Entanglement degradation as a tool to detect signatures of modified gravity

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We investigate entanglement degradation in the vicinity of a quantum corrected black hole. We consider a biprtite system (Alice-Rob) with Alice freely falling (radially) into the event horizon of a quantum corrected black hole and Rob being in the vicinity of the event horizon of the black hole. We consider a maximally entangled state (in the Fock basis) and start with the basic assumption that Rob is an uniformly accelerated observer. We then give a pedagogical analysis of the relation involving the Minkowski vaccum state and Rindler number states. Following the analogy given in Martín-Martínez *et al.* [Phys. Rev. D **82**, 064006 (2010)], we establish the relation between the Hartle-Hawking vacuum state and Boulware and anti-Bouware number states from the Minkowski-Rindler relation. We then write down the quantum corrected black hole metric by making use of the near horizon approximation in an appropriate form. Next, we obtain the analytical forms of logarithmic negativity and mutual information and plot as a function of Rob's distance from the $r = 0$ point. We observe that the entanglement degradation slows down because of the structural change in the lapse function of the metric via the incorporation of quantum gravity corrections in the Schwarzschild black hole. It is crucial to understand that any modified gravity theories that changes the metric structure results in a different rate of degradation of the entanglement. At the horizon radius, the entanglement degradation is always complete irrespective of the underlying theory. This observation may lead to the identification of the signature of modified gravity theories in a future generation of advanced observational scenarios. Such a modification can come from higher curvature corrections, higher dimensional gravity theories, quantum gravity corrections, etc. We can also interpret this effect as a noisy quantum channel with an operator sum representation of a completely positive and trace preserving map. We then finally obtain the entanglement fidelity using this operator sum representation.

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I. INTRODUCTION

Our day to day classical information theory is restricted to a binary system where we can make use of 0 and 1 as measures of information stored or communicated. With the advent of quantum mechanics in the first quarter of twentieth century, the idea of a quantum version of the classical information theory came as a by-product of the quantum superposition principle. This new branch of physics was later named as quantum information theory. The relativistic generalization of quantum information theory, which involves general relativity, quantum field theory and quantum information theory, is also known as relativistic quantum information theory. The study of quantum correlations in case of a noninertial perspective is a very interesting sector in the genre of relativistic quantum information technology [1–22]. In several of these works, the case of an entangled bipartite system was investigated when one of the observer was uniformly accelerated. The idea was to transport the stationary state to the Rindler space in order to truly investigate the effect of acceleration. In all of these cases, the entangled states were taken as Fock states and instead of entanglement between spins, entanglement between number states were considered. In [4], the generic Alice-Rob picture in the Minkowski-Rindler background was transferred to the black hole picture for bosonic fields. This study was inadequate in a sense that the Rindler horizon and the event horizon of a Schwarzschild black hole are very different in nature. The Rindler horizon can only be perceived by an accelerated observer whereas the event horizon exists for all observers. To deal with this problem, in [20], a one to one correspondence was observed among different vacuums from both the Minkowski and curved spacetimes. The system consists of two observers, Alice and Rob. Alice is freely falling into the

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event horizon of a Schwarzschild black hole, and Rob is at a fixed radial distance just outside the event horizon of the black hole. Both Alice and Rob are observing a bipartite quantum state, and the state is maximally entangled for the freely falling observer. Rob sees a degradation in the state due to the Hawking effect. In their analysis, it was shown that the major interesting entanglement behaviors are observed in the vicinity of the event horizon. In case of the black hole picture when the observer is on the event horizon of the black hole, it imitates the infinite acceleration case in the Rindler spacetime.

In our analysis, we shall consider Alice to be freely falling into the event horizon of a quantum corrected black hole and Rob to be at a fixed distance just outside the event horizon of the same. Our main motivation behind this analysis is to investigate the effects of modification of the lapse function infused via the "*asymptotic safety approach*" to quantum gravity on entanglement degradation. One can also investigate the same for any modified theories of gravity. For example, one can consider higher curvature corrections in the Einstein-Hilbert action as well as any higher dimensional gravity theories. The line element of the quantum corrected black hole spacetime following from a renormalization group approach of gravity is given by [23]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2, \tag{1}$$

where

$$f(r) = 1 - \frac{2G(r)M}{r}, \tag{2}$$

with

$$G(r) = \frac{G}{1 + \frac{\tilde{\omega}G}{r^2}}. \tag{3}$$

Throughout our analysis, we have used $\hbar = c = 1$. The metric structure, given above, originates from the well-known "*asymptotic safety approach*" to quantum gravity. This formalism revolves around an effective average action and by taking into consideration all loop effects, this effective action describes all gravitational phenomenon [24–27]. This action satisfies a renormalization group equation, which results in the flow of the Newton's gravitational constant as a function of this scale. Using the flow of the Newton's gravitational constant, the metric in Eq. (1) was obtained where the constant $\tilde{\omega}$ carries quantum gravity corrections to the black hole geometry as a result of this renormalization group approach.

At first, we have used the near horizon approximation to probe any static spherically symmetric black hole metric in the well-known Rindler form, and following the analysis in [20], we have then obtained three unique timelike Killing vectors. The positive frequency modes associated with these Killing vectors let one define three unique vacuum states (Hartle-Hawking, Boulware, and anti-Boulware). Finally, one can obtain the relation between the Hartle-Hawking vacuum state and Boulware—anti-Boulware Fock space basis. Using this relation, we then calculate the logarithmic negativity and mutual information for the reduced density matrix (where all the anti-Boulware states have been traced out). For the next part of our analysis, we have used the formalism in [22] and shown that the entanglement degradation due to the Hawking effect can be described via a quantum channel with a completely positive and trace preserving (CPTP) map. We then finally compute the entanglement fidelity to investigate how the quantum channel preserves the initial entanglement between the two parties of the bipartite state. It is important to note that we have considered only bosonic field modes in our analysis.

The construction of the paper goes as follows. In Sec. II, we give a brief preview of the Alice-Rob system and obtain the relation between the Minkowski vacuum and Rindler Fock state basis. In Sec. III, we express a static spherically symmetric black hole in the Rindler form and obtain the analogy between several vacuum states. In Sec. IV, we obtain the analytical forms of the logarithmic negativity and mutual information for a quantum corrected black hole and plotted against the distance of the observer from the $r = 0$ point. In Sec. V, we investigate the entire process as a quantum channel with a CPTP map and obtain the analytical form of the entanglement fidelity for a quantum corrected black hole. Finally, we conclude our analysis in Sec. VI.

II. MINKOWSKI-RINDLER IDENTIFICATION: A BRIEF REVIEW

In this section, we start by providing a detailed and pedagogical derivation of the expression connecting Minkowski vacuum state and the product of two mode squeezed states of the Rindler vacuum [4]. The Rindler coordinate system is the one describing an uniformly accelerated observer. The Minkowski coordinates in $3+1$- spacetime dimensions is given by $\{t, x, y, z\}$, and the Rindler coordinates are denoted by $\{\bar{t}, \bar{x}, \bar{y}, \bar{z}\}$. In region I (right Rindler wedge), we can express the Minkowski coordinates, in terms of the Rindler coordinates as

$$t = \bar{z}\sinh a\bar{t}, \quad x = \bar{x}, \quad y = \bar{y}, \quad z = \bar{z}\cosh a\bar{t}, \tag{4}$$

and in region IV (left Rindler wedge),

$$t = -\bar{z}\sinh a\bar{t}, \quad x = \bar{x}, \quad y = \bar{y}, \quad z = -\bar{z}\cosh a\bar{t}. \tag{5}$$

In order to interconnect, the coordinate transformation between the two coordinate systems, we have considered that the observer is uniformly accelerating in the z direction only with an uniform acceleration a. In order to proceed further, we now come down to a $1+1$-dimensional analysis involving (t, z) coordinates only. The massless Klein-Gordon equation for a scalar field in the Minkowski background reads

$$\partial_\mu \partial^\mu \phi(t,z) = 0 \Rightarrow (\partial_t^2 - \partial_z^2)\phi(t,z) = 0. \tag{6}$$

In order to define the normalization constant, we need to write down the Lorentz invariant inner product, which is given by

$$(\phi_1, \phi_2) = -i \int_\Sigma d\Sigma^\mu (-\phi_1^* \partial_\mu \phi_2 + \phi_2 \partial_\mu \phi_1^*), \tag{7}$$

where Σ is a spacelike hypersurface. Now for a constant time hypersurface, we can simplify the above inner product in the following form (in $1+1$-dimensions):

$$(\phi_1, \phi_2) = -i \int_z dz (-\phi_1^* \partial_t \phi_2 + \phi_2 \partial_t \phi_1^*). \tag{8}$$

