Superconformal interacting particles

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The free massless superparticle is reanalyzed, in particular by performing the Gupta-Bleuler quantization, using the first- and second-class constraints of the model, and obtaining as a result, the Weyl equation for the spinorial component of the chiral superfield. Then we construct a superconformal model of two interacting massless superparticles from the free case by the introduction of an invariant interaction. The interaction introduces an effective mass for each particle by modifying the structure of fermionic constraints, all becoming second class. The quantization of the model produces a bilocal chiral superfield. We also generalize the model by considering a system of superconformal interacting particles and its continuum limit.

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I. INTRODUCTION

The application of conformal invariance to classical interacting relativistic particles has recently been studied [1,2]. The motivation was to generalize the nonrelativistic one-dimensional case, as for example the Calogero-Moser rational model [3–5], which describes N interacting particles via two-body interactions. This model is very important in the context of integrable models. The other example, always in one dimension, is the conformal quantum mechanics [6]. Since there are also supersymmetric extensions of these models [7–9], we generalize the model contained in [1] to a superconformal one.

In this paper we have reanalyzed the free massless superparticle and its superconformal symmetries [10–13]. The superinversion is an important tool to study the superconformal special transformations and to build the invariants [14,15].

As is well-known, the massless Lagrangian implies a mixture of first-class and second-class fermionic constraints [13,16]. By using the light cone variables it is possible to disentangle the first- and second-class constraints in a noncovariant way and then perform the Gupta-Bleuler quantization of the system; as a result we obtain the Weyl equation for the spinorial component of the chiral superfield.

Then we construct a superconformal model of two interacting massless superparticles from the free case by using the einbein formulation for the action. The construction of the interaction term heavily uses the properties of the variables under superinversion. The interaction term is invariant under the diagonal superconformal group.

The interaction introduces an effective mass modifying the structure of fermionic constraints, all fermionic constraints are second class. The quantization of the model produces a bilocal chiral superfield.

We also generalize the model by considering a system of superconformal particles with nearest neighbor interaction and by studying its continuum limit.

The organization of the paper is as follows: In Sec. II we first review the classical and quantum theory of the superconformal particle, in Sec. III we propose a superconformal model for two interacting particles, in Sec. IV we generalize it to a system of particles on a one-dimensional lattice and we study its limit when the lattice spacing is sent to zero. In Sec. V we give an outlook.

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II. A SUPERCONFORMAL RELATIVISTIC PARTICLE

In this section we study the Lagrangian and the Hamiltonian formulation of a single superconformal relativistic particle [10–13] by analyzing the superconformal symmetries, the structure of the constraints and the Gupta-Bleuler quantization of the model. In particular we will show the appearance of the Weyl equation for the spinorial component of the chiral superfield.

The superconformal invariant action for a massless relativistic particle is given by

$$S = \int d\tau L = \int d\tau \frac{1}{2e}\dot{\omega}^\mu \dot{\omega}_\mu, \tag{1}$$

where

$$\dot{\omega}^\mu = \frac{d\omega^\mu}{d\tau} = \dot{x}^\mu + i\theta\sigma^\mu \dot{\bar{\theta}} - i\dot{\theta}\sigma^\mu\bar{\theta}, \tag{2}$$

and e is a Lagrange multiplier. We suppose to be in a $D = 4$ space-time with a flat metric $g^{\mu\nu} = (-,+,+,+)$ and we follow the spinor notations of the book of Wess and Bagger [17]. In particular $(\sigma^\mu)_{\alpha\dot{\alpha}} = (-1, \sigma^i), (\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}} = (-1, -\sigma^i), i = 1, 2, 3.$

The Lagrangian L is invariant under the following supersymmetry (SUSY) transformations:

$$\delta\theta = \epsilon, \qquad \delta\bar{\theta} = \bar{\epsilon},$$
$$\delta x^\mu = i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta}, \qquad \delta e = 0, \tag{3}$$

where ϵ and $\bar{\epsilon}$ are the SUSY parameters.

