Asymptotic symmetry algebra of $\mathcal{N} = 8$ supergravity

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The asymptotic symmetry algebra of $\mathcal{N} = 1$ supergravity was recently constructed using the well-known two-dimensional celestial conformal field (CFT) theory technique [A. Fotopoulos *et al.*, J. High Energy Phys. 09 (2020) 198]. In this paper, we extend the construction to the maximally supersymmetric four-dimensional $\mathcal{N} = 8$ supergravity theory in asymptotically flat spacetime and construct the extended asymptotic symmetry algebra, which we call $\mathcal{N} = 8$ $\mathfrak{sbms}_4$. We use the celestial CFT technique to find the appropriate currents for extensions of $\mathcal{N} = 8$ super-Poincaré and $\mathrm{SU}(8)_R$ R-symmetry current algebra on the celestial sphere $\mathcal{CS}^2$. We generalize the definition of shadow transformations and show that there is *no* infinite dimensional extension of the global $\mathrm{SU}(8)_R$ algebra in the theory.

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I. INTRODUCTION

The physical observables of a theory are encoded in the symmetries of that theory. This makes the study of symmetries very important. Furthermore, it has been observed that in gauge and gravity theories, there is an enhancement of symmetry at the boundaries. For asymptotic boundaries, such enhanced symmetries are known as the asymptotic symmetries. In four dimensions these asymptotic symmetries have been studied for both gauge and gravity theories including the $\mathcal{N} = 1$ supergravity [1–7]. These symmetries in the case of gravity and gauge theories are popularly known as BMS (Bondi-Metzner-Sachs) and large gauge symmetries, respectively [8–16]. More generic asymptotic symmetries have been studied in Refs. [17,18]. These infinite dimensional asymptotic symmetries also have experimental implications as in gauge and gravitational memory effects which are classical observables [19–23]. Moreover, a deeper understanding of these symmetries might help in understanding the black hole microscopics [24–28]. This necessitates the computation and analysis of asymptotic symmetries.

Another implication of the asymptotic symmetries is the soft theorems. It has been shown that soft theorems are the Ward identities of the asymptotic symmetries [12,15,29]. Let us discuss this relationship in a bit of detail. In general, every symmetry leads to constraints on physical observables such as the scattering amplitudes. Such constraints are known as Ward identities. For the Ward identities of asymptotic symmetries, we need to consider the amplitude in the soft limit. The soft limit of the amplitude is defined by taking the momenta of one or more external particles to zero. Quite generally, under the soft limit, the amplitude factorizes into a universal (soft) factor which contains the divergent part of the amplitude times the amplitude without the soft particle(s) insertions. This factorization is known as the soft theorem[1]. In other words, soft theorems are the Ward identities of asymptotic symmetries. The BMS and large gauge symmetries lead to a soft graviton and a soft photon theorem, respectively [16].

Another important limit of amplitudes is the collinear limit in which the momenta of two external particles are taken to be collinear. Again the amplitude factorizes into a collinear factor containing the divergence times the amplitude with the collinear particles replaced by another particle [35]. The collinear limit of amplitude turns into an operator product expansion (OPE) of conformal operators of the celestial conformal field theory (CCFT) [36–39] as identical momentum directions correspond to the same operator insertion points on the celestial sphere[2] (which we denote by $\mathcal{CS}^2$). An interesting fact is that the soft and collinear limits of scattering amplitudes can be used to read off the asymptotic symmetries in the context of CCFT [36,40]. It turns out that to calculate the asymptotic symmetries of a

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[1] Some progress on understanding of soft theorem in ads has been made in Refs. [30–34].

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theory, we need to probe the universal soft and collinear sectors of the scattering amplitudes. This has been used to reproduce the BMS algebra in [40,41] for pure gravity and large gauge algebra for the Einstein Yang-Mills theory [42]. Recently, it has also been used to compute the $\mathcal{N}=1$ supersymmetric extension of the BMS algebra [3].

In this paper, we calculate the asymptotic symmetries of the four-dimensional maximally supersymmetric $\mathcal{N}=8$ supergravity using the CCFT prescription. In celestial CFT of supergravity, the stress tensor is generated by the shadow transform of the soft graviton operator suitably modified to obtain the correct OPE [41],[3] while the supercurrent is generated by the soft gravitino operator [3]. For $\mathcal{N}>1$, the global symmetry algebra contains an additional R-symmetry, and hence naïvely one would expect that the asymptotic algebra would contain an infinite dimensional extension of the global R-symmetry algebra as well. It was shown in [43] that for $\mathcal{N}=2$, even for the $\mathrm{U}(1)^{\mathcal{N}}$ subgroup of the R-symmetry group $\mathrm{U}(\mathcal{N})$ which only scales the supercharges, such an infinite dimensional extension is mathematically inconsistent. For the present paper, we study the celestial amplitudes of $\mathcal{N}=8$ supergravity and use the soft and collinear limits calculated in a companion paper [44] to compute the Ward identities and the OPE of conformal operators in the corresponding CCFT. We then construct the stress tensor and the supercurrents of the theory using the shadow transforms of soft graviton and soft gravitino operators. Since the scalars and graviphotinos do not have soft divergences (see [44]), we are left with only soft graviphoton operators. The R-symmetry current (if any) can then only be constructed using the soft graviphoton operators. We construct the most general such operator present in the CCFT and show that the operator is trivial by requiring that the modes of this operator extend the $\mathrm{SU}(8)_R$ R-symmetry algebra.

The paper is organized as follows: in Sec. II we set up our notations and record some definitions and results about the soft and collinear limits in the CCFT of $\mathcal{N}=8$ supergravity used later in the paper. In Sec. III we construct the symmetry currents and compute their OPEs. We also construct the possible R-symmetry currents and show that the requirements of the R-symmetry extension make the current trivial. Finally, in Sec. IV we list the full $\mathcal{N}=8$ $\mathfrak{sbms}_4$ algebra. We conclude in Sec. V by summarising our results and emphasizing our future goals of the study. The appendixes contain the OPEs of various conformal operators in the Mellin basis computed from the results in [44] and a detailed calculation of the OPE of the possible R-symmetry currents.

II. NOTATIONS AND PRELIMINARIES

In this section, we set up the notations for celestial amplitudes and soft and collinear limits in supergravity.

[3]See Sec. III for more details.

A. Celestial amplitudes

Recall that helicity spinors are left- and right-handed representations of the Lorentz group $\mathrm{SO}(1,3)\sim\mathrm{SL}(2,\mathbb{C})$. We denote the left- and right-handed helicity spinors by λ_α and $\tilde{\lambda}^{\dot{\alpha}}$, respectively. A given null momentum p^μ can be written as a bispinor

$$p^{\alpha\dot{\alpha}}=\sigma_\mu^{\alpha\dot{\alpha}}p^\mu=\begin{pmatrix} p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & p^0-p^3\end{pmatrix}=\lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}, \qquad (2.1)$$

where $\sigma_\mu^{\alpha\dot{\alpha}}=(1,\sigma_x,\sigma_y,\sigma_z)$. For real physical momentum, the two spinors are related by complex conjugation $(\tilde{\lambda}^{\dot{\alpha}})^*=\lambda_\alpha$.

We now want to study scattering kinematics on the celestial sphere. We use the Bondi coordinates $(u,r,z,\bar{z})$ on the Minkowski space where $(z,\bar{z})$ parametrizes the celestial sphere $\mathcal{CS}^2$ at null infinity. The Lorentz group $\mathrm{SL}(2,\mathbb{C})$ acts on $\mathcal{CS}^2$ as follows:

$$(z,\bar{z})\mapsto\left(\frac{az+b}{cz+d},\frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right), \qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{C}).$$

A general null momentum vector p^μ can be parametrized as

$$p^\mu=\omega q^\mu, \qquad q^\mu=\frac{1}{2}(1+|z|^2,z+\bar{z},-i(z-\bar{z}),1-|z|^2),$$

where q^μ is a null vector, ω is identified with the light cone energy, and all the particles momenta are taken to be outgoing. Under the Lorentz group, the four-momentum transforms as a Lorentz vector $p^\mu\mapsto\Lambda^\mu_\nu p^\nu$. This induces the following transformation of ω and q^μ:

$$\omega\mapsto(cz+d)(\bar{c}\bar{z}+\bar{d})\omega,$$
$$q^\mu\mapsto q'^\mu=(cz+d)^{-1}(\bar{c}\bar{z}+\bar{d})^{-1}\Lambda^\mu_\nu q^\nu.$$

In the bispinor notation, we can write the basic null momentum vector q^μ as

$$q^{\alpha\dot{\alpha}}=\sigma_\mu^{\alpha\dot{\alpha}}q^\mu=\begin{pmatrix}1 & \bar{z}\\ z & z\bar{z}\end{pmatrix}=\begin{pmatrix}1\\ z\end{pmatrix}(1\quad\bar{z}), \qquad (2.2)$$

where $\sigma^\mu\equiv(1,\sigma_x,\sigma_y,\sigma_z)$ are two-dimensional identity and Pauli matrices. Further introducing the angle and square notations for the left- and right-handed momentum spinors, we have

$$\lambda^\alpha\equiv\langle p|^\alpha=\sqrt{\omega}\begin{pmatrix}1\\ z\end{pmatrix}=\sqrt{\omega}\langle q|^\alpha,$$
$$\tilde{\lambda}^{\dot{\alpha}}\equiv|p]^{\dot{\alpha}}=\sqrt{\omega}\begin{pmatrix}1\\ \bar{z}\end{pmatrix}=\sqrt{\omega}|q]^{\dot{\alpha}}, \qquad (2.3)$$

where we write

$$\langle q|^\alpha=\begin{pmatrix}1\\ z\end{pmatrix}, \qquad |q]^{\dot{\alpha}}=\begin{pmatrix}1\\ \bar{z}\end{pmatrix}. \qquad (2.4)$$

The inner product of momenta can then be written in terms of the angle and square brackets of the corresponding spinors which are now given by

$$\langle ij\rangle = -\sqrt{\omega_i\omega_j}z_{ij}, \qquad [ij] = \sqrt{\omega_i\omega_j}\bar{z}_{ij}, \tag{2.5}$$

where $z_{ij}=z_i-z_j, \bar{z}_{ij}=\bar{z}_i-\bar{z}_j$.

We can now Mellin transform the fields in the bulk to get conformal primaries on the celestial sphere. The massless conformal primary of conformal dimension Δ is given by

$$\begin{aligned}\varphi^{\pm}_{\Delta}(X^\mu,z,\bar{z}) &= \int_0^\infty d\omega\omega^{\Delta-1}e^{\pm i\omega q\cdot X-\epsilon\omega}\\ &= \frac{(\mp i)^\Delta\Gamma(\Delta)}{(-q\cdot X\mp i\epsilon)^\Delta}.\end{aligned} \tag{2.6}$$

The conformal primaries for nontrivial spins are then given by [3]

$$\begin{aligned}
\psi^{\pm}_{\Delta,\ell=-1/2;\alpha}(X,z,\bar{z}) &= |q\rangle_\alpha\varphi^{\pm}_{\Delta+\frac{1}{2}}(X,z,\bar{z}),\\
\psi^{\pm;\dot{\alpha}}_{\Delta,\ell=1/2}(X,z,\bar{z}) &= |q]^{\dot{\alpha}}\varphi^{\pm}_{\Delta+\frac{1}{2}}(X,z,\bar{z}),\\
V^{\mu\pm}_{\Delta,\ell=\pm1}(X,z,\bar{z}) &= \epsilon^\mu_{\ell=\pm1}(q,r)\varphi^{\pm}_{\Delta}(X,z,\bar{z}),\\
H^{\mu\nu\pm}_{\Delta,\ell=\pm2}(X,z,\bar{z}) &= \epsilon^\mu_{\ell=\pm1}(q,r)V^{\nu\pm}_{\Delta,\ell=\pm1}(X,z,\bar{z}),\\
\psi^{\mu\pm}_{\Delta,\ell=-3/2}(X,z,\bar{z}) &= \epsilon^\mu_{\ell=-1}(q,r)\psi^{\pm}_{\Delta,\ell=-1/2}(X,z,\bar{z}),\\
\bar{\psi}^{\mu\pm}_{\Delta,\ell=+3/2}(X,z,\bar{z}) &= \epsilon^\mu_{\ell=+1}(q,r)\bar{\psi}^{\pm}_{\Delta,\ell=+1/2}(X,z,\bar{z}),
\end{aligned} \tag{2.7}$$

where the polarizations are given by

$$\epsilon^\mu_{\ell=+1}(q,r) = \frac{\langle r|\sigma^\mu|q]}{\sqrt{2}\langle rq\rangle}, \qquad \epsilon^\mu_{\ell=-1}(q,r) = \frac{[r|\bar{\sigma}^\mu|q\rangle}{\sqrt{2}[qr]} \tag{2.8}$$

with r as a reference null vector and $\bar{\sigma}^\mu \equiv (1,-\sigma_x,-\sigma_y,-\sigma_z)$. One can further define the inner product of these conformal wave packets [3]. These conformal wave packets are normalizable only when the conformal dimension Δ belongs to the principle continuous series, that is, $\Delta = 1+i\lambda$ with $\lambda\in\mathbb{R}$. In a scattering process, we take all momenta to be outgoing. We now define *celestial amplitude* or *celestial correlator* on $\mathcal{CS}^2$ as the Mellin transform of the amplitudes:

$$\begin{aligned}\left\langle\prod_{n=1}^N\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle &\equiv \left(\prod_{n=1}^N\int d\omega_n\omega_n^{\Delta_n-1}\right)\delta^{(4)}\left(\sum_{n=1}^N\omega_n q_n\right)\\ &\quad\times A_{\ell_1\cdots\ell_N}(\omega_n,z_n,\bar{z}_n),\end{aligned} \tag{2.9}$$

where $A_{\ell_1,\ldots,\ell_n}$ is the bulk amplitude with external particles with helicities $\ell_1,\ldots,\ell_n$. The celestial correlators can be shown to transform as a conformal correlator under $\mathrm{SL}(2,\mathbb{C})$:

$$\begin{aligned}&\left\langle\prod_{n=1}^N\mathcal{O}_{\Delta_n,\ell_n}\left(\frac{az_n+b}{cz_n+d},\frac{\bar{a}\bar{z}_n+\bar{b}}{\bar{c}\bar{z}_n+\bar{d}}\right)\right\rangle\\ &\quad=\prod_{i=1}^N(cz_i+d)^{\Delta_i+\ell_i}(\bar{c}\bar{z}_i+\bar{d})^{\Delta_i-\ell_i}\left\langle\prod_{n=1}^N\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle,\end{aligned} \tag{2.10}$$

where

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{C}). \tag{2.11}$$

B. OPEs of celestial operators in $\mathcal{N}=8$ supergravity

Let $\{\eta_A\}_{A=1}^8$ be the Grassmann coordinates on the $\mathcal{N}=8$ superspace. We can package the on-shell degrees of freedom in $\mathcal{N}=8$ supergravity in an on-shell superfield defined as

$$\begin{aligned}\Psi(p,\eta) &= H^+(p)+\eta_A\psi_+^A(p)+\eta_{AB}G_+^{AB}(p)+\eta_{ABC}\chi_+^{ABC}(p)\\ &\quad+\eta_{ABCD}\Phi^{ABCD}(p)+\tilde{\eta}^{ABC}\chi^-_{ABC}(p)+\tilde{\eta}^{AB}G^-_{AB}(p)\\ &\quad+\tilde{\eta}^A\psi^-_A(p)+\tilde{\eta}H^-(p),\end{aligned} \tag{2.12}$$

where we have introduced the notation

$$\begin{aligned}\eta_{A_1\cdots A_n} &\equiv \frac{1}{n!}\eta_{A_1}\ldots\eta_{A_n},\\ \tilde{\eta}^{A_1\cdots A_n} &\equiv \epsilon^{A_1\cdots A_nB_1\cdots B_{8-n}}\eta_{B^1\ldots B^{8-n}},\\ \tilde{\eta} &\equiv \prod_{A=1}^8\eta^A.\end{aligned} \tag{2.13}$$