Using a separation of variables method, we can obtain a solution of the Klein-Gordon equation [Eq. (6)] and write down the analytical forms of the Minkowski field modes as

$$u_k^{\mathcal{M}}(t,z) = \mathcal{N}(k) e^{-i\omega t + i\omega z}, \tag{9}$$

where $\omega = k$ when the speed of light is set equal to unity and $\mathcal{N}(k)(= \mathcal{N}_\omega)$ is a real and undetermined normalization constant. We shall now make use of Eq. (8) to determine the undetermined normalization constant given by

$$\begin{aligned}
(u_k^{\mathcal{M}}, u_{k'}^{\mathcal{M}}) &= -i \int_{-\infty}^{\infty} dz \left[-u_k^{\mathcal{M}*} \partial_t u_{k'}^{\mathcal{M}} + u_{k'}^{\mathcal{M}} \partial_t u_k^{\mathcal{M}*} \right] \\
\Rightarrow \delta(\omega - \omega') &= 2\pi(\omega + \omega') \mathcal{N}_\omega \mathcal{N}_{\omega'} \delta(\omega - \omega') \\
&= 4\pi\omega \mathcal{N}_\omega^2 \delta(\omega - \omega') \\
\Rightarrow \mathcal{N}_\omega &= \frac{1}{\sqrt{4\pi\omega}}.
\end{aligned} \tag{10}$$

Using the above form of the normalization constant, we can finally write down the Minkowski mode solution as

$$u_k^{\mathcal{M}}(t,z) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega t + i\omega z}. \tag{11}$$

Next, we shall be calculating the Rindler modes in region I. We start by obtaining the Klein-Gordon equation in Rindler coordinates. To do this, we write down the relations among the partial derivatives corresponding to Minkowski and Rindler coordinates,

$$\begin{aligned}
\partial_t &= \frac{\partial \bar{t}}{\partial t} \partial_{\bar{t}} + \frac{\partial \bar{z}}{\partial t} \partial_{\bar{z}} \\
&= \frac{1}{a\bar{z}} \cosh a\bar{t} \partial_{\bar{t}} - \sinh a\bar{t} \partial_{\bar{z}},
\end{aligned} \tag{12}$$

$$\begin{aligned}
\partial_z &= \frac{\partial \bar{t}}{\partial z} \partial_{\bar{t}} + \frac{\partial \bar{z}}{\partial z} \partial_{\bar{z}} \\
&= -\frac{1}{a\bar{z}} \sinh a\bar{t} \partial_{\bar{t}} + \cosh a\bar{t} \partial_{\bar{z}}.
\end{aligned} \tag{13}$$

Using Eqs. (12) and (13) back in Eq. (6), we obtain the Klein-Gordon equation in the Rindler spacetime to be

$$(\partial_t^2 - \partial_z^2)\phi(t,z) = \frac{1}{a^2 \bar{z}^2} (\partial_{\bar{t}}^2 - a^2 \partial_{\ln \bar{z}}^2) \phi(\bar{t}, \bar{z}) = 0. \tag{14}$$

Solving Eq. (14) and making use of the inner product definition [Eq. (8)], we obtain the Rindler mode solution in region I to be

$$u_{k,\pm}^{\mathcal{R}_I} = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\bar{t} \pm \frac{i\omega}{a} \ln \bar{z}}. \tag{15}$$

In this analysis, we shall be mainly considering the $u_{k,+}^{\mathcal{R}_I}$ mode solutions. In terms of the Minkowski coordinates, the Rindler mode solution in Eq. (15) reads

$$u_{k,\pm}^{\mathcal{R}_I} = \sqrt{\frac{a}{4\pi\omega}} \left(\frac{z \mp t}{l_\omega} \right)^{\pm \frac{i\omega}{a}} = \frac{1}{\sqrt{4\pi\Omega}} \left(\frac{z \mp t}{l_\Omega} \right)^{\pm i\Omega} \equiv u_{\Omega,\pm}^I, \tag{16}$$

where Ω $(= \frac{\omega}{a})$ is a dimensionless constant, $l_\omega = l_\Omega$ has dimension of length in natural units, and $u_{\Omega,+}^I$ denotes field modes, which are propagating to the right direction along lines of constant $z - t$. Similarly, the Rindler mode solutions in region IV reads

$$u_{k,\pm}^{\mathcal{R}_{IV}} = \frac{1}{\sqrt{4\pi\Omega}} \left(\frac{\pm t - z}{l_\Omega} \right)^{\mp i\Omega} \equiv u_{\Omega,\pm}^{IV}. \tag{17}$$

As we shall mainly be considering the right moving modes, we shall be omitting the plus sign while writing down the mode solutions.

One can now do a second quantization of the classical field ϕ that satisfies the Klein-Gordon equation given by $\Box\hat{\phi} = 0$. In terms of the Minkowski mode solutions and the corresponding creation and annihilation operators, we can write down the quantized scalar field as

$$\hat{\phi} = \int dk (u_k^{\mathcal{M}}(t,z) \hat{a}_{k,\mathcal{M}} + u_k^{\mathcal{M}*}(t,z) \hat{a}_{k,\mathcal{M}}^\dagger), \tag{18}$$

where the creation and annihilation operators satisfy the following commutation relation

$$[\hat{a}_{k,\mathcal{M}}, \hat{a}_{k',\mathcal{M}}^\dagger] = \delta(k - k'). \tag{19}$$

The action of the annihilation operator on the vacuum state corresponding to a fixed field mode is defined as

$$\hat{a}_{k,\mathcal{M}}|0\rangle^k_{\mathcal{M}} = 0, \tag{20}$$

and the total Minkowski vacuum state is defined as a product of all the individual vacuum states corresponding to each field modes as

$$|0\rangle_{\mathcal{M}} = \prod_k |0\rangle^k_{\mathcal{M}}. \tag{21}$$

It is important to note that the mode solutions in regions I and IV provide a complete set of orthonormal solutions. As a result, one can express the field $\hat{\phi}$ in terms of the Rindler mode solutions as

$$\hat{\phi} = \int d\Omega (u^I_\Omega \hat{a}_{\Omega,I} + u^{I*}_\Omega \hat{a}^\dagger_{\Omega,I} + u^{IV}_\Omega \hat{a}_{\Omega,IV} + u^{IV*}_\Omega \hat{a}^\dagger_{\Omega,IV}), \tag{22}$$

where the creation and the annihilation operators act on the vacuum states of the two Rindler wedges, respectively, as

$$\hat{a}_{\Omega,I} \otimes \mathbb{1}_{IV}|0_I, 0_{IV}\rangle = (\hat{a}_{\Omega,I}|0_I\rangle) \otimes (\mathbb{1}_{IV}|0_{IV}\rangle) = 0, \tag{23}$$

$$\mathbb{1}_I \otimes \hat{a}_{\Omega,IV}|0_I, 0_{IV}\rangle = (\mathbb{1}_I|0_I\rangle) \otimes (\hat{a}_{\Omega,IV}|0_I V\rangle) = 0. \tag{24}$$

It is to be noted that region I and region IV are causally disconnected, and as a result, it is possible to write down the following commutation relations:

$$[\hat{a}_{\Omega,I}, \hat{a}^\dagger_{\Omega',I}] = [\hat{a}_{\Omega,IV}, \hat{a}^\dagger_{\Omega',IV}] = \delta(\Omega - \Omega'), \tag{25}$$

$$[\hat{a}_{\Omega,I}, \hat{a}_{\Omega',I}] = [\hat{a}^\dagger_{\Omega,I}, \hat{a}^\dagger_{\Omega',I}] = [\hat{a}_{\Omega,I}, \hat{a}^\dagger_{\Omega',IV}] = 0, \tag{26}$$

$$[\hat{a}_{\Omega,IV}, \hat{a}_{\Omega',IV}] = [\hat{a}^\dagger_{\Omega,IV}, \hat{a}^\dagger_{\Omega',IV}] = [\hat{a}^\dagger_{\Omega,I}, \hat{a}_{\Omega',IV}] = 0. \tag{27}$$

We now need to express the creation and annihilation operators of the Minkowski states in terms of the creation and annihilation operators of the Rindler states. Before proceeding with this analysis, we need to remember that the mode solutions $u^{\mathcal{M}}_k$ satisfies the following relation with respect to the inner product defined in Eq. (8) as

$$(u^{\mathcal{M}}_k, u^{\mathcal{M}}_{k'}) = \delta(k - k'), \qquad (u^{\mathcal{M}}_k, u^{\mathcal{M}*}_{k'}) = 0. \tag{28}$$

Taking the inner product of $\hat{\phi}$ (for the decomposition of $\hat{\phi}$ in terms of Minkowski field modes) with $u^{\mathcal{M}}_{k'}$, we obtain the following relation:

$$\begin{aligned}(u^{\mathcal{M}}_{k'}, \hat{\phi}) &= \int dk ((u^{\mathcal{M}}_{k'}, u^{\mathcal{M}}_k)\hat{a}_{k,\mathcal{M}} + (u^{\mathcal{M}}_{k'}, u^{\mathcal{M}*}_k)\hat{a}^\dagger_{k,\mathcal{M}}) \\ &= \int dk \delta(k - k')\hat{a}_{k,\mathcal{M}} \\ &= \hat{a}_{k',\mathcal{M}}.\end{aligned} \tag{29}$$