As in the case of conformal invariance, where invariance under Poincaré, dilatations and inversion is sufficient to ensure invariance under all the conformal group, also in the case of the superconformal invariance, Poincaré, dilatations, chiral

$$x \to x, \quad \theta \to e^{-i\Delta/2}\theta, \quad \bar{\theta} \to e^{i\Delta/2}\bar{\theta}, \tag{4}$$

SUSY transformations and superinversion are enough to guarantee the invariance under all the superconformal group [14,15]. Therefore, in our case, we need only to show that L is invariant under superinversion. The superinversion acts upon $\dot{\omega}^\mu$ as follows:

$$\dot{\omega}^\mu \to A(x)^\mu{}_\nu \dot{\omega}^\nu, \tag{5}$$

where [15]

$$A(x)^\mu{}_\nu = \frac{1}{x^2 + \theta^2\bar{\theta}^2}\left(\frac{x^2 - \theta^2\bar{\theta}^2}{x^2 + \theta^2\bar{\theta}^2} g^\mu_\nu - 2\frac{x^\mu x_\nu}{x^2 + \theta^2\bar{\theta}^2} + 2\epsilon^\mu{}_{\nu\lambda\rho}\frac{\theta\sigma^\lambda\bar{\theta}x^\rho}{x^2}\right). \tag{6}$$

The matrix $A(x)$ defining the superinversion satisfies the relation

$$A(x)^T g A(x) = \Omega^2(x)\eta. \tag{7}$$

By using this property we find

$$\dot{\omega}^2 \to \Omega^2(x)\dot{\omega}^2, \tag{8}$$

where

$$\Omega(x) = \frac{1}{x^2 + \theta^2\bar{\theta}^2}, \tag{9}$$

(for details see [15]). The Lagrangian L, in Eq. (1), is superconformal invariant, assuming the following transformation of the einbein under superinversion

$$e \to \Omega^2(x)e. \tag{10}$$

By evaluating the momenta from the Lagrangian we get

$$p^\mu = \frac{\partial L}{\partial \dot{x}_\mu} = \frac{1}{e}\dot{\omega}^\mu, \qquad \Pi_e = \frac{\partial L}{\partial \dot{e}} = 0, \tag{11}$$

$$\Pi_\alpha = \frac{\partial L}{\partial \dot{\theta}^\alpha} = -ip^\mu\sigma_{\mu\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}, \quad \bar{\Pi}_{\dot{\alpha}} = \frac{\partial L}{\partial \dot{\bar{\theta}}^{\dot{\alpha}}} = -i\theta^\alpha p^\mu \sigma_{\mu\alpha\dot{\alpha}}. \tag{12}$$

We therefore obtain the constraints

$$\Pi_e = 0, \tag{13}$$

and

$$D_\alpha = \Pi_\alpha + ip^\mu(\sigma_\mu\bar{\theta})_\alpha = 0, \quad \bar{D}_{\dot{\alpha}} = \bar{\Pi}_{\dot{\alpha}} + ip^\mu(\theta\sigma_\mu)_{\dot{\alpha}} = 0. \tag{14}$$

The canonical Hamiltonian is

$$H_c = \Pi_e \dot{e} + \frac{1}{2}ep^2. \tag{15}$$

The canonical Poisson brackets for boson and fermion variables are given by

$$\{x^\mu, p^\nu\} = g^{\mu\nu}, \quad \{\Pi_\alpha, \theta^\beta\} = -\delta^\beta_\alpha, \quad \{\Pi_{\dot{\alpha}}, \theta^{\dot{\beta}}\} = -\delta^{\dot{\beta}}_{\dot{\alpha}}, \tag{16}$$

The stability of the primary constraints gives the secondary constraint

$$p^2 = 0, \tag{17}$$

which is the mass-shell condition for a massless particle.

A. Analysis of the constraints and quantization

Let us now analyze the structure of the fermionic constraints, in particular their first and second class character. The Poisson brackets of D_α and $\bar{D}_{\dot{\alpha}}$ are given by

$$\{D_\alpha, D_\beta\} = 0 = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\}, \quad \{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2ip^\mu\sigma_{\mu\alpha\dot{\alpha}}, \tag{18}$$

with

$$p^\mu \sigma_{\mu\alpha\dot\alpha} = \begin{pmatrix} \sqrt{2}p^+ & p^{12} \\ (p^{12})^* & \sqrt{2}p^- \end{pmatrix}, \tag{19}$$

and where we have defined

$$p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^3), \quad p^{12} = p^1 - ip^2, \tag{20}$$

and $*$ indicates complex conjugation. The matrix (19) has zero determinant and its rank is one on the surface of the constraint $p^2 = 0$. Therefore, we have two first-class and two second-class constraints. The first-class constraints are

