Nonperturbative method for particle detectors with continuous interactions

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We show that detector switching profiles consisting of trains of delta couplings are a useful computational tool to efficiently approximate results involving continuous switching functions, both in setups involving a single detector and multiple ones. The rapid convergence to the continuous results at all orders in perturbation theory for sufficiently regular switchings means that this tool can be used to obtain nonperturbative results for general particle detector phenomena with continuous switching functions.

I. INTRODUCTION

Performing local operations on quantum fields is a challenging endeavor, both from the technical and the foundational point of view [1–8]. Among the several approaches used to implement operations and local measurements of a quantum field, particle detectors—nonrelativistic quantum systems locally coupled to quantum fields—have succeeded as models of local probes in quantum field theory (QFT), and more specifically in relativistic quantum information (RQI). Particle detector models have allowed us to make physical sense and build intuition for phenomena ranging from the concept of particle [9–14] to the entanglement structure of QFTs [15–39]. They also have provided a basis for modeling quantum information protocols in relativistic setups (e.g., [40–55]), as well as for the formulation of a measurement theory consistent with relativity [56–58].

Typically, particle detectors are coupled to a quantum field with some strength (determined by a parameter λ), and for a specific period of time (determined by some switching function χ). Depending on the specific configuration, particle detectors can be used either as emitters and receptors of information, or as measurement devices, i.e., local probes that gather information about the field through their interaction. In both cases, in order to make predictions we need to be able to compute the final state of the detectors after their interaction with the field. However, the full theory including the interaction between the detector and the field cannot be exactly solved in general, as is the case for most interacting field theories.

A frequent avenue to circumvent the complications mentioned above is the use of perturbation theory. In this approach, the final state of the detectors is expanded in a series of terms proportional to increasingly higher powers of the coupling parameter λ. Under the assumption that the detectors are weakly coupled to the field, the higher order contributions can be neglected, leaving a truncated series as the final result. Despite its popularity, the perturbative approach has its limitations. To start with, a proof that the perturbative series is convergent in all regimes is still missing for particle detector models. Even when we are in a regime of coupling strengths for which the perturbative series converges, the truncated result is an approximation whose error is not known *a priori*, and can only be trusted to be accurate for "small enough" coupling parameters. Furthermore, there are many physical situations where one needs to go beyond leading order in perturbation theory, which can be technically challenging. Finally, nonperturbative effects in the model cannot in general be understood using perturbative methods.

These limitations call for the use of nonperturbative techniques. One way to proceed is to work in specific scenarios where the evolution of the detectors and the field can be calculated exactly. One of those scenarios is found in the context of continuous-variable quantum mechanics (see, e.g., [59,60] for reviews). Specifically, when the quantum field has both infrared and ultraviolet cutoffs, it can be reduced to (or approximated by) a lattice, so that if the detectors and the field are initially in a so-called *Gaussian state* (i.e., the Wigner function describing their joint state is Gaussian), and if their coupling is linear, then the evolution can be solved exactly using Gaussian methods [61,62]. Another scenario where calculations can be carried out nonperturbatively is when the coupling between the

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detectors and the field consists of sudden interactions, i.e., one or several delta couplings [43,46,47,63–66]. In this case, as we will review in Secs. II A and III A, the evolution can always be written as a sequence of known unitaries, and, moreover, these unitaries can often be exactly evaluated (see, e.g., [52,67–69]).

In this paper we devise an efficient nonperturbative method that applies to particle detectors that couple to the quantum field continuously in time. Given the physical relevance of these scenarios, and the fact that so far they have been mostly analyzed perturbatively, these techniques may allow us to explore possible new nonperturbative phenomena in the study of the dynamics induced by time-extended couplings, such as, e.g., in the Unruh effect or entanglement harvesting, among many others.

The paper is organized as follows: in Secs. II and III we develop, respectively, the nonperturbative method for setups involving a single and multiple particle detectors. For each case, we first review the formalism of delta coupling interactions, we then proceed to show how they can be used to approximate the dynamics of continuous couplings, and conclude stating the approximation result that guarantees the efficiency of the method, whose proof can be found in Appendices A and B. In Sec. IV, we demonstrate the method by applying it to specific scenarios involving one and two detectors. Finally, our conclusions are presented in Sec. V.

II. SINGLE DETECTOR CASE

In this section we describe the setup of a single particle detector interacting with a quantum field through a sequence of delta couplings, and we explain how it can be used to approximate the phenomenology of setups involving continuous detector-field couplings.

A. Setup

We consider a localized nonrelativistic quantum system (i.e., a particle detector [9,10,70,71]), moving in a general $(1 + d)$-dimensional globally hyperbolic space-time $(\mathcal{M}, \mathsf{g})$, whose center of mass follows a trajectory $\mathsf{x}(\tau)$ parametrized by its proper time τ. The detector is modeled as a quantum system with free Hamiltonian $\hat{H}_{\text{free}}$. We also consider that we are within the regime in which the detector can be assumed to be Fermi-Walker rigid to a good degree of approximation [72], and that, therefore, in the reference frame defined by a set of Fermi normal coordinates for the detector's center of mass $\mathsf{z} = (\tau, z)$, its shape is kept constant. This assumption is commonly made in particle detector models [71,73–76], and it is regarded as a physically realistic approximation in many experimental setups [77,78].

The detector is weakly coupled to a quantum field $\hat{\phi}$, and its coupling can be described in the interaction picture by

the Hamiltonian weight[1] [76]

$$\hat{h}_{\text{int}}(\tau, z) = \lambda \chi(\tau)[F(z)\hat{\mu}_\alpha^\dagger(\tau)\hat{\mathcal{O}}^\alpha(\tau, z) + \text{H.c.}]. \quad (1)$$

Here, λ is the coupling strength of the interaction, and χ and F are the switching and smearing functions that modulate the coupling in time and space, where in particular F depends on the shape of the detector (and, in general, can be complex [21,71,79–81]). $\hat{\mu}^\alpha$ and $\hat{\mathcal{O}}^\alpha$ are arbitrary tensor operators of the detector and the field, respectively, and α is a general multi-index (for example made of several spacetime indices). This is the most general interaction Hamiltonian weight that we can write for a Fermi-Walker rigid particle detector coupled to a field. In particular, for a monopolar coupling with a real scalar field one recovers the Unruh-DeWitt model, but this interaction also covers multipolar couplings with real/complex, scalar/vector/tensor, bosonic/fermionic fields.

In the case of multiple sudden interactions, the detector couples to the field via a train of delta couplings at a sequence of times $\{\tau_j, j = 1, \dots, N\}$, $N \geq 1$, and the switching function is thus given by

$$\chi(\tau) = \sum_{j=1}^N \eta_j \delta(\tau - \tau_j), \quad (2)$$

where η_j is the strength with which the detector couples to the field at time τ_j. With the interaction Hamiltonian weight given by Eq. (1), the evolution operator in the interaction picture is given by

$$\begin{aligned}
\hat{U} &= \mathcal{T}_\tau \exp\left(-\mathrm{i} \int d\mathsf{z} \, \hat{h}_{\text{int}}(\mathsf{z})\right) \\
&= \mathcal{T}_\tau \exp\left\{-\mathrm{i}\lambda \sum_{j=1}^N \eta_j \left[\hat{\mu}_{\alpha,j}^\dagger \int d z \sqrt{-g_j} F(z)\hat{\mathcal{O}}_j^\alpha(z) + \text{H.c.}\right]\right\} \\
&= \mathcal{T}_\tau \prod_{j=1}^N \exp\left\{-\mathrm{i}\lambda \eta_j \left[\hat{\mu}_{\alpha,j}^\dagger \int d z \sqrt{-g_j} F(z)\hat{\mathcal{O}}_j^\alpha(z) + \text{H.c.}\right]\right\},
\end{aligned} \quad (3)$$

where $g_j \equiv \det g_{\mu\nu}(\tau_j, z)$ is the determinant of the space-time metric at (τ_j, z), and similarly $\hat{\mu}_{\alpha,j}^\dagger \equiv \hat{\mu}_\alpha^\dagger(\tau_j)$, and $\hat{\mathcal{O}}_j^\alpha(z) \equiv \hat{\mathcal{O}}^\alpha(\tau_j, z)$. $\mathcal{T}_\tau \exp$ and $\mathcal{T}_\tau$ respectively denote the time-ordered exponential and the time ordering operation with respect to the detector's proper time τ, which is an acceptable choice of time-ordering parameter as long as we are in the regime of validity of the detector

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[1]Notice that the Hamiltonian weight is a scalar, which upon multiplication by the geometric factor $\sqrt{-g(\mathsf{z})}$ yields the Hamiltonian *density*, where $g(\mathsf{z})$ is the determinant of the metric at z.

approximation, where the detector model is effectively covariant [82]. In particular, writing

$$\hat{U}_j := \exp\left\{-i\lambda\eta_j\left[\hat{\mu}^\dagger_{\alpha,j}\int dz\sqrt{-g_j}F(z)\hat{O}^\alpha_j(z) + \text{H.c.}\right]\right\}, \quad (4)$$

we can write

$$\hat{U} = \hat{U}_N \cdots \hat{U}_1. \quad (5)$$

Notice that the $\hat{U}_j$ operators do not commute with each other, but the action of the time ordering operation in Eq. (3) yields Eq. (5). After the interaction, the joint state of the detector and the field is

$$\hat{\rho} = \hat{U}\hat{\rho}_0\hat{U}^\dagger, \quad (6)$$

where $\hat{\rho}_0$ is the initial detector-field state. In particular, the final state of the detector after the interaction results from tracing over the field degrees of freedom, i.e.,

$$\hat{\rho}_\text{D} = \text{Tr}_\phi(\hat{U}\hat{\rho}_0\hat{U}^\dagger) = \text{Tr}_\phi(\hat{U}_N \cdots \hat{U}_1\hat{\rho}_0\hat{U}^\dagger_1 \cdots \hat{U}^\dagger_N). \quad (7)$$

B. Approximation results

Let us consider a single detector that couples to the field through a switching function $\xi(\tau)$ that is bounded, and continuous except for maybe a finite number of points (in the following, we call this switching function *regular*). For this regular scenario, the joint evolution of the field and the detector is given by the unitary

$$\hat{U}_\xi = \mathcal{T}_\tau\exp\left(-i\int dz\hat{h}^\xi_\text{int}(z)\right)$$

$$= \mathcal{T}_\tau\exp\left\{-i\lambda\int d\tau\xi(\tau)\right.$$

$$\left.\times\left[\hat{\mu}^\dagger_\alpha(\tau)\int dz\sqrt{-g}F(z)\hat{O}^\alpha(\tau,z) + \text{H.c.}\right]\right\}, \quad (8)$$

where $g = \det g_{\mu\nu}(\tau,z)$ is the determinant of the metric. The final state of the detector is thus

$$\hat{\rho}^\xi_\text{D} = \text{Tr}_\phi(\hat{U}_\xi\hat{\rho}_0\hat{U}^\dagger_\xi). \quad (9)$$

It turns out that this scenario can be efficiently approximated using a train of delta interactions. Specifically, let $[0, T]$ be the interval where $\xi(\tau)$ has its support, where in particular we have $\xi(0) = \xi(T) = 0$. The idea is to approximate the phenomenology of the regular switching function with that of a sequence of uniformly spaced Dirac delta pulses. Thus, we can define a train of N sudden

interactions associated with ξ and a uniform partition $\mathcal{P} = \{t_j = Tj/N, j = 0, ..., N\}$ of the interval $[0, T]$ as

$$\chi_\xi(\tau; N) = \frac{T}{N}\sum_{j=1}^N\xi\left(\frac{j-1/2}{N}T\right)\delta\left(\tau - \frac{j-1/2}{N}T\right). \quad (10)$$

Recalling Eq. (2), this amounts to setting

$$\tau_j = \frac{j-1/2}{N}T, \quad (11)$$

and

$$\eta_j = \xi(\tau_j)(t_j - t_{j-1}) = \xi(\tau_j)\frac{T}{N}. \quad (12)$$

The strength of each sudden interaction is thus determined by both the value of the switching ξ at $t = \tau_j$ and the duration T/N of the interval it is effectively substituting. Let us denote with $\hat{\rho}_\text{D}(N)$ the final state of the detector according to Eq. (7) after the train of N sudden interactions $\chi_\xi(\tau; N)$ in Eq. (10).