The fields $H^\pm$ represent positive and negative helicity gravitons, G_+^{AB} and G^-_{AB} represent positive and negative helicity graviphotons, ψ_+^A and ψ_A^- represent positive and negative helicity gravitinos, χ_+^{ABC} and χ^-_{ABC} represent positive and negative helicity graviphotinos, and finally Φ^{ABCD} represent the real scalars. The *superamplitude* is then defined by

$$\mathcal{M}_n(\{p_1,\eta^1\},\{p_n,\eta^n\}) = \langle\Psi_1(p_1,\eta^1)\cdots\Psi_n(p_n,\eta^n)\rangle. \tag{2.14}$$

This superfield can be Mellin transformed in the usual way to obtain a *celestial superfield* on $\mathcal{CS}^2$, but it turns out that the component fields will have the same conformal dimension [46]. This is not appropriate to work with since we want the component fields to have conformal dimensions according to their spin. Thus, we work with the so-called *quasi-on-shell superfield* [46] defined as

$$\Psi_\Delta(z,\bar z,\eta) = H^+_\Delta(z,\bar z) + \eta_A\psi^A_\Delta(z,\bar z) + \eta_{AB}G^{AB}_\Delta(z,\bar z) + \eta_{ABC}\chi^{ABC}_\Delta(z,\bar z) + \eta_{ABCD}\Phi^{ABCD}_\Delta(z,\bar z) + \tilde\eta^{ABC}\bar\chi_{ABC\Delta}(z,\bar z) + \tilde\eta^{AB}\bar G_{AB\Delta}(z,\bar z) + \tilde\eta^{A}\bar\psi_{A\Delta}(z,\bar z) + \tilde\eta H^-_\Delta(z,\bar z), \tag{2.15}$$

where the components are the Mellin transforms of the component fields of $\Psi(p,n)$, all with scaling dimension Δ as defined in (2.7). The celestial correlator for the component fields can then be defined as in (2.9). Using the collinear limit of the bulk amplitude, the OPEs of the celestial operators can be computed. To do this computation, we use the collinear limits computed in [44]. As an example, we calculate the OPE of two graviton operators. The celestial correlator is given by

$$\begin{aligned}\langle \mathcal{O}_{\Delta_1,+2}\mathcal{O}_{\Delta_2,+2}\cdots\mathcal{O}_{\Delta_n,\ell_n}\rangle &= \left(\prod_{j=1}^n\int_0^\infty d\omega_j\omega_j^{\Delta_j-1}\right)\delta^4\left(\sum_i\omega_iq_i\right)M_n(1^{+2},2^{+2},\ldots,n)\\ &= \left(\prod_{j=3}^n\int_0^\infty d\omega_j\omega_j^{\Delta_j-1}\int_0^\infty d\omega_1\int_0^\infty d\omega_2\omega_1^{\Delta_1-1}\omega_2^{\Delta_2-1}\right)\\ &\quad\times\delta^4\left(\sum_{i=3}^n\omega_iq_i+\omega_pq_p\right)\frac{\omega_p^2}{\omega_1\omega_2}\frac{\bar z_{12}}{z_{12}}M_{n-1}(p^{+2},\ldots,n),\end{aligned}\tag{2.16}$$

where M_n is the bulk amplitude of component fields and we used the collinear limit

$$M_n(1^{+2},2^{+2},\ldots,n) = \frac{\omega_p^2}{\omega_1\omega_2}\frac{\bar z_{12}}{z_{12}}M_{n-1}(p^{+2},\ldots,n). \tag{2.17}$$

Here $p_i=\omega_iq_i$, $i=1,2$, the momenta along the collinear channel is $p=p_1+p_2=\omega_pq_p$ with $\omega_p=\omega_1+\omega_2$. Now we use the following integral [46]:

$$\int_0^\infty d\omega_2\omega_2^{\Delta_2-1}\int_0^\infty d\omega_1\omega_1^{\Delta_1-1}\omega_1^\alpha\omega_2^\beta\omega_p^\gamma f(\omega_p) = B(\Delta_1+\alpha,\Delta_2+\beta)\int_0^\infty d\omega_p\omega_p^{\Delta_p-1}f(\omega_p), \tag{2.18}$$

where $\omega_p=\omega_1+\omega_2$, $\Delta_p=\Delta_1+\Delta_2+\alpha+\beta+\gamma$, and

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{2.19}$$

is the Euler beta function. We get

$$\begin{aligned}\langle \mathcal{O}_{\Delta_1,+2}\mathcal{O}_{\Delta_2,+2}\cdots\mathcal{O}_{\Delta_n,\ell_n}\rangle &= \frac{\bar z_{12}}{z_{12}}B(\Delta_1-1,\Delta_2-1)\left(\prod_{j=3}^n\int_0^\infty d\omega_j\omega_j^{\Delta_j-1}\int_0^\infty d\omega_p\omega_p^{\Delta_1+\Delta_2-1}\right)\\ &\quad\times\delta^4\left(\sum_{i=3}^n\omega_iq_i+\omega_pq_p\right)M_{n-1}(p^{+2},3,\ldots,n)\\ &= \frac{\bar z_{12}}{z_{12}}B(\Delta_1-1,\Delta_2-1)\langle\mathcal{O}_{\Delta_1+\Delta_2,+2}\mathcal{O}_{\Delta_3,\ell_3}\cdots\mathcal{O}_{\Delta_n,\ell_n}\rangle.\end{aligned}\tag{2.20}$$

This gives the OPE corresponding to the two positive helicity graviton operators,

$$\mathcal{O}_{\Delta_1,+2}(z_1,\bar z_1)\mathcal{O}_{\Delta_2,+2}(z_2,\bar z_2) \sim \frac{\bar z_{12}}{z_{12}}B(\Delta_1-1,\Delta_2-1)\mathcal{O}_{\Delta_1+\Delta_2,+2}(z_2,\bar z_2). \tag{2.21}$$

Similarly, for negative helicity gluon, we have the collinear amplitude

$$M_n(1^{-2},2^{-2},\ldots) = \frac{\omega_p^2}{\omega_1\omega_2}\frac{z_{12}}{\bar z_{12}}M_{n-1}(p^{-2},\ldots,n). \tag{2.22}$$

Hence, the OPE

$$\mathcal{O}_{\Delta_1,-2}(z_1,\bar{z}_1)\mathcal{O}_{\Delta_2,-2}(z_2,\bar{z}_2) \sim \frac{z_{12}}{\bar{z}_{12}}B(\Delta_1-1,\Delta_2-1)\mathcal{O}_{\Delta_1+\Delta_2,+2}(z_2,\bar{z}_2). \quad (2.23)$$

The collinear limit of two opposite helicity gravitons is

$$M_n(1^{+2},2^{-2},3,\ldots,n) = \frac{\omega_1^3}{\omega_p^2\omega_2}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}(p^{-2},3,\ldots,n) + \frac{\omega_2^3}{\omega_p^2\omega_1}\frac{z_{12}}{\bar{z}_{12}}M_{n-1}(p^{+2},3,\ldots,n), \quad (2.24)$$

which gives us the OPE

$$\mathcal{O}_{\Delta_1,+2}(z_1,z_1)\mathcal{O}_{\Delta_2,-2}(z_2,\bar{z}_2) = B(\Delta_1+3,\Delta_2-1)\frac{\bar{z}_{12}}{z_{12}}\mathcal{O}_{\Delta_1+\Delta_2,-2}(z_2,\bar{z}_2) + B(\Delta_1-1,\Delta_2+3)\frac{z_{12}}{\bar{z}_{12}}\mathcal{O}_{\Delta_1+\Delta_2,+2}(z_2,\bar{z}_2). \quad (2.25)$$

One can calculate the OPEs of all other component fields in a similar way using the collinear limits. The results are listed in Appendix A.

C. Soft operators in $\mathcal{N}=8$ supergravity

In the last section, we discussed the collinear limits of amplitudes. In this section, we are looking at their soft limits. As we know, soft momentum $p\to 0$ can be written as $\omega_p \to 0$ on the celestial sphere, and hence an amplitude written in the celestial coordinates can be analyzed in the soft limit of any of the external momenta. The result is a soft theorem that expresses an n-point amplitude with soft external momentum p in terms of an $(n-1)$-point amplitude along with a soft factor given by powers of ω_p^{-1}. The various powers of ω_p^{-1} then correspond to leading, subleading, subsubleading soft theorems, and so on. Let us first define the *celestial superamplitude* as the Mellin transform of superamplitude:

$$\left\langle \prod_{n=1}^{N}\mathcal{O}_{\Delta_n}(z_n,\bar{z}_n,\eta^n)\right\rangle \equiv \left(\prod_{n=1}^{N}\int d\omega_n\omega_n^{\Delta_n-1}\right)\delta^{(4)}\left(\sum_{n=1}^{N}\omega_n q_n\right) \times \mathcal{M}_N(\{\omega_1,z_1,\bar{z}_1,\eta^1\},\ldots,\{\omega_N,z_N,\bar{z}_N,\eta^N\}), \quad (2.26)$$

where $\mathcal{M}_N(\{\omega_1,z_1,\bar{z}_1,\eta^1\},\ldots,\{\omega_N,z_N,\bar{z}_N,\eta^N\})$ is the superamplitude (2.14) written in the celestial basis. We also denote it simply by $\mathcal{M}_N(1,2,\ldots,N)$. The above expression is identical to that of (2.9), with the explicit incorporation of the Grassmann factors in the scattering amplitudes.

One can now expand both sides of (2.26) in the Grassmann parameter η_i and compare the coefficients to get the celestial amplitude of various component fields. This has been used in Appendix B to calculate the celestial correlator with a soft graviton and a soft gravitino. The celestial correlator of the leading soft graviton operator is given by

$$\left\langle J_1(z,\bar{z})\prod_{n=1}^{N}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \sum_{i=1}^{N}\frac{(\bar{z}-\bar{z}_i)}{(z-z_i)}\frac{(\xi-z_i)^2}{(\xi-z)^2}\langle\mathcal{O}_{\Delta_1,\ell_1}(z_1,\bar{z}_1),\ldots, \mathcal{O}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\ldots,\mathcal{O}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle \quad (2.27)$$

and

$$\left\langle \bar{J}_1(z,\bar{z})\prod_{n=1}^{N}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \sum_{i=1}^{N}\frac{(z-z_i)}{(\bar{z}-\bar{z}_i)}\frac{(\bar{\xi}-\bar{z}_i)^2}{(\bar{\xi}-\bar{z})^2}\langle\mathcal{O}_{\Delta_1,\ell_1}(z_1,\bar{z}_1),\ldots, \mathcal{O}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\ldots,\mathcal{O}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle, \quad (2.28)$$

where

$$J_1(z,\bar{z}) = \lim_{\Delta\to 1}(\Delta-1)\mathcal{O}_{\Delta,+2}(z,\bar{z}),$$
$$\bar{J}_1(z,\bar{z}) = \lim_{\Delta\to 1}(\Delta-1)\mathcal{O}_{\Delta,-2}(z,\bar{z}) \quad (2.29)$$

are the $\Delta=1$ soft graviton operators and $\xi\in\mathcal{CS}^2$ is a reference point. The celestial correlator of the subleading soft graviton operator is

$$\left\langle J_0(z,\bar{z})\prod_{n=1}^{N}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \sum_{i=1}^{N}\frac{(\bar{z}-\bar{z}_i)}{(z-z_i)}\frac{(\xi-z_i)}{(\xi-z)}((\bar{z}-\bar{z}_i)\partial_{\bar{z}_i}-2\bar{h}_i) \times\langle\cdots\mathcal{O}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\rangle, \quad (2.30)$$

and

$$\left\langle \bar{J}_0(z,\bar{z})\prod_{n=1}^{N}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \sum_{i=1}^{N}\frac{(z-z_i)}{(\bar{z}-\bar{z}_i)}\frac{(\bar{\xi}-\bar{z}_i)}{(\bar{\xi}-\bar{z})}((z-z_i)\partial_{z_i}-2h_i) \times\langle\cdots\mathcal{O}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\rangle, \quad (2.31)$$

where

$$J_0(z,\bar{z}) = \lim_{\Delta\to 0}\Delta\mathcal{O}_{\Delta,+2}(z,\bar{z}),$$
$$\bar{J}_0(z,\bar{z}) = \lim_{\Delta\to 0}\Delta\mathcal{O}_{\Delta,-2}(z,\bar{z}) \tag{2.32}$$

are the $\Delta = 0$ soft graviton operators and $h_i = \frac{\Delta_i+\ell_i}{2}$, $\bar{h}_i = \frac{\Delta_i-\ell_i}{2}$ are the conformal weights of the operator $\mathcal{O}_{\Delta_i,\ell_i}(z,\bar{z})$.

The celestial correlator of the soft gravitino operator is given by (cf. [3])

$$\left\langle J^A_{1/2}(z,\bar{z})\prod_{n=1}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \sum_{i=1}^{N} f(A,\ell_i,*_i,*_i')(-1)^{\sigma_i}\frac{(\bar{z}-\bar{z}_i)}{(z-z_i)}\frac{(\xi-z_i)}{(\xi-z)} \times\langle\cdots\mathcal{O}^{*_i'}_{\Delta_i+\frac{1}{2},\ell^c_i}(z_i,\bar{z}_i)\cdots\rangle \tag{2.33}$$

and

$$\left\langle \bar{J}_{1/2A}(z,\bar{z})\prod_{n=1}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell^c_n}(z_n,\bar{z}_n)\right\rangle = \sum_{i=1}^{N} \bar{f}(A,\ell^c_i,*_i,*_i')(-1)^{\sigma_i}\frac{(z-z_i)}{(\bar{z}-\bar{z}_i)}\frac{(\bar{\xi}-\bar{z}_i)}{(\bar{\xi}-\bar{z})} \times\langle\cdots\mathcal{O}^{*_i'}_{\Delta_i+\frac{1}{2},\ell_i}(z_i,\bar{z}_i)\cdots\rangle, \tag{2.34}$$

where

$$J^A_{1/2}(z,\bar{z}) = \lim_{\Delta\to\frac{1}{2}}\left(\Delta-\frac{1}{2}\right)\mathcal{O}^A_{\Delta,+\frac{3}{2}}(z,\bar{z}),$$
$$\bar{J}_{1/2A}(z,\bar{z}) = \lim_{\Delta\to\frac{1}{2}}\left(\Delta-\frac{1}{2}\right)\mathcal{O}_{\Delta,-\frac{3}{2},A}(z,\bar{z}) \tag{2.35}$$

are soft gravitino operators. Here the superscripts $*_i$ indicate the R-symmetry index of the operator. We have put the R-symmetry index $*_i$ as a superscript for brevity but it can also be on subscript depending on the helicity of the operator. Here the number of fermions preceding particle i, $\sigma_i = 1$ if $\ell_i \in \mathbb{Z}+\frac{1}{2}$ and 0 otherwise (see [3] for detailed explanation).

As explained in Appendix B, the positive helicity soft gravitino operator only acts on celestial operators $\mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)$ with

$$\ell_i \in \{-3/2,-1,-1/2,0,+1/2,+1,+3/2,+2\}, \tag{2.36}$$

while the negative helicity soft gravitino operator acts on celestial operators $\mathcal{O}^{*_i}_{\Delta_i,\ell^c_i}(z_i,\bar{z}_i)$ with

$$\ell^c_i \in \{-2,-3/2,-1,-1/2,0,+1/2,+1,+3/2\}. \tag{2.37}$$

The factors $f(A,\ell_i,*_i,*_i')$, $\bar{f}(A,\ell^c_i,*_i,*_i')$ are the R-symmetry factors that we can determine using the collinear limits given above. From (2.33) and (2.34) it is clear that the first argument of f is the R-symmetry index of the soft gravitino operator itself, the second and third arguments are the helicity ℓ_i and R-symmetry index $*_i$, respectively, of the operator $\mathcal{O}^{*_i}_{\Delta_i,\ell_i}$ which the soft gravitino will act on. Last, the fourth argument will be the R-symmetry index $*_i'$ of the resultant operator. Similarly, it goes for $\bar{f}$. As an example, we can see from the OPE in Eq. (A1) that when $\ell_i = -\frac{3}{2}$, $f(A,-3/2,B,*_i') = \delta^A_B$. Since the resulting particle $\ell = -2$ has no R-symmetry index, the $*_i'$ entry is empty.

