{Li}_3(z)+\frac{16z^3}{81}-\frac{47z^2}{81}+\left(\frac{4z^2}{3}-\frac{8z^3}{3}\right)\ln^3(1-z)+\left(\frac{4z^3}{9}-\frac{2z^2}{9}\right)\ln^3(z)+\left(\frac{8z^3}{3}-\frac{20z^2}{3}-\frac{1}{3}\right)\ln^2(1-z)\\
&+\left(\frac{4z^3}{3}-\frac{10z^2}{9}+\left(\frac{10z^2}{3}-\frac{20z^3}{3}\right)\ln(1-z)+\frac{1}{24}\right)\ln^2(z),
\end{aligned}
\tag{B6}
$$

$$
\begin{aligned}
F_{44}(z) = &-\frac{7577}{3456}+\left(2z^2-\frac{4z^3}{3}\right)\ln^4(1-z)+\left(\frac{8z^3}{9}-\frac{4z^2}{3}+\frac{8z}{3}+\frac{10}{9}\right)\ln^3(1-z)\\
&+\left(\zeta_2(8z^3-12z^2)-\frac{8z^3}{27}-\frac{47z^2}{9}+\frac{25z}{3}+(12z^2-8z^3)\mathrm{Li}_2(z)+\frac{35}{12}+\frac{1}{3z}\right)\ln^2(1-z)\\
&+\left(-\frac{157z^2}{54}+\frac{725z}{108}+\zeta_2\left(-\frac{8z^3}{3}+4z^2-8z-\frac{10}{3}\right)+\left(-\frac{8z^3}{3}-8z^2+8z+\frac{10}{3}\right)\mathrm{Li}_2(1-z)+(24z^2-16z^3)\right.\\
&\left.\times\mathrm{Li}_3(1-z)+\left(\frac{16z^3}{3}-8z^2\right)\mathrm{Li}_3(z)+\left(32z^2-\frac{64z^3}{3}\right)\zeta_3+\frac{41}{72}+\frac{10}{27z}\right)\ln(1-z)\\
&+\left(-\frac{4z^3}{27}-\frac{z^2}{9}-\frac{z}{24}-\frac{5}{288}\right)\ln^3(z)+\frac{278z^2}{81}+\left(-\frac{4z^3}{27}-\frac{43z^2}{18}+\frac{5z}{96}+\zeta_2\left(\frac{8z^3}{3}-4z^2\right)\right.\\
&\left.+\left(\frac{4z^3}{9}-\frac{2z^2}{3}+\frac{4z}{3}+\frac{5}{9}\right)\ln(1-z)-\frac{607}{1728}\right)\ln^2(z)+\left(8z^2-\frac{16z^3}{3}\right)\mathrm{Li}_2(z)^2+\frac{191z}{576}\\
&+\zeta_2\left(\frac{16z^3}{27}+10z^2-\frac{34z}{9}-\frac{73}{108}\right)+\zeta_4\left(70z^2-\frac{140z^3}{3}\right)+4(3-2z)z^2\mathrm{H}(3,0,z)+4(3-2z)z^2\mathrm{H}(1,2,0,z)\\
&+\left(8z^2-\frac{16z^3}{3}\right)\mathrm{H}(2,0,0,z)+(24z^2-16z^3)\mathrm{H}(2,1,0,z)+\left(2z^2-\frac{4z^3}{3}\right)\mathrm{H}(0,0,0,0z)\\
&+\left(2z^2-\frac{4z^3}{3}\right)\mathrm{H}(1,0,0,0z)+\left(8z^2-\frac{16z^3}{3}\right)\mathrm{H}(1,1,0,0z)+(24z^2-16z^3)\mathrm{H}(1,1,1,0z)
\end{aligned}
$$