It is now possible to substitute the mode expansion of $\hat{\phi}$ from Eq. (22) in the left-hand side of the above equation, and we can recast Eq. (29) in the following form:

$$\begin{aligned}\hat{a}_{k,\mathcal{M}} = \int d\Omega ((u^{\mathcal{M}}_k, u^I_\Omega)\hat{a}_{\Omega,I} + (u^{\mathcal{M}}_k, u^{I}_\Omega{}^*)\hat{a}^\dagger_{\Omega,I} \\ + (u^{\mathcal{M}}_k, u^{IV}_\Omega)\hat{a}_{\Omega,IV} + (u^{\mathcal{M}}_k, u^{IV*}_\Omega)\hat{a}^\dagger_{\Omega,IV}).\end{aligned} \tag{30}$$

We shall now evaluate all of the four inner products in the above equation. Applying the definition of the inner product, the first inner product turns out to be

$$\begin{aligned}(u^{\mathcal{M}}_k, u^I_\Omega) &= -i \int dz (u^{\mathcal{M}*}_k \partial_t u^I_\Omega + u^I_\Omega \partial_t u^{\mathcal{M}*}_k) \\ &= \frac{1}{4\pi l^{i\Omega}_\Omega \sqrt{\omega\Omega}} \int dz (\Omega (z - t)^{i\Omega - 1} \\ &\quad + \omega (z - t)^{i\Omega}) e^{-i\omega(z - t)}.\end{aligned} \tag{31}$$

We shall now make a change of coordinates given by $z - t = \zeta$ and in the Rindler wedge I, $(z - t) > 0$. Hence, ζ will range from 0 to ∞. We can recast Eq. (31) in the following form:

$$\begin{aligned}(u^{\mathcal{M}}_k, u^I_\Omega) &= \frac{1}{4\pi l^{i\Omega}_\Omega \sqrt{\omega\Omega}} \int_0^\infty d\zeta (\Omega \zeta^{i\Omega - 1} + \omega \zeta^{i\Omega}) e^{-i\omega\zeta} \\ &= \frac{\Omega (i\omega)^{-i\Omega}}{2\pi l^{i\Omega}_\Omega \sqrt{\omega\Omega}} \Gamma[i\Omega] \\ &= \frac{\Omega (i l_\Omega \omega)^{-i\Omega}}{2\pi \sqrt{\omega\Omega}} \sqrt{\frac{\pi}{\Omega}} \sqrt{\frac{2}{e^{\pi\Omega} - e^{-\pi\Omega}}} e^{i \arg[\Gamma[i\Omega]]} \\ &= \frac{1}{\sqrt{2\pi\omega}} (l_\Omega e^{-\frac{\phi}{\Omega}} \omega)^{-i\Omega} \frac{1}{\sqrt{1 - e^{-2\pi\Omega}}} \\ &= \frac{1}{\sqrt{2\pi\omega}} (l\omega)^{-i\Omega} \frac{1}{\sqrt{1 - e^{-2\pi\Omega}}},\end{aligned} \tag{32}$$

where $\phi \equiv \mathrm{Arg}[\Gamma[i\Omega]]$, $l \equiv l_\Omega e^{-\frac{\phi}{\Omega}}$, and $(i)^{-i\Omega} = (e^{\frac{i\pi}{2}})^{-i\Omega} = e^{\frac{\pi\Omega}{2}}$. The next inner product of $u^{\mathcal{M}}_k$ with $u^I_\Omega{}^*$ is given as follows:

$$(u^{\mathcal{M}}_k, u^I_\Omega{}^*) = -\frac{1}{\sqrt{2\pi\omega}} (l\omega)^{i\Omega} \frac{e^{-\pi\Omega}}{\sqrt{1 - e^{-2\pi\Omega}}}. \tag{33}$$

The final two inner products have the forms given by

$$(u^{\mathcal{M}}_k, u^{IV}_\Omega) = \frac{1}{\sqrt{2\pi\omega}} (l\omega)^{i\Omega} \frac{1}{\sqrt{1 - e^{-2\pi\Omega}}}, \tag{34}$$

$$(u^{\mathcal{M}}_k, u^{IV*}_\Omega) = -\frac{1}{\sqrt{2\pi\omega}} (l\omega)^{-i\Omega} \frac{e^{-\pi\Omega}}{\sqrt{1 - e^{-2\pi\Omega}}}. \tag{35}$$

With a new redefinition $e^{-\pi\Omega} \equiv \tanh r_\Omega$, we can recast Eq. (30) as

$$\hat{a}_{k,\mathcal{M}} = \int_0^\infty d\Omega(\alpha^R_{\omega,\Omega}{}^*(\cosh r_\Omega \hat{a}_{\Omega,I} - \sinh r_\Omega \hat{a}^\dagger_{\Omega,IV}) + \alpha^L_{\omega,\Omega}{}^*(-\sinh r_\Omega \hat{a}^\dagger_{\Omega,I} + \cosh r_\Omega \hat{a}_{\Omega,IV})), \quad (36)$$

where $\alpha^R_{\omega,\Omega}{}^* = \frac{1}{\sqrt{2\pi\omega}}(l\omega)^{-i\Omega}$ and $\alpha^L_{\omega,\Omega}{}^* = \frac{1}{\sqrt{2\pi\omega}}(l\omega)^{i\Omega}$ are the Bogoliubov coefficients.

We can now express the right and left moving Unruh annihilation operators as

$$\hat{a}^R_\Omega = \cosh r_\Omega \hat{a}_{\Omega,I} - \sinh r_\Omega \hat{a}^\dagger_{\Omega,IV}, \quad (37)$$

$$\hat{a}^L_\Omega = -\sinh r_\Omega \hat{a}^\dagger_{\Omega,I} + \cosh r_\Omega \hat{a}_{\Omega,IV}. \quad (38)$$

The Bogoliubov coefficients in the context of Minkowski-Rindler transformations has been calculated in a plethora of literatures [28–34]. By means of Eqs. (37) and (38), we can indeed reexpress the Minkowski annihilation operator in Eq. (36) as

$$\hat{a}_{k,\mathcal{M}} = \int_0^\infty d\Omega(\alpha^R_{\omega,\Omega}{}^*\hat{a}^R_\Omega + \alpha^L_{\omega,\Omega}{}^*\hat{a}^L_\Omega). \quad (39)$$

It is important to note from Eq. (39) that the Minkowski annihilation operator can be expressed as a combination of the Unruh annihilation operators only. As a result, the Unruh annihilation operator will annihilate the Minkowski vacuum as well. Hence, we can write down the following relation:

$$\hat{a}_{\omega,\mathcal{M}}|0\rangle_\mathcal{M} = \hat{a}^R_\Omega|0\rangle_\mathcal{M} = \hat{a}^L_\Omega|0\rangle_\mathcal{M} = 0. \quad (40)$$

From Eq. (40), it is straightforward to conclude that the Minkowski vacuum and Unruh vacuum are identical, which can be represented in the following form:

$$|0\rangle_\mathcal{M} = |0\rangle_U = \prod_\Omega |0\rangle^\Omega_U, \quad (41)$$

where $|0\rangle^\Omega_U$ is the Unruh vacuum corresponding to an individual field mode with frequency Ω.

Now, we take an ansatz given by

$$|0\rangle^\Omega_U = \sum_n f_\Omega(n)|n\rangle^\Omega_I \otimes |n\rangle^\Omega_{IV}, \quad (42)$$

where $f_\Omega(n)$ is an unknown normalization factor, dependent on the dimensionless number Ω. Before acting with $\hat{a}^R_\Omega$ on the both sides of Eq. (42), we need to express $\hat{a}^R_\Omega$ rigorously as

$$\hat{a}^R_\Omega = \cosh r_\Omega \hat{a}_{\Omega,I} \otimes \mathbb{1}_{IV} - \sinh r_\Omega \mathbb{1}_I \otimes \hat{a}^\dagger_{\Omega,IV}. \quad (43)$$

Action of $\hat{a}^R_\Omega$ from Eq. (43) on the both sides of Eq. (42) is given by

$$\begin{aligned} 0 = \hat{a}^R_\Omega|0\rangle^\Omega_U &= \sum_n f_\Omega(n)(\cosh r_\Omega \hat{a}_{\Omega,I}|n\rangle^\Omega_I \otimes |n\rangle^\Omega_{IV} - \sinh r_\Omega |n\rangle^\Omega_I \hat{a}^\dagger_{\Omega,IV}|n\rangle^\Omega_{IV}) \\ &= \sum_n f_\Omega(n)(\cosh r_\Omega \sqrt{n}|n-1\rangle^\Omega_I \otimes |n\rangle^\Omega_{IV} - \sinh r_\Omega \sqrt{n+1}|n\rangle^\Omega_I \otimes |n+1\rangle^\Omega_{IV}). \end{aligned} \quad (44)$$