The soft graviphoton limit can be calculated using the OPEs of the graviphoton operator with various conformal operators. These OPEs are listed in Appendix A. Soft limits correspond to the values of scaling dimension Δ of the graviphoton operator for which the beta functions appearing in the OPEs have poles. From Appendix A we see that the OPEs of the graviphoton operator with various other operators involve[4] $B(\Delta,*)$. Since $B(\Delta,*)$ has poles at all nonpositive integer values of Δ, the leading soft limit of the graviphoton operator is $\Delta\to 0$ and all other negative integral values are subleading. In Sec. III B, we will need the leading soft graviphoton limit.

Finally, as noted in [44], graviphotino and scalars are trivial in the soft limit and hence do not correspond to any global symmetry [3]. So we do not consider them further.

III. ASYMPTOTIC SYMMETRY GENERATORS IN $\mathcal{N}=8$ SUGRA

Let us first consider the obvious global symmetries of $\mathcal{N}=8$ supergravity. The global symmetry algebra consists of the Poincaré algebra and the $\mathcal{N}=8$ supersymmetry algebra, together called the $\mathcal{N}=8$ super-Poincaré algebra and $SU(8)_R$ R-symmetry algebra. At null infinity, we expect to obtain infinite dimensional extensions of these algebras. Following previous works [3,40,42], we can easily construct the currents that extend the super-Poincaré algebra, and we call this algebra the $\mathcal{N}=8$ $\mathfrak{sbms}_4$ algebra. We start by constructing the currents for the $\mathcal{N}=8$ $\mathfrak{sbms}_4$ algebra.

A. $\mathcal{N}=8$ $\mathfrak{sbms}_4$ algebra currents

The $\mathfrak{bms}_4$ part of the $\mathcal{N}=8$ $\mathfrak{sbms}_4$ algebra is known to be generated [40] by the shadow transform of the $\Delta=0$ graviton operator suitably modified as discussed below. This is called the generator of superrotations, and the level one descendant of the $\Delta=1$ graviton operator is called the generator of supertranslations on the celestial sphere. Let us define the shadow transforms $T_0(z,\bar{z})$ and $\bar{T}_0(z,\bar{z})$ as

[4] The OPE of two graviphoton operators with opposite helicity involves another term; see Eq. (A2). One of the terms in the OPE vanishes depending on which of the two helicities of the graviphoton we take to be soft. See Appendix C for such calculations.

$$T_0(z,\bar{z}) = \lim_{\Delta\to 0}\frac{3!\Delta}{2\pi}\int d^2z'\frac{1}{(z-z')^4}\mathcal{O}_{\Delta,-2}(z',\bar{z}'),$$
$$\bar{T}_0(z,\bar{z}) = \lim_{\Delta\to 0}\frac{3!\Delta}{2\pi}\int d^2z'\frac{1}{(\bar{z}-\bar{z}')^4}\mathcal{O}_{\Delta,+2}(z',\bar{z}'). \tag{3.1}$$

It has been argued in [41] that the above shadow transform operator does not satisfy the usual OPE of a stress tensor. In particular, the T_0T_0 OPE has an extra term that does not vanish as shown in [41] unless we modify the stress tensor appropriately. The origin of the problem is the observation that $T_0(z)$ is not holomorphic:

$$\bar{\partial}T_0 = -\frac{1}{2}\partial^3\bar{J}_0(z,\bar{z}), \tag{3.2}$$

where J_0 is the $\Delta=0$ soft graviton operator defined in (2.32). Hence, the modified stress tensor can be defined as follows:

$$T_{\rm mod} \coloneqq T_0 + \frac{1}{2}\partial^3\epsilon_{\bar{J}_0}, \tag{3.3}$$

where

$$\epsilon_{\bar{J}_0} \coloneqq \int_{\bar{z}_0}^{\bar{z}} d\bar{w}\bar{J}_0(z,\bar{w}) \tag{3.4}$$

with z_0 as a reference point. Then it has been shown that the modified stress tensor satisfies the correct $T_{\rm mod}T_{\rm mod}$ OPE [41]. From now on we omit the subscript "mod" and $T,\bar{T}$ will denote the modified stress tensor. Using the soft limits (2.27), (2.28), (2.30), and (2.31) and performing the same calculations as in [40], we arrive at the OPE

$$T(z)\mathcal{O}_{\Delta,\ell}(w,\bar{w}) = \frac{h}{(z-w)^2}\mathcal{O}_{\Delta,\ell}(w,\bar{w}) + \frac{1}{z-w}\partial_w\mathcal{O}_{\Delta,\ell}(w,\bar{w}) + \text{regular},$$
$$\bar{T}(\bar{z})\mathcal{O}_{\Delta,\ell}(w,\bar{w}) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\mathcal{O}_{\Delta,\ell}(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\mathcal{O}_{\Delta,\ell}(w,\bar{w}) + \text{regular}. \tag{3.5}$$

The supertranslations generator $P(z)$, $\bar{P}(z)$ are defined as

$$P(z) = \lim_{\Delta\to 1}\frac{(\Delta-1)}{4}\partial_{\bar{z}}\mathcal{O}_{\Delta,+2}(z,\bar{z}),$$
$$\bar{P}(\bar{z}) = \lim_{\Delta\to 1}\frac{(\Delta-1)}{4}\partial_{z}\mathcal{O}_{\Delta,-2}(z,\bar{z}). \tag{3.6}$$

For $P(z)$ we have

$$P(z)\mathcal{O}_{\Delta,\ell}(w,\bar{w}) = \frac{1}{z-w}\mathcal{O}_{\Delta+1,\ell}(w,\bar{w}) + \text{regular}, \tag{3.7}$$

and similar OPEs hold for $\bar{P}(\bar{z})$ with conjugated poles. These operators are related to the supertranslation generator $\mathcal{P}(z,\bar{z})$, which is a primary field operator of conformal weight $(\frac{3}{2},\frac{3}{2})$. By contour integrals [40]

$$P(z) = \frac{1}{2\pi i}\oint d\bar{z}\mathcal{P}(z,\bar{z}),$$
$$\bar{P}(\bar{z}) = \frac{1}{2\pi i}\oint dz\mathcal{P}(z,\bar{z}). \tag{3.8}$$

The supertranslation satisfies the OPE

$$\mathcal{P}(z,\bar{z})\mathcal{O}_{\Delta,\ell}(w,\bar{w}) = \frac{1}{z-w}\frac{1}{\bar{z}-\bar{w}}\mathcal{O}_{\Delta+1,\ell}(w,\bar{w}) + \text{regular}. \tag{3.9}$$

The supercurrent for $\mathcal{N}=1$ supersymmetry was constructed in [3]. We will see that the same construction will give us the eight supercurrents for $\mathcal{N}=8$ supersymmetry. We thus define the supercurrents as the shadow transform of the $\Delta=\frac{1}{2}$ gravitino operator:

$$S_A(z) = \lim_{\Delta\to\frac{1}{2}}\frac{\Delta-\frac{1}{2}}{\pi}\int d^2z'\frac{1}{(z-z')^3}\mathcal{O}_{A;\Delta,-\frac{3}{2}}(z',\bar{z}'),$$
$$\bar{S}^A(\bar{z}) = \lim_{\Delta\to\frac{1}{2}}\frac{\Delta-\frac{1}{2}}{\pi}\int d^2z'\frac{1}{(\bar{z}-\bar{z}')^3}\mathcal{O}^A_{\Delta,+\frac{3}{2}}(z',\bar{z}'). \tag{3.10}$$

Note that the above operators are also not holomorphic since

$$\bar{\partial}S_A(z,\bar{z}) = \lim_{\Delta\to 1/2}\left(\Delta-\frac{1}{2}\right)\partial_z^2\mathcal{O}_{A;\Delta,-\frac{3}{2}}(z,\bar{z}) = \partial^2\bar{J}_{1/2A}(z,\bar{z}) \neq 0, \tag{3.11}$$

where $\bar{J}_{1/2A}(z,\bar{z})$ is the leading soft gravitino operator defined in (2.35). One can modify it in a similar way as in Eq. (3.3). Put

$$\epsilon_{\bar{J}_{1/2}A}(z,\bar{z}) \coloneqq \int_{\bar{z}_0}^{z} d\bar{w}\bar{J}_{1/2A}(w,\bar{w}), \tag{3.12}$$

where z_0 is a reference point and define

$$S^A_{\rm mod} \coloneqq S^A - \partial^2\epsilon^A_{\bar{J}_{1/2}}. \tag{3.13}$$

We emphasize that this modification is not required at the quantum level since the OPEs of S^A are as expected for a supercurrent. So we continue to use the shadow transform of the leading soft gravitino operator as the supercurrent without any modification.

Following the calculations of [3], it is straightforward to see that

$$T(z)S_A(w) = \frac{3}{2}\frac{S_A(w)}{(z-w)^2} + \frac{\partial S_A(w)}{z-w} + \text{regular},$$
$$\bar{T}(\bar{z})\bar{S}^A(\bar{w}) = \frac{3}{2}\frac{\bar{S}^A(\bar{z})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{S}^A(\bar{w})}{\bar{z}-\bar{w}} + \text{regular}. \qquad (3.14)$$

and the OPEs $T\bar{S}^A$ and $\bar{T}S_A$ are regular. These OPEs confirm the conformal weights of S_A and $\bar{S}_A$ as $(\frac{3}{2},0)$ and $(0,\frac{3}{2})$ respectively. We now want to show that

$$:\{S_B(z),\bar{S}^A(\bar{z})\}: = :S_B(z)\bar{S}^A(\bar{z}) + \bar{S}^A(\bar{z})S_B(z): = \delta^A_B\mathcal{P}(z,\bar{z}). \qquad (3.15)$$

Using the gravitino soft limit (2.33) and (2.34) and the leading graviton limits (2.27) and (2.28) and following the calculations in [3], we get[5]

$$\begin{aligned}\left\langle S_B(z)\bar{S}^A(\bar{w})\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle &= \delta^A_B\sum_{i=3}^{N}\left[\frac{1}{(\bar{w}-\bar{z})^2}\frac{\bar{z}-\bar{z}_i}{z-z_i} + \frac{1}{\bar{z}-\bar{w}}\frac{1}{z-z_i} + \frac{1}{\bar{w}-\bar{z}_i}\frac{1}{z-z_i}\right]\langle\cdots\mathcal{O}^{*_i}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i)\cdots\rangle\\ &\quad - \sum_{i=3}^{N} f(A,\ell_i,*_i,*'_i)\bar{f}(B,\ell_i-1/2,*'_i,*''_i)\frac{1}{z-z_i}\frac{1}{\bar{w}-\bar{z}_i}\langle\cdots\mathcal{O}^{*''_i}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i)\cdots\rangle\\ &\quad + \sum_{\substack{i,j=3\\ i\neq j}}^{N}(-1)^{\sigma_i+\sigma_j}f(A,\ell_i,*_i,*'_i)\bar{f}(B,\ell_j,*_j,*'_j)\frac{1}{z-z_i}\frac{1}{\bar{w}-\bar{z}_j}\langle\cdots\mathcal{O}^{*'_i}_{\Delta_i+\frac{1}{2},\ell_i-\frac{1}{2}}(z_i,\bar{z}_i),\ldots,\\ &\quad \mathcal{O}^{*'_j}_{\Delta_j+\frac{1}{2},\ell_j+\frac{1}{2}}(z_j,\bar{z}_j)\cdots\rangle, \end{aligned} \qquad (3.16)$$

where the factors $f(A,\ell_i,*_i,*'_i), \bar{f}(B,\ell_j,*_j,*'_j)$ are the R-symmetry factors that appear on taking the soft or collinear limit depending on the spins and helicities of the soft and collinear particles. In this notation, the first argument of f is the R-symmetry index of the positive helicity soft gravitino, the second argument is the spin (and helicity) of one of the hard[6] particles, the third argument is the R-symmetry index of that hard particle (left implicit for generality), and the fourth argument is the resulting R-symmetry index of the hard particle after the soft limit is taken (again left implicit for generality). The notation for $\bar{f}$ is similar. It is understood that if the spins do not belong to the required range specified in (B15) and (B16), then $f,\bar{f}=0$. Similarly

$$\begin{aligned}\left\langle \bar{S}^A(\bar{z})S_B(w)\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle &= \delta^A_B\sum_{i=3}^{N}\left[\frac{1}{(w-z)^2}\frac{z-z_i}{\bar{z}-\bar{z}_i} + \frac{1}{z-w}\frac{1}{\bar{z}-\bar{z}_i} + \frac{1}{\bar{z}-\bar{z}_i}\frac{1}{w-z_i}\right]\langle\cdots\mathcal{O}^{*_i}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i)\cdots\rangle\\ &\quad - \sum_{i=3}^{N}\bar{f}(B,\ell_i,*_i,*'_i)f(A,\ell_i+1/2,*'_i,*''_i)\frac{1}{w-z_i}\frac{1}{\bar{z}-\bar{z}_i}\langle\cdots\mathcal{O}^{*''_i}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i)\cdots\rangle\\ &\quad - \sum_{\substack{i,j=3\\ i\neq j}}^{N}(-1)^{\sigma_i+\sigma_j}\bar{f}(B,\ell_i,*_i,*'_i)f(A,\ell_j,*_j,*'_j)\frac{1}{w-z_i}\frac{1}{\bar{z}-\bar{z}_j}\langle\cdots\mathcal{O}^{*'_i}_{\Delta_i+\frac{1}{2},\ell_i+\frac{1}{2}}(z_i,\bar{z}_i),\ldots,\\ &\quad \mathcal{O}^{*'_j}_{\Delta_j+\frac{1}{2},\ell_j-\frac{1}{2}}(z_j,\bar{z}_j)\cdots\rangle. \end{aligned} \qquad (3.17)$$

Thus, the anticommutator is

[5]Note that we do not separate the operators in the correlator according to their spins ℓ, ℓ^c unlike [3] since there is an overlap in the ranges of the two spins. So in the correlators in this calculation, the spins are assumed to be arbitrary.