$$+\left(-\frac{8z^3}{27}-\frac{43z^2}{9}-\frac{41z}{9}+\zeta_2\left(\frac{32z^3}{3}-16z^2\right)-\frac{121}{54}-\frac{1}{3z}\right)\mathrm{Li}_2(z)+\left(\frac{8z^3}{3}+8z^2-8z-\frac{10}{3}\right)\mathrm{Li}_3(1-z)$$
$$+\left(\frac{16z^3}{9}+\frac{8z^2}{3}-\frac{4z}{3}-\frac{5}{9}\right)\mathrm{Li}_3(z)+(32z^3-48z^2)\mathrm{Li}_4(1-z)+\left(8z^2-\frac{16z^3}{3}\right)\mathrm{Li}_4(z)+(16z^3-24z^2)\mathrm{S}_{2,2}(z)$$
$$+\ln(z)\left[\left(4z^2-\frac{8z^3}{3}\right)\ln^3(1-z)+\left(-\frac{8z^3}{3}-2z^2\right)\ln^2(1-z)+\left(\frac{8z^3}{27}+\frac{47z^2}{9}-\frac{25z}{3}+\zeta_2\left(8z^2-\frac{16z^3}{3}\right)-\frac{35}{12}-\frac{1}{3z}\right)\right.$$
$$\times\ln(1-z)+\frac{23z^2}{18}-\frac{6415z}{864}+\zeta_2\left(\frac{8z^3}{3}+\frac{8z^2}{3}-\frac{4z}{3}-\frac{5}{9}\right)+\left(-\frac{16z^3}{9}-\frac{10z^2}{3}+\frac{8z}{3}+\frac{10}{9}\right)\mathrm{Li}_2(z)-\frac{2}{27z}$$
$$\left.+\left(\frac{16z^3}{3}-8z^2\right)\zeta_3-\frac{997}{576}\right]-\frac{37}{162z}+\left(-\frac{16z^2}{3}+\frac{20z}{3}+\frac{25}{9}\right)\zeta_3, \tag{B7}$$