We now act with ${}^\Omega_I\langle m| \otimes {}^\Omega_{IV}\langle m'|$ from the left in the above equation, and we can then recast Eq. (44) as

$$\begin{aligned} 0 &= \sum_n f_\Omega(n)(\cosh r_\Omega \sqrt{n}\,{}^\Omega_I\langle m|n-1\rangle^\Omega_I\,{}^\Omega_{IV}\langle m'|n\rangle^\Omega_{IV} - \sinh r_\Omega \sqrt{n+1}\,{}^\Omega_I\langle m|n\rangle^\Omega_I\,{}^\Omega_{IV}\langle m'|n+1\rangle^\Omega_{IV}) \\ &= \sum_n f_\Omega(n)(\cosh r_\Omega \sqrt{n}\delta_{m,n-1}\delta_{m',n} - \sinh r_\Omega \sqrt{n+1}\delta_{m,n}\delta_{m',n+1}) \\ &= (f_\Omega(m')\cosh r_\Omega - f_\Omega(m'-1)\sinh r_\Omega)\sqrt{m'}\delta_{m',m+1}. \end{aligned} \quad (45)$$

From Eq. (45), we can write down a recursion relation in $f_\Omega(n)$ as follows:

$$f_\Omega(n) = \tanh^n r_\Omega f_\Omega(0). \quad (46)$$

We need to determine the constant $f_\Omega(0)$ by imposing the normalization condition of $|0\rangle^\Omega_U$ as follows:

$$\begin{aligned} 1 &= {}^\Omega_U\langle 0|0\rangle^\Omega_U \\ &= f^2_\Omega(0)\sum_{n,m}(\tanh r_\Omega)^{n+m}\,{}^\Omega_I\langle m|n\rangle^\Omega_I\,{}^\Omega_{IV}\langle m|n\rangle^\Omega_{IV} \\ &= f^2_\Omega(0)\sum_n \tanh^{2n} r_\Omega \\ &= f^2_\Omega(0)\cosh^2 r_\Omega \\ \Rightarrow f_\Omega(0) &= \frac{1}{\cosh r_\Omega}. \end{aligned} \quad (47)$$

Using the form of $f_\Omega(n)$ and $f_\Omega(0)$ from Eqs. (46) and (47), we can recast Eq. (42) as

$$|0\rangle^\Omega_U = \frac{1}{\cosh r_\Omega}\sum_n \tanh^n r_\Omega |n\rangle^\Omega_I |n\rangle^\Omega_{IV}, \quad (48)$$

where for simplicity, we have omitted the tensor product sign. The Unruh vacuum state and the Minkowski vacuum state coincides; hence, from Eq. (48), we can write down the following relation:

$$|0\rangle^k_\mathcal{M} = \frac{1}{\cosh r_\Omega}\sum_n \tanh^n r_\Omega |n\rangle^\Omega_I |n\rangle^\Omega_{IV}. \quad (49)$$

We shall also need to calculate the first excited state corresponding to the single mode Minkowski vacuum state. As the Minkowski and Unruh states can be mapped with one another, we start by calculating the first excited state in the Unruh vacuum corresponding to a single mode only. The complete Unruh raising operator is given by a linear combination of the raising operators corresponding to the left and right moving Unruh modes as follows:

$$\hat{a}_U^{\Omega\dagger} = \mathcal{A}_L \hat{a}_\Omega^{L\dagger} + \mathcal{A}_R \hat{a}_\Omega^{R\dagger}, \tag{50}$$

where $|\mathcal{A}_L|^2 + |\mathcal{A}_R|^2 = 1$. A very convenient choice is to take $\mathcal{A}_R = 1$ and $\mathcal{A}_L = 0$. We can hence write down the first excited state by using the operator action of $\hat{a}_U^{\Omega\dagger}$ on $|0\rangle_U^\Omega$ along with the determination of an appropriate normalization constant (following the earlier procedure) as

$$|1\rangle_U^\Omega = |1\rangle_{\mathcal{M}}^k = \sum_n \frac{\sqrt{n+1}}{\cosh^2 r_\Omega} \tanh^n r_\Omega |n+1\rangle_I^\Omega |n\rangle_{IV}^\Omega. \tag{51}$$

With Eqs. (49) and (51) in hand, we can now move towards the analysis of entangled Fock states in a curved background.

III. NEAR HORIZON ANALYSIS AND THE RINDLER-KRUSKAL IDENTIFICATION

In this section, we shall consider the quantum corrected black hole geometry and apply the near horizon approximation to recast the metric in a form which will help us to use the usual quantum information theoretic wisdom to analyze the entanglement degradation for a maximally entangled state on this black hole geometry. In this section, we shall follow the analysis used in [20].

The line element for a static spherically symmetric black hole geometry with a lapse function $f(r)$ in $3+1$-spacetime dimensions is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2, \tag{52}$$

where we have used the $\{-,+,+,+\}$ signature for the metric. We shall now make use of the near horizon approximation and write down the lapse function in the following form:

$$f(r) \simeq (r - r_+) f'(r_+), \tag{53}$$

where r_+ is the event horizon radius of the black hole. We now make a change of coordinates given by

$$\zeta = 2\sqrt{\frac{r - r+}{f'(r_+)}} \Rightarrow r - r_+ = \frac{\zeta^2}{4} f'(r_+). \tag{54}$$

Using Eqs. (53) and (54), we can recast the line element in Eq. (52) in the following form (in $1+1$-spacetime dimensions):

$$ds^2 = -\frac{\zeta^2 f'^2(r_+)}{4} dt^2 + d\zeta^2. \tag{55}$$

In terms of the surface gravity $\kappa = \frac{f'(r_+)}{2}$, we can recast the above equation in the following form:

$$ds^2 = -\kappa^2 \zeta^2 dt^2 + d\zeta^2. \tag{56}$$

We now consider an observer sitting at a distance $\mathscr{r}$ where the proper time of the observer is denoted by τ. We can then write down the following relation:

$$-d\tau^2 = -f(r)|_{r=\mathscr{r}} dt^2 + \frac{1}{f(r)}\bigg|_{r=\mathscr{r}} d\mathscr{r}^2 = -f(\mathscr{r})dt^2$$

$$\text{or,} \quad \frac{dt}{d\tau} = \frac{1}{\sqrt{f(\mathscr{r})}} \Rightarrow t = \frac{\tau}{\sqrt{f(\mathscr{r})}}. \tag{57}$$

In terms of the proper time τ, we can recast Eq. (56) as follows:

$$ds^2 = -\frac{\kappa^2 \zeta^2}{f(\mathscr{r})} d\tau^2 + d\zeta^2. \tag{58}$$

Now the value of the proper acceleration for an accelerated observer at some r is defined as

$$a = \sqrt{a_\mu a^\mu}, \tag{59}$$

where $a^\mu = \frac{\xi^\beta}{|\xi|} \nabla_\beta \left(\frac{\xi^\mu}{|\xi|}\right)$ gives the four acceleration with $v^\mu = \frac{\xi^\mu}{|\xi|}$ denoting the four velocity of the observer and $\xi^\mu = \{1, 0, 0, 0\}$ being a timelike Killing vector. It is now straightforward to evaluate the four acceleration of the observer,

$$a^\mu = \left\{0, \frac{1}{2}\partial_r f, 0, 0\right\}, \quad a_\mu = g_{\kappa\mu} a^\kappa = \left\{0, \frac{1}{2f}\partial_r f, 0, 0\right\}. \tag{60}$$

Using Eq. (60), we can obtain the proper acceleration of the observer to be of the form,

$$a(r) = \sqrt{a_\mu a^\mu} = \frac{\partial_r f}{2\sqrt{f(r)}}. \tag{61}$$

From Eq. (61), it is straightforward to infer that the acceleration becomes infinite when $r = r_+$. In the near horizon approximation, we can evaluate the following relation:

$$\partial_r f(r) \simeq \partial_r((r - r_+) f'(r_+)) = f'(r_+) = 2\kappa. \tag{62}$$

Hence, we can write down the proper acceleration for an observer sitting at a distance $\mathcal{r}$ from the $r = 0$ point to be

$$\mathcal{a} = a(\mathcal{r}) = \frac{\kappa}{\sqrt{f(\mathcal{r})}}. \tag{63}$$

Using Eq. (63), we can recast Eq. (58) as

$$ds^2 = -\mathcal{a}^2\zeta^2 d\tau^2 + d\zeta^2. \tag{64}$$

Equation (63) depicts the fact that any static spherically symmetric black hole metric can be expressed in the Rindler form by means of near horizon approximation, and the constant acceleration in the Rindler case is now replaced by the proper acceleration of an observer sitting at a fixed radial distance outside but in the vicinity of the event horizon of the black hole.