[6]That is, not soft.

$$\begin{aligned}&\left\langle(\bar{S}^A(\bar{z})S_B(w)+S_B(z)\bar{S}^A(\bar{w}))\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\\&=\delta^A_B\sum_{i=3}^{N}\left[\frac{1}{(w-z)^2}\frac{z-z_i}{\bar{z}-\bar{z}_i}+\frac{1}{z-w}\frac{1}{\bar{z}-\bar{z}_i}+\frac{1}{\bar{z}-\bar{z}_i}\frac{1}{w-z_i}\right]\langle\cdots\mathcal{O}^{*_i}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i)\cdots\rangle\\&\quad+\delta^A_B\sum_{i=3}^{N}\left[\frac{1}{(\bar{w}-\bar{z})^2}\frac{\bar{z}-\bar{z}_i}{z-z_i}+\frac{1}{\bar{z}-\bar{w}}\frac{1}{z-z_i}+\frac{1}{\bar{w}-\bar{z}_i}\frac{1}{z-z_i}\right]\langle\cdots\mathcal{O}^{*_i}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i)\cdots\rangle\\&\quad-\sum_{i=3}^{N}f(A,\ell_i,*_i,*'_i)\bar{f}(B,\ell_i-1/2,*'_i,*_i'')\frac{1}{z-z_i}\frac{1}{\bar{w}-\bar{z}_i}\langle\cdots\mathcal{O}^{*_i''}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i)\cdots\rangle\\&\quad-\sum_{i=3}^{N}\bar{f}(B,\ell_i,*_i,*'_i)f(A,\ell_i+1/2,*'_i,*_i'')\frac{1}{w-z_i}\frac{1}{\bar{z}-\bar{z}_i}\langle\cdots\mathcal{O}^{*_i''}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i)\cdots\rangle.\end{aligned}\tag{3.18}$$

Here in the last terms in Eqs. (3.16) and (3.17) we have relative signs; hence, both terms cancel. One can notice that the relative sign is due to the action of S and $\bar{S}$ on different clusters for $i<j$ and $i>j$ in both terms. Then we see that the normal ordered current $:\{S_B(z),\bar{S}^A(\bar{z})\}:$ satisfies

$$\begin{aligned}\left\langle:\{S_B(z),\bar{S}^A(\bar{z})\}:\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle&=2\delta^A_B\sum_{i=3}^{N}\frac{1}{\bar{z}-\bar{z}_i}\frac{1}{z-z_i}\langle\cdots\mathcal{O}^{*_i}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i)\cdots\rangle\\&\quad-\sum_{i=3}^{N}f(A,\ell_i,*_i,*'_i)\bar{f}(B,\ell_i-1/2,*'_i,*_i'')\frac{1}{z-z_i}\frac{1}{\bar{z}-\bar{z}_i}\langle\cdots\mathcal{O}^{*_i''}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i)\cdots\rangle\\&\quad-\sum_{i=3}^{N}\bar{f}(B,\ell_i,*_i,*'_i)f(A,\ell_i+1/2,*'_i,*_i'')\frac{1}{z-z_i}\frac{1}{\bar{z}-\bar{z}_i}\langle\cdots\mathcal{O}^{*_i''}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i)\cdots\rangle.\end{aligned}\tag{3.19}$$

We now show that for any ℓ_i, R-symmetry factors in the last two sums reduce to δ^A_B. Let us start with $\ell_i=+2$ in which case $*_i,*_i''$ is empty. Moreover, in this case $\bar{f}(B,+2,*_i,*'_i)=0$ so that we only have one term to analyze. From the OPEs in Appendix A, we see that $*'_i=A$ and

$$\begin{aligned}&f(A,+2_{,-},*'_i)\bar{f}(B,+3/2,*'_{i,-})\mathcal{O}_{\Delta_i+1,+2}(z_i,\bar{z}_i)\\&\quad=\delta^A_B\mathcal{O}_{\Delta_i+1,+2}(z_i,\bar{z}_i).\end{aligned}\tag{3.20}$$

The case $\ell_i=+\frac{3}{2}$ is more interesting. Suppose $*_i=C$; then from the OPEs, we can easily see that $*'_i=AC$ for the second term and $*'_i$ is empty for the last term. We then have

$$\begin{aligned}&f(A,+3/2,C,*'_i)\bar{f}(B,+1,*'_i,*_i'')\mathcal{O}^{*_i''}_{\Delta_i+1,+3/2}(z_i,\bar{z}_i)\\&\quad=2!\delta^{[A}_B\mathcal{O}^{C]}_{\Delta_i+1,+3/2}(z_i,\bar{z}_i)\end{aligned}\tag{3.21}$$

and similarly

$$\begin{aligned}&\bar{f}(B,+3/2,C_{,-})f(A,+2_{,-},*_i'')\mathcal{O}^{*_i''}_{\Delta_i+1,+3/2}(z_i,\bar{z}_i)\\&\quad=\delta^C_B\mathcal{O}^A_{\Delta_i+1,+3/2}(z_i,\bar{z}_i).\end{aligned}\tag{3.22}$$

We can clearly see that the sum of the last two terms is simply $\delta^A_B\mathcal{O}^C_{\Delta_i+1,+3/2}(z_i,\bar{z}_i)$. The case $\ell_i=-\frac{3}{2}$ is similar. Let us now analyze the case $\ell_i=+1$ in which case $*_i=CD$. We get

$$\begin{aligned}&f(A,+1,CD,*'_i)\bar{f}(B,+1/2,*'_i,*_i'')\mathcal{O}^{*_i''}_{\Delta_i+1,+2}(z_i,\bar{z}_i)\\&\quad=\bar{f}(B,+1/2,*'_i,ACD)\mathcal{O}^{ACD}_{\Delta_i+\frac{1}{2},+1/2}(z_i,\bar{z}_i)\\&\quad=3\delta^{[A}_B\mathcal{O}^{CD]}_{\Delta_i+1,+1}(z_i,\bar{z}_i).\end{aligned}\tag{3.23}$$

Similarly,

$$\begin{aligned}&\bar{f}(B,+1,CD,*'_i)f(A,+3/2,*'_i,*_i'')\mathcal{O}^{*_i''}_{\Delta_i+1,+3/2}(z_i,\bar{z}_i)\\&\quad=-2!\delta^{[C}_B\mathcal{O}^{D]A}_{\Delta_i+1,+1}(z_i,\bar{z}_i),\end{aligned}\tag{3.24}$$

which finally implies

$$3\delta_B^{[A}\mathcal{O}^{CD]} - 2!\delta_B^{[C}\mathcal{O}^{D]A} = \frac{1}{2}[(\delta_B^A\mathcal{O}^{CD} - \delta_B^A\mathcal{O}^{DC}) + (\delta_B^C\mathcal{O}^{DA} - \delta_B^C\mathcal{O}^{AD}) + (\delta_B^D\mathcal{O}^{AC} - \delta_B^D\mathcal{O}^{CA})] - \delta_B^C\mathcal{O}^{DA} + \delta_B^D\mathcal{O}^{CA} = \delta_B^A\mathcal{O}^{CD}. \tag{3.25}$$

The case $\ell_i = -1$ is similar. The same calculation as in $\ell_i = 1$ recurs for the cases $\ell_i = 1/2,\,0$.

These calculations simplify the OPE (3.19). We get

$$\left\langle :\{S_B(z), \bar{S}^A(\bar{z})\}: \prod_{n=3}^{N} \mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \delta_B^A \sum_{i=3}^{N} \frac{1}{\bar{z}-\bar{z}_i}\frac{1}{z-z_i}\langle\cdots\mathcal{O}^{*_i}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i)\cdots\rangle. \tag{3.26}$$

In particular,

$$:\{S_B(z),\bar{S}^A(\bar{z})\}:\mathcal{O}_{\Delta,\ell}(w,\bar{w}) = \delta_B^A\frac{1}{z-w}\frac{1}{\bar{z}-\bar{w}}\mathcal{O}_{\Delta+1,\ell}(w,\bar{w}) + \text{regular}. \tag{3.27}$$

Comparing this OPE with (3.9) readily implies the desired result

$$:\{S_B(z),\bar{S}^A(\bar{z})\} \coloneqq \delta_B^A\mathcal{P}(z,\bar{z}). \tag{3.28}$$

B. Possible R-symmetry current

Recall that R-symmetry acts on supercharges Q_α^A and $\bar{Q}_{\dot{\alpha}A}, A=1,\ldots,8$, by multiplying a unitary matrix $U\in U(8)$. This means that the supercharges transform in the fundamental representation of the R-symmetry group. At the level of Lie algebra, we can identify the R-symmetry group as simply $\mathfrak{su}(8)\oplus\mathfrak{u}(1)$ since $\mathrm{U}(8)\cong(\mathrm{SU}(8)\times\mathrm{U}(1))/\mathbb{Z}_8$. Thus, we can label the generators of R-symmetry to be T_B^A and R, where T_B^A are generators of the fundamental representation of SU(8) satisfying the $\mathfrak{su}(8)$ algebra:

$$[T_B^A, T_D^C] = \delta_D^A T_B^C - \delta_B^C T_D^A, \tag{3.29}$$

and R is the generator of the scaling U(1). A suitable matrix representation for the generators is [47]

$$(T_B^A)^C{}_D = \delta_D^A\delta_B^C - \frac{1}{8}\delta_B^A\delta_D^C. \tag{3.30}$$

T_B^A acts on the supercharges as

$$[T_B^A, Q_\alpha^C] = (T_B^A)_D^C Q_\alpha^D,\quad [T_B^A, \bar{Q}_{\dot{\alpha}C}] = -(T_B^A)_C^D\bar{Q}_{\dot{\alpha}D}. \tag{3.31}$$

We now want to construct a current $\tilde{\mathcal{G}}_B^A(z,\bar{z})$ whose modes will extend the generators T_B^A. As will be shown in Sec. IV, the modes of the supercurrents $S_A, \bar{S}^A$ will extend the supercharges. Since the OPE of currents directly translates to the commutator of their modes within radial quantization, our currents must satisfy the OPE:

$$\tilde{\mathcal{G}}_B^A(z,\bar{z})S_C(w) \sim ((z-w)\ \text{singularity})\ (T_B^A)_C^D S_D(w),$$
$$\tilde{\mathcal{G}}_B^A(z,\bar{z})\bar{S}^C(w) \sim -((\bar{z}-\bar{w})\ \text{singularity})\ (T_B^A)_D^C\bar{S}^D(w). \tag{3.32}$$

Note that S_A and $\bar{S}^A$ are holomorphic and antiholomorphic currents, respectively; this imposes the condition that the singularities in (3.32) be holomorphic and antiholomorphic, respectively. As will be shown in Sec. IV, nonholomorphic [holomorphic] singularity in the OPE of $\mathcal{G}_B^A(z,\bar{z})$ with $S_C(w)$ [$\bar{S}^C(w)$] results in nonsensical algebra. This requirement will be crucial.

The only conformal operator we are left with is the graviphoton operator. Moreover, the leading soft graviphoton operator corresponds to $\Delta = 0$ as can be inferred from the poles of the beta function in the OPEs of graviphoton operators with other operators that are summarized in Appendix A. It is clear that we must consider the order independent graviphoton double soft limit with opposite helicity, which is their normal ordered commutator (since they are bosonic). Since it contains the factor δ_{CD}^{AB}, as can be seen from the collinear limits, this can be manipulated properly to obtain the SU(8) generators. Here we consider the most general integral transform corresponding to negative and positive helicity soft graviphotons, respectively, as

$$G_{AB}(z,\bar{z}) = \lim_{\Delta\to 0}\frac{\Delta}{\pi}\int d^2z'\frac{1}{(z-z')^a}\frac{1}{(\bar{z}-\bar{z}')^b}\mathcal{O}_{AB;\Delta,-1}(z',\bar{z}'),$$
$$\bar{G}^{CD}(z,\bar{z}) = \lim_{\Delta\to 0}\frac{\Delta}{\pi}\int d^2z'\frac{1}{(\bar{z}-\bar{z}')^{a'}}\frac{1}{(z-z')^{b'}}\mathcal{O}^{CD}_{\Delta,+1}(z',\bar{z}'). \tag{3.33}$$

One can easily see that we can recover the usual shadow transformation [48] by taking specific values of a and b. The operators $\mathcal{O}_{AB;0,-1}$ and $\mathcal{O}^{CD}_{0,+1}$ have conformal weights $(-\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},-\frac{1}{2})$, respectively. Hence, the scaling transformation reveals the conformal weights of the currents G_{AB} and $\bar{G}^{CD}$ to be $(a-\frac{3}{2}, b-\frac{1}{2})$ and $(b'-\frac{1}{2}, a'-\frac{3}{2})$, respectively. Let us start with the OPE of our new currents G_{AB} and $\bar{G}^{CD}$ with any conformal primary operators,

$$\left\langle G_{AB}(z,\bar{z})\prod_{n=2}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle$$
$$=\lim_{\Delta_1\to 0}\frac{\Delta_1}{\pi}\int d^2z_1\frac{1}{(z-z_1)^a}\frac{1}{(\bar{z}-\bar{z}_1)^b}\left\langle \mathcal{O}_{AB\Delta_1,-1}(z_1,\bar{z}_1)\prod_{n=2}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle$$
$$=\frac{1}{\pi}\int d^2z_1\frac{1}{(z-z_1)^a}\frac{1}{(\bar{z}-\bar{z}_1)^b}\left[\sum_{n=2}^{N}f(A,B,\ell_i,*_n,*'_n)\frac{z_1-z_n}{\bar{z}_1-\bar{z}_n}\langle\cdots\mathcal{O}^{*'_n}_{\Delta_n,\ell_n+1}(z_n,\bar{z}_n)\rangle\right], \tag{3.34}$$

where $f(A,B,\ell_i,*_n,*'_n)$ contains the R-symmetry index of the operators in the correlation function that appears on taking the collinear limit. We used the fact that $\lim_{\Delta\to 0}\Delta B(\Delta,*)=1$. Now we use two basic integrals (see [49] for proof):

$$\int d^2z_1\frac{1}{(z-z_1)^A}\frac{1}{(\bar{z}-\bar{z}_1)^B}\frac{(\bar{z}_1-\bar{z}_j)^s}{z_1-z_j}=C_s(A,B)\frac{1}{(z_j-z)^A(\bar{z}_j-\bar{z})^{B-s-1}},$$
$$\int d^2z_1\frac{1}{(\bar{z}-\bar{z}_1)^A}\frac{1}{(z_1-z_1)^B}\frac{(z_1-z_j)^s}{\bar{z}_1-\bar{z}_j}=C_s(A,B)\frac{1}{(\bar{z}_j-\bar{z})^A}\frac{1}{(z_j-z)^{B-s-1}}, \tag{3.35}$$

where

$$C_s(A,B)=\frac{(-1)^{s+A+B}(-\pi)s!}{(-B+1)(-B+2)\cdots(-B+s+1)}. \tag{3.36}$$

Now performing the shadow integral for $n\neq 1$ and $s=1$,

$$\int d^2z_1\frac{1}{(z-z_1)^a}\frac{1}{(\bar{z}-\bar{z}_1)^b}\frac{z_1-z_n}{\bar{z}_1-\bar{z}_n}=C_1(b,a)\frac{1}{(\bar{z}_n-\bar{z})^b}\frac{1}{(z_n-z)^{a-2}}. \tag{3.37}$$

We have

$$\left\langle G_{AB}(z,\bar{z})\prod_{n=2}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle=\sum_{i=2}^{N}f(A,B,\ell_i,*_i,*'_i)C_1(b,a)\frac{1}{(\bar{z}_i-\bar{z})^b}\frac{1}{(z_i-z)^{a-2}}\langle\cdots\mathcal{O}^{*'_i}_{\Delta_i,\ell_i+1}(z_i,\bar{z}_i)\rangle. \tag{3.38}$$

Here the helicities of the conformal operators inside the correlator are restricted to $\ell_n\in\{-2,-\frac{3}{2},-1,-\frac{1}{2},0,+\frac{1}{2},+1\}$. This can be verified from the beta function singularities in the OPEs in Appendix A. Similarly, we can have the OPE for antiholomorphic current $\bar{G}^{CD}$ which act on the conformal primaries with helicities restricted in the range $\ell'_n\in\{-1,-\frac{1}{2},0,+\frac{1}{2},+1,+\frac{3}{2},+2\}$,

$$\left\langle \bar{G}^{CD}(z,\bar{z})\prod_{n=2}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell'_n}(z_n,\bar{z}_n)\right\rangle=\sum_{n=2}^{N}\bar{f}(C,D,\ell'_i,*_n,*'_n)C(b',a')\frac{1}{(\bar{z}_n-\bar{z})^{b'}}\frac{1}{(z_n-z)^{a'-2}}\langle\cdots\mathcal{O}^{*'_n}_{\Delta_n,\ell'_n-1}(z_n,\bar{z}_n)\rangle. \tag{3.39}$$

Here we can pair ℓ' with $\ell=\ell'+1$. Hence, we can write the OPEs as

$$G_{AB}(z)\mathcal{O}^{*}_{\Delta,\ell}(w,\bar{w})\sim f(A,B,\ell,*,*')C_1(b,a)\frac{1}{(\bar{w}-\bar{z})^b}\frac{1}{(w-z)^{a-2}}\mathcal{O}^{*'}_{\Delta,\ell'}(z,\bar{w}),$$
$$\bar{G}^{CD}(z)\mathcal{O}^{*}_{\Delta,\ell'}(w,\bar{w})\sim\bar{f}(C,D,\ell',*,*')C_1(b',a')\frac{1}{(\bar{w}-\bar{z})^{b'}}\frac{1}{(z-z)^{a'-2}}\mathcal{O}^{*'}_{\Delta,\ell}(w,\bar{w}). \tag{3.40}$$

1. The composite current

To construct a suitable current for R-symmetry, we need to use double soft limits of the graviphoton operators. As is well known, the double soft limit of opposite helicity operators depends on the order of the soft limit. For this reason, as in [42] we consider the following operator:

$$\mathcal{G}^{CD}_{AB}(z,\bar{z};w,\bar{w})\coloneqq G_{AB}(z)\bar{G}^{CD}(\bar{w})-\bar{G}^{CD}(\bar{w})G_{AB}(z)\equiv[G_{AB}(z),\bar{G}^{CD}(\bar{w})]. \tag{3.41}$$