Now, let us assume that the detector and the field are initially uncorrelated, so that the initial joint state is of the form

$$\hat{\rho}_0 = \hat{\rho}_{\text{D},0} \otimes \hat{\rho}_\phi. \quad (13)$$

We can use perturbation theory to write

$$\hat{\rho}_\text{D}(N) = \hat{\rho}_{\text{D},0} + \sum_{k=1}^\infty\hat{\rho}^{(k)}_\text{D}(N) \quad (14)$$

for the train of deltas, and

$$\hat{\rho}^\xi_\text{D} = \hat{\rho}_{\text{D},0} + \sum_{k=1}^\infty\hat{\rho}^{(k)\xi}_\text{D} \quad (15)$$

for the regular scenario, where $\hat{\rho}^{(k)}_\text{D}(N)$ and $\hat{\rho}^{(k)\xi}_\text{D}$ are proportional to λ^k. Then it can be shown that, for each $k \in \mathbb{Z}^+$, under the assumption that the switching function ξ, the smearing function F, and the initial state of the field $\hat{\rho}_\phi$ are *sufficiently regular*,[2]

$$\lim_{N\to\infty}\hat{\rho}^{(k)}_\text{D}(N) = \hat{\rho}^{(k)\xi}_\text{D}, \quad (16)$$

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[2]To expand on what is meant by sufficiently regular, please see Appendix A. In short, for most choices of switching and smearing functions, and for the initial states of the field usually employed in the literature (e.g., Hadamard states, both in flat and curved spacetimes), we expect the regularity condition to be satisfied.

in the weak operator topology. In fact, the convergence is *at least* as fast as $1/N$. Specifically, for any linear functional G,

$$G[\hat{\rho}_{\mathrm{D}}^{(k)\xi}] = G[\hat{\rho}_{\mathrm{D}}^{(k)}(N)] + \mathcal{O}(1/N). \tag{17}$$

Now, when ξ, F, and $\hat{\rho}_\phi$ are such that the conditions for Eq. (17) are fulfilled *for all $k \in \mathbb{Z}^+$*, then the train of deltas approximates the regular scenario *at all orders in perturbation theory*, since

$$G[\hat{\rho}_{\mathrm{D}}^{\xi}] = G[\hat{\rho}_{\mathrm{D}}(N)] + \mathcal{O}(1/N) \tag{18}$$

for any linear functional G, as long as λ is within the radius of convergence of the Dyson series in Eqs. (14) and (15). In particular,

$$\lim_{N \to \infty} \hat{\rho}_{\mathrm{D}}(N) = \hat{\rho}_{\mathrm{D}}^{\xi} \tag{19}$$

in the weak operator topology. Notice that since it is possible to perform nonperturbative calculations with delta couplings (see, e.g., [67–69]), Eq. (18) allows to efficiently approximate nonperturbative results, as opposed to the usual perturbative approximations based on truncating the Dyson expansion. While this is still an approximation (that can be made arbitrarily precise by increasing the density of delta couplings), it approximates the full sum of the perturbative expansion, and not only up to a given order. This is why we refer to this technique as a nonperturbative approximation.

III. MULTIPLE DETECTORS CASE

In this section we will generalize the framework and results of Sec. II to setups involving multiple particle detectors coupling to a quantum field.

A. Setup

We consider D particle detectors in the general globally hyperbolic spacetime $(\mathcal{M}, \mathsf{g})$, whose centers of mass follow trajectories $\mathsf{x}_n(\tau_n)$ parametrized by their proper times τ_n, within the regime in which each one of them can be fairly assumed to be Fermi-Walker rigid.

Each detector is weakly coupled to a quantum field $\hat{\phi}$. In order to jointly describe the interaction of all the detectors, we will assume that there exists a set of coordinates (τ, z) of $\mathcal{M}$ such that, for each detector, there is a neighborhood of its trajectory $\mathsf{x}_n(\tau_n)$ where the coordinates (τ, z) are a set of Fermi normal coordinates of the detector's center of mass. In particular, if these neighborhoods include the supports of each detector's smearing function, then we can write the interaction Hamiltonian weight as

$$\hat{h}_{\mathrm{int}}(\tau, z) = \sum_{n=1}^{D} \lambda_n \chi_n(\tau)[F_n(z)\hat{\mu}_{n,\alpha}^\dagger(\tau)\hat{\mathcal{O}}_n^\alpha(\tau, z) + \mathrm{H.c.}]. \tag{20}$$

In complete analogy with Eq. (1), here λ_n, χ_n, and F_n are the coupling strength, the switching, and the smearing functions of the nth detector, while $\hat{\mu}_n^\alpha$ is an arbitrary tensor operator of the nth detector, and $\hat{\mathcal{O}}_n^\alpha$ is the generic tensor field operator it couples to.

In the case of multiple delta couplings, the switching functions can be written as

$$\chi_n(\tau) = \sum_{j=1}^{N_n} \eta_{n,j}\delta(\tau - \tau_{n,j}), \tag{21}$$

where N_n is the number of times that the nth detector couples to the field, and $\eta_{n,j}$ is the strength of the coupling at time $\tau_{n,j}$. With the interaction Hamiltonian weight given by Eq. (20), the evolution operator in the interaction picture is given by

$$\begin{aligned}
\hat{U} &= \mathcal{T}_\tau \exp\left(-\mathrm{i}\int \mathrm{d}z\,\hat{h}_{\mathrm{int}}(\mathsf{z})\right) \\
&= \mathcal{T}_\tau \exp\left\{-\mathrm{i}\sum_{n=1}^{D}\lambda_n\sum_{j=1}^{N_n}\eta_{n,j}\right. \\
&\quad \left.\times\left[\hat{\mu}_{n,j,\alpha}^\dagger\int \mathrm{d}z\sqrt{-g_{n,j}}F(z)\hat{\mathcal{O}}_{n,j}^\alpha(z) + \mathrm{H.c.}\right]\right\}, \\
&= \mathcal{T}_\tau \prod_{n,j}\exp\left\{-\mathrm{i}\lambda_n\eta_{n,j}\left[\hat{\mu}_{n,j,\alpha}^\dagger\int \mathrm{d}z\sqrt{-g_{n,j}}F(z)\hat{\mathcal{O}}_{n,j}^\alpha(z)\right.\right. \\
&\quad \left.\left.+\,\mathrm{H.c.}\right]\right\},
\end{aligned} \tag{22}$$

where in the last equality, n is running from 1 to D, and j runs from 1 to N_n, for each n. Here, in analogy with the notation used for one detector in Sec. II A, $g_{n,j} \equiv \det g_{\mu\nu}(\tau_{n,j}, z)$ is the determinant of the metric at $(\tau_{n,j}, z)$, and similarly $\hat{\mu}_{n,j,\alpha}^\dagger \equiv \hat{\mu}_{n,\alpha}^\dagger(\tau_{n,j})$, and $\hat{\mathcal{O}}_{n,j}^\alpha(z) \equiv \hat{\mathcal{O}}^\alpha(\tau_{n,j}, z)$. Notice that Eq. (22) implies that we can write $\hat{U}$ as a product of unitaries in analogy with Eq. (5), where each unitary of the sequence is of the form

$$\begin{aligned}
\hat{U}_{n,j} = \exp\left\{-\mathrm{i}\lambda_n\eta_{n,j}\left[\hat{\mu}_{n,j,\alpha}^\dagger\int \mathrm{d}z\sqrt{-g_{n,j}}F(z)\hat{\mathcal{O}}_{n,j}^\alpha(z)\right.\right. \\
\left.\left.+\,\mathrm{H.c.}\right]\right\},
\end{aligned} \tag{23}$$

for some n, j. Notice that this amounts to ordering all the $\tau_{n,j}$ and applying the $\hat{U}_{n,j}$ following the same order. If two times $\tau_{n,j}$ and $\tau_{n',j'}$ coincide, for some $n \neq n'$, then, since we can assume that both detectors do not overlap, the

corresponding unitaries $\hat{U}_{n,j}$ and $\hat{U}_{n',j'}$ commute, and therefore it does not matter how we decide to order them. This is because if the supports of detectors n and n' at $\tau_{n,j} = \tau_{n',j'}$ do not overlap, then the field operators in $\hat{U}_{n,j}$ and $\hat{U}_{n',j'}$ have support in spacelike separated regions, and thus commute. Since the detector operators also commute for being associated with different detectors (and then defined over different Hilbert spaces), the exponents of the exponentials that define $\hat{U}_{n,j}$ and $\hat{U}_{n',j'}$ commute, and hence so do $\hat{U}_{n,j}$ and $\hat{U}_{n',j'}$ themselves.

Finally, after the interaction, the joint state of the D detectors and the field is

$$\hat{\rho} = \hat{U}\hat{\rho}_0\hat{U}^{\dagger}, \tag{24}$$

where $\hat{\rho}_0$ is the initial state of the whole system. In particular, the final joint state of the D detectors after the interaction is obtained by tracing over the field degrees of freedom,

$$\hat{\rho}_{\mathrm{D}} = \mathrm{Tr}_{\phi}(\hat{U}\hat{\rho}_0\hat{U}^{\dagger}) = \mathrm{Tr}_{\phi}(\hat{U}_N \cdots \hat{U}_1\hat{\rho}_0\hat{U}_1^{\dagger} \cdots \hat{U}_N^{\dagger}). \tag{25}$$

B. Approximation results

Let us consider D particle detectors under the same conditions as in the previous subsection, except for their coupling to the field, which now happens through switching functions $\xi_n(\tau)$ that are bounded, and continuous except for maybe a finite number of points (*regular*, as we called this kind of switching in Sec. II B). For this regular scenario, the joint evolution of all the particle detectors and the field is given by the unitary

$$\hat{U}_{\xi} = \mathcal{T}_{\tau} \exp\left(-\mathrm{i} \int \mathrm{d}z \hat{h}_{\mathrm{int}}^{\xi}(\mathbf{z})\right)$$

$$= \mathcal{T}_{\tau} \exp\left\{-\mathrm{i} \sum_{n=1}^{D} \lambda_n \int \mathrm{d}\tau \xi_n(\tau)\right.$$

$$\left. \times \left[\hat{\mu}_{n,\alpha}^{\dagger}(\tau) \int \mathrm{d}z\sqrt{-g}F_n(z)\hat{O}_n^{\alpha}(\tau,z) + \mathrm{H.c.}\right]\right\}, \tag{26}$$

where as before $g = \det g_{\mu\nu}(\tau, z)$ is the determinant of the metric. The final joint state of the D detectors is

$$\hat{\rho}_{\mathrm{D}}^{\xi} = \mathrm{Tr}_{\phi}(\hat{U}_{\xi}\hat{\rho}_0\hat{U}_{\xi}^{\dagger}). \tag{27}$$

As in the single detector case, it turns out that this scenario can be efficiently approximated using a train of delta interactions. Specifically, let us denote with $[T_n, T_n + \Delta T_n]$ the interval where $\xi_n(\tau)$ has its support, and in particular $\xi_n(T_n) = \xi_n(T_n + \Delta T_n) = 0$. We can then define the train of N sudden interactions associated with ξ_n and a uniform partition

$\mathcal{P}_n = \{t_{n,j} = T_n + j\Delta T_n/N, j = 1, ..., N\}$ of the interval $[T_n, T_n + \Delta T_n]$ as

$$\chi_{\xi_n}(\tau; N) = \frac{\Delta T_n}{N} \sum_{j=1}^{N} \xi_n\left(T_n + \frac{j - 1/2}{N}\Delta T_n\right)$$

$$\times \delta\left(\tau - T_n - \frac{j - 1/2}{N}\Delta_n T\right). \tag{28}$$

From Eq. (21), this means having $N_n = N$ for all n, setting

$$\tau_{n,j} = T_n + \frac{j - 1/2}{N}\Delta T_n, \tag{29}$$

and

$$\eta_{n,j} = \xi_n(\tau_{n,j})(t_{n,j} - t_{n,j-1}) = \xi_n(\tau_{n,j})\frac{\Delta T_n}{N}. \tag{30}$$

This is in complete analogy with the single detector case presented in Sec. II B, i.e., the strength of each sudden interaction is determined by both the value of the switching ξ_n at the corresponding time, and the duration of the interval it stands for.

Now, let us assume that, initially, the detectors and the field are mutually uncorrelated, so that the initial joint state is of the form

$$\hat{\rho}_0 = \hat{\rho}_{\mathrm{D},0} \otimes \hat{\rho}_{\phi}, \tag{31}$$

where $\hat{\rho}_{\mathrm{D},0}$ is the initial joint state of the D detectors (which in particular can be correlated). Then, we can use perturbation theory to write

$$\hat{\rho}_{\mathrm{D}}(N) = \sum_{\boldsymbol{l} \in \mathbb{N}_0^D} \hat{\rho}_{\mathrm{D}}^{(\boldsymbol{l})}(N) \tag{32}$$

for the train of deltas (where $\mathbb{N}_0 = \mathbb{Z}^+ \cup \{0\}$), and

$$\hat{\rho}_{\mathrm{D}}^{\xi} = \sum_{\boldsymbol{l} \in \mathbb{N}_0^D} \hat{\rho}_{\mathrm{D}}^{(\boldsymbol{l})\xi} \tag{33}$$

for the regular scenario, where $\hat{\rho}_{\mathrm{D}}^{(\boldsymbol{l})}(N)$ and $\hat{\rho}_{\mathrm{D}}^{(\boldsymbol{l})\xi}$ are proportional to

$$\prod_{n=1}^{D} \lambda_n^{l_n} \tag{34}$$

for each $\boldsymbol{l} \in \mathbb{N}_0^D$, and the first term for both series is just the initial state, i.e.,

$$\hat{\rho}_{\mathrm{D}}^{(\boldsymbol{0})}(N) = \hat{\rho}_{\mathrm{D}}^{(\boldsymbol{0})\xi} = \hat{\rho}_{\mathrm{D},0}. \tag{35}$$

It can be shown that, for each $l \in \mathbb{N}_0^D$, under the assumption that the switching functions ξ_n, the smearing functions F_n, and the initial state of the field $\hat{\rho}_\phi$ are *sufficiently regular*,[3]

$$\lim_{N \to \infty} \hat{\rho}_{\mathrm{D}}^{(l)}(N) = \hat{\rho}_{\mathrm{D}}^{(l)\xi} \tag{36}$$

in the weak operator topology, and the convergence is *at least* as fast as $1/N$, i.e., for any linear functional G,

$$G[\hat{\rho}_{\mathrm{D}}^{(l)\xi}] = G[\hat{\rho}_{\mathrm{D}}^{(l)}(N)] + \mathcal{O}(1/N). \tag{37}$$

In particular, when ξ_n, F_n, and $\hat{\rho}_\phi$ are such that the conditions for Eq. (37) are fulfilled *for all* $l \in \mathbb{N}_0^D$, then

$$G[\hat{\rho}_{\mathrm{D}}^{\xi}] = G[\hat{\rho}_{\mathrm{D}}(N)] + \mathcal{O}(1/N) \tag{38}$$

for any linear functional G, as long as λ is within the radius of convergence of the Dyson series in Eqs. (32) and (33). In particular, the dynamics induced by the train of deltas approximates those of the regular scenario *at all orders in perturbation theory*, i.e.,

$$\lim_{N \to \infty} \hat{\rho}_{\mathrm{D}}(N) = \hat{\rho}_{\mathrm{D}}^{\xi} \tag{39}$$

in the weak operator topology. This means that the same conclusions that we obtained for the single detector scenario—and in particular the possibility of efficiently approximating nonperturbative results—can be extended to setups involving multiple detectors as well. Notice that the density matrix that we get to approximate describes the joint state of all the detectors, and not just one of them. This means that we can use this technique to extract conclusions regarding not only the state of each individual detector, but also the correlations between them.