Our next aim is to define timelike vectors. We start by writing down the null Kruskal-Szekeres coordinates as

$$\mathcal{u} = -\frac{1}{\kappa}e^{-\kappa(t-\int\frac{dr}{f(r)})}, \qquad \mathcal{v} = \frac{1}{\kappa}e^{\kappa(t+\int\frac{dr}{f(r)})}. \tag{65}$$

Using Eq. (65), one can write down the radial part of the black hole metric in the following form:

$$ds^2 = -f(r)e^{-2\kappa\int\frac{dr}{f(r)}}d\mathcal{u}d\mathcal{v}. \tag{66}$$

Very near the horizon, Eq. (66) can be expressed as (keeping only leading constant terms and setting the integration constant to $\frac{1}{2\kappa}$)

$$ds^2 \simeq -e^{-1}d\mathcal{u}d\mathcal{v}. \tag{67}$$

This analysis shows (following [20]) that there are three possible timelike Killing vectors. The first timelike Killing vector is $\partial_{\bar{t}} \propto \partial_{\mathcal{u}} + \partial_{\mathcal{v}}$, where this timelike vector is similar to the timelike Killing vector in the Minkowski spacetime. One can construct a vacuum state out of positive frequency modes associated with this timelike Killing vector, and this vacuum state is also known as the Hartle-Hawking vacuum state. As a result of the analogy between the Killing vectors, we can also claim that the Hartle-Hawking vacuum state is analogous to the Minkowski vacuum state. The Hawking-Hartle vacuum state is generally written as $|0\rangle_H$ and $|0\rangle_H \leftrightarrow |0\rangle_{\mathcal{M}}$. The second Killing vector is ∂_t, and it is straightforward to obtain a relation in terms of the $\{\mathcal{u}, \mathcal{v}\}$ coordinate system as follows:

$$\begin{aligned}\partial_t &= \frac{\partial\mathcal{u}}{\partial t}\frac{\partial}{\partial\mathcal{u}} + \frac{\partial\mathcal{v}}{\partial t}\frac{\partial}{\partial\mathcal{v}} \\ &= (-\kappa)\left[-\frac{1}{\kappa}e^{-\kappa(t-\int\frac{dr}{f(r)})}\right]\frac{\partial}{\partial\mathcal{u}} + \kappa\left[\frac{1}{\kappa}e^{\kappa(t+\int\frac{dr}{f(r)})}\right]\frac{\partial}{\partial\mathcal{v}} \\ &= -\kappa(\mathcal{u}\partial_{\mathcal{u}} - \mathcal{v}\partial_{\mathcal{v}}).\end{aligned} \tag{68}$$

From the above calculation, we deduce that $\partial_t \propto \mathcal{u}\partial_{\mathcal{u}} - \mathcal{v}\partial_{\mathcal{v}}$. ∂_t is a timelike Killing vector for any static spherically symmetric black hole geometry, and the positive frequency modes associated with this timelike Killing vector results in a vacuum state known as the Boulware vacuum state. The Boulware vacuum state is denoted by $|0\rangle_B$ and $|0\rangle_B \leftrightarrow |0\rangle_I$, which indicates the Boulware vacuum state is analogous to the Rindler vacuum state in region I. Another timelike Killing vector which can be defined is $-\partial_t$, and the positive frequency modes associated with this time like Killing vectors results in the $|0\rangle_{\bar{B}}$, also known as the anti-Boulware vacuum state. The anti-Boulware vacuum state is analogous to $|0\rangle_{IV}$. From the analogy among the Hartle-Hawking (Boulware, anti-Boulware) and Minkowski (Rindler I, Rindler IV) vacuum states, we can rewrite Eq. (49) in a static spherically black hole geometry as

$$|0\rangle_H^{\omega_i} = \frac{1}{\cosh\sigma_{\omega_i}}\sum_n \tanh^n\sigma_{\omega_i}|n\rangle_B^{\omega_i}|n\rangle_{\bar{B}}^{\omega_i}, \tag{69}$$

where $|0\rangle_H = \otimes_j|0\rangle_H^{\omega_j}$ and

$$\tanh\sigma_{\omega_i} = e^{-\frac{\pi\omega_i}{\mathcal{a}}} = \exp\left(-\frac{\pi\omega_i\sqrt{f(\mathcal{r})}}{\kappa}\right). \tag{70}$$

The above result comes from direct analogy with the corresponding result in the Minkowski-Rindler scenario. For a quantum corrected black hole metric, we can recast Eq. (70) as

$$\tanh\sigma_{\omega_i} = e^{-\frac{2\pi\omega_i GM\left(1-\frac{2GM\mathcal{r}}{\mathcal{r}^2+\tilde{\omega}G}\right)\left(GM+\sqrt{G^2M^2-\tilde{\omega}G}\right)^2}{G^2M^2+GM\sqrt{G^2M^2-\tilde{\omega}G}-\tilde{\omega}G}}. \tag{71}$$

As $\tilde{\omega}$ is a quantum gravity correction (which is very small), we can recast Eq. (71) in a much simpler form given by

$$\tanh\sigma_{\omega_i} \simeq e^{-4\pi\omega_i GM\sqrt{1-\frac{2GM}{\mathcal{r}}}\left(1+\frac{\tilde{\omega}}{4GM^2}+\frac{\tilde{\omega}G^2M}{\mathcal{r}^2(\mathcal{r}-2GM)}\right)}. \tag{72}$$

For the quantum corrected black hole metric, we redefine the σ_{ω_i} term as $\mathcal{r}_{\tilde{\omega},i}$. The one particle Hartle-Hawking state takes the form given as (for a quantum corrected black hole)

$$|1\rangle_H^{\omega_i} = \frac{1}{\cosh^2\mathcal{r}_{\tilde{\omega},i}}\sum_{n=0}^{\infty}\tanh^n\mathcal{r}_{\tilde{\omega},i}\sqrt{n+1}|n+1\rangle_B^{\omega_i}|n\rangle_{\bar{B}}^{\omega_i}. \tag{73}$$

With this background in place, we would now like to consider a maximally entangled bipartite state in the basis of an observer freely falling into the event horizon of a black hole. Before writing this state we would like to recall that the maximally entangled state in the Minkowski-Rindler analysis takes the form [4,20],

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_{A,\mathcal{M}}^{\omega_i}|0\rangle_{R,\mathcal{M}}^{\omega_i} + |1\rangle_{A,\mathcal{M}}^{\omega_i}|1\rangle_{R,\mathcal{M}}^{\omega_i}), \tag{74}$$

where $|0\rangle^{\omega_i}_{A,\mathcal{M}}$ denotes the single mode Minkowski vacuum state for Alice and $|0\rangle^{\omega_i}_{R,\mathcal{M}}$ denotes single mode Minkowski vacuum state for Rob. With the analogy established earlier between the flat spacetime case and the black hole scenario, we know that $|0\rangle^{\omega_i}_{\mathcal{M}} \leftrightarrow |0\rangle^{\omega_i}_{H}$. Hence, using this correspondence, we can write down the maximally entangled bipartite state in the basis of an observer freely falling into the event horizon of a black hole as [20]

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\omega_i}_{A,H}|0\rangle^{\omega_i}_{R,H} + |1\rangle^{\omega_i}_{A,H}|1\rangle^{\omega_i}_{R,H}). \tag{75}$$

The suffix "A" in the first part of the system (out of the two subsystems) denotes freely falling Alice and the second subsystem is for Rob, who is at a distance r near the event horizon of the quantum corrected black hole. In Eq. (75), the states corresponding to Alice ($|\rangle_{A,H}$) are the number states of the scalar field theory in the Hartle-Hawking basis and the Rob states ($|\rangle_{R,H}$) are the number states in the Hartle-Hawking basis as well [20]. As the underlying theory is that of scalar quantum electrodynamics, the number states corresponds to the number of scalar photons with energy equal to the angular frequency (according to the label of the state) multiplied by the reduced Planck's constant.

IV. LOGARITHMIC NEGATIVITY AND MUTUAL INFORMATION

In this section, we shall obtain the logarithmic negativity and mutual information corresponding to the maximally entangled biparite state given in Eq. (75). Our main aim is to do a side by side comparison for the case of a Schwarzschild and a quantum corrected black hole to truly investigate the effect of the change in the event horizon radius due to the underlying quantum nature of the black hole. Before proceeding further, it is important to note that $|0\rangle_A \leftrightarrow |0\rangle_H$ and $|0\rangle_R$ for a fixed frequency value is described by Eq. (69). Boulware and anti-Boulware states are causally disconnected, and Rob is causally disconnected from accessing the anti-Boulware states. As a result, we shall be tracing over the anti-Boulware states which shall lead to a mixed state. The reduced density matrix is given by

$$\begin{aligned}\rho_{AR} &= \sum_{m=0}^{\infty} {}_{\bar{B}}\langle m|\psi\rangle\langle\psi|m\rangle_{\bar{B}} \\ &= \frac{1}{2\cosh^2 r_{\tilde{\omega},i}}\sum_{n=0}^{\infty}\tanh^{2n} r_{\tilde{\omega},i}\Big[|0\,n\rangle\langle 0\,n| \\ &\quad + \frac{\sqrt{n+1}}{\cosh r_{\tilde{\omega},i}}(|1\,n+1\rangle\langle 0\,n| + |0\,n\rangle\langle 1\,n+1|) \\ &\quad + \frac{n+1}{\cosh^2 r_{\tilde{\omega},i}}|1\,n+1\rangle\langle 1\,n+1|\Big].\end{aligned} \tag{76}$$