To construct a local operator, one needs to consider the normal order of this operator evaluated at $z = w, \bar{z} = \bar{z}$. We thus define

$$\mathcal{G}_{AB}^{CD}(z,\bar{z}) =: \mathcal{G}_{AB}^{CD}(z,\bar{z};z,\bar{z}) \coloneqq :G_{AB}(z)\bar{G}^{CD}(\bar{z}) - \bar{G}^{CD}(\bar{z})G^{AB}(z): \equiv :[G_{AB}(z),\bar{G}^{CD}(\bar{z})]:. \tag{3.42}$$

We show in Appendix C that subject to the requirement of the R-symmetry current explained above (3.32), the current $\mathcal{G}_{AB}^{CD}(z,\bar{z})$ satisfies the following OPE:

$$\begin{aligned}\left\langle \mathcal{G}_{AB}^{CD}(z,\bar{z})\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle &= (-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a')\Big[f(A,B,\ell_j,*_j,*'_j)\bar{f}(C,D,\ell_j+1,*'_j,*_j{}'')\\ &\times\frac{1}{(z-z_j)^{a+b'-2}}\frac{1}{(\bar{z}-\bar{z}_j)^{a'+b-2}}\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle\\ &-\bar{f}(C,D,\ell_j,*_j,*'_j)f(A,B,\ell_j-1,*'_j,*_j{}'')\\ &\times\frac{1}{(z-z_j)^{a'+b-2}}\frac{1}{(\bar{z}-\bar{z}_j)^{a+b'-2}}\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle\Big].\end{aligned} \tag{3.43}$$

In particular,

$$\begin{aligned}\mathcal{G}_{AB}^{CD}(z,\bar{z})\mathcal{O}_{E\Delta,-\frac{3}{2}}(w,\bar{w}) &\sim -\delta_{AB}^{CD}\frac{(-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a')}{(z-w)^{a+b'-2}(\bar{z}-\bar{w})^{a'+b-2}}\mathcal{O}_{E\Delta,-\frac{3}{2}}(w,\bar{w}),\\ \mathcal{G}_{AB}^{CD}(z,\bar{z})\mathcal{O}^{E}_{\Delta,+\frac{3}{2}}(w,\bar{w}) &\sim \delta_{AB}^{CD}\frac{(-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a')}{(z-w)^{a'+b-2}(\bar{z}-\bar{w})^{a+b'-2}}\mathcal{O}^{E}_{\Delta,+\frac{3}{2}}(w,\bar{w}),\end{aligned} \tag{3.44}$$

where we used the fact that for the gravitino operator, the R-symmetry factor in the double soft limit is $-\delta_{AB}^{CD}$. Indeed,

$$\lim_{\Delta_1\to 0}\Delta_1\mathcal{O}_{AB;\Delta_1,-1}(z,\bar{z})\mathcal{O}_{E,\Delta,-\frac{3}{2}}(z_1,\bar{z}_1) = \frac{z-z_1}{\bar{z}-\bar{z}_1}\mathcal{O}_{ABE;\Delta,-\frac{1}{2}}(z_1,\bar{z}_1).$$

If $E = r$, $A = a$, $B = b$, $C = c$, $D = d$, then

$$\lim_{\Delta_2\to 0}\Delta_2\mathcal{O}^{cd}_{\Delta_{2,+1}}(w,\bar{w})\mathcal{O}_{abr;\Delta,-\frac{1}{2}}(z_1,\bar{z}_1) = -\delta_{ab}^{cd}\frac{\bar{w}-\bar{z}_1}{w-z_1}\mathcal{O}_{r,\Delta,-\frac{3}{2}}(z_1,\bar{z}_1).$$

In all other cases, one can check from the collinear limit in Appendix A that the R-symmetry factor is $-\delta_{AB}^{CD}$. Let us construct a new current as a linear combination of our previous currents as follows:

$$(\tilde{\mathcal{G}}^{A}_{B})^{C}_{D}(z,\bar{z}) \coloneqq -\left(\frac{1}{7}\delta^{A}_{D}\sum_{E=1}^{8}\mathcal{G}^{EC}_{BE}(z,\bar{z}) + \frac{1}{56}\delta^{A}_{B}\sum_{E=1}^{8}\mathcal{G}^{EC}_{ED}(z,\bar{z})\right). \tag{3.45}$$

Using the definition of generalized Kronecker delta

$$\delta^{a_1\ldots a_n}_{b_1\ldots b_n} = \sum_{\sigma\in S_n}\mathrm{sign}(\sigma)\delta^{a_1}_{b_{\sigma(1)}}\cdots\delta^{a_n}_{b_{\sigma(n)}}, \tag{3.46}$$

we see that

$$\sum_{E=1}^{8}\delta^{EC}_{BE} = -7\delta^{C}_{B}, \qquad \sum_{E=1}^{8}\delta^{EC}_{ED} = 7\delta^{C}_{D}. \tag{3.47}$$

This gives us the OPE

$$(\tilde{\mathcal{G}}_A^C)_B^D(z,\bar{z})\mathcal{O}_{D\Delta,-\frac{3}{2}}(w,\bar{w}) \sim \frac{(-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a')}{(z-w)^{a+b'-2}(\bar{z}-\bar{w})^{a'+b-2}}(T_A^C)_B^D\mathcal{O}_{D\Delta,-\frac{3}{2}}(w,\bar{w}),$$
$$(\tilde{\mathcal{G}}_A^C)_B^D(z,\bar{z})\mathcal{O}^B_{\Delta,+\frac{3}{2}}(w,\bar{w}) \sim -\frac{(-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a')}{(z-w)^{a'+b-2}(\bar{z}-\bar{w})^{a+b'-2}}(T_A^C)_B^D\mathcal{O}^B_{\Delta,+\frac{3}{2}}(w,\bar{w}). \quad (3.48)$$

Hence, $\tilde{\mathcal{G}}_A^C$ is a candidate which can extend the R-symmetry algebra. But we see an immediate problem. The OPE of $\tilde{\mathcal{G}}_A^C$ with supercurrents $S_D(w), \bar{S}^B(\bar{w})$ is given by

$$(\tilde{\mathcal{G}}_A^C)_B^D(z,\bar{z})S_D(w) \sim (T_A^C)_B^D \lim_{\Delta\to\frac{1}{2}} \frac{\Delta-\frac{1}{2}}{\pi}(-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a') \times \int d^2z_1 \frac{1}{(w-z_1)^3}\frac{1}{(z-z_1)^{a+b'-2}}\frac{1}{(\bar{z}-\bar{z}_1)^{a'+b-2}}\mathcal{O}_{D,\Delta,-\frac{3}{2}}(z_1,\bar{z}_1) \quad (3.49)$$

and

$$(\tilde{\mathcal{G}}_A^C)_B^D(z,\bar{z})\bar{S}^B(\bar{w}) \sim -(T_A^C)_B^D \lim_{\Delta\to\frac{1}{2}} \frac{\Delta-\frac{1}{2}}{\pi}(-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a') \times \int d^2z_1 \frac{1}{(\bar{w}-\bar{z}_1)^3}\frac{1}{(z-z_1)^{a'+b-2}}\frac{1}{(\bar{z}-\bar{z}_1)^{a+b'-2}}\mathcal{O}^B_{\Delta,+\frac{3}{2}}(z_1,\bar{z}_1). \quad (3.50)$$

The requirement (3.32) forces $a'+b-2=0$ in (3.49) and (3.50). But then in view of (C11) we get

$$a+b'-2=0 \quad \text{and} \quad a'+b-2=0 \quad (3.51)$$

and conclude that the OPE is trivial:

$$(\tilde{\mathcal{G}}_A^C)_B^D(z,\bar{z})S_D(w) \sim \text{regular},$$
$$(\tilde{\mathcal{G}}_A^C)_B^D(z,\bar{z})\bar{S}^B(\bar{w}) \sim \text{regular}. \quad (3.52)$$

IV. THE $\mathcal{N}=8$ $\mathfrak{sbms}_4$ ALGEBRA

Let us now find the asymptotic symmetries of the theory. The usual symmetry currents in the theory are the stress tensors $T(z)$ and $\bar{T}(\bar{z})$, which are the superrotation generators, and $\mathcal{P}(z,\bar{z})$, which is the supertranslation generator. The modes of these currents generate the $\mathfrak{bms}_4$ algebra as described in [40]. As usual the generators of $\mathfrak{bms}_4$ are the modes of $T(z), \bar{T}(\bar{z})$, and $\mathcal{P}(z,\bar{z})$. Let us expand these currents in modes:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \qquad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}; \quad (4.1)$$

$$\mathcal{P}(z,\bar{z}) \equiv \sum_{n,m\in\mathbb{Z}} P_{n-\frac{1}{2},m-\frac{1}{2}} z^{-n-1}\bar{z}^{-m-1}. \quad (4.2)$$

As discussed in [40], the modes $P_{n-\frac{1}{2},m-\frac{1}{2}}$ can be obtained from the modes of the current $P(z)$ or $\bar{P}(\bar{z})$. If we write

$$P(z) = \sum_{n\in\mathbb{Z}} P_{n-\frac{1}{2}} z^{-n-1}, \qquad \bar{P}(\bar{z}) = \sum_{m\in\mathbb{Z}} \bar{P}_{m-\frac{1}{2}} \bar{z}^{-m-1}, \quad (4.3)$$

then

$$P_{n-\frac{1}{2},-\frac{1}{2}} = P_{n-\frac{1}{2}}, \qquad P_{-\frac{1}{2},m-\frac{1}{2}} = \bar{P}_{m-\frac{1}{2}}, \quad (4.4)$$

and

$$P_{n-\frac{1}{2},m-\frac{1}{2}} = \frac{1}{i\pi(m+1)}\oint d\bar{w}\bar{w}^{m+1}[\bar{T}(\bar{w}), P_{n-\frac{1}{2},-\frac{1}{2}}] = \frac{1}{i\pi(m+1)}\oint dw w^{n+1}[T(w), \bar{P}_{-\frac{1}{2},m-\frac{1}{2}}]. \quad (4.5)$$

These modes satisfy the usual $\mathfrak{bms}_4$ algebra:

$$[L_m, L_n] = (m-n)L_{m+n}, \qquad [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n},$$
$$[L_n, P_{kl}] = \left(\frac{1}{2}n-k\right)P_{n+k,l}, \qquad [\bar{L}_n, P_{kl}] = \left(\frac{1}{2}n-l\right)P_{k,n+l}, \quad (4.6)$$

where $m,n\in\mathbb{Z}$ and $k,l\in\mathbb{Z}+\frac{1}{2}$. In addition, an infinite dimensional extension of the $\mathcal{N}=1$ supersymmetry algebra was constructed in [3]. The supercurrent was shown to be the shadow transform of the gravitino operator. In our theory, we have eight supercurrents $S^A(z)$ and their antiholomorphic counterpart $\bar{S}_A(\bar{z})$. The OPEs (3.14) show that $S^A(z)$ and $\bar{S}_A(\bar{z})$ are conformal primaries of dimensions

$(\frac{3}{2},0)$ and $(0,\frac{3}{2})$, respectively. Consequently, if we expand the supercurrents as

$$S_A(z)=\sum_{k\in\mathbb{Z}+\frac{1}{2}}\frac{(S_A)_k}{z^{k+\frac{3}{2}}},\quad\text{with}\quad (S_A)_k=\frac{1}{2\pi i}\oint dz z^{k+\frac{1}{2}}S_A(z),$$
$$\bar{S}^A(\bar{z})=\sum_{l\in\mathbb{Z}+\frac{1}{2}}\frac{\bar{S}^A_l}{z^{l+\frac{3}{2}}},\quad\text{with}\quad \bar{S}^A_l=\frac{1}{2\pi i}\oint d\bar{z}\bar{z}^{l+\frac{1}{2}}\bar{S}^A(\bar{z}),\tag{4.7}$$

then we can write the commutator of these modes with the Virasoro generators as

$$[L_n,(S_A)_m]=\left(\frac{n}{2}-m\right)(S_A)_{m+n},\qquad [\bar{L}_n,(S_A)_m]=0,$$
$$[L_n,\bar{S}^A_m]=0,\qquad [\bar{L}_n,\bar{S}^A_m]=\left(\frac{n}{2}-m\right)\bar{S}^A_{m+n}.\tag{4.8}$$

The operator relation (3.15) gives the anticommutator

$$\{(S_B)_m,\bar{S}^A_n\}=\delta^A_B P_{mn},\quad m,n\in\mathbb{Z}+\frac{1}{2}.\tag{4.9}$$

Let us now discuss the requirement of (anti)holomorphicity of the singularity in (3.32). Suppose $(\tilde{\mathcal{G}}^A_B)^C_D(z,\bar{z})$ has conformal weights[7] $(h,\bar{h})$. Then we can expand the current as

$$(\tilde{\mathcal{G}}^A_B)^C_D(z,\bar{z})=\sum_{n,m\in\mathbb{Z}}\{(\tilde{\mathcal{G}}^A_B)^C_D\}_{mn}z^{-m-h}\bar{z}^{-n-\bar{h}},\tag{4.10}$$

with

$$\{(\tilde{\mathcal{G}}^A_B)^C_D\}_{mn}=\frac{1}{(2\pi i)^2}\oint dz\oint d\bar{z}z^{m+h-1}\bar{z}^{n+\bar{h}+1}(\tilde{\mathcal{G}}^A_B)^C_D(z,\bar{z}).\tag{4.11}$$

Suppose we had an OPE of the form[8]

$$(\tilde{\mathcal{G}}^A_B)^C_D(z,\bar{z})S_C(w)\sim\frac{1}{z-w}\frac{1}{\bar{z}-\bar{w}}(T^A_B)^C_D S_C(w).\tag{4.12}$$

One can readily check that this would give us the commutator

$$[(\tilde{\mathcal{G}}^A_B)^C_D\}_{mn},(S_C)_k]=\bar{w}^n(T^A_B)^C_D\{S_C\}_{m+k}.\tag{4.13}$$

This is nonsensical since we do not have any $\bar{w}$ dependence on the left-hand side. Similarly, one can justify the second part of (3.32). So we conclude that the $\mathcal{N}=8$ $\mathfrak{sbms}_4$ algebra does not contain the extension of global R-symmetry algebra. The final algebra is then given by

$$[L_m,L_n]=(m-n)L_{m+n},\qquad [\bar{L}_m,\bar{L}_n]=(m-n)\bar{L}_{m+n},$$
$$[L_n,P_{kl}]=\left(\frac{1}{2}n-k\right)P_{n+k,l},$$
$$[\bar{L}_n,P_{kl}]=\left(\frac{1}{2}n-l\right)P_{k,n+l},$$
$$[L_n,(S_A)_m]=\left(\frac{n}{2}-m\right)(S_A)_{m+n},\qquad [\bar{L}_n,(S_A)_m]=0,$$
$$[L_n,\bar{S}^A_m]=0,\qquad [\bar{L}_n,\bar{S}^A_m]=\left(\frac{n}{2}-m\right)\bar{S}^A_{m+n},$$
$$\{(S_B)_m,\bar{S}^A_n\}=\delta^A_B P_{mn}.\tag{4.14}$$

V. CONCLUSION

In this paper, we have used the CCFT technique to compute the asymptotic symmetry algebra of $\mathcal{N}=8$ supergravity in asymptotically flat spacetime. The crucial part of our result is the nonextension of the global $SU(8)_R$ R-symmetry algebra. The purely mathematical considerations [43] for $\mathcal{N}=2$ theory suggests that the infinite-dimensional extension of R-symmetry is fraught with mathematical inconsistencies. Here, performing a direct asymptotic symmetry analysis of the supergravity theory using the CCFT prescription, we have confirmed that indeed supergravity does not result in such an extension. The rest of the symmetry algebra is as expected and is presented in (4.14).

It is instructive to note that our results are consistent with the usual expectation of symmetry enhancement at the boundary for gauge symmetries. In the case of ordinary gravity and minimal supergravity theories, the corresponding symmetries are local in nature, and hence, they have a natural infinite extension at the boundary. For extended supergravity, the R-symmetry is primarily a global symmetry, and in our study, we find that the symmetry group is not extended at the asymptotic boundary. In the CCFT language, this result comes from the regularity of the OPEs between the R-symmetry charges and the supersymmetry, which signifies the absence of collinear divergences. It would be nice to check the fate of R-symmetry in the context of gauged supergravity theories (where the R-symmetries are also local) by performing a direct asymptotic symmetry analysis of those theories. On the other hand, global noncompact symmetries, such as translation, also have a local counterpart in the theory of dynamical gravity, and hence, it does get an infinite extension at the asymptotic null boundary.