IV. EXAMPLES

After introducing the approximation method in Secs. II and III, here we will use it in two specific scenarios, involving, respectively, one and two detectors. For each case, we will also analyze the rate of convergence of the approximate density matrices toward the exact ones, showing explicitly that the approximations are at least as efficient as guaranteed by the results stated in Secs. II B and III B (and proved in Appendix A).

A. Single detector example

In this example, we consider a two-level Unruh-DeWitt particle detector [9,70] at rest in a $(1+3)$-dimensional

Minkowski spacetime, with ground and excited states $|g\rangle$ and $|e\rangle$, separated by an energy gap Ω, and a Gaussian shape,

$$F(\boldsymbol{x}) = \frac{1}{\sqrt{\pi^3}\sigma^3} e^{-x^2/\sigma^2}. \tag{40}$$

The detector is coupled to a massless scalar quantum field via an interaction Hamiltonian

$$\hat{H}_{\mathrm{int}}(t) = \lambda \xi(t)\hat{\mu}(t)\int \mathrm{d}\boldsymbol{x}\, F(\boldsymbol{x})\hat{\phi}(t,\boldsymbol{x}), \tag{41}$$

where λ is the coupling strength as in Eq. (1), and $\hat{\mu}$ is the monopole moment of the detector in the interaction picture,

$$\hat{\mu}(t) = |g\rangle\langle e|e^{-\mathrm{i}\Omega t} + |e\rangle\langle g|e^{\mathrm{i}\Omega t}. \tag{42}$$

The field operator $\hat{\phi}$ can be expanded in plane-wave modes as

$$\hat{\phi}(t,\boldsymbol{x}) = \int \frac{\mathrm{d}\boldsymbol{k}}{\sqrt{(2\pi)^3 2|\boldsymbol{k}|}} (\hat{a}_{\boldsymbol{k}} e^{-\mathrm{i}(|\boldsymbol{k}|t - \boldsymbol{k}\cdot\boldsymbol{x})} + \mathrm{H.c.}). \tag{43}$$

Finally, ξ is a switching function supported in $[0, T]$ that is bounded and continuous except for maybe a finite number of points, as in Sec. II B.

We take the detector and the field to be initially uncorrelated and in their respective ground states,

$$\hat{\rho}_0 = \hat{\rho}_{\mathrm{D},0} \otimes |0\rangle\langle 0|, \qquad \hat{\rho}_{\mathrm{D},0} = |g\rangle\langle g|. \tag{44}$$

Using perturbation theory, we can write the state of the detector at the end of the interaction process as

$$\hat{\rho}_{\mathrm{D}} = \hat{\rho}_{\mathrm{D},0} + \hat{\rho}_{\mathrm{D}}^{(2)} + \mathcal{O}(\lambda^4), \tag{45}$$

where the terms of odd order in the perturbative expansion are zero because the vacuum $|0\rangle$ is a zero-mean Gaussian state, and therefore its odd-point functions are zero. Thus, the final state of the detector is diagonal in the $\{|g\rangle, |e\rangle\}$ basis, and can be completely characterized by the excitation probability,

$$P_{\mathrm{e}} = \langle e|\hat{\rho}_{\mathrm{D}}|e\rangle = \langle e|\hat{\rho}_{\mathrm{D}}^{(2)}|e\rangle + \mathcal{O}(\lambda^4). \tag{46}$$

For a massless scalar quantum field, up to second order in λ (see, e.g., [83]),

$$P_{\mathrm{e}} = \lambda^2 \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3 2|\boldsymbol{k}|} |\tilde{\xi}(|\boldsymbol{k}| + \Omega)|^2 |\tilde{F}(\boldsymbol{k})|^2, \tag{47}$$

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[3]As in Sec. II B, see Appendix A for details on what is meant by sufficiently regular.
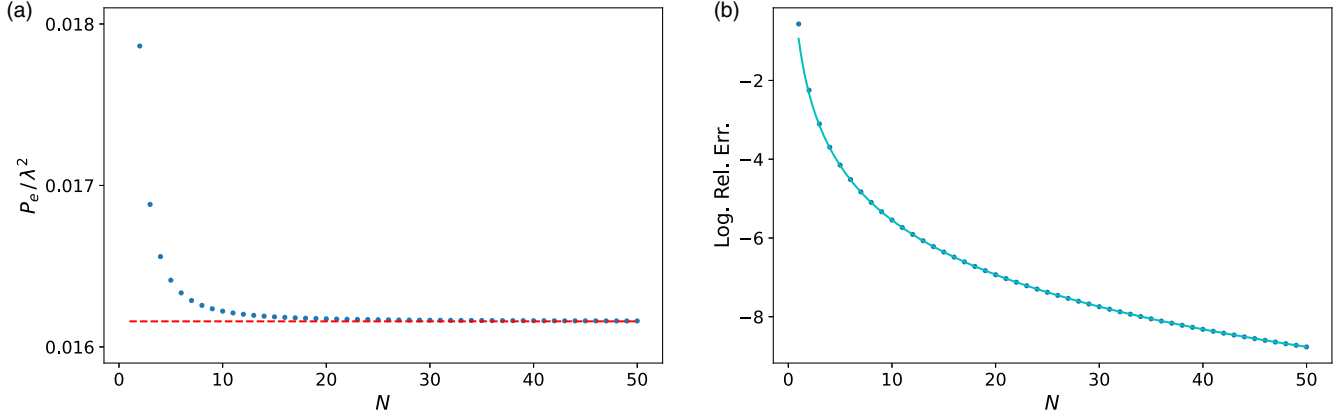
FIG. 1. (a) Excitation probability of a two-level UDW detector coupled to a quantum scalar field through a train of delta couplings approximating a Heaviside switching, as a function of the number of couplings N. Here we set the relevant parameters γ and s to 1. The constant red dashed line marks the value of the excitation probability for the exact Heaviside switching. (d) Logarithmic relative error of the excitation probability for the Heaviside switching approximated by delta couplings, as a function of the number of couplings N. The function represented by the solid cyan line is proportional to $1/N^2$, marking the rate of convergence of the approximation.

where $\tilde{\xi}$ and $\tilde{F}$ are the Fourier transforms[4] of ξ and F, respectively. For the Gaussian shape specified before, which has spherical symmetry, we get

$$P_{\mathrm{e}} = \frac{\lambda^2}{4\pi^2} \int_0^\infty \mathrm{d}|\mathbf{k}| |\mathbf{k}| e^{-|\mathbf{k}|^2 \sigma^2 / 2} |\tilde{\xi}(|\mathbf{k}| + \Omega)|^2. \quad (49)$$

As a particular application of the result stated in Sec. II B, Eq. (49) can be approximated by the transition probability for a scenario where the detector couples to the field through the train of N sudden interactions given by

$$\chi_\xi(t; N) = \frac{T}{N} \sum_{j=1}^N \xi\left(\frac{j-1/2}{N}T\right) \delta\left(t - \frac{j-1/2}{N}T\right). \quad (50)$$

In this case, the excitation probability is given (up to second order in λ) by

$$P_{\mathrm{e}}(N) = \frac{\lambda^2 T^2}{N^2} \sum_{j,j'=1}^N \xi\left(\frac{j-1/2}{N}T\right) \xi\left(\frac{j'-1/2}{N}T\right)$$
$$\times \frac{e^{i\Omega T(j-j')/N}}{4\pi^2} \int_0^\infty \mathrm{d}|\mathbf{k}| |\mathbf{k}| e^{-|\mathbf{k}|^2 \sigma^2 / 2} e^{i|\mathbf{k}|T(j-j')/N}. \quad (51)$$

In the following we will see, for specific switching functions ξ, how Eq. (51) approximates Eq. (49) increasingly well as N increases, and does so with the predicted efficiency.

---

[4]The convention we followed here for the definition of the Fourier transform is that

$$\tilde{G}(\mathbf{v}) := \int \mathrm{d}\mathbf{u}\, G(\mathbf{u}) e^{-i\mathbf{u}\cdot\mathbf{v}}. \quad (48)$$

1. Heaviside switching

The first example that we showcase is a switching function that is constant in its support,

$$\xi(t) = \mathrm{I}_{(0,T)}(t), \quad (52)$$

where $\mathrm{I}_{(0,T)}$ is the indicator function that equals 1 in the interval $(0, T)$, and zero everywhere else. In this case,

$$\tilde{\xi}(k) = \frac{2e^{-ikT/2}}{k} \sin\left(\frac{kT}{2}\right). \quad (53)$$

From Eq. (49), we have

$$P_{\mathrm{e}} = \frac{\lambda^2}{\pi^2} \int_0^\infty \mathrm{d}|\mathbf{k}| \frac{|\mathbf{k}| e^{-|\mathbf{k}|^2 \sigma^2 / 2}}{(|\mathbf{k}| + \Omega)^2} \sin^2\left[\frac{(|\mathbf{k}| + \Omega)T}{2}\right] \quad (54)$$

$$= \frac{\lambda^2}{\pi^2} \int_0^\infty \mathrm{d}\kappa \frac{\kappa e^{-\kappa^2 s^2 / 2}}{(\kappa + \gamma)^2} \sin^2\left(\frac{\kappa + \gamma}{2}\right), \quad (55)$$

where the second line explicitly shows that the result only depends on the dimensionless parameters $\gamma = \Omega T$, and $s = \sigma/T$, and we also denoted $\kappa = |\mathbf{k}|T$. Similarly, from Eq. (51) we get the excitation probability associated with the train of sudden interactions χ_ξ that approximates the Heaviside switching,

$$P_{\mathrm{e}}(N) = \frac{\lambda^2}{N^2} \sum_{j,j'=1}^N \frac{e^{i\gamma(j-j')/N}}{4\pi^2} \int_0^\infty \mathrm{d}\kappa\, \kappa e^{-\kappa^2 s^2 / 2} e^{i\kappa(j-j')/N}. \quad (56)$$

In Fig. 1a we see how $P_{\mathrm{e}}(N)$ as given in Eq. (56) converges to the exact value P_{e} given in Eq. (55), for the case $\gamma = s = 1$. Moreover, Fig. 1(b) shows that the rate of convergence of the approximation is as fast as $1/N^2$, which
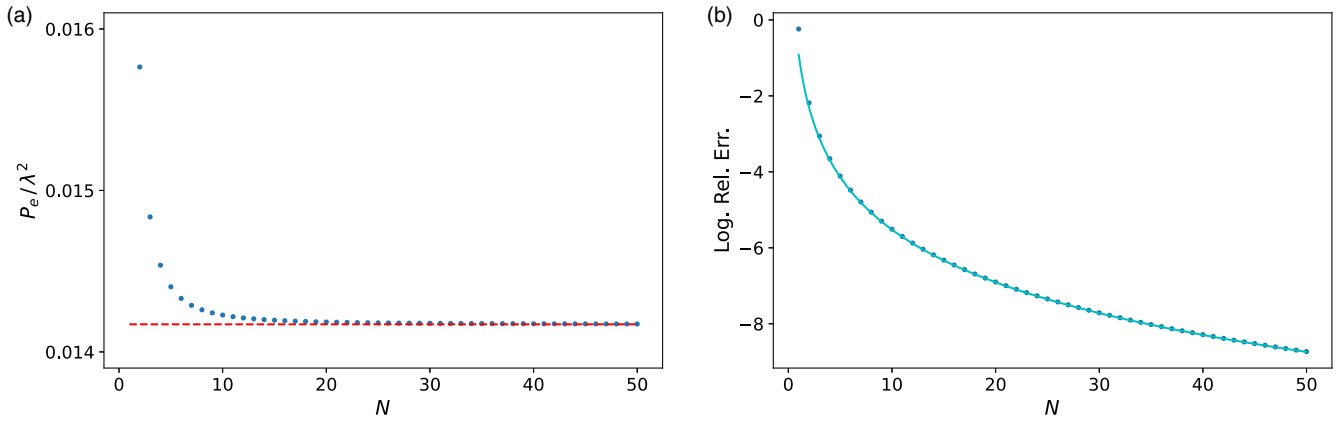
FIG. 2.   (a) Excitation probability of a two-level UDW detector coupled to a quantum scalar field through a train of delta couplings approximating a Gaussian switching truncated at $q = 1$ variances, as a function of the number of couplings N. Here we set the relevant parameters γ and s to 1. The constant red dashed line marks the value of the excitation probability for the exact truncated Gaussian switching. (b) Logarithmic relative error of the excitation probability for the Gaussian switching approximated by delta couplings, as a function of the number of couplings N. The function represented by the solid cyan line is proportional to $1/N^2$, marking the rate of convergence of the approximation.

is better than the minimum rate guaranteed by the approximation result of Sec. II B.