$$\hat D_1 = -D_1 + \frac{p^{12}}{\sqrt{2}p^-} D_2 = -\Pi_1 + \frac{p^{12}}{\sqrt{2}p^-}\Pi_2,$$
$$\hat{\bar D}_{\dot 1} = -\bar D_{\dot 1} + \frac{(p^{12})^*}{\sqrt{2}p^-}\bar D_{\dot 2} = -\bar\Pi_{\dot 1} + \frac{(p^{12})^*}{\sqrt{2}p^-}\bar\Pi_{\dot 2}, \tag{21}$$

and we have

$$\{\hat D_1, \hat D_1\} = 0 = \{\hat{\bar D}_{\dot 1}, \hat{\bar D}_{\dot 1}\}, \tag{22}$$

$$\{\hat D_1, \hat{\bar D}_{\dot 1}\} = -2\sqrt{2}ip^+ - i\sqrt{2}\frac{|p^{12}|^2}{p^-} + i\sqrt{2}\frac{|p^{12}|^2}{p^-} + i\sqrt{2}\frac{|p^{12}|^2}{p^-} = 0, \tag{23}$$

where use has been made of the bosonic first-class constraint $2p^+p^- = |p^{12}|^2$. The second-class constraints are

$$D_2 = \Pi_2 + i[(p^{12})^*\bar\theta_{\dot 1} + \sqrt{2}p^-\bar\theta_{\dot 2}],$$
$$\bar D_{\dot 2} = \bar\Pi_{\dot 2} + i[p^{12}\theta_1 + \sqrt{2}p^-\theta_2], \tag{24}$$

since

$$\{D_2, \bar D_{\dot 2}\} = -2i\sqrt{2}p^-. \tag{25}$$

The extended Hamiltonian includes the first class constraints Π_e, p^2, $\hat D_1$ and $\hat{\bar D}_{\dot 1}$

$$H_E = \Pi_e \dot e + \frac{1}{2}ep^2 + \mu_1 \hat D_1 + \bar\mu_{\dot 1}\hat{\bar D}_{\dot 1}, \tag{26}$$

where μ_1 and $\bar\mu_{\dot 1}$ are arbitrary Grassmann multipliers.

Notice that there are two Grassmann constraints of first class. This corresponds to the invariance of our action under an additional local symmetry, the kappa symmetry (see [18]). It is given by (k is an arbitrary time dependent two-component anticommuting spinor)

$$\delta x^\mu = \frac{i}{2}\bar\theta\tilde\sigma^\mu\sigma\cdot p\bar k - \frac{i}{2}k\sigma\cdot p\tilde\sigma^\mu\theta,$$
$$\delta\theta = \frac{1}{2}\bar k\,\tilde\sigma\cdot p, \qquad \delta\bar\theta = \frac{1}{2}\tilde\sigma\cdot pk,$$
$$\delta e = 2ik\dot\theta - 2i\dot{\bar\theta}\bar k, \qquad \delta p^\mu = 0, \tag{27}$$

where p^μ is given in Eq. (11).

The invariance of the model under the kappa symmetry shows that only half of the Grassmann variables are physical.

At this point one of the possibilities to develop the quantum mechanics of the model is the standard procedure that consists in computing the Dirac brackets and quantizing with them. However, commutators of canonical operators are in general modified by the presence of second-class constraints by making cumbersome the quantization.

Instead of using Dirac brackets we can do the weak quantization by using standard commutation relations between canonical operators,

$$[x^\mu, p^\nu] = ig^{\mu\nu}, \quad [\Pi_\alpha, \theta^\beta] = -i\delta^\beta_\alpha, \quad [\Pi_{\dot\alpha}, \theta^{\dot\beta}] = -i\delta^{\dot\beta}_{\dot\alpha}, \tag{28}$$

and by imposing the first class constraints as the operatorial conditions

$$p^2|\Phi>=0, \tag{29}$$

$$\hat D_1|\Phi>=0, \quad \hat{\bar D}_{\dot 1}|\Phi>=0. \tag{30}$$

For the second class constraints we use the Gupta-Bleuler procedure in the following way:

$$\bar D_{\dot 2}|\Phi>=0, \quad <\Phi|D_2 = 0 \tag{31}$$

with $p^\mu = -i\partial/\partial x_\mu$, $\Pi_\alpha = -i\partial/\partial\theta^\alpha$ and $\bar\Pi_{\dot\alpha} = -i\partial/\partial\bar\theta^{\dot\alpha}$. By using Eqs. (21) and (31), we have

$$\hat{\bar D}_{\dot 1}|\Phi>=0 \to \bar D_{\dot 1}|\Phi>=0. \tag{32}$$

So $\langle\theta_\alpha, \bar\theta_{\dot\alpha}, x|\Phi\rangle$ is a chiral superfield $\Phi = \Phi(\theta, y)$, where $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$:

$$\Phi(\theta, y) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y). \tag{33}$$