Let us end the paper with relevant open problems. In the seminal work of Hawking *et al.* [51] the importance of the infinite number of soft hairs in the context of black hole microscopics was discussed. The study was further taken

[7] The scaling dimensions of $(\tilde{\mathcal{G}}^A_B)^C_D(z,\bar{z})$ can be calculated from those of $G_{AB},\bar{G}^{CD}$. It is $(a+b'-2,a'+b-2)$.

[8] Exactly the same argument works if we have higher power singularities.

forward in [50–54] and beautifully reviewed in [16]. They emphasized the importance of symmetry enhancements at the future horizon $\mathcal{H}^+$ of the black holes and how both the hypersurfaces[9] $\mathcal{H}^+$ and $\mathcal{I}^+$ carry information of conserved charges that are in turn important for understanding black hole microscopics. The study of the present paper indicates that the asymptotic soft hairs of the supergravity theories will not have distinct infinite R-charges; rather, they will only carry the global fixed number of R-charges. An interesting question that remains to be studied is the effect, if any, of these R-charges at the horizon and finally their importance in the black hole microscopics. We hope to return to this question in the future.

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APPENDIX A: OPEs OF COMPONENT FIELDS

Using (2.18), we can extract the rest of the OPEs from the collinear singularities of the amplitudes calculated in [44]. In the following, the zero, one, two, three, and four index operators are, respectively, graviton, gravitino, graviphoton, graviphotino, and scalar operators.

1. Same spin OPEs

$$\begin{aligned}
\mathcal{O}^A_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\mathcal{O}^B_{\Delta_2,+\frac{3}{2}}(z_2,\bar{z}_2) &\sim \frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-\frac{1}{2},\Delta_2-\frac{1}{2}\right)\mathcal{O}^{AB}_{\Delta_1+\Delta_2,+1}(z_2,\bar{z}_2\Big),\\
\mathcal{O}^A_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\mathcal{O}_{B\Delta_2,-\frac{3}{2}}(z_2,\bar{z}_2) &\sim \frac{\bar{z}_{12}}{z_{12}}\delta^A_B B\left(\Delta_1-\frac{1}{2},\Delta_2+\frac{5}{2}\right)\mathcal{O}_{\Delta_1+\Delta_2,-2}(z_2,\bar{z}_2\Big)\\
&\quad+\frac{z_{12}}{\bar{z}_{12}}\delta^A_B B\left(\Delta_1+\frac{5}{2},\Delta_2-\frac{1}{2}\right)\mathcal{O}_{\Delta_1+\Delta_2,+2}(z_2,\bar{z}_2\Big),
\end{aligned}\tag{A1}$$

$$\begin{aligned}
\mathcal{O}^{AB}_{\Delta_1,+1}(z_1,\bar{z}_1)\mathcal{O}^{CD}_{\Delta_2,+1}(z_2,\bar{z}_2) &\sim \frac{\bar{z}_{12}}{z_{12}}B(\Delta_1,\Delta_2)\mathcal{O}^{ABCD}_{\Delta_1+\Delta_2,0}(z_2,\bar{z}_2),\\
\mathcal{O}^{AB}_{\Delta_1,+1}(z_1,\bar{z}_1)\mathcal{O}_{CD;\Delta_2,-1}(z_2,\bar{z}_2) &\sim -\delta^{AB}_{CD}\left[\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1,\Delta_2+2)\mathcal{O}_{\Delta_1+\Delta_2,-2}(z_2,\bar{z}_2)\right.\\
&\quad\left.+\frac{z_{12}}{\bar{z}_{12}}B(\Delta_1+2,\Delta_2)\mathcal{O}_{\Delta_1+\Delta_2,+2}(z_2,\bar{z}_2)\right].
\end{aligned}\tag{A2}$$

In the following, the notation is $a,b,c,\ldots\in\{1,2,3,4\}$ and $r,s,t,\ldots\in\{5,6,7,8\}$. See [44] for details.

$$\begin{aligned}
\mathcal{O}^{ars}_{\Delta_1,+\frac{1}{2}}(z_1,\bar{z}_1)\mathcal{O}^{btu}_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2) &\sim \epsilon^{rstu}\epsilon^{abcd}\frac{z_{12}}{\bar{z}_{12}}B\left(\Delta_1+\frac{1}{2},\Delta_2+\frac{1}{2}\right)\mathcal{O}_{cd;\Delta_1+\Delta_2,-1}(z_2,\bar{z}_2\Big),\\
\mathcal{O}^{ars}_{\Delta_1,+\frac{1}{2}}(z_1,\bar{z}_1)\mathcal{O}^{bct}_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2) &\sim \epsilon^{rstu}\epsilon^{abcd}\frac{z_{12}}{\bar{z}_{12}}B\left(\Delta_1+\frac{1}{2},\Delta_2+\frac{1}{2}\right)\mathcal{O}_{ud;\Delta_1+\Delta_2,-1}(z_2,\bar{z}_2\Big),\\
\mathcal{O}^{rst}_{\Delta_1,+\frac{1}{2}}(z_1,\bar{z}_1)\mathcal{O}^{abc}_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2) &\sim \epsilon^{rstu}\epsilon^{abcd}\frac{z_{12}}{\bar{z}_{12}}B\left(\Delta_1+\frac{1}{2},\Delta_2+\frac{1}{2}\right)\mathcal{O}_{ud;\Delta_1+\Delta_2,-1}(z_2,\bar{z}_2\Big),\\
\mathcal{O}^{ars}_{\Delta_1,+\frac{1}{2}}(z_1,\bar{z}_1)\mathcal{O}_{btu;\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2) &\sim \epsilon_{tuvw}\epsilon^{rsvw}\delta^a_b\left[\frac{z_{12}}{\bar{z}_{12}}B\left(\Delta_1+\frac{3}{2},\Delta_2+\frac{1}{2}\right)\mathcal{O}_{\Delta_1+\Delta_2,+2}(z_2,\bar{z}_2\Big)\right.\\
&\quad\left.+\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1+\frac{1}{2},\Delta_2+\frac{3}{2}\right)\mathcal{O}_{\Delta_1+\Delta_2,-2}(z_2,\bar{z}_2\Big)\right],
\end{aligned}\tag{A3}$$

[9] $\mathcal{I}^+$ denotes the future null horizon.

$$\mathcal{O}^{abrs}_{\Delta_1,0}(z_1,\bar{z}_1)\mathcal{O}^{cdtu}_{\Delta_2,0}(z_2,\bar{z}_2)\sim\epsilon^{abcd}\epsilon^{rstu}B(\Delta_1+1,\Delta_2+1)\left[\frac{z_{12}}{\bar{z}_{12}}\mathcal{O}_{\Delta_1+\Delta_2,+2}(z_2,\bar{z}_2)+\frac{\bar{z}_{12}}{z_{12}}\mathcal{O}_{\Delta_1+\Delta_2,-2}(z_2,\bar{z}_2)\right]. \tag{A4}$$

2. Different spins

$$\begin{aligned}\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\mathcal{O}^{A}_{\Delta_2,+\frac{3}{2}}(z_2,\bar{z}_2)&\sim\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-1,\Delta_2-\frac{1}{2}\right)\mathcal{O}^{A}_{\Delta_1+\Delta_2,+\frac{3}{2}}(z_2,\bar{z}_2),\\ \mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\mathcal{O}_{A;\Delta_2,-\frac{3}{2}}(z_2,\bar{z}_2)&\sim\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-1,\Delta_2+\frac{5}{2}\right)\mathcal{O}_{A;\Delta_1+\Delta_2,-\frac{3}{2}}(z_2,\bar{z}_2),\end{aligned} \tag{A5}$$

$$\begin{aligned}\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\mathcal{O}^{AB}_{\Delta_2,+1}(z_2,\bar{z}_2)&\sim\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1-1,\Delta_2)\mathcal{O}^{AB}_{\Delta_1+\Delta_2,+1}(z_2,\bar{z}_2),\\ \mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\mathcal{O}_{AB;\Delta_2,-1}(z_2,\bar{z}_2)&\sim\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1-1,\Delta_2+2)\mathcal{O}_{AB;\Delta_1+\Delta_2,-1}(z_2,\bar{z}_2),\end{aligned} \tag{A6}$$

$$\begin{aligned}\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\mathcal{O}^{abr}_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2)&\sim\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-1,\Delta_2+\frac{1}{2}\right)\mathcal{O}^{abr}_{\Delta_1+\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2),\\ \mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\mathcal{O}^{abc}_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2)&\sim\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-1,\Delta_2+\frac{1}{2}\right)\mathcal{O}^{abc}_{\Delta_1+\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2),\\ \mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\mathcal{O}_{abc;\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2)&\sim-\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-1,\Delta_2+\frac{3}{2}\right)\mathcal{O}_{abc;\Delta_1+\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2),\end{aligned} \tag{A7}$$

$$\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\mathcal{O}^{ABCD}_{\Delta_2,0}(z_2,\bar{z}_2)\sim\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1-1,\Delta_2+1)\mathcal{O}^{ABCD}_{\Delta_1+\Delta_2,0}(z_2,\bar{z}_2), \tag{A8}$$

$$\begin{aligned}\mathcal{O}^{A}_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\mathcal{O}^{BC}_{\Delta_2,+1}(z_2,\bar{z}_2)&\sim\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-\frac{1}{2},\Delta_2\right)\mathcal{O}^{ABC}_{\Delta_1+\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2),\\ \mathcal{O}^{A}_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\mathcal{O}_{BC;\Delta_2,-1}(z_2,\bar{z}_2)&\sim 2!\delta^{A}_{[B}\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-\frac{1}{2},\Delta_2\right)\mathcal{O}_{C];\Delta_1+\Delta_2,-\frac{3}{2}}(z_2,\bar{z}_2),\end{aligned} \tag{A9}$$

$$\begin{aligned}\mathcal{O}^{A}_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\mathcal{O}^{BCD}_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2)&\sim\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-\frac{1}{2},\Delta_2+\frac{1}{2}\right)\mathcal{O}^{ABCD}_{\Delta_1+\Delta_2,0}(z_2,\bar{z}_2),\\ \mathcal{O}^{A}_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\mathcal{O}_{BCD\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2)&\sim 3\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-\frac{1}{2},\Delta_2+\frac{3}{2}\right)\delta^{A}_{[B}\mathcal{O}_{CD];\Delta_1+\Delta_2,-1}(z_2,\bar{z}_2),\end{aligned} \tag{A10}$$

$$\begin{aligned}\mathcal{O}^{A}_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\mathcal{O}^{BCDE}_{\Delta_2,0}(z_2,\bar{z}_2)&\sim-\frac{1}{6}\epsilon^{ABCDEFGH}\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-\frac{1}{2},\Delta_2+1\right)\mathcal{O}_{FGH;\Delta_1+\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2),\\ \mathcal{O}^{A}_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\mathcal{O}_{BCDE\Delta_2,0}(z_2,\bar{z}_2)&\sim 3!\delta^{A}_{[B}\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-\frac{1}{2},\Delta_2+1\right)\mathcal{O}_{CDE];\Delta_1+\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2),\end{aligned} \tag{A11}$$

$$\begin{aligned}\mathcal{O}^{ab}_{\Delta_1,+1}(z_1,\bar{z}_1)\mathcal{O}^{cdr}_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2)&\sim\frac{1}{3!}\epsilon^{rstu}\epsilon^{abcd}\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1,\Delta_2+\frac{1}{2}\right)\mathcal{O}_{stu;\Delta_1+\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2),\\ \mathcal{O}^{AB}_{\Delta_1,+1}(z_1,\bar{z}_1)\mathcal{O}_{CDE;\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2)&\sim-\delta^{AB}_{CD}\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1,\Delta_2+\frac{3}{2}\right)\mathcal{O}_{E;\Delta_1+\Delta_2,-\frac{3}{2}}(z_2,\bar{z}_2),\end{aligned} \tag{A12}$$

$$\begin{aligned}\mathcal{O}^{ab}_{\Delta_1,+1}(z_1,\bar{z}_1)\mathcal{O}^{cdrs}_{\Delta_2,0}(z_2,\bar{z}_2)&\sim\epsilon^{rstu}\epsilon^{abcd}\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1,\Delta_2+1)\mathcal{O}_{tu;\Delta_1+\Delta_2,-1}(z_2,\bar{z}_2),\\ \mathcal{O}^{ab}_{\Delta_1,+1}(z_1,\bar{z}_1)\mathcal{O}^{cdef}_{\Delta_2,0}(z_2,\bar{z}_2)&\sim-\epsilon^{cdef}\epsilon^{abgh}\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1,\Delta_2+1)\mathcal{O}_{gh;\Delta_1+\Delta_2,-1}(z_2,\bar{z}_2),\end{aligned} \tag{A13}$$

$$\mathcal{O}^{abr}_{\Delta_1,+\frac{1}{2}}(z_1,\bar{z}_1)\mathcal{O}^{cdst}_{\Delta_2,0}(z_2,\bar{z}_2)\sim\epsilon^{rstu}\epsilon^{abcd}\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1+\frac{1}{2},\Delta_2+1\right)\mathcal{O}_{u;\Delta_1+\Delta_2,-\frac{3}{2}}(z_2,\bar{z}_2),$$
$$\mathcal{O}^{abr}_{\Delta_1,+\frac{1}{2}}(z_1,\bar{z}_1)\mathcal{O}^{cstu}_{\Delta_2,0}(z_2,\bar{z}_2)\sim-\epsilon^{rstu}\epsilon^{abcd}\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1+\frac{1}{2},\Delta_2+1\right)\mathcal{O}_{d;\Delta_1+\Delta_2,-\frac{3}{2}}(z_2,\bar{z}_2). \quad \text{(A14)}$$

Similarly, all other OPEs can be extracted from the amplitudes given in [44].

APPENDIX B: SOFT GRAVITON AND SOFT GRAVITINO OPERATORS

We will use the soft limit of the superamplitude and then perform an expansion in the Grassmann odd coordinate of the superspace to obtain the soft graviton and gravitino limits in an amplitude.