2. Truncated Gaussian switching

The second example that we consider is a Gaussian switching where the Gaussian tails are cut beyond q times its variance:

$$\xi(t) = \exp\left[-\left(\frac{t}{qT} - \frac{1}{2q}\right)^2\right] \mathrm{I}_{(0,T)}(t). \tag{57}$$

Notice that the Gaussian reaches its maximum at $t = T/2$, i.e., the center of the switching's compact support. Its Fourier transform reads

$$\tilde{\xi}(k) = \sqrt{\pi} qT e^{-ikT/2} e^{-k^2 q^2 T^2/4} \mathrm{Re}\left[\mathrm{erf}\left(\frac{1}{2q} + \frac{ikqT}{2}\right)\right]. \tag{58}$$

From Eq. (49), we have

$$P_{\mathrm{e}} = \frac{\lambda^2 q^2 T^2}{4\pi} \int_0^\infty \mathrm{d}|\boldsymbol{k}| |\boldsymbol{k}| e^{-|\boldsymbol{k}|^2 \sigma^2/2} e^{-(|\boldsymbol{k}|+\Omega)^2 q^2 T^2/2}$$
$$\times \mathrm{Re}\left\{\mathrm{erf}\left[\frac{1}{2q} + \frac{i(|\boldsymbol{k}|+\Omega)qT}{2}\right]\right\}^2 \tag{59}$$

$$= \frac{\lambda^2 q^2}{4\pi} \int_0^\infty \mathrm{d}\kappa\, \kappa e^{-\kappa^2 s^2/2} e^{-(\kappa+\gamma)^2 q^2/2}$$
$$\times \mathrm{Re}\left\{\mathrm{erf}\left[\frac{1}{2q} + \frac{i(\kappa+\gamma)q}{2}\right]\right\}^2, \tag{60}$$

where, as before, we have shown the explicit dependence on the dimensionless parameters $\gamma = \Omega T$, and $s = \sigma/T$, and the integration is performed with respect to the dimensionless variable $\kappa = |\boldsymbol{k}|T$. Again, from Eq. (51) we get the excitation probability associated with the train of delta couplings χ_ξ that yields a transition probability that approximates that of the truncated Gaussian switching,

$$P_{\mathrm{e}}(N) = \frac{\lambda^2}{N^2} \sum_{j,j'=1}^N e^{-(2j-N-1)^2/4q^2 N^2} e^{-(2j'-N-1)^2/4q^2 N^2}$$
$$\times \frac{e^{i\gamma(j-j')/N}}{4\pi^2} \int_0^\infty \mathrm{d}\kappa\, \kappa e^{-\kappa^2 s^2/2} e^{i\kappa(j-j')/N}. \tag{61}$$

In Fig. 2(a) we see how $P_{\mathrm{e}}(N)$—as given in Eq. (61)—converges to the value of P_{e} given in Eq. (60), for the case $\gamma = s = q = 1$. Moreover, Fig. 2(b) shows that the rate of convergence of the approximation is once more as fast as $1/N^2$, which is again better than the minimum rate guaranteed by the approximation result of Sec. II B.

3. Smooth bump switching

As a third example, let us consider the switching to be the C^∞ bump function given by

$$\xi(t) = \begin{cases} \exp\left[-\dfrac{T^2}{4t(T-t)}\right] & \text{if } 0 < t < T, \\ 0 & \text{otherwise,} \end{cases} \tag{62}$$

which is compactly supported in $[0, T]$, infinitely differentiable on $\mathbb{R}$, and reaches its maximum at $t = T/2$.
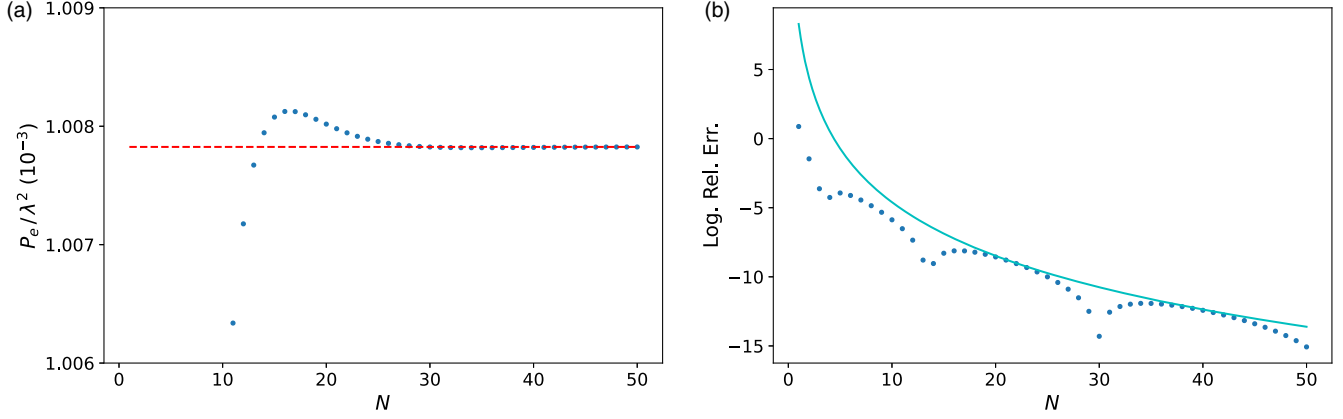
FIG. 3. (a) Excitation probability of a two-level UDW detector coupled to a quantum scalar field through a train of delta couplings approximating a bump switching, as a function of the number of couplings N. Here we set the relevant parameters γ and s to 1. The constant red dashed line marks the value of the excitation probability for the exact bump switching. (b) Logarithmic relative error of the excitation probability for the bump switching approximated by delta couplings, as a function of the number of couplings N. The function represented by the solid cyan line is proportional to $1/N^{5.6}$, upper bounding the rate of convergence of the approximation.

Its Fourier transform does not admit a closed form. However, let us define

$$\beta(t) = \begin{cases} \exp\left[-\frac{1}{4t(1-t)}\right] & \text{if } 0 < t < 1, \\ 0 & \text{otherwise,} \end{cases} \quad (63)$$

which does not depend on T. This function is related to ξ by $\xi(t) = \beta(t/T)$. Thus, their Fourier transforms satisfy

$$\tilde{\xi}(k) = T\tilde{\beta}(kT), \quad (64)$$

and hence, from Eq. (49), we can write

$$P_e = \frac{\lambda^2}{4\pi^2} \int_0^\infty d\kappa\, \kappa e^{-\kappa^2 s^2/2} |\tilde{\beta}(\kappa + \gamma)|^2, \quad (65)$$

which shows once again the complete dependence of P_e on the dimensionless parameters γ and s. Meanwhile, from Eq. (51) we get the excitation probability associated with the train of delta couplings χ_ξ used to approximate that of the bump switching,

$$P_e(N) = \frac{\lambda^2}{N^2} \sum_{j,j'=1}^{N} \exp\left[-\frac{N^2}{4(j-1/2)(N-j+1/2)}\right]$$

$$\times \exp\left[-\frac{N^2}{4(j'-1/2)(N-j'+1/2)}\right]$$

$$\times \frac{e^{i\gamma(j-j')/N}}{4\pi^2} \int_0^\infty d\kappa\, \kappa e^{-\kappa^2 s^2/2} e^{i\kappa(j-j')/N}. \quad (66)$$

In Fig. 3(a) we see how, for the case $\gamma = s = 1$, the approximated values $P_e(N)$ [as given in Eq. (66)] converge to the exact value of P_e [given in Eq. (65)]. Moreover, Fig. 3(b) shows that the error incurred by the approximation

is upper bounded by a function that decays faster than $1/N^5$, much faster than the minimum rate guaranteed by the result of Sec. II B. In fact, the increase of the rate of convergence of the approximation in this example with respect to the two previous ones is not surprising, since the approximation results in [84] suggest that switching functions of higher differentiability class should admit tighter convergence bounds.

B. Two detectors example

The setup considered for this example consists of two identical two-level Unruh-DeWitt detectors, A and B, both at rest at positions x_A and x_B in a $(1+3)$-dimensional Minkowski spacetime. We consider the ground and excited states of both detectors to be separated by the same energy gap Ω, and their shapes to be (normalized) hard spheres of radius R,

$$F_i(\boldsymbol{x}) = \frac{3}{4\pi R^3}\theta(R - |\boldsymbol{x} - \boldsymbol{x}_i|), \quad i = \text{A}, \text{B}. \quad (67)$$

As in the single detector example, here we consider a linear coupling between the detectors and a massless scalar quantum field, given in the interaction picture by the interaction Hamiltonian

$$\hat{H}_{\text{int}}(t) = \lambda\xi(t)\left[\hat{\mu}_A(t)\int d\boldsymbol{x}F_A(\boldsymbol{x})\hat{\phi}(t,\boldsymbol{x}) \right.$$

$$\left. + \hat{\mu}_B(t)\int d\boldsymbol{x}F_B(\boldsymbol{x})\hat{\phi}(t,\boldsymbol{x})\right], \quad (68)$$

where λ is the coupling strength as in Eq. (1), $\hat{\mu}_i$ is the monopole moment of detector i, which can be written in the interaction picture as

$$\hat{\mu}_i(t) = |g_i\rangle\langle e_i| e^{-i\Omega t} + |e_i\rangle\langle g_i| e^{i\Omega t}, \qquad (69)$$

the field operator $\hat{\phi}$ can be expanded in plane-wave modes as in Eq. (43), and finally ξ is again a switching function supported in $[0, T]$ that is bounded and continuous except for maybe a finite number of points, as in Sec. III B. Notice that, for simplicity, in this example we have chosen the coupling strengths and the switching functions to be the same for both detectors, although this is of course not necessary in the most general case.

Initially, we consider the detectors and the field to be uncorrelated and in their ground states,

$$\hat{\rho}_0 = |g_A\rangle\langle g_A| \otimes |g_B\rangle\langle g_B| \otimes |0\rangle\langle 0|. \qquad (70)$$

Perturbation theory allows us to write the joint state of the two detectors after the interaction as

$$\hat{\rho}_{AB} = \hat{\rho}_{AB,0} + \hat{\rho}_{AB}^{(2)} + \mathcal{O}(\lambda^4), \qquad (71)$$

where $\hat{\rho}_{AB,0} = |g_A\rangle\langle g_A| \otimes |g_B\rangle\langle g_B|$ and, as in the single detector case, the terms of odd order are zero because the odd-point functions of the vacuum state of the field are zero. In the ordered basis $\{|g_A g_B\rangle, |g_A e_B\rangle, |e_A g_B\rangle, |e_A e_B\rangle\}$, the second order contribution can be written in matrix form as (see, e.g., [19,85])

$$\hat{\rho}_{AB}^{(2)} = \begin{pmatrix} -\mathcal{L}_{AA} - \mathcal{L}_{BB} & 0 & 0 & \mathcal{M}^* \\ 0 & \mathcal{L}_{BB} & \mathcal{L}_{AB}^* & 0 \\ 0 & \mathcal{L}_{AB} & \mathcal{L}_{AA} & 0 \\ \mathcal{M} & 0 & 0 & 0 \end{pmatrix}. \qquad (72)$$

Now, let us restrict ourselves to the case in which the interaction of the two detectors happens in regions of spacetime that are spacelike separated, i.e., when

$$T < D - 2R, \qquad (73)$$

where $D = |\boldsymbol{x}_A - \boldsymbol{x}_B|$ denotes the spatial distance between the two detectors. Then, we get the following expressions for the terms in Eq. (72):

$$\mathcal{L}_{AA} = \mathcal{L}_{BB} = \frac{\lambda^2}{4\pi^2} \int_0^\infty d|\boldsymbol{k}| |\boldsymbol{k}| |\tilde{\xi}(|\boldsymbol{k}| + \Omega)|^2 |\tilde{F}(|\boldsymbol{k}|)|^2, \qquad (74)$$

$$\mathcal{L}_{AB} = \frac{\lambda^2}{4\pi^2 D} \int_0^\infty d|\boldsymbol{k}| \sin(|\boldsymbol{k}|D) |\tilde{\xi}(|\boldsymbol{k}| + \Omega)|^2 |\tilde{F}(|\boldsymbol{k}|)|^2, \qquad (75)$$

$$\mathcal{M} = -\frac{\lambda^2}{4\pi^2 D} \int_0^\infty d|\boldsymbol{k}| \sin(|\boldsymbol{k}|D) \tilde{\xi}(|\boldsymbol{k}| - \Omega) \\ \times \tilde{\xi}(|\boldsymbol{k}| + \Omega)^* |\tilde{F}(|\boldsymbol{k}|)|^2, \qquad (76)$$

where $\tilde{\xi}$ and $\tilde{F}$ are the Fourier transforms of the switching function ξ and

$$F(\boldsymbol{x}) = \frac{3}{4\pi R^3} \theta(R - |\boldsymbol{x}|), \qquad (77)$$

respectively. Notice that to arrive at Eqs. (74)–(76) we used that

$$F(\boldsymbol{x}) = F_A(\boldsymbol{x} + \boldsymbol{x}_A) = F_B(\boldsymbol{x} + \boldsymbol{x}_B), \qquad (78)$$

as well as the spherical symmetry of F, which implies

$$\tilde{F}(\boldsymbol{k}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}_A}\tilde{F}_A(\boldsymbol{k}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}_B}\tilde{F}_B(\boldsymbol{k}) = \tilde{F}(|\boldsymbol{k}|), \qquad (79)$$

where

$$\tilde{F}(|\boldsymbol{k}|) = \frac{3}{(|\boldsymbol{k}|R)^3} [\sin(|\boldsymbol{k}|R) - |\boldsymbol{k}|R\cos(|\boldsymbol{k}|R)]. \qquad (80)$$