We shall now make use of the partial transpose criteria [35] which shall provide us with sufficient criteria for entanglement. The $\{n, n+1\}$ block of the reduced density matrix in Eq. (76) is given by

$$\left(\frac{1}{2\cosh^2 r_{\tilde{\omega},i}}\right)\left[\begin{array}{cc|cc} \mathcal{B}^{0n}_{0n} & \mathcal{B}^{0n}_{1n} & \mathcal{B}^{0n}_{0n+1} & \mathcal{B}^{0n}_{1n+1} \\ \mathcal{B}^{1n}_{0n} & \mathcal{B}^{1n}_{1n} & \mathcal{B}^{1n}_{0n+1} & \mathcal{B}^{1n}_{1n+1} \\ \hline \mathcal{B}^{0n+1}_{0n} & \mathcal{B}^{0n+1}_{1n} & \mathcal{B}^{0n+1}_{0n+1} & |\mathcal{B}^{0n+1}_{1n+1} \\ \mathcal{B}^{1n+1}_{0n} & \mathcal{B}^{1n+1}_{1n} & \mathcal{B}^{1n+1}_{0n+1} & \mathcal{B}^{1n+1}_{1n+1} \end{array}\right], \tag{77}$$

where $\mathcal{B}^{ab}_{cd}$ denotes the coefficient associated with the $|ab\rangle\langle cd|$ state. After taking partial transpose of the matrix in Eq. (77), we obtain the following matrix:

$$\left(\frac{1}{2\cosh^2 r_{\tilde{\omega},i}}\right)\left[\begin{array}{cc|cc} \mathcal{B}^{0n}_{0n} & \mathcal{B}^{0n}_{1n} & \mathcal{B}^{0n}_{0n+1} & \mathcal{B}^{1n}_{0n+1} \\ \mathcal{B}^{1n}_{0n} & \boxed{\mathcal{B}^{1n}_{1n}} & \boxed{\mathcal{B}^{0n}_{1n+1}} & \mathcal{B}^{1n}_{1n+1} \\ \hline \mathcal{B}^{0n+1}_{0n} & \boxed{\mathcal{B}^{1n+1}_{0n}} & \boxed{\mathcal{B}^{0n+1}_{0n+1}} & |\mathcal{B}^{0n+1}_{1n+1} \\ \mathcal{B}^{0n+1}_{1n} & \mathcal{B}^{1n+1}_{1n} & \mathcal{B}^{1n+1}_{0n+1} & \mathcal{B}^{1n+1}_{1n+1} \end{array}\right]. \tag{78}$$

The new matrix consisting of the boxed elements from Eq. (78) is given by

$$\mathcal{P}_{n,n+1} = \frac{\tanh^{2n} r_{\tilde{\omega},i}}{2\cosh^2 r_{\tilde{\omega},i}}\begin{bmatrix} \frac{n}{\sinh^2 r_{\tilde{\omega},i}} & \frac{\sqrt{n+1}}{\cosh r_{\tilde{\omega},i}} \\ \frac{\sqrt{n+1}}{\cosh r_{\tilde{\omega},i}} & \tanh^2 r_{\tilde{\omega},i}. \end{bmatrix}. \tag{79}$$

The eigenvalues of the $\mathcal{P}_{n,n+1}$ matrix are given as

$$\xi_{n,\pm} = \frac{\tanh^{2n} r_{\tilde{\omega},i}}{4\cosh^2 r_{\tilde{\omega},i}}\left[\left(\tanh^2 r_{\tilde{\omega},i} + \frac{n}{\sinh^2 r_{\tilde{\omega},i}}\right) \pm \sqrt{\left(\tanh^2 r_{\tilde{\omega},i} + \frac{n}{\sinh^2 r_{\tilde{\omega},i}}\right)^2 + \frac{4}{\cosh^2 r_{\tilde{\omega},i}}}\right]. \tag{80}$$

From Eq. (80), it is straightforward to infer that $\xi_{n,-} < 0$. The logarithmic negativity is obtained as [4]

$$\begin{aligned} N(\rho_{AR}) &= \log_2 ||\rho_{AR}^T|| \\ &= \log_2\left[1 + \sum_{n=0}^{\infty}(|\xi_{n,-}| - \xi_{n,-})\right] \\ &= \log_2\left[1 - 2\sum_{n=0}^{\infty}\xi_{n,-}\right] \\ &= \log_2\left[1 + \sum_{n=0}^{\infty}\frac{\tanh^{2n} r_{\tilde{\omega},i}}{2\cosh^2 r_{\tilde{\omega},i}}\sqrt{\left(\tanh^2 r_{\tilde{\omega},i} + \frac{n}{\sinh^2 r_{\tilde{\omega},i}}\right)^2 + \frac{4}{\cosh^2 r_{\tilde{\omega},i}}} - \sum_{n=0}^{\infty}\frac{\tanh^{2n} r_{\tilde{\omega},i}}{2\cosh^2 r_{\tilde{\omega},i}}\left(\frac{n}{\sinh^2 r_{\tilde{\omega},i}} + \tanh^2 r_{\tilde{\omega},i}\right)\right] \\ &= \log_2\left[\frac{1}{2\cosh^2 r_{\tilde{\omega},i}} + \sum_{n=0}^{\infty}\frac{\tanh^{2n} r_{\tilde{\omega},i}}{2\cosh^2 r_{\tilde{\omega},i}}\sqrt{\left(\tanh^2 r_{\tilde{\omega},i} + \frac{n}{\sinh^2 r_{\tilde{\omega},i}}\right)^2 + \frac{4}{\cosh^2 r_{\tilde{\omega},i}}}\right] \\ &= \log_2\left[\frac{1}{2\cosh^2 r_{\tilde{\omega},i}} + \Lambda(r_{\tilde{\omega},i})\right], \end{aligned} \tag{81}$$

where

$$\Lambda(r_{\tilde{\omega},i}) = \sum_{n=0}^{\infty}\frac{\tanh^{2n} r_{\tilde{\omega},i}}{2\cosh^2 r_{\tilde{\omega},i}} \times \sqrt{\left(\tanh^2 r_{\tilde{\omega},i} + \frac{n}{\sinh^2 r_{\tilde{\omega},i}}\right)^2 + \frac{4}{\cosh^2 r_{\tilde{\omega},i}}}. \tag{82}$$

For an observer at infinite distance, $a(r \to \infty) = 0$ leading to $N(\rho_{AR}) = 1$. When the observer is on the event horizon of the black hole then $a(r_+) \to \infty$ which is identical to the condition $r_{\tilde{\omega},i} \to \infty$. To obtain the value of the entanglement negativity at this point, we need to truly investigate the bound on the negativity value in this limit. It is straightforward to obtain a bound on the summation term in the above equation. We know that $a^2 + b^2 < (a+b)^2$, and we can write down the following inequality:

$$\begin{aligned} \Lambda(r_{\tilde{\omega},i}) &< \sum_{n=0}^{\infty}\frac{\tanh^{2n} r_{\tilde{\omega},i}}{2\cosh^2 r_{\tilde{\omega},i}}\left[\tanh^2 r_{\tilde{\omega},i} + \frac{n}{\sinh^2 r_{\tilde{\omega},i}} + \frac{2}{\cosh r_{\tilde{\omega},i}}\right] \\ &= \frac{1}{2}\left(1 + \frac{2}{\cosh r_{\tilde{\omega},i}} + \tanh^2 r_{\tilde{\omega},i}\right) < 1 + \frac{1}{\cosh r_{\tilde{\omega},i}}. \end{aligned} \tag{83}$$

Now in the $r_{\tilde{\omega},i} \to \infty$ limit Λ goes to 1. Hence, in the infinite acceleration case or when the observer is on the event horizon of the black hole, the logarithmic negativity becomes 0. In the $\tilde{\omega} \to 0$ limit, $r_{\tilde{\omega},i} \to r_{\mathrm{Sch.},i}$ with "*Sch.*" denotes the case for a Schwarzschild black hole. For the next part of our analysis, we shall be denoting $r_{\tilde{\omega},i}$ as $\mathcal{R}_i$. The logarithmic negativity from Eq. (81) can be expressed in terms of $\mathcal{R}_i$ as

$$\begin{aligned} N(\rho_{AR}) \simeq \log_2\Bigg[&\frac{1}{2\cosh^2\mathcal{R}_i}(1 + \tilde{\omega}\mathcal{K}_i\sinh^2\mathcal{R}_i) + \sum_{n=0}^{\infty}\frac{\tanh^{2n}\mathcal{R}_i}{2\cosh^2\mathcal{R}_i}\sqrt{\left(\frac{n}{\sinh^2\mathcal{R}_i} + \tanh^2\mathcal{R}_i\right)^2 + \frac{4}{\cosh^2\mathcal{R}_i}} \\ &\times\left(1 + \tilde{\omega}\mathcal{K}_i\left(\sinh^2\mathcal{R}_i - n + \left[2\tanh^2\mathcal{R}_i + \left(\frac{n}{\sinh^2\mathcal{R}_i} + \tanh^2\mathcal{R}_i\right)\left[\frac{n}{\sinh^2\mathcal{R}_i} + n - \tanh^2\mathcal{R}_i\right]\right]\Big/\left[\left(\frac{n}{\sinh^2\mathcal{R}_i}\right.\right.\right.\right. \\ &\left.\left.\left.\left. + \tanh^2\mathcal{R}_i\right)^2 + \frac{4}{\cosh^2\mathcal{R}_i}\right]\right)\right)\Bigg], \end{aligned} \tag{84}$$