The leading and subleading soft factors in a superamplitude corresponding to ω_p^{-1} and ω_p^{0} were calculated in [44] using double copy relations. Here we only present relevant results and refer the readers to [44] for further details. In the celestial basis, the leading soft factor is given by

$$\mathcal{M}_N(\ldots,j-1,j,j+1,\ldots)\xrightarrow{\omega_j\to0}\frac{1}{\omega_j}\sum_{\substack{i=1\\i\neq j}}^{N}\omega_i z_{ji}\bar{z}_{ji}\left[\frac{z^2_{j-1,i}}{z^2_{j-1,j}z^2_{j,i}}+\frac{\bar{z}^2_{j-1,i}}{\bar{z}^2_{j-1}\bar{z}^2_{j,i}}\delta^4(\eta^j)\right]\mathcal{M}_{N-1}(\ldots,j-1,j+1,\ldots). \quad \text{(B1)}$$

One can now get the soft limit in terms of the celestial superamplitude in a straightforward way. We have

$$\begin{aligned}\left\langle\prod_{n=1}^{N}\lim_{\Delta_j\to1}(\Delta_j-1)\mathcal{O}_{\Delta_n}(z_n,\bar{z}_n,\eta^n)\right\rangle&=\left(\prod_{\substack{n=1\\n\neq j}}^{N}\int d\omega_n\omega_n^{\Delta_n-1}\right)\lim_{\Delta_j\to1}\int_0^\infty d\omega_j(\Delta_j-1)\omega_j^{\Delta_j-1}\delta^{(4)}\left(\sum_{\substack{k=1\\k\neq j}}^{N}\omega_k q_k\right)\mathcal{M}_N(1,\ldots,n,\ldots,N)\\&=\left(\prod_{\substack{n=1\\n\neq j}}^{N}\int d\omega_n\omega_n^{\Delta_n-1}\right)\int_0^\infty d\omega_j\frac{d}{d\omega_j}\left(\lim_{\Delta_j\to1}\omega_j^{\Delta_j-1}\right)\delta^{(4)}\left(\sum_{\substack{k=1\\k\neq j}}^{N}\omega_k q_k\right)\omega_j\mathcal{M}_N(1,\ldots,n,\ldots,N).\end{aligned} \quad \text{(B2)}$$

Using the fact that

$$\frac{d}{d\omega_j}\left(\lim_{\Delta_j\to1}\omega_j^{\Delta_j-1}\right)=\frac{d}{d\omega_j}\theta(\omega_j)=\delta(\omega_j), \quad \text{(B3)}$$

where $\theta(\omega)$ is the Heaviside step function, we see that the integral on ω_j on the right-hand side gives us

$$\left\langle\prod_{n=1}^{N}\lim_{\Delta_j\to1}(\Delta_j-1)\mathcal{O}_{\Delta_n}(z_n,\bar{z}_n,\eta^n)\right\rangle=\left(\prod_{\substack{n=1\\n\neq j}}^{N}\int d\omega_n\omega_n^{\Delta_n-1}\right)\delta^{(4)}\left(\sum_{\substack{k=1\\k\neq j}}^{N}\omega_k q_k\right)\lim_{\omega_j\to0}\omega_j\mathcal{M}_N(1,\ldots,n,\ldots,N). \quad \text{(B4)}$$

Using the soft limit (B1) we get

$$\begin{aligned}&\left\langle\prod_{n=1}^{N}\lim_{\Delta_j\to1}(\Delta_j-1)\mathcal{O}_{\Delta_n}(z_n,\bar{z}_n,\eta^n)\right\rangle\\&=\sum_{\substack{i=1\\i\neq j}}^{N}\omega_i\left\{\frac{z^2_{j-1,i}\bar{z}_{ji}}{z^2_{j-1,j}z_{ji}}+\frac{\bar{z}^2_{j-1,i}z_{ji}}{\bar{z}^2_{j-1,j}\bar{z}_{ji}}\delta^4(\eta^j)\right\}\left(\prod_{\substack{n=1\\n\neq j}}^{N}\int_0^\infty d\omega_k\omega_k^{\Delta_k-1}\right)\delta^{(4)}\left(\sum_{\substack{k=1\\k\neq j}}^{N}\omega_k q_k\right)\mathcal{M}_{N-1}(1,\ldots,i,\ldots,N)\\&=\sum_{\substack{i=1\\i\neq j}}^{n}\left\{\frac{z^2_{j-1,i}\bar{z}_{ji}}{z^2_{j-1,j}z_{ji}}+\frac{\bar{z}^2_{j-1,i}z_{ji}}{\bar{z}^2_{j-1,j}\bar{z}_{ji}}\delta^8(\eta^j)\right\}\left[\prod_{\substack{n=1\\n\neq i}}^{N}\int_0^\infty d\omega_k\omega_k^{\Delta_k-1}\int_0^\infty d\omega_i\omega_i^{\Delta_i}\delta^{(4)}\left(\sum_{\substack{k=1\\k\neq j}}^{N}\omega_k q_k\right)\mathcal{M}_{N-1}(1,\ldots,i,\ldots,N)\right]\\&=\sum_{\substack{i=1\\i\neq j}}^{n}\left\{\frac{z^2_{j-1,i}\bar{z}_{ji}}{z^2_{j-1,j}z_{ji}}+\frac{\bar{z}^2_{j-1,i}z_{ji}}{\bar{z}^2_{j-1,j}\bar{z}_{ji}}\delta^8(\eta^j)\right\}\langle\mathcal{O}_{\Delta_1}(z_1,\bar{z}_1,\eta_1),\ldots,\mathcal{O}_{\Delta_i+1}(z_i,\bar{z}_i,\eta^i),\ldots\rangle.\end{aligned}$$

The super-Ward identity that we get from the conformally supersoft theorem is

$$\langle J_1(z,\bar{z},\eta)\mathcal{O}_{\Delta_1}(z_1,\bar{z}_1,\eta^1)\cdots\mathcal{O}_{\Delta_N}(z_N,\bar{z}_N,\eta^N)\rangle = \sum_{i=1}^{N}\left\{\frac{(\bar{z}-\bar{z}_i)}{(z-z_i)}\frac{(z_N-z_i)^2}{(z_N-z)^2}+\frac{(z-z_i)}{(\bar{z}-\bar{z}_i)}\frac{(\bar{z}_N-\bar{z}_i)^2}{(\bar{z}_N-\bar{z})^2}\delta^8(\eta)\right\} \times\langle\mathcal{O}_{\Delta_1}(z_1,\bar{z}_1,\eta^1),\ldots,\mathcal{O}_{\Delta_i+1}(z_i,\bar{z}_i,\eta^i),\ldots,\mathcal{O}_{\Delta_N}(z_N,\bar{z}_N,\eta^N)\rangle, \quad (B5)$$

where

$$J_1(z,\bar{z},\eta)=\lim_{\Delta\to 1}(\Delta-1)\mathcal{O}_\Delta(z,\bar{z},\eta)$$

is the $\Delta\to 1$ soft operator. In the above soft factor, we chose the reference vector for polarization of the soft particle to be the momentum vector of the nth particle. We leave this reference vector arbitrary, which corresponds to a point $\xi\in\mathcal{CS}^2$. The super-Ward identity then takes the form

$$\langle J_1(z,\bar{z},\eta)\mathcal{O}_{\Delta_1}(z_1,\bar{z}_1,\eta^1)\cdots\mathcal{O}_{\Delta_N}(z_N,\bar{z}_N,\eta^N)\rangle = \sum_{i=1}^{N}\left\{\frac{(\bar{z}-\bar{z}_i)}{(z-z_i)}\frac{(\xi-z_i)^2}{(\xi-z)^2}+\frac{(z-z_i)}{(\bar{z}-\bar{z}_i)}\frac{(\bar{\xi}-\bar{z}_i)^2}{(\bar{\xi}-\bar{z})^2}\delta^8(\eta)\right\} \times\langle\mathcal{O}_{\Delta_1}(z_1,\bar{z}_1,\eta^1),\ldots,\mathcal{O}_{\Delta_i+1}(z_i,\bar{z}_i,\eta^i),\ldots,\mathcal{O}_{\Delta_N}(z_N,\bar{z}_N,\eta^N)\rangle. \quad (B6)$$

When we expand both sides in the Grassmann variables η^i and compare coefficients, we get the Ward identity for the soft graviton operator:

$$\left\langle J_1(z,\bar{z})\prod_{n=1}^{N}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \sum_{i=1}^{N}\frac{(\bar{z}-\bar{z}_i)}{(z-z_i)}\frac{(\xi-z_i)^2}{(\xi-z)^2}\langle\mathcal{O}_{\Delta_1,\ell_1}(z_1,\bar{z}_1),\ldots,\mathcal{O}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\ldots,\mathcal{O}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle \quad (B7)$$

and

$$\left\langle \bar{J}_1(z,\bar{z})\prod_{n=1}^{N}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \sum_{i=1}^{N}\frac{(z-z_i)}{(\bar{z}-\bar{z}_i)}\frac{(\bar{\xi}-\bar{z}_i)^2}{(\bar{\xi}-\bar{z})^2}\langle\mathcal{O}_{\Delta_1,\ell_1}(z_1,\bar{z}_1),\ldots,\mathcal{O}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\ldots,\mathcal{O}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle, \quad (B8)$$

where

$$J_1(z,\bar{z})=\lim_{\Delta\to 1}(\Delta-1)\mathcal{O}_{\Delta,+2}(z,\bar{z}),\qquad \bar{J}_1(z,\bar{z})=\lim_{\Delta\to 1}(\Delta-1)\mathcal{O}_{\Delta,-2}(z,\bar{z}) \quad (B9)$$

are the $\Delta=1$ soft graviton operators. The subleading soft factor was also calculated in [44]. It turns out that it is the same as the subleading soft factor for positive and negative helicity gravitons in pure gravity [45]. We then write the super-Ward identity following the calculations in [55]:

$$\langle J_0(z,\bar{z},\eta)\mathcal{O}_{\Delta_1}(z,\bar{z},\eta^1)\cdots\mathcal{O}_{\Delta_N}(z_N,\bar{z}_N,\eta^N)\rangle = \sum_{i=1}^{N}\left\{\frac{(\bar{z}-\bar{z}_i)}{(z-z_i)}\frac{(\xi-z_i)}{(\xi-z)}((\bar{z}-\bar{z}_i)\partial_{\bar{z}_i}-2\bar{h}_i)+\frac{(z-z_i)}{(\bar{z}-\bar{z}_i)}\frac{(\bar{\xi}-\bar{z}_i)}{(\bar{\xi}-\bar{z})}\delta^8(\eta)((z-z_i)\partial_{z_i}-2h_i)\right\} \times\langle\mathcal{O}_{\Delta_1}(z_1,\bar{z}_1,\eta^1),\ldots,\mathcal{O}_{\Delta_i}(z_i,\bar{z}_i,\eta^i),\ldots,\mathcal{O}_{\Delta_N}(z_N,\bar{z}_N,\eta^N)\rangle, \quad (B10)$$

where

$$J_0(z,\bar{z},\eta)=\lim_{\Delta\to 0}\Delta[\mathcal{O}_{\Delta,+2}(z,\bar{z})+\delta^8(\eta)\mathcal{O}_{\Delta,-2}(z,\bar{z})]$$

only contains the $\Delta = 0$ soft graviton operators. This immediately gives us the subleading soft graviton limit:

$$\left\langle J_0(z,\bar{z})\prod_{n=1}^{N}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \sum_{i=1}^{N}\frac{(\bar{z}-\bar{z}_i)}{(z-z_i)}\frac{(\xi-z_i)}{(\xi-z)}((\bar{z}-\bar{z}_i)\partial_{\bar{z}_i}-2\bar{h}_i)\langle\cdots\mathcal{O}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\rangle \tag{B11}$$

and

$$\left\langle \bar{J}_0(z,\bar{z})\prod_{n=1}^{N}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \sum_{i=1}^{N}\frac{(z-z_i)}{(\bar{z}-\bar{z}_i)}\frac{(\bar{\xi}-\bar{z}_i)}{(\bar{\xi}-\bar{z})}((z-z_i)\partial_{z_i}-2h_i)\langle\cdots\mathcal{O}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\rangle, \tag{B12}$$

where

$$J_0(z,\bar{z}) = \lim_{\Delta\to 0}\Delta\mathcal{O}_{\Delta,+2}(z,\bar{z}), \qquad \bar{J}_0(z,\bar{z}) = \lim_{\Delta\to 0}\Delta\mathcal{O}_{\Delta,-2}(z,\bar{z}) \tag{B13}$$

are the $\Delta = 0$ soft graviton operators and $h_i = \frac{\Delta_i+\ell_i}{2}$ and $\bar{h}_i = \frac{\Delta_i-\ell_i}{2}$ are the conformal weights of the operator $\mathcal{O}_{\Delta_i,\ell_i}(z,\bar{z})$.

Next, we move on to the soft gravitino operator. The leading soft gravitino limit for superamplitudes is given by

$$\mathcal{M}_{N+1}(\psi^A_{s+},\{p_1,\eta^1\},\ldots,\{p_N,\eta^N\}) = \sum_{i=1}^{N}\frac{[si]\langle ri\rangle}{\langle si\rangle\langle rs\rangle}\frac{\partial}{\partial\eta_{iA}}\mathcal{M}_N(\{p_1,\eta^1\},\ldots,\{p_N,\eta^N\}), \tag{B14}$$

where r is the reference vector corresponding to point $\xi\in\mathcal{CS}^2$. The negative helicity soft gravitino limit can be obtained by conjugating the soft factor. We can expand both sides in η^i and get the soft theorem in terms of component fields. Note that because of $\partial/\partial_{\eta_{iA}}$, the soft gravitino operator changes the spin of the particle $\ell_i\to\ell_i^c\equiv\ell_i-\frac{1}{2}$. Thus, we can only have

$$\ell_i\in\{-3/2,-1,-1/2,0,+1/2,+1,+3/2,+2\}. \tag{B15}$$

For negative helicity gravitino $\ell_i^c\to\ell_i$ and clearly

$$\ell_i^c\in\{-2,-3/2,-1,-1/2,0,+1/2,+1,+3/2\}. \tag{B16}$$

The explicit soft theorem in terms of celestial amplitudes is given by

$$\left\langle J^A_{1/2}(z,\bar{z})\prod_{n=1}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \sum_{i=1}^{N}f(A,\ell_i,*_i,*'_i)(-1)^{\sigma_i}\frac{(\bar{z}-\bar{z}_i)}{(z-z_i)}\frac{(\xi-z_i)}{(\xi-z)}\langle\cdots\mathcal{O}^{*'_i}_{\Delta_i+\frac{1}{2},\ell_i^c}(z_i,\bar{z}_i)\cdots\rangle \tag{B17}$$

and

$$\left\langle \bar{J}_{1/2A}(z,\bar{z})\prod_{n=1}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n^c}(z_n,\bar{z}_n)\right\rangle = \sum_{i=1}^{N}\bar{f}(A,\ell_i^c,*_i,*'_i)(-1)^{\sigma_i}\frac{(z-z_i)}{(\bar{z}-\bar{z}_i)}\frac{(\bar{\xi}-\bar{z}_i)}{(\bar{\xi}-\bar{z})}\langle\cdots\mathcal{O}^{*'_i}_{\Delta_i+\frac{1}{2},\ell_i}(z_i,\bar{z}_i)\cdots\rangle, \tag{B18}$$

where

$$J^A_{1/2}(z,\bar{z}) = \lim_{\Delta\to\frac{1}{2}}\left(\Delta-\frac{1}{2}\right)\mathcal{O}^A_{\Delta,+\frac{3}{2}}(z,\bar{z}), \qquad \bar{J}_{1/2A}(z,\bar{z}) = \lim_{\Delta\to\frac{1}{2}}\left(\Delta-\frac{1}{2}\right)\mathcal{O}_{\Delta,-\frac{3}{2},A}(z,\bar{z}) \tag{B19}$$

are the soft gravitino operators.