We still have to impose the first condition of Eqs. (30). Let us first change the basis from $\theta, \bar\theta, x$ to $\theta, \bar\theta, y$:

$$\frac{\partial}{\partial\theta^\alpha} = \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar\theta)_\alpha\frac{\partial}{\partial y^\mu}, \quad \frac{\partial}{\partial\bar\theta^{\dot\alpha}} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i(\theta\sigma^\mu)_{\dot\alpha}\frac{\partial}{\partial y^\mu}, \tag{34}$$

$$\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial y^\mu}. \tag{35}$$

Therefore,

$$\frac{\partial}{\partial\theta^1} = \frac{\partial}{\partial\theta^1} - i(\sqrt{2}p^+\bar{\theta}_{\dot{1}} + p^{12}\bar{\theta}_{\dot{2}}),$$
$$\frac{\partial}{\partial\theta^2} = \frac{\partial}{\partial\theta^2} - i((p^{12})^*\bar{\theta}_{\dot{1}} + \sqrt{2}p^-\bar{\theta}_{\dot{2}}). \quad (36)$$

The operator $\hat{D}_1$ can be written as

$$\hat{D}_1 = -i\left(-\frac{\partial}{\partial\theta_1} + \frac{p^{12}}{\sqrt{2}p^-}\frac{\partial}{\partial\theta_2}\right). \quad (37)$$

We have therefore

$$0 = \hat{D}_1\Phi(\theta, y) = -i\left(-\frac{\partial}{\partial\theta_1} + \frac{p^{12}}{\sqrt{2}p^-}\frac{\partial}{\partial\theta_2}\right)\Phi(\theta, y), \quad (38)$$

which implies, for the superfield components, the equations of motion

$$-\psi_1(y) + \frac{p^{12}}{\sqrt{2}p^-}\psi_2(y) = 0, \quad (39)$$

$$F(y) = 0. \quad (40)$$

Equation (39) can be rewritten as the Weyl equation,

$$p^\mu\tilde{\sigma}_\mu\psi(y) = 0, \quad (41)$$

using

$$p^\mu\tilde{\sigma}_\mu = \begin{pmatrix} \sqrt{2}p^- & -p^{12} \\ -(p^{12})^* & \sqrt{2}p^+ \end{pmatrix}, \quad (42)$$

and in Eq. (41) $p^\mu = -i\partial/\partial x_\mu = -i\partial/\partial y_\mu$.

III. TWO MASSLESS INTERACTING SUPERPARTICLES

In order to construct the model, let us first consider the case of two free massless superparticles:

$$L_2 = \frac{1}{2e_1}(\dot{\omega}_1^\mu)^2 + \frac{1}{2e_2}(\dot{\omega}_2^\mu)^2 \quad (43)$$

with

$$\dot{\omega}_i^\mu = \dot{x}_i^\mu + i\theta_i\sigma^\mu\dot{\bar{\theta}}_i - i\dot{\theta}_i\sigma^\mu\bar{\theta}_i, \qquad i = 1, 2, \quad (44)$$

where x_i^μ, θ_i are the space-time coordinates and Grassmann variables of the two particles.

This Lagrangian is invariant under the two superconformal groups acting on the variables of each particle.

Let us write the SUSY transformations as

$$\delta\theta_i = \epsilon_i, \quad \delta\bar{\theta}_i = \bar{\epsilon}_i$$
$$\delta x_i^\mu = i\theta_i\sigma^\mu\bar{\epsilon}_i - i\epsilon_i\sigma^\mu\bar{\theta}_i, \quad \delta e_i = 0, \quad i = 1, 2. \quad (45)$$

In order to introduce the superconformal interactions, following the bosonic case [1], let us define a space-time relative variable:

$$x_{12}^\mu = x_1^\mu - x_2^\mu - i\theta_1\sigma^\mu\bar{\theta}_2 + i\theta_2\sigma^\mu\bar{\theta}_1, \quad (46)$$

and the relative spinors

$$\theta_{12} = \theta_1 - \theta_2, \quad \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2. \quad (47)$$