Finally, we choose the switching function for this example to be of the Heaviside type,

$$\xi(t) = I_{(0,T)}(t), \qquad (81)$$

with Fourier transform

$$\tilde{\xi}(k) = \frac{2e^{-ikT/2}}{k} \sin\left(\frac{kT}{2}\right). \qquad (82)$$

Equations (74)–(76) can then be written in terms of dimensionless parameters as

$$\mathcal{L}_{AA} = \mathcal{L}_{BB} = \frac{\lambda^2}{\pi^2} \int_0^\infty d\kappa \frac{\sin^2[(\kappa+\gamma)/2]}{\kappa} |\tilde{F}(\kappa)|^2, \qquad (83)$$

$$\mathcal{L}_{AB} = \frac{\lambda^2}{\pi^2 d} \int_0^\infty d\kappa \sin(\kappa d) \frac{\sin^2[(\kappa+\gamma)/2]}{\kappa^2} |\tilde{F}(\kappa)|^2, \qquad (84)$$

$$\mathcal{M} = -\frac{\lambda^2 e^{i\gamma}}{2\pi^2 d} \int_0^\infty d\kappa \frac{\sin(\kappa d)(\cos\gamma - \cos\kappa)}{\kappa^2 - \gamma^2} |\tilde{F}(\kappa)|^2, \qquad (85)$$

where $\gamma = \Omega T$, $d = D/T$, $\kappa = |\boldsymbol{k}|T$, and

$$\tilde{F}(\kappa) = \frac{3}{(\kappa r)^3} [\sin(\kappa r) - \kappa r\cos(\kappa r)], \qquad (86)$$

with $r = R/T$.

As a particular application of the approximation result of Sec. III B, we know that Eqs. (83)–(85) can be approximated by their counterparts in the scenario where the detectors couple to the field through the train of sudden interactions given by
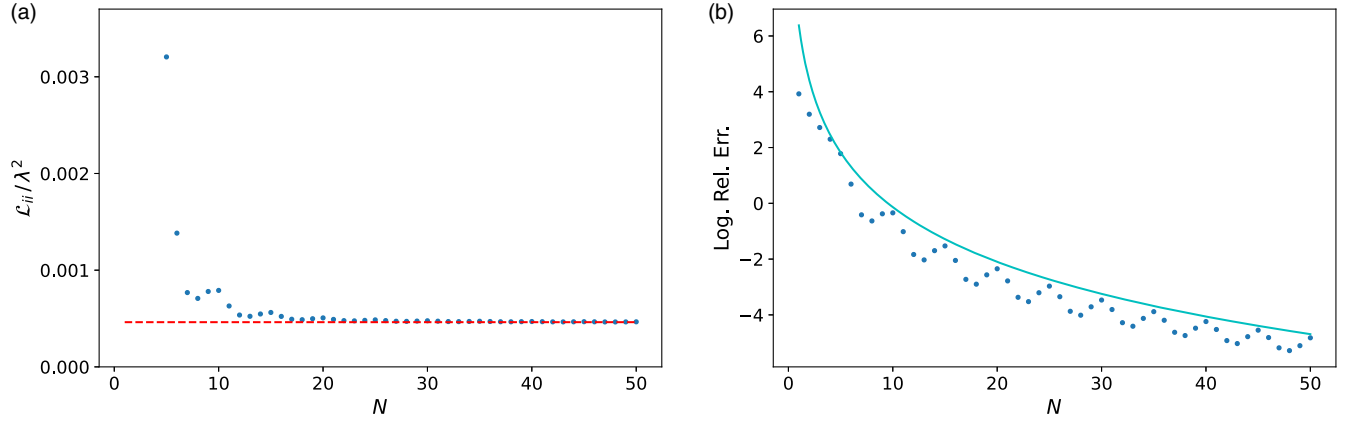
FIG. 4. (a) $\mathcal{L}_{AA} = \mathcal{L}_{BB} \equiv \mathcal{L}_{ii}$ term for a pair of identical two-level UDW detectors coupled to a quantum scalar field in spacelike separated regions through a train of delta couplings approximating a Heaviside switching, as a function of the number of couplings N. Here we set the relevant parameters to $\gamma = 1$, $d = 1.2$, and $r = 0.1$. The constant red dashed line marks the value of the $\mathcal{L}_{ii}$ term for the exact Heaviside switching. (b) Logarithmic relative error of the $\mathcal{L}_{ii}$ term for the Heaviside switching approximated by delta couplings, as a function of the number of couplings N. The function represented by the solid cyan line is proportional to $1/N^{2.8}$, upper bounding the rate of convergence of the approximation.

$$\chi_\xi(t; N) = \frac{T}{N} \sum_{j=1}^{N} \delta\left(t - \frac{j - 1/2}{N} T\right). \quad (87)$$

The terms that approximate Eqs. (83)–(85) can then be written as

$$\mathcal{L}_{AA}(N) = \mathcal{L}_{BB}(N) = \frac{\lambda^2}{N^2} \sum_{j,j'=1}^{N} \frac{1}{4\pi^2} \int_0^\infty d\kappa\, \kappa |\tilde{F}(\kappa)|^2$$
$$\times \cos\left[\frac{(\kappa + \gamma)(j - j')}{N}\right], \quad (88)$$

$$\mathcal{L}_{AB}(N) = \frac{\lambda^2}{N^2} \sum_{j,j'=1}^{N} \frac{1}{4\pi^2 d} \int_0^\infty d\kappa\, \sin(\kappa d) |\tilde{F}(\kappa)|^2$$
$$\times \cos\left[\frac{(\kappa + \gamma)(j - j')}{N}\right], \quad (89)$$

$$\mathcal{M}(N) = -\frac{\lambda^2}{N^2} \sum_{j,j'=1}^{N} \frac{e^{i\gamma(j+j')/T}}{4\pi^2 d} \int_0^\infty d\kappa\, \sin(\kappa d) |\tilde{F}(\kappa)|^2$$
$$\times \cos\left[\frac{\kappa(j - j')}{N}\right]. \quad (90)$$

In Figs. 4–6, we see how, for the case $\gamma = 1$, $d = 1.2$, and $r = 0.1$, the approximated values given in Eqs. (88)–(90) converge to the exact values given in Eqs. (83)–(85). In particular, Figs. 4(b), 5(b), and 6(d) show that the errors committed by the approximations are upper bounded by functions decaying at least as fast as $1/N$, in agreement with the approximation result of Sec. III B.

V. CONCLUSION

We have shown how to efficiently approximate non-perturbatively the time evolution of particle detectors interacting with a quantum field for very general compactly supported switching functions and arbitrary smearing functions.

Specifically, we have described how to approximate the detectors' dynamics when they couple through compactly supported bounded switching functions, continuous except for maybe a finite number of points, with that of a detector coupled through a sequence of delta couplings, which can be evaluated nonperturbatively. The approximation converges at least as fast as $1/N$ (where N is the number of delta pulses), and often much faster than this for regular enough switchings.

The fast convergence of the approximation is guaranteed at all orders in perturbation theory as long as the detectors' switching and smearing functions and the field's state are "regular enough." This, in particular, includes the cases in which the state of the field is Hadamard, and the switching and smearing functions are smooth except for maybe a finite number of points, which covers most of the relevant scenarios in both flat and curved spacetimes.

Since this approximation scheme can be easily (and efficiently) evaluated in current computers, we expect the method presented here to pave the way for a number of nonperturbative analyses of phenomena involving particle detectors and measurements in quantum field theory.
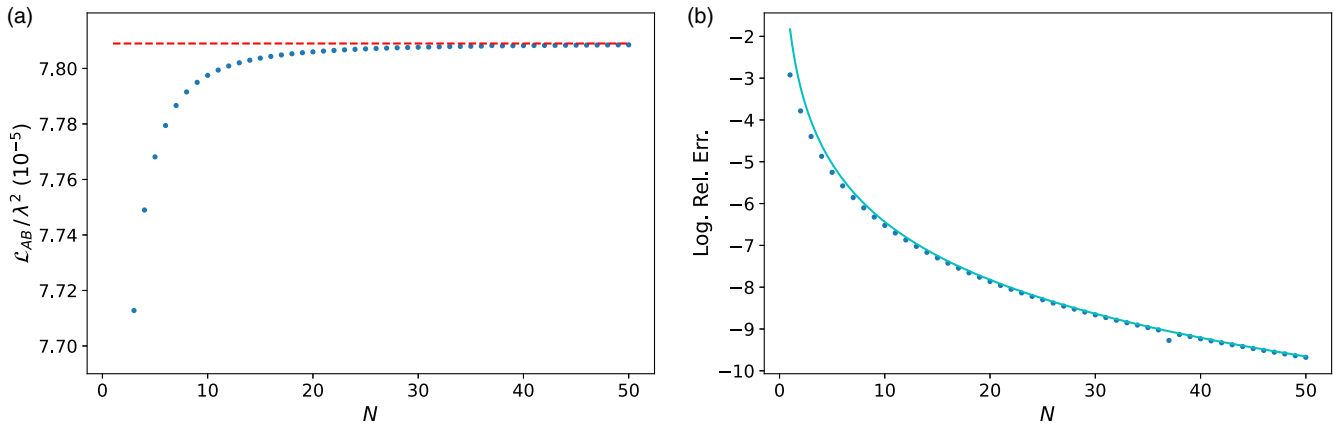
FIG. 5.   (a) $\mathcal{L}_{AB}$ term for a pair of identical two-level UDW detectors coupled to a quantum scalar field in spacelike separated regions through a train of delta couplings approximating a Heaviside switching, as a function of the number of couplings N. Here we set the relevant parameters to $\gamma = 1$, $d = 1.2$, and $r = 0.1$. The constant red dashed line marks the value of the $\mathcal{L}_{AB}$ term for the exact Heaviside switching. (b) Logarithmic relative error of the $\mathcal{L}_{AB}$ term for the Heaviside switching approximated by delta couplings, as a function of the number of couplings N. The function represented by the solid cyan line is proportional to $1/N^2$, marking the rate of convergence of the approximation.
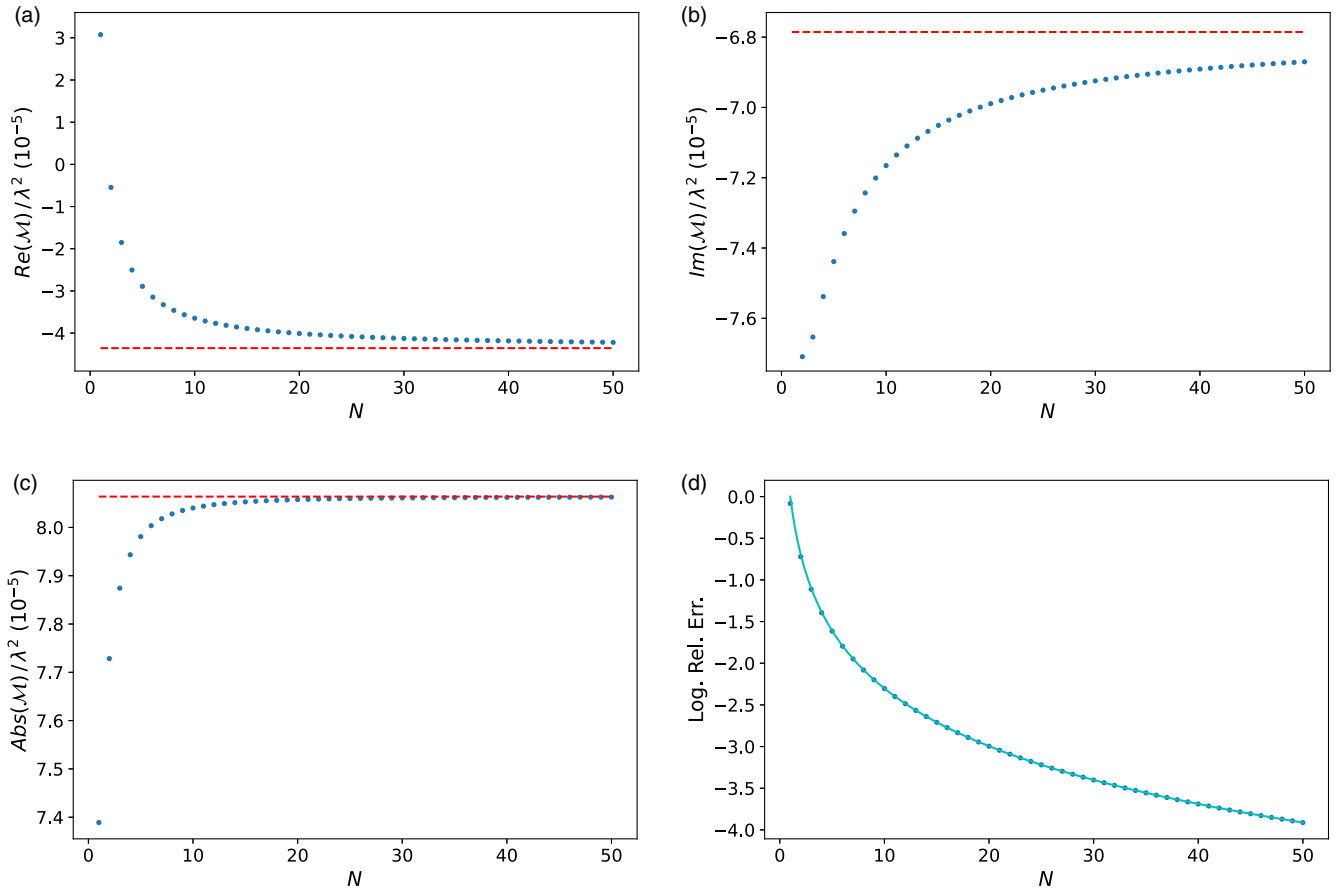

FIG. 6.   (a), (b), and (c) Respectively, real part, imaginary part, and absolute value of the $\mathcal{M}$ term for a pair of identical two-level UDW detectors coupled to a quantum scalar field in spacelike separated regions through a train of delta couplings approximating a Heaviside switching, as a function of the number of couplings N. Here we set the relevant parameters to $\gamma = 1$, $d = 1.2$, and $r = 0.1$. The constant red dashed lines mark the values of the real part, imaginary part, and absolute value of the $\mathcal{M}$ term for the exact Heaviside switching. (d) Logarithmic relative error of the $\mathcal{M}$ term for the Heaviside switching approximated by delta couplings, as a function of the number of couplings N. The function represented by the solid cyan line is proportional to $1/N$, marking the rate of convergence of the approximation.