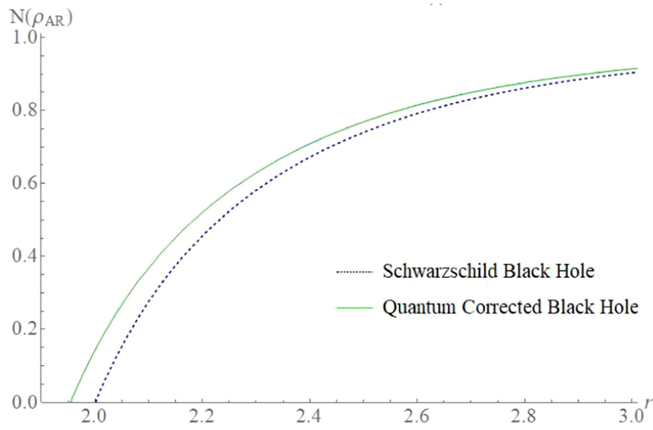

FIG. 1. Logarithmic negativity vs radial distance (of the observer) plot for a Schwarzschild and quantum corrected black hole.

where

$$\mathcal{K}_i \equiv 8\pi\omega_i GM\sqrt{1-\frac{2GM}{r}}\left[\frac{1}{4GM^2}+\frac{G^2M}{r^2(r-2GM)}\right]. \quad (85)$$

Equation (84) is one of the main results in our paper. In order to plot Fig. 1, we set $G = 0.1\ell_0^2$, $M = 1.0\ell_0^{-1}$, $\omega_i = \frac{1}{\pi}\ell_0^{-1}$, and $\tilde{\omega} = 0.9$ with respect to some arbitrary length scale ℓ_0. Here, the values are chosen in a manner such that the quantum effects get amplified. From Fig. 1, we observe that the negativity has a slower rate of decreasing than the Schwarzschild black hole. Hence, if an observer finds out that at the Schwarzschild radius the logarithmic negativity does not drop to zero, then it will indicate towards the existence of a modified gravity theory. The reason behind such a slower rate of degradation is due to the change in the event horizon radius of the black hole as a result of the embedded quantum correction. In principle, just from an observer's perspective, it will be impossible to tell whether this change in the rate of entanglement degradation is because of underlying quantum gravity correction or any other corrections to the Einstein-Hilbert action. Such corrections can be incorporated by higher curvature corrections, extra dimensional black holes, etc. It is also important to note that the negativity goes to zero on the event horizon radius of the black hole [the points for each curve where they meet on the $N(\rho_{AR}) = 0$ axis], which signifies that the states does not possess any distillable entanglement anymore. It is important to note that we have made use of Eq. (71) instead of Eq. (72) to obtain Fig. 1 (and later Figs. 2 and 3).

Our next aim is to calculate the mutual information and compare between the Shwarzschild and quantum corrected case. The mutual information gives one the idea of the total amount of correlation. The mutual information is given by [4,36]

$$I(\rho_{AR}) = S(\rho_A) + S(\rho_R) - S(\rho_{AR}), \quad (86)$$

where $S(\rho) = -\mathrm{tr}(\rho\log_2\rho) = -\sum_n \rho_{n,n}\log_2\rho_{n,n}$. In Eq. (86), ρ_A denotes Alice's density matrix while Rob's states are traced out. The values of the individual entropies can be obtained as follows:

$$S(\rho_A) = 1, \quad (87)$$

$$S(\rho_R) = -\sum_{n=0}^{\infty}\frac{\tanh^{2n} r_{\tilde{\omega},i}}{2\cosh^2 r_{\tilde{\omega},i}}\left(1+\frac{n}{\sinh^2 r_{\tilde{\omega},i}}\right) \times \log_2\left[1+\frac{n}{\sinh^2 r_{\tilde{\omega},i}}\right], \quad (88)$$

$$S(\rho_{AR}) = -\sum_{n=0}^{\infty}\frac{\tanh^{2n} r_{\tilde{\omega},i}}{2\cosh^2 r_{\tilde{\omega},i}}\left(1+\frac{n+1}{\cosh^2 r_{\tilde{\omega},i}}\right) \times \log_2\left[1+\frac{n+1}{\cosh^2 r_{\tilde{\omega},i}}\right]. \quad (89)$$

Substituting Eqs. (87)–(89) in Eq. (86), we obtain the following relation:

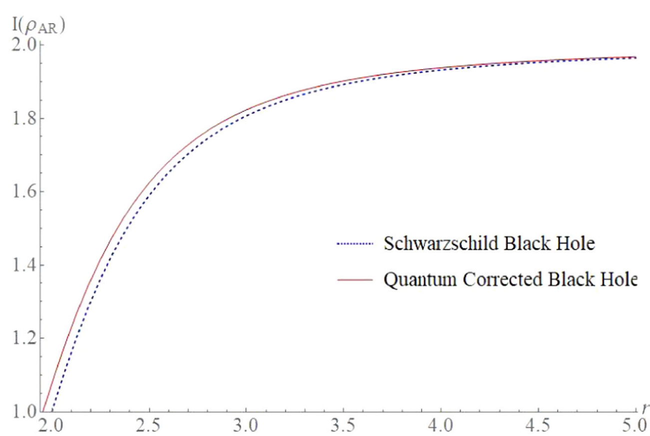

FIG. 2. Mutual information vs radial distance (of the observer) plot for a Schwarzschild and quantum corrected black hole.

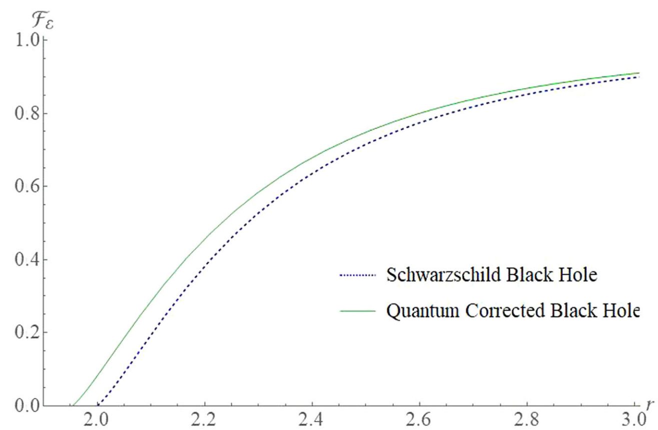

FIG. 3. Entanglement fidelity vs r plot for a Schwarzschild and a quantum corrected black hole.

$$I(\rho_{AR}) = 1 - \sum_{n=0}^{\infty} \frac{\tanh^{2n} r_{\tilde{\omega},i}}{2\cosh^2 r_{\tilde{\omega},i}} \left[\left(1+\frac{n}{\sinh^2 r_{\tilde{\omega},i}}\right)\log_2\left[1+\frac{n}{\sinh^2 r_{\tilde{\omega},i}}\right] - \left(1+\frac{n+1}{\cosh^2 r_{\tilde{\omega},i}}\right)\log_2\left[1+\frac{n+1}{\cosh^2 r_{\tilde{\omega},i}}\right]\right]$$
$$\simeq 1 - \sum_{n=0}^{\infty} \frac{\tanh^{2n}\mathcal{R}_i}{2\cosh^2\mathcal{R}_i}(1+\tilde{\omega}\mathcal{K}_i(\sinh^2\mathcal{R}_i - n))\left[\left(1+\frac{n}{\sinh^2\mathcal{R}_i}\left(1+\tilde{\omega}\mathcal{K}_i(1+\sinh^2\mathcal{R}_i)\right)\right)\log_2\left[1+\frac{n}{\sinh^2\mathcal{R}_i}\right.\right.$$
$$\left.\left.\times(1+\tilde{\omega}\mathcal{K}_i(1+\sinh^2\mathcal{R}_i))\right] - \left(1+\frac{n+1}{\cosh^2\mathcal{R}_i}(1+\tilde{\omega}\mathcal{K}_i\sinh^2\mathcal{R}_i)\right)\log_2\left[1+\frac{n+1}{\cosh^2\mathcal{R}_i}(1+\tilde{\omega}\mathcal{K}_i\sinh^2\mathcal{R}_i)\right]\right]. \quad (90)$$

Equation (90) is also one of the main results in our paper.