It is easily verified that x_{12}^μ, θ_{12} and $\bar{\theta}_{12}$ are invariant under the SUSY transformations (45) with $\epsilon_1 = \epsilon_2 = \epsilon$, i.e., the diagonal supersymmetry. The transformation properties of these variables under superinversion are complicated. Instead the quantity

$$d_{12}^2 = x_{12}^2 + \theta_{12}^2\bar{\theta}_{12}^2 \to \Omega(x_1)\Omega(x_2)d_{12}^2, \quad (48)$$

is invariant up to a superconformal factor, or, using (9)

$$x_{12}^2 + \theta_{12}^2\bar{\theta}_{12}^2 \to \frac{1}{(x_1^2 + \theta_1^2\bar{\theta}_1^2)}(x_{12}^2 + \theta_{12}^2\bar{\theta}_{12}^2)\frac{1}{(x_2^2 + \theta_2^2\bar{\theta}_2^2)}. \quad (49)$$

Equations (8) and (49) generalize the transformation properties of $\dot{x}^2$ and $(x_1 - x_2)^2$ of the nonsupersymmetric case [1],

$$\dot{x}^2 \to \frac{1}{x^4}\dot{x}^2, \quad (x_1 - x_2)^2 \to \frac{1}{x_1^2}(x_1 - x_2)^2\frac{1}{x_2^2}. \quad (50)$$

In other words, the conformal factor $1/x^2$ goes into the superconformal factor $\Omega(x)$ (9).

We are now in the position to write down a two superconformal particle interactions. A possible superconformal model for two interacting superparticles is given by the action,

$$S_2 = \int d\tau L_2 = \int d\tau\left(\frac{1}{2e_1}\dot{\omega}_1^2 + \frac{1}{2e_2}\dot{\omega}_2^2 + \frac{\alpha^2}{4}\frac{\sqrt{e_1e_2}}{d_{12}^2}\right), \quad (51)$$

where d_{12}^2 is given in Eq. (48). The transformation properties of the variables under dilatations are given by

$$x_i^\mu \to \lambda x_i^\mu, \quad \theta_i \to \lambda^{1/2}\theta_i, \quad e_i \to \lambda^2 e_i, \quad i = 1, 2. \quad (52)$$

The SUSY transformations are contained in (45). Instead under superinversions, Eq. (5), we have

$$\dot{\omega}_i^2 \to \frac{\dot{\omega}_i^2}{(x^2 + \theta_i^2\bar{\theta}_i^2)^2} \equiv \Omega^2(x_i)\dot{\omega}_i^2, \quad i = 1, 2, \quad (53)$$

and for the einbeins:

$$e_i \to \Omega^2(x_i)e_i, \quad i = 1, 2. \tag{54}$$

The action S_2 is superconformal invariant.

In order to obtain the action in terms of the superconfiguration variables $x^\mu{}_i, \theta_i$ we compute the equation of motion of the einbein variables e_i

$$\frac{\partial L_2}{\partial e_1} = -\frac{\dot{\omega}_1^2}{2e_1^2} + \frac{\alpha^2}{8}\sqrt{\frac{e_2}{e_1}}\frac{1}{d_{12}^2} = 0,$$
$$\frac{\partial L_2}{\partial e_2} = -\frac{\dot{\omega}_2^2}{2e_2^2} + \frac{\alpha^2}{8}\sqrt{\frac{e_1}{e_2}}\frac{1}{d_{12}^2} = 0. \tag{55}$$

Solving these equations in e_1 and e_2 (the choice of the minus signs is for later convenience)

$$\frac{1}{e_1} = -\frac{\alpha}{2\dot{\omega}_1^2}\left(\frac{\omega_1^2\omega_2^2}{d_{12}^4}\right)^{1/4}, \quad \frac{1}{e_2} = -\frac{\alpha}{2\dot{\omega}_2^2}\left(\frac{\omega_1^2\omega_2^2}{d_{12}^4}\right)^{1/4}, \tag{56}$$

and substituting into Eq. (51) we obtain the action in the superconfiguration space

$$S_2 = -\alpha\int d\tau\left(\frac{\dot{\omega}_1^2\dot{\omega}_2^2}{d_{12}^4}\right)^{1/4}. \tag{57}$$

Notice that this action can be obtained from the bosonic configuration action of [1]

$$S_2 = -\alpha\int d\tau\left(\frac{\dot{x}_1^2\dot{x}_2^2}{(x_1 - x_2)^4}\right)^{1/4}, \tag{58}$$

by the supersymmetric substitution

$$\dot{x}_i^\mu \to \dot{\omega}_i^\mu, \quad (\dot{x}_1 - \dot{x}_2)^2 \to d_{12}^2. \tag{59}$$