$$G_{44}(z)=\frac{1015}{10368}+\left(\frac{2z^2}{3}-\frac{4z^3}{3}\right)\ln^4(1-z)+\left(\frac{8z^3}{9}-4z^2-\frac{2}{9}\right)\ln^3(1-z)$$
$$+\left(-\frac{8z^3}{27}-\frac{119z^2}{27}-\frac{2z}{3}+\zeta_2(8z^3-4z^2)+(4z^2-8z^3)\mathrm{Li}_2(z)-\frac{7}{12}-\frac{1}{9z}\right)$$
$$\times\ln^2(1-z)+\left[\zeta_2\left(-\frac{8z^3}{3}+12z^2+\frac{2}{3}\right)-\frac{349z^2}{162}-\frac{77z}{36}+\left(-\frac{8z^3}{3}-\frac{8z^2}{3}-\frac{2}{3}\right)\mathrm{Li}_2(1-z)+(8z^2-16z^3)\right.$$
$$\left.\times\mathrm{Li}_3(1-z)+\left(\frac{16z^3}{3}-\frac{8z^2}{3}\right)\mathrm{Li}_3(z)+\left(\frac{32z^2}{3}-\frac{64z^3}{3}\right)\zeta_3-\frac{71}{216}-\frac{10}{81z}\right]\ln(1-z)$$
$$+\left(-\frac{4z^3}{27}+\frac{z^2}{9}+\frac{1}{288}\right)\ln^3(z)+\frac{26z^2}{81}+\left(-\frac{4z^3}{27}-\frac{25z^2}{18}-\frac{41z}{288}+\zeta_2\left(\frac{8z^3}{3}-\frac{4z^2}{3}\right)\right.$$
$$\left.+\left(\frac{4z^3}{9}-2z^2-\frac{1}{9}\right)\ln(1-z)+\frac{107}{1728}\right)\ln^2(z)+\left(\frac{8z^2}{3}-\frac{16z^3}{3}\right)\mathrm{Li}_2(z)^2-\frac{9485z}{5184}+\zeta_4\left(\frac{70z^2}{3}-\frac{140z^3}{3}\right)$$
$$+\zeta_2\left(\frac{16z^3}{27}+\frac{62z^2}{9}+\frac{z}{3}+\frac{5}{108}\right)+4(1-2z)z^2(\mathrm{H}(3,0,z)+\mathrm{H}(1,2,0,z))+\left(\frac{8z^2}{3}-\frac{16z^3}{3}\right)\mathrm{H}(2,0,0,z)$$
$$+\left.(8z^2-16z^3\right)\mathrm{H}(2,1,0,z)+\left(\frac{2z^2}{3}-\frac{4z^3}{3}\right)\mathrm{H}(0,0,0,0,z)+\left(\frac{2z^2}{3}-\frac{4z^3}{3}\right)\mathrm{H}(1,0,0,0z)$$
$$+\left(\frac{8z^2}{3}-\frac{16z^3}{3}\right)\mathrm{H}(1,1,0,0z)+(8z^2-16z^3)\mathrm{H}(1,1,1,0z)$$
$$+\left(-\frac{8z^3}{27}-\frac{67z^2}{27}+\frac{z}{3}+\zeta_2\left(\frac{32z^3}{3}-\frac{16z^2}{3}\right)+\frac{29}{54}+\frac{1}{9z}\right)\mathrm{Li}_2(z)+\left(\frac{8z^3}{3}+\frac{8z^2}{3}+\frac{2}{3}\right)\mathrm{Li}_3(1-z)$$
$$+\left(\frac{16z^3}{9}-\frac{8z^2}{3}+\frac{1}{9}\right)\mathrm{Li}_3(z)+(32z^3-16z^2)\mathrm{Li}_4(1-z)+\left(\frac{8z^2}{3}-\frac{16z^3}{3}\right)\mathrm{Li}_4(z)+(16z^3-8z^2)\mathrm{S}_{2,2}(z)$$
$$+\ln(z)\left[\left(\frac{4z^2}{3}-\frac{8z^3}{3}\right)\ln^3(1-z)+\left(\frac{14z^2}{3}-\frac{8z^3}{3}\right)\ln^2(1-z)+\left(\frac{8z^3}{27}+\frac{119z^2}{27}+\frac{2z}{3}+\frac{2}{81z}+\zeta_2\left(\frac{8z^2}{3}-\frac{16z^3}{3}\right)\right.\right.$$
$$\left.+\frac{7}{12}+\frac{1}{9z}\right)\ln(1-z)+\frac{71z^2}{54}+\frac{19z}{32}+\zeta_2\left(\frac{8z^3}{3}-\frac{8z^2}{3}+\frac{1}{9}\right)+\left(-\frac{16z^3}{9}+\frac{2z^2}{3}-\frac{2}{9}\right)\mathrm{Li}_2(z)+\left(\frac{16z^3}{3}-\frac{8z^2}{3}\right)\zeta_3$$
$$\left.+\frac{475}{1728}\right]+\frac{11}{162z}-\left(\frac{16z^2}{3}+\frac{5}{9}\right)\zeta_3. \tag{B8}$$

APPENDIX C: HARMONIC POLYLOGARITHMS

The functions $H(a_1,\ldots,a_k;z)$ are harmonic polylogarithms [23,28]. The dimension of the vector $a=(a_1,\ldots,a_k)$ is called the weight of the harmonic polylogarithm (HPL). They are defined through the functions

$$f_1(z) = \frac{1}{1-z}, \tag{C1}$$

$$f_0(z) = \frac{1}{z}, \tag{C2}$$

$$f_{-1}(z) = \frac{1}{1+z}. \tag{C3}$$

We can get the HPLs recursively through the integration of the functions (C1)–(C3):

$$H(1;z) = \int_0^z f_1(y)dy = -\ln(1-z), \tag{C4}$$

$$H(0;z) = \ln z, \tag{C5}$$

$$H(-1;z) = \int_0^z f_{-1}(y)dy = -\ln(1+z), \tag{C6}$$

$$H({}^n0;z) = \frac{1}{n!}\ln^n z, \tag{C7}$$

$$H(a,a_1,\ldots,a_k;z) = \int_0^z f_a(y)H(a_1,\ldots,a_k;y)dy, \tag{C8}$$

where

$${}^n0 = \underbrace{0,\ldots,0}_{n}. \tag{C9}$$

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