We shall now investigate the entanglement degradation for a quantum corrected black hole and compare it with that of the Schwarzschild black hole. Using the same parameters as before, we plot mutual information vs r in Fig. 2. In order to obtain Fig. 2, we have used the value of $r_{\tilde{\omega}_i}$ from Eq. (71) ($\sigma_{\omega_i} = r_{\tilde{\omega}_i}$ in this equation) instead of Eq. (72). It is straightforward to observe that the entanglement degradation gets significant as the observer approaches the event horizons of the respective black holes. It is again important to notice that for the quantum corrected black the mutual information degrades at a slower rate. When the mutual information becomes 1, there are no distillable entanglement left between the two states. As before such advanced observations cannot indicate certainly towards quantum gravity signatures but it does indicate towards modified gravity theories if for the observer being at the Schwarzschild radius, the mutual information does not go to unity. This implies that the entanglement does not degrade completely at the Schwarzschild radius. It is very difficult to costruct such experimental scenarios where the degradation in mutual information is directly observed, but it may be possible to do the same in future with advanced experimental setups. In the next section, we shall demonstrate the entire setup as a quantum channel with a completely positive and trace preserving map and try to obtain the entanglement fidelity for the same channel.

V. NOISY QUANTUM CHANNEL AND ENTANGLEMENT FIDELITY

We start with the initial density matrix (anti-Boulware states traced out),

$$\rho^{\mathcal{I}}_{AR} = |\phi\rangle\langle\phi| = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|), \quad (91)$$

where $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. We need to construct a map such that we obtain ρ_{AR} in Eq. (76) from the above equation. We consider a map of the following form [37]:

$$\rho_{AR} = \mathcal{E}(\rho^{\mathcal{I}}_{AR}) = \sum_n \mathcal{S}_n \rho^{\mathcal{I}}_{AR} \mathcal{S}_n^\dagger = \sum_n \mathcal{S}_n |\phi\rangle\langle\phi| \mathcal{S}_n^\dagger. \quad (92)$$

One can obtain the analytical form of $\mathcal{S}_n$ as

$$\mathcal{S}_n = \frac{1}{\sqrt{n!}} \frac{\tanh^n r_{\tilde{\omega},i}}{\cosh r_{\tilde{\omega},i}} (\mathrm{sech} r_{\tilde{\omega},i})^{\hat{N}_A} \otimes (\hat{a}_B^\dagger)^n, \quad (93)$$

with $\hat{N}_A$ being the number operator whose action is defined on Alice's Hilbert space (Hawking-Hartle states) and $\hat{a}_B^\dagger$ being the raising operator for the states measured by Rob (Boulware states). Now the operator $\mathcal{S}_n$ is an operator of the Hilbert space where the density matrix $\rho^{\mathcal{I}}_{AR}$ is prepared. Hence, the map $\mathcal{E}$ is a positive map [37,38]. It is also straightforward to check that $\mathrm{tr}(\rho^{\mathcal{I}}_{AR}) = \mathrm{tr}(\rho_{AR})$. Hence, the map $\mathcal{E}$ is a CPTP map. Our final aim is to investigate how this quantum channel preserves the initial entanglement. For this, we need to calculate entanglement fidelity given by [37]

$$\mathcal{F}_{\mathcal{E}} = \sum_{n=0}^{\infty} \mathrm{tr}[\rho^{\mathcal{I}}_{AR}\mathcal{S}_n]\mathrm{tr}[\rho^{\mathcal{I}}_{AR}\mathcal{S}_n^\dagger]. \quad (94)$$

The analytical forms of the two traces are given by

$$\mathrm{tr}[\rho^{\mathcal{I}}_{AR}\mathcal{S}_n] = \frac{\tanh^n r_{\tilde{\omega},i}}{2\cosh r_{\tilde{\omega},i}}\left(1+\frac{\sqrt{n+1}}{\cosh r_{\tilde{\omega},i}}\right)\delta_{n,0} = \frac{1}{2\cosh r_{\tilde{\omega},i}}\left(1+\frac{1}{\cosh r_{\tilde{\omega},i}}\right)\delta_{n,0} = \mathrm{tr}[\rho^{\mathcal{I}}_{AR}\mathcal{S}_n^\dagger]. \quad (95)$$

Using Eq. (95) in Eq. (94), we obtain the entanglement fidelity given by

$$\mathcal{F}_{\mathcal{E}} = \sum_{n=0}^{\infty} \mathrm{tr}[\rho^{\mathcal{I}}_{AR}\mathcal{S}_n]\mathrm{tr}[\rho^{\mathcal{I}}_{AR}\mathcal{S}_n^\dagger] = \sum_{n=0}^{\infty}\frac{\tanh^{2n} r_{\tilde{\omega},i}}{4\cosh^2 r_{\tilde{\omega},i}}\left(1+\frac{\sqrt{n+1}}{\cosh r_{\tilde{\omega},i}}\right)^2(\delta_{n,0})^2 = \frac{1}{4\cosh^2 r_{\tilde{\omega},i}}\left(1+\frac{1}{\cosh r_{\tilde{\omega},i}}\right)^2 \simeq \mathcal{F}^{\mathrm{Sch.}}_{\mathcal{E}}\left(1+\tilde{\omega}\mathcal{K}_i\sinh^2\mathcal{R}_i\left(1+\frac{1}{1+\cosh\mathcal{R}_i}\right)\right). \quad (96)$$

where

$$\mathcal{F}_{\mathcal{E}}^{\text{Sch.}} = \frac{1}{4\cosh^2 \mathcal{R}_i}\left(1 + \frac{1}{\cosh \mathcal{R}_i}\right)^2 \quad (97)$$

denotes the entanglement fidelity for a Schwarzschild black hole. In Fig. 3, we plot the entanglement fidelity vs the distance of the observer from $r = 0$. It is important to observe from Fig. 3 that near the event horizon of the black hole (which depicts the infinite acceleration limit in the flat spacetime case), the entanglement fidelity approaches zero and the rate of degradation is slower for the quantum corrected black hole (compared to the Schwarzschild black hole), which shows a similar behavior as shown by logarithmic negativity and mutual information. It is important to note that the delayed nature of the degradation of the information theoretical measures are completely dependent upon the change in the event horizon radius. As in the quantum corrected case, the event horizon radius is smaller than that of the Schwarzschild radius, the primary measures (logarithmic negativity, mutual information, and entanglement fidelty) decay slowly than that of the simple Schwarzschild case. It is therefore not quite feasible to claim that the quantum gravity corrections are the only reasons for such a delayed fall-off behavior from a observer's perspective. It does indicate that the underlying theory is that of a modified general relativity theory.

VI. CONCLUSION

We investigate the phenomenon of entanglement degradation for a quantum corrected black hole, in the vicinity of the event horizon of the same. We observe that in the near horizon approximation, it is possible to write down any static and spherically symmetric metric in a Rindler form, which helps later in identifying three timelike Killing vectors and ultimately, in identifying the vacuum modes and their analogy with the flat spacetime case. For the next part of our analysis, we obtain the logarithmic negativity for the quantum corrected black hole using the partial transpose criterion of the reduced density matrix and expressed it in terms of the Schwarzschild parameters. Then we have plotted logarithmic negativity with respect to the change in the position of the observer r for a quantum corrected black hole and compared it with that of the Schwarzshild black hole. We observe that the logarithmic negativity asymptotically reaches unity when the observer is sitting very far away from each of the black holes and attains a zero value for an observer sitting on the event horizon of the black hole. It is although important to note that when the logarithmic negativity reaches zero value for observer sitting at the event horizon radius of the Schwarzschild black hole, it still would have been nonzero if there are underlying quantum gravity corrections in the black hole. One important thing to keep in mind is that the logarithmic negativity being nonzero in the quantum corrected case is a direct consequence of the event horizon radius being smaller in this case than the Schwarzsdchild radius. Hence, such an outcome does not guarantee an embedded quantum correction from an observer's perspective. It rather indicates towards the existence of a modified gravity theory, which may have resulted in a smaller event horizon radius than the Schwarzschild radius. For example, it can be any gravity theory with higher curvature corrections or theories with the existence of extra dimensions. Next, we have calculated the mutual information for the Alice-Rob bipartite state for the quantum corrected black hole. We have then plotted the mutual information with respect to r for both the black holes and observe that very near the event horizon radius, mutual information drops from 2 to very close to unity and reaches unity while the observer is sitting on the event horizon of the black hole. Similar to previous case the mutual information has a slower rate of fall for the quantum corrected black hole. It affirms that if a black hole has underlying quantum gravity corrections (which is almost impossible to notice for any observer) then even if the observer is at the Schwarzschild radius there will still be some distillable entanglement left. Again from an observational perspective, it can not be considered uniquely as a quantum gravity signature. But, there is always a possibility that the correction has emerged from an underlying quantum theory of gravity. Finally, we consider the entire procedure as a quantum channel and obtained a completely positive trace preserving map which translates the initial stationary entangled state to a mixed state in the black hole spacetime. We finally calculate the entanglement fidelity to investigate how the quantum channel preserves entanglement. We find out that the entanglement fidelity degrades near the vicinity of the event horizon of the black hole and as per the earlier cases the rate of fall is slower in case of the quantum corrected black hole. It is then important to conclude that quantum gravity corrections reduces the event horizon of the black hole than the Schwarzschild radius, which delays entanglement degradation. Our future plan involves doing a rigorous calculation of the Bogoliubov coefficients and obtain the Hawking-Hartle and Boulware vacuum connection considering the effects of the curved background rather by using an analogy.

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