A. Constraint analysis

In order to do the constraint analysis here we consider the superconfiguration Lagrangian (57). The conjugated momenta to $\dot{x}_i$ are given by

$$p_1^\mu = \frac{\partial L}{\partial \dot{x}_{1\mu}} = \frac{1}{2}\left(\frac{\dot{\omega}_2^2}{d_{12}^4}\right)^{1/4}\frac{\dot{\omega}_1^\mu}{(\dot{\omega}_1^2)^{3/4}},$$
$$p_2^\mu = \frac{\partial L}{\partial \dot{x}_{2\mu}} = \frac{1}{2}\left(\frac{\dot{\omega}_1^2}{d_{12}^4}\right)^{1/4}\frac{\dot{\omega}_2^\mu}{(\dot{\omega}_2^2)^{3/4}}, \tag{60}$$

from which we obtain the primary constraint

$$\phi = p_1^2 p_2^2 - \frac{\alpha^4}{16 d_{12}^4} = 0. \tag{61}$$

The fermionic momenta are given by

$$\Pi_i = \frac{\partial L_2}{\partial \dot{\theta}_i} = -i p_{i\mu}\sigma^\mu\bar{\theta}_i,$$
$$\bar{\Pi}_i = \frac{\partial L_2}{\partial \dot{\bar{\theta}}_i} = -i\theta_i p_{i\mu}\sigma^\mu, \quad i = 1, 2, \tag{62}$$

which imply four primary fermionic constraints

$$D_i = \Pi_i + i p_{i\mu}\sigma^\mu\bar{\theta}_i = 0,$$
$$\bar{D}_i = \bar{\Pi}_i + i\theta_i p_{i\mu}\sigma^\mu = 0, \quad i = 1, 2. \tag{63}$$

The Poisson brackets of the constraints (63) are

$$\{D_i, D_j\} = \{\bar{D}_i, \bar{D}_j\} = 0, \quad i, j = 1, 2, \tag{64}$$

and

$$\{D_i, \bar{D}_j\} = -2i\delta_{ij} p_i \cdot \sigma, \quad i, j = 1, 2. \tag{65}$$

Furthermore we have

$$det|\{D_i, \bar{D}_j\}| = 16 p_1^2 p_2^2 = \frac{\alpha^4}{x_{12}^2 + \theta_{12}^2\bar{\theta}_{12}^2}, \quad i, j = 1, 2. \tag{66}$$

The determinant of the matrix of the fermionic constraint Poisson brackets given in Eq. (66) is different from zero, unless one considers $r_{12} \to \infty$, and therefore the set of constraints $D_i, \bar{D}_j$ is second class.

Notice that the presence of the interaction term modifies the structure of the constraint algebra with respect to the case of the free superconformal particle, giving a sort of effective mass to the two superconformal particles; all fermionic constraints $D_i, \bar{D}_j$ becomes second class as for the massive superparticle [11].

The Dirac Hamiltonian is given by

$$H_D = \lambda\phi + \sum_{i=1,2}\mu_i D_i + \sum_{i=1,2}\bar{\mu}_i\bar{D}_i, \tag{67}$$

and the stability of the primary constraints gives

$$0 = \{\phi, H_D\} = \sum_{i=1,2}\mu_i\{\phi, D_i\} + \sum_{i=1,2}\bar{\mu}_i\{\phi, \bar{D}_i\}, \tag{68}$$

$$0 = \{D_i, H_D\} = \lambda\{D_i, \phi\} - \bar{\mu}_i\{D_i, \bar{D}_i\}, \quad i = 1, 2, \tag{69}$$

$$0 = \{\bar{D}_i, H_D\} = \lambda\{\bar{D}_i, \phi\} - \mu_i\{\bar{D}_i, D_i\}, \quad i = 1, 2. \tag{70}$$

By solving Eqs. (69) and (70) for μ_i and $\bar{\mu}_i$ and substituting in Eq. (68) we obtain the first-class Dirac Hamiltonian

$$H_D = \lambda\Big[\phi + \sum_{i=1,2}\{\bar{D}_i, \phi\}\{D_i, \bar{D}_i\}^{-1}D_i$$
$$+ \sum_{i=1,2}\{D_i, \phi\}\{D_i, \bar{D}_i\}^{-1}\bar{D}_i\Big]. \tag{71}$$

In conclusion we have a first-class constraint

$$\tilde{\phi} = \phi + \sum_{i=1,2} \{\bar{D}_i, \phi\}\{D_i, \bar{D}_i\}^{-1} D_i + \sum_{i=1,2} \{D_i, \phi\}\{D_i, \bar{D}_i\}^{-1} \bar{D}_i \tag{72}$$

and four second-class constraints

$$D_{i\alpha}, \quad \bar{D}_{i\dot{\alpha}}, \quad i, j = 1, 2. \tag{73}$$

Since in this case there is only one primary constraint that generates worldline diffeomorphism, there is no kappa symmetry.