APPENDIX A: PROOF OF THE APPROXIMATION RESULT

In this Appendix, we prove the approximation results stated in Eqs. (16)–(19) for a single detector, and in Eqs. (37)–(39) for multiple detectors. The claim establishes that each term of the Dyson series of the time-evolved density matrix resulting from couplings with trains of sudden interactions converges (as we increase the number of sudden interactions) to the contribution of the same order for the density matrix resulting from couplings with (sufficiently regular) switching functions that the delta interactions aim to approximate, both for setups involving one or more particle detectors.

Let us first consider a one detector scenario with the interaction Hamiltonian weight given by Eq. (1), i.e.,

$$\hat{h}_{\text{int}}(\tau, z) = \lambda \chi(\tau)[F(z)\hat{\mu}_\alpha^\dagger(\tau)\hat{\mathcal{O}}^\alpha(\tau, z) + \text{H.c.}], \quad (A1)$$

where the switching function χ can be either the regular switching, ξ, or its approximation with N sudden interactions, $\chi_\xi(N)$, given in Eq. (10). For the regular scenario, the joint evolution of the field and the detector is given by the unitary

$$
\begin{aligned}
\hat{U}_\xi &= \mathcal{T}_\tau \exp\left(-\mathrm{i}\int \mathrm{d}z \hat{h}_{\text{int}}^\xi(z)\right) \\
&= \mathcal{T}_\tau \exp\Big\{-\mathrm{i}\lambda \int \mathrm{d}\tau \xi(\tau) \\
&\quad \times \Big[\hat{\mu}_\alpha^\dagger(\tau)\int \mathrm{d}z\sqrt{-g}F(z)\hat{\mathcal{O}}^\alpha(\tau, z) + \text{H.c.}\Big]\Big\}, \quad (A2)
\end{aligned}
$$

with the final state of the detector being

$$\hat{\rho}_{\text{D}}^\xi = \text{Tr}_\phi(\hat{U}_\xi \hat{\rho}_0 \hat{U}_\xi^\dagger). \quad (A3)$$

Expanding in series of λ, we get

$$\hat{U}_\xi = \mathbb{1} + \sum_{k=1}^\infty \hat{U}_\xi^{(k)} \quad \text{and} \quad \hat{\rho}_{\text{D}}^\xi = \hat{\rho}_{\text{D},0} + \sum_{k=1}^\infty \hat{\rho}_{\text{D}}^{(k)\xi}, \quad (A4)$$

where $\hat{U}_\xi^{(k)}$ and $\hat{\rho}_{\text{D}}^{(k)\xi}$ are proportional to λ^k, and

$$\hat{\rho}_{\text{D}}^{(k)\xi} = \sum_{r+s=k} \text{Tr}_\phi(\hat{U}_\xi^{(r)}\hat{\rho}_0 \hat{U}_\xi^{(s)\dagger}). \quad (A5)$$

Now,

$$
\begin{aligned}
\text{Tr}_\phi(\hat{U}_\xi^{(r)}\hat{\rho}_0 \hat{U}_\xi^{(s)\dagger}) &= (-\mathrm{i})^r \mathrm{i}^s \lambda^k \int \mathrm{d}\tau_1\cdots\mathrm{d}\tau_r \mathrm{d}\tau_1'\cdots\mathrm{d}\tau_s' \prod_{p=1}^{r-1}\theta(\tau_{p+1}-\tau_p)\prod_{q=1}^{s-1}\theta(\tau_{q+1}'-\tau_q') \\
&\quad \times \prod_{l=1}^r \xi(\tau_l)\prod_{m=1}^s \xi(\tau_m')[W_{\boldsymbol{\alpha}'}^{\boldsymbol{\alpha}}(\tau_1',\ldots,\tau_s';\tau_1,\ldots,\tau_r;F)\hat{\mu}_{\alpha_r}^\dagger(\tau_r)\cdots\hat{\mu}_{\alpha_1}^\dagger(\tau_1)\hat{\rho}_{\text{D},0}\hat{\mu}^{\alpha_1'}(\tau_1')\cdots\hat{\mu}^{\alpha_s'}(\tau_s') \\
&\quad + W_{\boldsymbol{\alpha}}^{\boldsymbol{\alpha}'}(\tau_1',\ldots,\tau_s';\tau_1,\ldots,\tau_r)\hat{\mu}^{\alpha_r}(\tau_r)\cdots\hat{\mu}^{\alpha_1}(\tau_1)\hat{\rho}_{\text{D},0}\hat{\mu}_{\alpha_1'}^\dagger(\tau_1')\cdots\hat{\mu}_{\alpha_s'}^\dagger(\tau_s') \\
&\quad + W_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}(\tau_1',\ldots,\tau_s';\tau_1,\ldots,\tau_r)\hat{\mu}^{\alpha_r}(\tau_r)\cdots\hat{\mu}^{\alpha_1}(\tau_1)\hat{\rho}_{\text{D},0}\hat{\mu}^{\alpha_1'}(\tau_1')\cdots\hat{\mu}^{\alpha_s'}(\tau_s') \\
&\quad + W^{\boldsymbol{\alpha},\boldsymbol{\alpha}'}(\tau_1',\ldots,\tau_s';\tau_1,\ldots,\tau_r)\hat{\mu}_{\alpha_r}^\dagger(\tau_r)\cdots\hat{\mu}_{\alpha_1}^\dagger(\tau_1)\hat{\rho}_{\text{D},0}\hat{\mu}_{\alpha_1'}^\dagger(\tau_1')\cdots\hat{\mu}_{\alpha_s'}^\dagger(\tau_s')],
\end{aligned}
\quad (A6)
$$

where we denoted

$$W_{\boldsymbol{\alpha}'}^{\boldsymbol{\alpha}}(t_1',\ldots,t_s';t_1,\ldots,t_r;F) := \text{Tr}_\phi[\hat{\rho}_\phi \hat{\mathcal{O}}_{\alpha_1'}^\dagger(t_1',F)\cdots\hat{\mathcal{O}}_{\alpha_s'}^\dagger(t_s',F)\hat{\mathcal{O}}^{\alpha_r}(t_r,F)\cdots\hat{\mathcal{O}}^{\alpha_1}(t_1,F)], \quad (A7)$$

$$W_{\boldsymbol{\alpha}}^{\boldsymbol{\alpha}'}(t_1',\ldots,t_s';t_1,\ldots,t_r;F) := \text{Tr}_\phi[\hat{\rho}_\phi \hat{\mathcal{O}}^{\alpha_1'}(t_1',F)\cdots\hat{\mathcal{O}}^{\alpha_s'}(t_s',F)\hat{\mathcal{O}}_{\alpha_r}^\dagger(t_r,F)\cdots\hat{\mathcal{O}}_{\alpha_1}^\dagger(t_1,F)], \quad (A8)$$

$$W_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}(t_1',\ldots,t_s';t_1,\ldots,t_r;F) := \text{Tr}_\phi[\hat{\rho}_\phi \hat{\mathcal{O}}_{\alpha_1'}^\dagger(t_1',F)\cdots\hat{\mathcal{O}}_{\alpha_s'}^\dagger(t_s',F)\hat{\mathcal{O}}_{\alpha_r}^\dagger(t_r,F)\cdots\hat{\mathcal{O}}_{\alpha_1}^\dagger(t_1,F)], \quad (A9)$$

$$W^{\boldsymbol{\alpha},\boldsymbol{\alpha}'}(t_1',\ldots,t_s';t_1,\ldots,t_r;F) := \text{Tr}_\phi[\hat{\rho}_\phi \hat{\mathcal{O}}^{\alpha_1'}(t_1',F)\cdots\hat{\mathcal{O}}^{\alpha_s'}(t_s',F)\hat{\mathcal{O}}^{\alpha_r}(t_r,F)\cdots\hat{\mathcal{O}}^{\alpha_1}(t_1,F)], \quad (A10)$$

and

$$\hat{\mathcal{O}}^\alpha(t, F) := \int dz \sqrt{-g} F(z) \hat{\mathcal{O}}^\alpha(t, z). \tag{A11}$$

We can rewrite Eq. (A6) as

$$\text{Tr}_\phi(\hat{U}_\xi^{(r)} \hat{\rho}_0 \hat{U}_\xi^{(s)\dagger}) = (-i)^r i^s \lambda^k \int_{\Delta_r \times \Delta_s'} d\boldsymbol{\tau} d\boldsymbol{\tau}' \Xi(\boldsymbol{\tau}; \boldsymbol{\tau}') \hat{\omega}(\boldsymbol{\tau}; \boldsymbol{\tau}'; F). \tag{A12}$$

Here,

$$\Delta_r := \{\boldsymbol{\tau} = (\tau_1, ..., \tau_r) : \tau_1 \in [0, T], \tau_i \in [0, \tau_{i-1}], \ \forall \ i = 2, ..., r\}, \tag{A13}$$

$$\Delta_s' := \{\boldsymbol{\tau}' = (\tau_1', ..., \tau_s') : \tau_1' \in [0, T], \tau_i \in [0, \tau_{i-1}], \ \forall \ i = 2, ..., s\}, \tag{A14}$$

are triangular domains; the function Ξ is an abbreviation,

$$\Xi(\boldsymbol{\tau}; \boldsymbol{\tau}') := \prod_{l=1}^r \xi(\tau_l) \prod_{m=1}^s \xi(\tau_m'), \tag{A15}$$

and $\hat{\omega}$ is the operator corresponding to the expression in square brackets of Eq. (A6).

On the other hand, if we consider the evolution produced by the coupling through the train of sudden interactions described by $\chi_\xi(N)$, from Eqs. (3) and (10) we have that the joint field-detector time-evolution is given by

$$\hat{U}(N) = \mathcal{T}_\tau \exp\left\{-i\lambda \frac{T}{N} \sum_{j=1}^N \xi\left(\frac{j-1/2}{N} T\right) \left[\hat{\mu}_\alpha^\dagger\left(\frac{j-1/2}{N} T\right) \int dz \sqrt{-g_j} F(z) \hat{\mathcal{O}}^\alpha\left(\frac{j-1/2}{N}, z\right) + \text{H.c.}\right]\right\}, \tag{A16}$$

with the final state of the detector thus being

$$\hat{\rho}_D(N) = \text{Tr}_\phi(\hat{U}(N) \hat{\rho}_0 \hat{U}(N)^\dagger). \tag{A17}$$

Under the same conditions as before, we write

$$\hat{U} = \mathbb{1} + \sum_{k=1}^\infty \hat{U}^{(k)}(N) \quad \text{and} \quad \hat{\rho}_D(N) = \hat{\rho}_{D,0} + \sum_{k=1}^\infty \hat{\rho}_D^{(k)}(N), \tag{A18}$$

where again $\hat{U}^{(k)}(N)$ and $\hat{\rho}_D^{(k)}(N)$ are proportional to λ^k, and

$$\hat{\rho}_D^{(k)}(N) = \sum_{r+s=k} \text{Tr}_\phi(\hat{U}^{(r)}(N) \hat{\rho}_0 \hat{U}^{(s)}(N)^\dagger). \tag{A19}$$