APPENDIX C: OPE OF THE COMPOSITE CURRENT $\mathcal{G}_{AB}^{CD}(z,\bar{z})$

We begin by calculating the OPEs $G\bar{G}$. We have[10]

$$\left\langle G_{AB}(z,\bar{z})\bar{G}^{CD}(w,\bar{w})\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \lim_{\substack{\Delta_1\to 0\\ \Delta_2\to 0}}\frac{\Delta_1\Delta_2}{\pi^2}\int d^2z_1\,\frac{1}{(z-z_1)^a}\frac{1}{(\bar{z}-\bar{z}_1)^b}\frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}} \times\left\langle \mathcal{O}_{AB\Delta_1,-1}(z_1,\bar{z}_1)\mathcal{O}^{CD}_{\Delta_2,+1}(z_2,\bar{z}_2)\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle. \tag{C1}$$

By taking the soft limit of the first operator $\Delta_1\to 0$,

$$\left\langle G_{AB}(z)\bar{G}^{CD}(\bar{w})\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \lim_{\Delta_2\to 0}\frac{\Delta_2}{\pi^2}\int d^2z_1\int d^2z_2\frac{1}{(z-z_1)^a}\frac{1}{(\bar{z}-\bar{z}_1)^b}\frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}} \times\left[-\delta_{AB}^{CD}\frac{z_1-z_2}{\bar{z}_1-\bar{z}_2}\left\langle \mathcal{O}_{\Delta_2,+2}(z_2,\bar{z}_2)\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle +\sum_{j=3}^{N}f(A,B,\ell_j,*_j,*'_j)\frac{z_1-z_j}{\bar{z}_1-\bar{z}_j}\left\langle \mathcal{O}^{CD}_{\Delta_2,+1}(z_2,\bar{z}_2)\cdots\mathcal{O}^{*'_j}_{\Delta_j,\ell_j+1}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle\right]. \tag{C2}$$

Now doing the first integral using (3.35), we get

$$\left\langle G_{AB}(z)\bar{G}^{CD}(\bar{w})\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \lim_{\Delta_2\to 0}\frac{\Delta_2}{\pi}\int d^2z_2\,\frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}} \times\left[-\delta_{AB}^{CD}C_1(b,a)\frac{1}{(z_2-z)^{a-2}(\bar{z}_2-\bar{z})^b}\left\langle \mathcal{O}_{\Delta_2,+2}(z_2,\bar{z}_2)\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle +\sum_{j=3}^{N}f(A,B,\ell_j,*_j,*'_j)C_1(b,a)\frac{1}{(z_j-z)^{a-2}}\frac{1}{(\bar{z}_j-\bar{z})^b} \times\left\langle \mathcal{O}^{CD}_{\Delta_2,+1}(z_2,\bar{z}_2)\cdots\mathcal{O}^{*'_j}_{\Delta_j,\ell_j+1}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle\right]. \tag{C3}$$

We now use the collinear limits of the graviton operator with other fields in the first term and take the conformally soft limit $\Delta_2\to 0$ in the second term. The first term becomes

$$\lim_{\Delta_2\to 0}\frac{\Delta_2}{\pi}\int d^2z_2\,\frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}}\left[-\delta_{AB}^{CD}C_1(b,a)\frac{1}{(z_2-z)^{a-2}(\bar{z}_2-\bar{z})^b}\left\langle \mathcal{O}_{\Delta_2,+2}(z_2,\bar{z}_2)\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\right]$$
$$= -\delta_{AB}^{CD}C_1(b,a)\lim_{\Delta_2\to 0}\frac{\Delta_2}{\pi}\int d^2z_2\,\frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}}\frac{1}{(z_2-z)^{a-2}(\bar{z}_2-\bar{z})^b} \times\sum_{i=3}^{N}B(\Delta_2-1,f(\Delta_i))\frac{\bar{z}_2-\bar{z}_i}{z_2-z_i}\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle$$
$$= -\delta_{AB}^{CD}C_1(b,a)\frac{1}{\pi}\int d^2z_2\,\frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}}\frac{1}{(z_2-z)^{a-2}(\bar{z}_2-\bar{z})^b} \times\sum_{i=3}^{N}(1-f(\Delta_i))\frac{\bar{z}_2-\bar{z}_i}{z_2-z_i}\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle, \tag{C4}$$

[10]Since there is also overlap as there is in the case of soft gravitino currents in Sec. III A, we do not separate the operators in the correlator according to their spins ℓ,ℓ' and keep the spins to be arbitrary here as well.

where we used

$$\lim_{\Delta_2\to 0}\Delta_2 B(\Delta_2-1, f(\Delta_i)) = 1 - f(\Delta_i).$$

In the second term in (C3) now we can take the $\Delta_2\to 0$ limit,

$$\begin{aligned}
&\lim_{\Delta_2\to 0}\frac{\Delta_2}{\pi}\sum_{j=3}^{N} f(A,B,\ell_j,*_j,*'_j)\frac{1}{(z_j-z)^{a-2}}\frac{1}{(\bar z_j-\bar z)^{b}}\int d^2z_2\frac{1}{(\bar w-\bar z_2)^{a'}}\frac{1}{(w-z_2)^{b'}}\\
&\quad\times\langle\mathcal{O}^{CD}_{\Delta_2,+1}(z_2,\bar z_2)\cdots\mathcal{O}^{*'_j}_{\Delta_j,\ell_j+1}(z_j,\bar z_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar z_N)\rangle\\
&=\frac{1}{\pi}\sum_{j=3}^{N} f(A,B,\ell_j,*_j,*'_j)\bar f(C,D,\ell_j+1,*'_j,*''_j)C_1(b,a)\frac{1}{(z_j-z)^{a-2}}\frac{1}{(\bar z_j-\bar z)^{b}}\\
&\quad\times\int d^2z_2\frac{1}{(\bar w-\bar z_2)^{a'}}\frac{1}{(w-z_2)^{b'}}\frac{\bar z_2-\bar z_j}{z_2-z_j}\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar z_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar z_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar z_N)\rangle\\
&\quad+\frac{1}{\pi}\sum_{\substack{i,j=3\\ i\neq j}}^{N} f(A,B,\ell_j,*_j,*'_j)\bar f(C,D,\ell_i,*_i,*'_i)C_1(b,a)\frac{1}{(z_j-z)^{a-2}}\frac{1}{(\bar z_j-\bar z)^{b}}\\
&\quad\times\int d^2z_2\frac{1}{(\bar w-\bar z_2)^{a'}}\frac{1}{(w-z_2)^{b'}}\frac{\bar z_2-\bar z_i}{z_2-z_i}\langle\cdots\mathcal{O}^{*'_j}_{\Delta_j,\ell_j+1}(z_j,\bar z_j)\cdots\mathcal{O}^{*'_i}_{\Delta_i,\ell_i-1}(z_i,\bar z_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar z_N)\rangle\\
&=\sum_{j=3}^{N} f(A,B,\ell_j,*_j,*'_j)\bar f(C,D,\ell_j+1,*'_j,*''_j)C_1(b',a')\frac{1}{(z_j-z)^{a-2}}\frac{1}{(\bar z_j-\bar z)^{b}}\\
&\quad\times\frac{1}{(z_j-w)^{b'}}\frac{1}{(\bar z_j-\bar w)^{a'-2}}\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar z_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar z_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar z_N)\rangle\\
&\quad+\sum_{\substack{i,j=3\\ i\neq j}}^{N} f(A,B,\ell_j,*_j,*'_j)\bar f(C,D,\ell_i,*_i,*'_i)C_1(b,a)C_1(b',a')\frac{1}{(z_j-z)^{a-2}}\frac{1}{(\bar z_j-\bar z)^{b}}\\
&\quad\times\frac{1}{(z_i-w)^{b'}}\frac{1}{(\bar z_i-\bar w)^{a'-2}}\langle\cdots\mathcal{O}^{*'_j}_{\Delta_j,\ell_j+1}(z_j,\bar z_j)\cdots\mathcal{O}^{*'_i}_{\Delta_i,\ell_i-1}(z_i,\bar z_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar z_N)\rangle.
\end{aligned}\tag{C5}$$

Combining the two integrals we get

$$\begin{aligned}
&\left\langle G_{AB}(z,\bar z)\bar G^{CD}(w,\bar w)\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar z_n)\right\rangle=-\delta^{CD}_{AB}C_1(b,a)\frac{1}{\pi}\int d^2z_2\frac{1}{(\bar w-\bar z_2)^{a'}}\frac{1}{(w-z_2)^{b'}}\frac{1}{(z_2-z)^{a-2}(\bar z_2-\bar z)^{b}}\\
&\quad\times\sum_{i=3}^{N}(1-f(\Delta_i))\frac{\bar z_2-\bar z_i}{z_2-z_i}\times\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar z_3)\cdots\mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar z_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar z_N)\rangle\\
&\quad+\sum_{j=3}^{N} f(A,B,\ell_j,*_j,*'_j)\bar f(C,D,\ell_j+1,*'_j,*''_j)C_1(b',a')\frac{1}{(z_j-z)^{a-2}}\frac{1}{(\bar z_j-\bar z)^{b}}\\
&\quad\times\frac{1}{(z_j-w)^{b'}}\frac{1}{(\bar z_j-\bar w)^{a'-2}}\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar z_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar z_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar z_N)\rangle\\
&\quad+\sum_{\substack{i,j=3\\ i\neq j}}^{N} f(A,B,\ell_j,*_j,*'_j)\bar f(C,D,\ell_i,*_i,*'_i)C_1(b,a)C_1(b',a')\frac{1}{(z_j-z)^{a-2}}\frac{1}{(\bar z_j-\bar z)^{b}}\\
&\quad\times\frac{1}{(z_i-w)^{b'}}\frac{1}{(\bar z_i-\bar w)^{a'-2}}\langle\cdots\mathcal{O}^{*'_j}_{\Delta_j,\ell_j+1}(z_j,\bar z_j)\cdots\mathcal{O}^{*'_i}_{\Delta_i,\ell_i-1}(z_i,\bar z_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar z_N)\rangle.
\end{aligned}\tag{C6}$$

Here when we take the normal order of this composite current, we only need to care about the nonsingular terms in the above OPE. The nonsingular term in the integral above can be obtained by taking $z \to w$ limit in the integral. The integral can then be evaluated as

$$\int d^2 z_2 \frac{1}{(\bar{w}-\bar{z}_2)^{a'}} \frac{1}{(w-z_2)^{b'}} \frac{1}{(z_2-z)^{a-2}} \frac{1}{(\bar{z}_2-\bar{z})^{b}} \frac{\bar{z}_2-\bar{z}_i}{z_2-z_i} \xrightarrow{z=w} = (-1)^{a+b-2} \int d^2 z_2 \frac{1}{(\bar{z}-\bar{z}_2)^{a'+b}} \frac{1}{(z-z_2)^{b'+a-2}} \frac{\bar{z}_2-\bar{z}_i}{z_2-z_i}$$
$$= (-1)^{a+b} C_1(a+b'-2, a'+b) \frac{1}{(z_i-z)^{a'+b-2}} \frac{1}{(\bar{z}_i-\bar{z})^{a+b'-2}}.$$

Hence,

$$\begin{aligned}
&\left\langle :G_{AB}(z,\bar{z})\bar{G}^{CD}(z,\bar{z}): \prod_{n=3}^{N} \mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle \\
&= \frac{(-1)^{a+b+1}}{\pi} \delta^{CD}_{AB} C_1(b,a) C(a+b'-2, a'+b) \sum_{i=3}^{N} (1-f(\Delta_i)) \\
&\quad \times \frac{1}{(z_i-z)^{a'+b-2}} \frac{1}{(\bar{z}_i-\bar{z})^{a+b'-2}} \langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3) \cdots \mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar{z}_i) \cdots \mathcal{O}^{*_N}_{\Delta_N,\ell'_N}(z_N,\bar{z}_N) \rangle \\
&\quad + \sum_{j=3}^{N} f(A,B,\ell_j,*_j,*'_j) \bar{f}(C,D,\ell_j+1,*'_j,*''_j) C_1(b,a) C_1(b',a') (-1)^{a+b+a'+b'} \\
&\quad \times \frac{1}{(z-z_j)^{a+b'-2}} \frac{1}{(\bar{z}-\bar{z}_j)^{a'+b-2}} \langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3) \cdots \mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j) \cdots \mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N) \rangle \\
&\quad + \sum_{\substack{i,j=3 \\ i\neq j}}^{N} f(A,B,\ell_j,*_j,*'_j) \bar{f}(C,D,\ell_i,*_i,*'_i) C_1(b,a) C_1(b',a') (-1)^{a+b+a'+b'} \frac{1}{(z-z_j)^{a-2}} \frac{1}{(\bar{z}-\bar{z}_j)^{b}} \\
&\quad \times \frac{1}{(z-z_i)^{b'}} \frac{1}{(\bar{z}-\bar{z}_i)^{a'-2}} \langle \cdots \mathcal{O}^{*'_j}_{\Delta_j,\ell_j+1}(z_j,\bar{z}_j) \cdots \mathcal{O}^{*'_i}_{\Delta_i,\ell_i-1}(z_i,\bar{z}_i) \cdots \mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N) \rangle.
\end{aligned} \tag{C7}$$

Similarly, we have

$$\begin{aligned}
&\left\langle :\bar{G}^{CD}(z,\bar{z}) G_{AB}(z,\bar{z}): \prod_{n=3}^{N} \mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle \\
&= \frac{(-1)^{a'+b'+1}}{\pi} \delta^{CD}_{AB} C_1(b',a') C(a'+b-2, a+b') \sum_{i=3}^{N} (1-f(\Delta_i)) \\
&\quad \times \frac{1}{(z_i-z)^{a+b'-2}} \frac{1}{(\bar{z}_i-\bar{z})^{a'+b-2}} \langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3) \cdots \mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar{z}_i) \cdots \mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N) \rangle \\
&\quad + \sum_{j=3}^{N} \bar{f}(C,D,\ell_j,*_j,*'_j) f(A,B,\ell_j-1,*'_j,*''_j) C_1(b',a') C_1(b,a) (-1)^{a+b+a'+b'} \\
&\quad \times \frac{1}{(z-z_j)^{a'+b-2}} \frac{1}{(\bar{z}-\bar{z}_j)^{a+b'-2}} \langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3) \cdots \mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j) \cdots \mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N) \rangle \\
&\quad + \sum_{\substack{i,j=3 \\ i\neq j}}^{N} \bar{f}(C,D,\ell_j,*_j,*'_j) f(A,B,\ell_i,*_i,*'_i) C_1(b,a) C_1(b',a') (-1)^{a+b+a'+b'} \frac{1}{(\bar{z}-\bar{z}_j)^{a'-2}} \frac{1}{(z-z_j)^{b'}} \\
&\quad \times \frac{1}{(\bar{z}-\bar{z}_i)^{b}} \frac{1}{(z-z_i)^{a-2}} \langle \cdots \mathcal{O}^{*'_j}_{\Delta_j,\ell_j-1}(z_j,\bar{z}_j) \cdots \mathcal{O}^{*'_i}_{\Delta_i,\ell_i+1}(z_i,\bar{z}_i) \cdots \mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N) \rangle.
\end{aligned} \tag{C8}$$

We have the correlator of the normalized current with any conformal primary as

$$\begin{aligned}\left\langle \mathcal{G}_{AB}^{CD}(z,\bar{z})\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle =& -\delta_{AB}^{CD}\sum_{i=3}^{N}(1-f(\Delta_i))\left[\frac{(-1)^{a+b}}{\pi}C_1(b,a)C(a+b'-2,a'+b)\frac{1}{(z_i-z)^{a'+b-2}}\frac{1}{(\bar{z}_i-\bar{z})^{a+b'-2}}\right.\\ &\times\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle\\ &-\frac{(-1)^{a'+b'}}{\pi}C_1(b',a')C_1(a'+b-2,a+b')\frac{1}{(z_i-z)^{a+b'-2}}\frac{1}{(\bar{z}_i-\bar{z})^{a'+b-2}}\\ &\left.\times\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle\right]\\ &+(-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a')\left[f(A,B,\ell_j,*_j,*'_j)\bar{f}(C,D,\ell_j+1,*'_j,*''_j)\right.\\ &\times\frac{1}{(z-z_j)^{a+b'-2}}\frac{1}{(\bar{z}-\bar{z}_j)^{a'+b-2}}\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle\\ &-\bar{f}(C,D,\ell_j,*_j,*'_j)f(A,B,\ell_j-1,*'_j,*''_j)\\ &\left.\times\frac{1}{(z-z_j)^{a'+b-2}}\frac{1}{(\bar{z}-\bar{z}_j)^{a+b'-2}}\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle\right].\end{aligned}\tag{C9}$$

The last term in both (C7) and (C8) cancels when we take the commutator. Now in the above OPE, we can see that the first term which has the graviton soft limits does not satisfy our requirement explained in (3.32). Hence, we require that the two terms in the first expression be the same so that they cancel once we take the commutator. This is equivalent to the requirement

$$C_1(b',a')C_1(a'+b-2,a+b')(-1)^{a+b}=C_1(b,a)C_1(a+b'-2,a'+b)(-1)^{a'+b'}\tag{C10}$$

and

$$a'+b-2=a+b'-2.\tag{C11}$$

Now in (C10) by substituting the explicit expression from (3.36) we have

$$(-a'-b+1)(-a'-b+2)(-a+1)(-a+2)=(-a'+1)(-a'+2)(-a-b'+1)(-a-b'+2).$$

which after using (C11) gives

$$(-a+1)(-a+2)=(-a'+1)(-a'+2),\tag{C12}$$

which clearly has solutions. Hence, the correlator corresponding to this normal order current is

$$\begin{aligned}\left\langle \mathcal{G}_{AB}^{CD}(z,\bar{z})\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle =& (-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a')\left[f(A,B,\ell_j,*_j,*'_j)\bar{f}(C,D,\ell_j+1,*'_j,*''_j)\right.\\ &\times\frac{1}{(z-z_j)^{a+b'-2}}\frac{1}{(\bar{z}-\bar{z}_j)^{a'+b-2}}\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle\\ &-\bar{f}(C,D,\ell_j,*_j,*'_j)f(A,B,\ell_j-1,*'_j,*''_j)\\ &\left.\times\frac{1}{(z-z_j)^{a'+b-2}}\frac{1}{(\bar{z}-\bar{z}_j)^{a+b'-2}}\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle\right].\end{aligned}\tag{C13}$$

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