B. Quantization

Quantization can be performed *à la* Gupta-Bleuler by requiring the following operatorial conditions on the "ket" vectors

$$\tilde{\phi}|\Phi >= 0, \quad \bar{D}_{i\dot{\alpha}}|\Phi >= 0 \tag{74}$$

and the following ones on the "bra":

$$< \Phi|D_{i\alpha} = 0. \tag{75}$$

Note that the solution to the second one of Eqs. (74) implies that the bilocal field $\Phi(x_i, \theta_i, \bar{\theta}_i) = \langle x_i, \theta_i, \bar{\theta}_i|\Phi\rangle$ is a bilocal chiral superfield

$$\Phi(x_i, \theta_i, \bar{\theta}_i) = \Phi(\theta_i, y_i), \tag{76}$$

where

$$y_i^\mu = x_i^\mu + i\theta_i\sigma^\mu\bar{\theta}_i, \quad i = 1, 2. \tag{77}$$

By using Eq. (74) we have

$$\tilde{\phi}|\Phi >= \left[\phi + \sum_{i=1,2} \{\bar{D}_i, \Phi\}\{D_i, \bar{D}_i\}^{-1} D_i\right]|\Phi >= 0. \tag{78}$$

Note that $\tilde{\phi}|\Phi >$ is also a chiral superfield. Indeed

$$[\bar{D}_{i\alpha}, \tilde{\phi}] = 0 \tag{79}$$

implies

$$\bar{D}_{i\alpha}\tilde{\phi}|\Phi >= 0. \tag{80}$$

Chiral bilocal superfield can be expanded as

$$\Phi(\theta_i, y_i) = \phi(y_i) + \theta_i^\alpha\psi_\alpha^i(y_i) + \theta_i^\alpha\theta_j^\beta\left[\epsilon^{ij}F_{\mu\nu}(y_i)\sigma_{\alpha\beta}^{\mu\nu} + \epsilon_{\alpha\beta}C^{ij}(y_i)\right] + \theta_2^2\theta_1^\alpha\chi_{1\alpha}(y_i) + \theta_1^2\theta_2^\alpha\chi_{2\alpha}(y_i) + \theta_1^2\theta_2^2F(y_i), \tag{81}$$

and contains five scalars; ϕ, C^{ij}, F, a 3-component antisymmetric tensor $F_{\mu\nu}$ and eight fermionic fields $\psi_\alpha^i, \chi_i^\alpha$.

Wave equations for the component fields can be evaluated by expanding Eq. (78) in series of Grassmann variables θ_i. For the scalar field ϕ one recovers the field equation of the purely bosonic case [1], while for the fermionic and the other bosonic fields additional terms are present. This analysis is beyond the aim of the present paper and deserves further studies.

IV. NEAREST-NEIGHBOR INTERACTIONS AND CONTINUUM LIMIT

In this section we generalize the model by considering a system of superconformal particles in which each particle interacts with its nearest neighbors. In other words we consider the $N+1$ particles as an ordered set labeled by an index i running from 1 to $N+1$ on a one-dimensional lattice with a lattice spacing denoted by a.

We assume the following action, containing only two-body interactions of the type that we have already proposed in Sec. III,

$$S = \int d\tau \left[\sum_{i=1}^{N+1} \frac{\dot{\omega}_i^2}{2e_i} + \frac{\alpha^2}{4}\sum_{i=1}^{N} \frac{\sqrt{e_ie_{i+1}}}{d_{i,i+1}^2}\right], \tag{82}$$

with

$$d_{i,i+1}^2 = x_{i,i+1}^2 + \theta_{i,i+1}^2\bar{\theta}_{i,i+1}^2 \tag{83}$$

and

$$x_{i,i+1}^\mu = x_i^\mu - x_{i+1}^\mu - i\theta_i\sigma^\mu\bar{\theta}_{i+1} + i\theta_{i+1}\sigma^\mu\bar{\theta}_i, \qquad \theta_{i,i+1} = \theta_i - \theta_{i+1}, \quad \bar{\theta}_{i,i+1} = \bar{\theta}_i - \bar{\theta}_{i+1}. \tag{84}$$

Instead of considering a linear lattice one could identify the two ends $x_1 = x_{N+1}$, and close the lattice to a circle. Let us notice that the physical dimensions of the various quantities appearing in this Lagrangian are $[x] = [\tau] = [e] = \ell, [\theta] = [\bar{\theta}] = \ell^{1/2}, [\alpha] = \ell^0$.