In this case, for $r + s = k$, we have

$$
\begin{aligned}
\text{Tr}_\phi(\hat{U}^{(r)}(N) \hat{\rho}_0 \hat{U}^{(s)}(N)^\dagger) = {}& (-i)^r i^s \frac{\lambda^k T^k}{N^k} \sum_{\boldsymbol{j} \in \mathcal{J}_r} \sum_{\boldsymbol{j}' \in \mathcal{J}_s'} \prod_{n=1}^r \xi\left(\frac{j_n - 1/2}{N} T\right) \prod_{m=1}^s \xi\left(\frac{j_m' - 1/2}{N} T\right) \\
& \times \left[W_{\boldsymbol{\alpha}'}^{\boldsymbol{\alpha}}\left(\frac{j_1' - 1/2}{N} T, ..., \frac{j_s' - 1/2}{N} T; \frac{j_1 - 1/2}{N} T, ..., \frac{j_r - 1/2}{N} T; F\right) \right. \\
& \times \hat{\mu}_{\alpha_r}^\dagger\left(\frac{j_r - 1/2}{N} T\right) \cdots \hat{\mu}_{\alpha_1}^\dagger\left(\frac{j_1 - 1/2}{N} T\right) \hat{\rho}_{D,0} \hat{\mu}^{\alpha_1'}\left(\frac{j_1' - 1/2}{N} T\right) \cdots \hat{\mu}^{\alpha_s'}\left(\frac{j_s' - 1/2}{N} T\right) \\
& \left. + W_{\boldsymbol{\alpha}}^{\boldsymbol{\alpha}'}\left(\frac{j_1' - 1/2}{N} T, ..., \frac{j_s' - 1/2}{N} T; \frac{j_1 - 1/2}{N} T, ..., \frac{j_r - 1/2}{N} T; F\right) \right.
\end{aligned}
$$

$$\times \hat{\mu}^{\alpha_r}\left(\frac{j_r - 1/2}{N}T\right)\cdots\hat{\mu}^{\alpha_1}\left(\frac{j_1 - 1/2}{N}T\right)\hat{\rho}_{\text{D},0}\hat{\mu}^{\dagger}_{\alpha'_1}\left(\frac{j'_1 - 1/2}{N}T\right)\cdots\hat{\mu}^{\dagger}_{\alpha'_s}\left(\frac{j'_s - 1/2}{N}T\right)$$

$$+ W_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}\left(\frac{j'_1 - 1/2}{N}T, ..., \frac{j'_s - 1/2}{N}T; \frac{j_1 - 1/2}{N}T, ..., \frac{j_r - 1/2}{N}T; F\right)$$

$$\times \hat{\mu}^{\alpha_r}\left(\frac{j_r - 1/2}{N}T\right)\cdots\hat{\mu}^{\alpha_1}\left(\frac{j_1 - 1/2}{N}T\right)\hat{\rho}_{\text{D},0}\hat{\mu}^{\alpha'_1}\left(\frac{j'_1 - 1/2}{N}T\right)\cdots\hat{\mu}^{\alpha'_s}\left(\frac{j'_s - 1/2}{N}T\right)$$

$$+ W^{\boldsymbol{\alpha},\boldsymbol{\alpha}'}\left(\frac{j'_1 - 1/2}{N}T, ..., \frac{j'_s - 1/2}{N}T; \frac{j_1 - 1/2}{N}T, ..., \frac{j_r - 1/2}{N}T; F\right)$$

$$\times \hat{\mu}^{\dagger}_{\alpha_r}\left(\frac{j_r - 1/2}{N}T\right)\cdots\hat{\mu}^{\dagger}_{\alpha_1}\left(\frac{j_1 - 1/2}{N}T\right)\hat{\rho}_{\text{D},0}\hat{\mu}^{\dagger}_{\alpha'_1}\left(\frac{j'_1 - 1/2}{N}T\right)\cdots\hat{\mu}^{\dagger}_{\alpha'_s}\left(\frac{j'_s - 1/2}{N}T\right)\bigg], \tag{A20}$$

where we have used the same notation as in Eq. (A6), and

$$\mathcal{J}_r := \{\boldsymbol{j} = (j_1, ..., j_r) : j_1 \in \{1, ..., N\}, j_i \in \{1, ..., j_{i-1}\}, \ \forall \ i = 2, ..., r\}, \tag{A21}$$

$$\mathcal{J}'_s := \{\boldsymbol{j}' = (j'_1, ..., j'_s) : j'_1 \in \{1, ..., N\}, j'_i \in \{1, ..., j'_{i-1}\}, \ \forall \ i = 2, ..., s\}. \tag{A22}$$

Using the notation employed in Eq. (A12), we can rewrite Eq. (A20) as

$$\text{Tr}_\phi(\hat{U}^{(r)}(N)\hat{\rho}_0\hat{U}^{(s)}(N)^{\dagger}) = (-\text{i})^r \text{i}^s \frac{\lambda^k T^k}{N^k} \sum_{(\boldsymbol{j},\boldsymbol{j}') \in \mathcal{J}_r \times \mathcal{J}'_s} \Xi(\boldsymbol{\tau}_{\boldsymbol{j}}; \boldsymbol{\tau}'_{\boldsymbol{j}'})\hat{\omega}(\boldsymbol{\tau}_{\boldsymbol{j}}; \boldsymbol{\tau}'_{\boldsymbol{j}'}; F), \tag{A23}$$

where

$$\boldsymbol{\tau}_{\boldsymbol{j}} := \left(\frac{j_1 - 1/2}{N}T, ..., \frac{j_r - 1/2}{N}T\right)$$

and $\quad \boldsymbol{\tau}'_{\boldsymbol{j}'} := \left(\frac{j'_1 - 1/2}{N}T, ..., \frac{j'_s - 1/2}{N}T\right). \tag{A24}$

To prove the approximation result presented in Sec. II, we follow a generalization of the strategy used by C. K. Chui in [84,86]. First, let us define $H_{ab}^{(r,s)} : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{C}$ as

$$H_{ab}^{(r,s)}(\boldsymbol{\tau}; \boldsymbol{\tau}') = (-\text{i})^r \text{i}^s \Xi(\boldsymbol{\tau}; \boldsymbol{\tau}')\omega_{ab}(\boldsymbol{\tau}; \boldsymbol{\tau}'; F)I_{\Delta_r \times \Delta'_s}, \tag{A25}$$

where the dependence on r and s for Ξ and $\hat{\omega}$ manifests itself in the vectors $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$ having r and s components, respectively. Here, ω_{ab} is the (a, b)th matrix component of $\hat{\omega}$ in some basis of the detector Hilbert space $\mathcal{H}_{\text{D}}$, and I_X is the indicator function for the set X, i.e., a function that equals 1 when the argument belongs to X, and 0 otherwise. With this definition, from Eq. (A12) we have that

$$[\text{Tr}_\phi(\hat{U}_\xi^{(r)}\hat{\rho}_0\hat{U}_\xi^{(s)\dagger})]_{ab} = \lambda^k \int_{[0,T]^k} \text{d}\boldsymbol{z}H_{ab}^{(r,s)}(\boldsymbol{z}), \tag{A26}$$

where the subindices ab again denote the (a, b)th matrix element of the trace in the chosen basis of $\mathcal{H}_{\text{D}}$. It is worth remarking that the presence of the indicator function in the definition of H_{ab} allowed us to have here the hypercube $[0, T]^k$ as integration domain, instead of the product of triangular domains of Eq. (A12). Meanwhile, from Eq. (A23) we also have that

$$[\text{Tr}_\phi(\hat{U}^{(r)}(N)\hat{\rho}_0\hat{U}^{(s)}(N)^{\dagger})]_{ab} = \lambda^k R_N^{(k)}[H_{ab}^{(r,s)}], \tag{A27}$$

where $R_N^{(k)}$ is the Riemann sum associated with the uniform partition of $[0, T]^k$ into N^k identical cubes with tags placed in their centers, i.e., for an arbitrary $h : \mathbb{R}^k \to \mathbb{C}$,

$$R_N^{(k)}[h] := \sum_{\boldsymbol{j}} h\left(\frac{j_1 - 1/2}{N}T, ..., \frac{j_k - 1/2}{N}T\right) \cdot \frac{T^k}{N^k}, \tag{A28}$$

with each component of $\boldsymbol{j}$ running over $\{1, ..., N\}$. Let us now define $v_N^{(k)} : \mathbb{R}^k \to \mathbb{C}$ as

$$v_N^{(k)}(\boldsymbol{z}) = \sum_{\boldsymbol{j}} T^k \prod_{i=1}^{k} I_{[\frac{j_i - 1/2}{N}, 1]}\left(\frac{z_i}{T}\right) - N^k \prod_{i=1}^{k} z_i, \tag{A29}$$

where again each component of j runs over $\{1, ..., N\}$. With this definition, we can write

$$R_N^{(k)}[H_{ab}^{(r,s)}] - \int_{[0,T]^k} \mathrm{d}z H_{ab}^{(r,s)}(z) = \frac{1}{N^k} \int_{[0,T]^k} \mathrm{d}v_N^{(k)} H_{ab}^{(r,s)},$$

(A30)

where the right-hand side should be understood as a k-dimensional Riemann-Stieltjes integral of $H_{ab}^{(r,s)}$ over $[0, T]^k$ with respect to $v_N^{(k)}$, as defined in Appendix B. Now, $H_{ab}^{(r,s)}$ is a product of functions including Ξ [defined in Eq. (A15)], which is zero whenever one of its arguments is 0 or T. In particular, $H_{ab}^{(r,s)}$ is zero along the boundary of $[0, T]^k$, so by Lemma 1 in Appendix B,

$$R_N^{(k)}[H_{ab}^{(r,s)}] - \int_{[0,T]^k} \mathrm{d}z H_{ab}^{(r,s)}(z) = \frac{(-1)^k}{N^k} \int_{[0,T]^k} \mathrm{d}H_{ab}^{(r,s)} v_N^{(k)}.$$

(A31)

Since $v_N^{(k)}$ is bounded in $[0, T]^k$, with bound

$$\|v_N^{(k)}\|_\infty \le \frac{kT^k}{2} N^{k-1},$$

(A32)

assuming $H_{ab}^{(r,s)}$ is of bounded variation, we can use Lemma 2 in Appendix B to conclude that

$$\left| R_N^{(k)}[H_{ab}^{(r,s)}] - \int_{[0,T]^k} \mathrm{d}z H_{ab}^{(r,s)}(z) \right| \le \frac{k\tilde{C}_{ab}^{(r,s)}}{2N} T^k \equiv \frac{C_{ab}^{(r,s)}}{N},$$

(A33)

where $\tilde{C}_{ab}^{(r,s)}$ is the variation of $H_{ab}^{(r,s)}$ in $[0, T]^k$. If $H_{ab}^{(r,s)}$ is of bounded variation for all pairs (r, s) such that $r + s = k$, from Eqs. (A5) and (A19) we get that

$$|[\hat{\rho}_\mathrm{D}^{(k)}(N)]_{ab} - [\hat{\rho}_\mathrm{D}^{(k)\xi}]_{ab}| \le \lambda^k \sum_{k=r+s} \frac{C_{ab}^{(r,s)}}{N} = \lambda^k \frac{C_{ab}^{(k)}}{N}, \quad (A34)$$

for some $C_{ab}^{(k)}$, which proves Eq. (17). This also implies

$$\lim_{N\to\infty} [\hat{\rho}_\mathrm{D}^{(k)}(N)]_{ab} = [\hat{\rho}_\mathrm{D}^{(k)\xi}]_{ab}, \quad (A35)$$

which proves Eq. (16). In particular, if Eq. (A33) holds at all orders, Eqs. (18) and (19) are also satisfied for all values of the coupling strength for which the Dyson series converges.

In order to arrive at the bound given in Eq. (A33), we needed to assume that $H_{ab}^{(r,s)}$ is of bounded variation as a function of its (time) arguments. Since at the end of the day we want to use this nonperturbative tool for physical

scenarios, it is worth discussing how exotic this condition is.

Notice first that $H_{ab}^{(r,s)}$ depends on Ξ, which is a product of the values of the switching functions at different times, and the matrix elements ω_{ab}, which involve a linear combination of smeared k-point correlation functions of the field operator $\hat{\mathcal{O}}^\alpha$ and the corresponding matrix element of an operator that consists of products of $\hat{\mu}$ with $\hat{\rho}_{\mathrm{D},0}$. If the dependence of $\hat{\mu}$ on time in the interaction picture only comes from the free evolution of the detector, as it is often the case, then we can assume that it is a smooth function of time. That way, whether $H_{ab}^{(r,s)}$ is sufficiently regular for Eq. (A33) to hold or not ends up depending on the switching function (through Ξ) and the state of the field (through the smeared k-point correlation functions for the operator $\hat{\mathcal{O}}$). By Lemma 3 in Appendix B, one sufficient (but not necessary) condition for $H_{ab}^{(r,s)}$ to be of bounded variation is that it is of class $\mathcal{C}^k$ in $[0, T]^k$. In particular, this can be straightforwardly extended to the case in which $H_{ab}^{(r,s)}$ is bounded, and $\mathcal{C}^k$ except for maybe a finite number of points in $[0, T]^k$.

Now, the (continuous) switching functions typically employed in particle detector setups (e.g., Heaviside, Gaussian, and Lorentzian couplings) are smooth except for maybe a finite number of points. Therefore, when ξ is one of these common switching functions, it would suffice that the field's k-point correlation functions be of class $\mathcal{C}^k$ in $[0, T]^k$ for $H_{ab}^{(r,s)}$ to be bounded and of class $\mathcal{C}^k$ except for maybe a finite number of points, for all (r, s) such that $r + s = k$.