Here, we will not discuss this action but rather its continuum limit. To this end, let us define a variable σ to identify the lattice points,

$$\sigma_i = ia, \quad i = 1, \ldots, N+1. \tag{85}$$

In the continuum limit we have

$$\frac{1}{a}x_{i,i+1}^\mu \to -\left[\frac{\partial x^\mu}{\partial\sigma} + i\theta\sigma^\mu\frac{\partial\bar{\theta}}{\partial\sigma} - i\frac{\partial\theta}{\partial\sigma}\sigma^\mu\bar{\theta}\right] \equiv -\omega^{\mu\prime}, \qquad \frac{1}{a}\theta_{i,i+1} \to -\frac{\partial\theta}{\partial\sigma} \equiv -\theta' \tag{86}$$

and analogously for $\bar{\theta}_{i,i+1}$. Notice that ω' transforms under superconformal inversion exactly as $\dot{\omega}$, that is

$$\omega'^2 \to \frac{\omega'^2}{(x^2+\theta^2\bar{\theta}^2)^2} \equiv \Omega^2(x)\omega'^2. \tag{87}$$

Furthermore, the sum must be transformed as follows:

$$\sum_i \to \frac{1}{a}\int d\sigma. \tag{88}$$

The expression (82) becomes [assuming $a = \pi/(N+1)$ or σ to vary in the range $(0,\pi)$]

$$S \to -\int d\tau \int_0^\pi d\sigma\left[\frac{1}{a}\frac{\dot{\omega}^2(\sigma,\tau)}{2e(\sigma,\tau)} + \frac{1}{a^3}\frac{\alpha^2}{4}\frac{e(\sigma,\tau)}{\omega'^2(\sigma,\tau)}\right]. \tag{89}$$

In order to eliminate the divergence we redefine the einbein field $e(\sigma,\tau)$ and the coupling α as

$$\tilde{e} = ae, \qquad \frac{\tilde{\alpha}^2}{2} = \frac{1}{a^4}\frac{\alpha^2}{4}, \tag{90}$$

where the factor $1/2$ has been chosen for later convenience. Then, by denominating e and α as before, we obtain the action in the continuum limit:

$$S = \int d\tau \int_0^\pi d\sigma\left[\frac{1}{2}\frac{\dot{\omega}^2(\sigma,\tau)}{e(\sigma,\tau)} + \frac{\alpha^2}{2}\frac{e(\sigma,\tau)}{\omega'^2(\sigma,\tau)}\right]. \tag{91}$$

By varying the action with respect to $e(\sigma,\tau)$ we get

$$\frac{1}{2}\frac{\dot{\omega}^2}{e^2} = \frac{\alpha^2}{2}\frac{1}{\omega'^2} \tag{92}$$

or

$$e = \frac{1}{\alpha}\sqrt{\dot{\omega}^2\omega'^2}. \tag{93}$$

By substituting the expression of the einbein inside Eq. (91) we get

$$S = -\alpha\int d\tau \int_0^\pi d\sigma\sqrt{\frac{\dot{\omega}^2}{\omega'^2}}. \tag{94}$$

Notice that the action is trivially conformal invariant, since $\dot{\omega}^2$ and ω'^2 transform in the same way under inversion. It is also invariant under diffeomorphism in τ but not in σ. In this paper we do not perform the constraint analysis and their physical consequences.

V. OUTLOOK

For future investigations it would be interesting to analyze several aspects that we did not consider in this paper, starting, for example by a study of the equations of motion for the components of the bilocal chiral superfield and their solutions. It would also be interesting to compare the results of the predictions of the weak and the reduced space quantization. As already noted in the paper, another subject which deserves further work is the analysis of the constraints and their physical consequences in the continuum limit of the model. The study of the Killing equation would be interesting to find if, by any chance, the model contains some accidental symmetry. Finally, future investigations will be devoted to the Carroll and nonrelativistic limits of the model.

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