Let us consider, for instance, the vacuum of a real scalar field in $(1 + 3)$-dimensional Minkowski spacetime. For this case, the odd correlation functions vanish, while the even correlation functions can all be expressed as sums of products of the two-point function

$$w_2(\mathsf{x}_1, \mathsf{x}_2) = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3 2|\boldsymbol{k}|} e^{\mathrm{i}\boldsymbol{k}\cdot(\mathsf{x}_1 - \mathsf{x}_2)}. \quad (A36)$$

If we have a linear coupling with the field amplitude, as in the Unruh-DeWitt model [9,70], $\hat{\mathcal{O}} = \hat{\phi}$, and the relevant k-point functions can be given in terms of the smeared two-point function

$$W(t; t'; F) = \int \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{x}' F(\boldsymbol{x}) F(\boldsymbol{x}') w_2(\mathsf{x}_1, \mathsf{x}_2)$$

$$= \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^n 2|\boldsymbol{k}|} |\tilde{F}(\boldsymbol{k})|^2 e^{-\mathrm{i}|\boldsymbol{k}|(t-t')}, \quad (A37)$$

for a given smearing function F. One very popular choice of smearing function in the literature is the Gaussian of variance σ^2,

$$G(\boldsymbol{x}) = \frac{1}{\sqrt{\pi^3}\sigma^3} e^{-x^2/\sigma^2}. \tag{A38}$$

With a Gaussian smearing, a straightforward calculation using spherical coordinates yields

$$W(t;t';G) = \frac{1}{4\pi^2} \int_0^\infty d|\boldsymbol{k}| |\boldsymbol{k}| e^{-|\boldsymbol{k}|^2\sigma^2/2 - i|\boldsymbol{k}|(t-t')}$$
$$= \frac{1 - \sqrt{\frac{\pi}{2\sigma^2}}(t-t')e^{-(t-t')^2/2\sigma^2}[i + \mathrm{erfi}(\frac{t-t'}{\sqrt{2}\sigma})]}{\sigma^2}, \tag{A39}$$

which is a smooth function of t and t'. Another common choice is the hard sphere of radius $R > 0$,

$$S(\boldsymbol{x}) = \theta(R - |\boldsymbol{x}|). \tag{A40}$$

Notice that its Fourier transform

$$\tilde{S}(\boldsymbol{k}) = \frac{4\pi}{k^3}[\sin(kR) - kR\cos(kR)] \tag{A41}$$

converges to a constant when $k \to 0$, and behaves like $1/k^2$ as $k \to \infty$, so that

$$W(t;t';S) = 4\int_0^\infty d|\boldsymbol{k}| \frac{[\sin(|\boldsymbol{k}|R) - |\boldsymbol{k}|R\cos(|\boldsymbol{k}|R)]^2}{|\boldsymbol{k}|^5} e^{-i|\boldsymbol{k}|(t-t')} \tag{A42}$$

is convergent. It can be shown, moreover, that it is of bounded variation, even though in this case

$$\frac{\partial^2}{\partial t \partial t'}W(t;t';S)$$
$$= 4\int_0^\infty d|\boldsymbol{k}| \frac{[\sin(|\boldsymbol{k}|R) - |\boldsymbol{k}|R\cos(|\boldsymbol{k}|R)]^2}{|\boldsymbol{k}|^3} e^{-i|\boldsymbol{k}|(t-t')} \tag{A43}$$

fails to converge and therefore we cannot use Lemma 3 to reach that conclusion. Thus, we see that for two of the most common smearing functions,[5] the vacuum of a real scalar field in $(1 + 3)$-dimensional Minkowski satisfies the condition for Eq. (A35) to hold, *at all orders* in perturbation theory. Since, for two or more spatial dimensions,[6] the possible nonsmoothness of the field correlation functions can only arise from the ultraviolet behavior of the field state, we can argue that any Hadamard state, both in flat and curved spacetime (barring pathological geometries), will essentially have the same ultraviolet behavior as the

Minkowski vacuum, and therefore should in principle satisfy the assumptions of the approximation result as well.

Finally, we can turn to the multiple detector scenario. The proof proceeds in just the same way as for the single detector case. For the sake of brevity and simplicity we will not give its details here, but we argue that the procedure is completely analogous. Specifically, the similar structure of Eqs. (4) and (23) is revealing of how close both scenarios are from the technical point of view. The case of multiple detectors will in general involve products of different switching functions and different detector operators, as well as field correlation functions associated to potentially different field operators, smeared over different spatial profiles. However, one can give homologous definitions of Ξ [cf. Eq. (A15)] and $\hat{\omega}$ [cf. Eq. (A6)] that lead to the multiple detectors version of Eq. (A12). The steps that follow are exactly those followed above for the one detector scenario, adapted to the new integration regions, which for the case of multiple detectors are general rectangles, instead of the hypercubes we dealt with in the case of a single detector.

APPENDIX B: THE n-DIMENSIONAL RIEMANN-STIELTJES INTEGRAL

Even though the Riemann-Stieltjes integral is a basic tool in real analysis, its generalization to the n-dimensional case is not as popular in classical texts. Because of that, in this appendix we present a very brief introduction to the n-dimensional Riemann-Stieltjes integral—or rather, its generalization to functions of complex argument—and we prove the three Lemmas we referred to in Appendix A.

Let $f, g : \mathbb{R}^n \to \mathbb{C}$, and let $R = [x_1, y_1] \times \cdots \times [x_n, y_n] \subset \mathbb{R}^n$ be a closed rectangle in $\mathbb{R}^n$. Following [87], we define

$$\Delta_R^{(n)}g = \sum_{\boldsymbol{z}}(-1)^{\#x(\boldsymbol{z})}g(\boldsymbol{z}), \tag{B1}$$

where the sum runs over all $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$ such that $z_i = x_i$ or y_i, for each $i = 1, \ldots, n$, and $\#x(\boldsymbol{z})$ is the number of components of $\boldsymbol{z}$ that are equal to x_i and not y_i. This definition is motivated by the fact that for the function $p(\boldsymbol{z}) = \prod_{i=1}^n z_i$, we have

$$\Delta_R^{(n)}p = \prod_{i=1}^n(y_i - x_i) = \mathrm{Vol}(R). \tag{B2}$$

Now, for each $i = 1, \ldots, n$, consider a partition of the interval $[x_i, y_i]$,

$$P_i = \{t_0^{(i)} = x_i, \ldots, t_{M_i}^{(i)} = y_i\}, \tag{B3}$$

with a set of tags

$$T_i = \{u_1, \ldots, u_{M_i} : u_j^{(i)} \in [t_{j-1}^{(i)}, t_j^{(i)}], \forall j = 1, \ldots, M_i\}. \tag{B4}$$

---

[5]For a more detailed discussion of when and how the smearing function regularizes the two-point distribution, see [74].

[6]Also for one spatial dimension, assuming that the infrared divergences are properly regularized.

All these partitions together induce an n-dimensional rectangular partition $P \equiv P_1 \times \cdots \times P_n$ of the rectangle R, formed by subrectangles

$$R_j = [t_{j_1-1}^{(1)}, t_{j_1}^{(1)}] \times \cdots \times [t_{j_n-1}^{(n)}, t_{j_n}^{(n)}] \subset R \qquad \text{(B5)}$$

that can only overlap on their edges. Each subrectangle also has an associated tag $\boldsymbol{u_j} = (u_{j_1}, \ldots, u_{j_n})$, and we can define the norm of P as the maximum among the norms of each P_i, i.e.,

$$\|P\| := \max_{i=1,\ldots,n} \|P_i\| = \max\{|t_j^{(i)} - t_{j-1}^{(i)}| :$$

$$j = 1, \ldots, M_i, i = 1, \ldots, n\}. \qquad \text{(B6)}$$

We define the Riemann-Stieltjes sum associated with f, g, and the n-dimensional tagged partition P of the rectangle R as

$$S_R(f, g; P) = \sum_j f(\boldsymbol{u_j}) \Delta_{R_j}^{(n)} g. \qquad \text{(B7)}$$

Hence, we say that f is Riemann-Stieltjes integrable with respect to g in R when there exists $I \in \mathbb{C}$ such that, for every $\epsilon > 0$, there is some $\delta > 0$ for which

$$|S_R(f, g; P) - I| < \epsilon \qquad \text{(B8)}$$

holds whenever $\|P\| < \delta$. In that case, we call I the Riemann-Stieltjes integral of f with respect to g over R,

$$I \equiv \int_R \mathrm{d}g \, f. \qquad \text{(B9)}$$

Lemma 1. Let $f, g : \mathbb{R}^n \to \mathbb{C}$, and let $R = [x_1, y_1] \times \cdots \times [x_n, y_n]$ be a rectangle in $\mathbb{R}^n$, such that f is integrable over R with respect to g, and either $f = 0$ or $g = 0$ on the boundary ∂R. Then, g is also integrable over R with respect to f, and

$$\int_R \mathrm{d}f \, g = (-1)^n \int_R \mathrm{d}g \, f. \qquad \text{(B10)}$$

Proof. Given $\epsilon > 0$, since f is integrable over R with respect to g, there exists $\delta > 0$ such that if $P = P_1 \times \cdots \times P_n$ is an n-dimensional tagged partition of R with $\|P\| < \delta$, then

$$\left| S_R(f, g; P) - \int_R \mathrm{d}g \, f \right| < \epsilon. \qquad \text{(B11)}$$

Consider now a rectangular partition of R, $P = P_1 \times \cdots \times P_n$, such that $\|P\| < \delta/2$. For each one-dimensional partition

$P_i = \{t_0^{(i)} = x_i, \ldots, t_{M_i}^{(i)} = y_i\}$ of the interval $[x_i, y_i]$, with tags $\{u_1, \ldots, u_{M_i}\}$, consider another partition

$$\bar{P}_i := \{t_0^{(i)} = x_i, u_1, \ldots, u_{M_i}, t_{M_i} = y_i\}, \qquad \text{(B12)}$$

with set of tags $\bar{T}_i = P_i$, i.e.,

$$\bar{T}_i = \{v_1, \ldots, v_{M_i}, v_{M_i+1} : v_1 = t_0, v_j = t_{j-1}, \forall \, j > 1\}. \qquad \text{(B13)}$$

Notice in particular that the set of tags of $\bar{P}_i$ includes the limits of the interval, x_i and y_i. We can then define a new partition $\bar{P} = \bar{P}_1 \times \cdots \times \bar{P}_n$, with set of tags $\bar{T} = \bar{T}_1 \times \cdots \times \bar{T}_n$, which by construction satisfies $\|P\| < \delta$. Moreover, a simple calculation shows that

$$S(f, g; \bar{P}) = (-1)^n S(g, f; P) + \text{Boundary terms}, \qquad \text{(B14)}$$

where the boundary terms only include sums of products of f and g at points of the boundary ∂R. Since either $f = 0$ or $g = 0$ on the boundary, the associated terms cancel out. Equations (B11) and (B14) thus allow us to conclude that

$$\left| (-1)^n S_R(g, f; P) - \int_R \mathrm{d}g \, f \right| < \epsilon, \qquad \text{(B15)}$$

and therefore that g is integrable over R with respect to f, with

$$\int_R \mathrm{d}f \, g = (-1)^n \int_R \mathrm{d}g \, f, \qquad \text{(B16)}$$

∎

Lemma 2. Let $f, g : \mathbb{R}^n \to \mathbb{C}$, and let $R = [x_1, y_1] \times \cdots \times [x_n, y_n]$ be a rectangle in $\mathbb{R}^n$. If f is bounded in R and g is of bounded variation in R, then we have that

$$\left| \int_R \mathrm{d}g \, f \right| \le \|f\|_\infty \text{Var}_R(g), \qquad \text{(B17)}$$

where

$$\|f\|_\infty = \sup_R |f| \qquad \text{(B18)}$$

is the optimal bound for f in R, and

$$\text{Var}_R(g) := \sup_P \sum_j |\Delta_{R_j}^{(n)} g| \qquad \text{(B19)}$$

is the variation of g in R, with the supremum taken over all rectangular partitions P of the rectangle R, and the notation used as in Eq. (B7).

Proof. Since f is integrable over R with respect to g, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| S_R(f,g;P) - \int_R \mathrm{d}g\, f \right| < \epsilon \tag{B20}$$

whenever $\|P\| < \delta$. Now,

$$|S_R(f,g;P)| = \left| \sum_j f(\boldsymbol{u}_j) \Delta_{R_j}^{(n)} g \right| \le \sum_j |f(\boldsymbol{u}_j)| |\Delta_{R_j}^{(n)} g|$$

$$\le \|f\|_\infty \sum_j |\Delta_{R_j}^{(n)} g| \le \|f\|_\infty \mathrm{Var}_R(g), \tag{B21}$$

and

$$\left| S_R(f,g;P) - \int_R \mathrm{d}g\, f \right| \ge \left| \int_R \mathrm{d}g\, f \right| - |S_R(f,g;P)|, \tag{B22}$$

so we conclude from Eq. (B20) that

$$\left| \int_R \mathrm{d}g\, f \right| \le \|f\|_\infty \mathrm{Var}_R(g) + \epsilon. \tag{B23}$$

Since Eq. (B23) holds for all $\epsilon > 0$, this proves the claim. ∎

Lemma 3. Let $g \colon \mathbb{R}^n \to \mathbb{C}$, and let $R = [x_1, y_1] \times \cdots \times [x_n, y_n]$ be a rectangle in $\mathbb{R}^n$. If g is of class $\mathcal{C}^n(R)$, then g is of bounded variation in R, and

$$\mathrm{Var}_R(g) = \int_R \mathrm{d}\boldsymbol{z} \left| \frac{\partial^n g}{\partial z_1 \cdots \partial z_n} \right|. \tag{B24}$$

Proof. Consider a rectangle $K = [v_1, w_1] \times \cdots \times [v_n, w_n] \subset R$. Singling out the first coordinate, we can write

$$\Delta_K^{(n)}(g) = \sum_{\boldsymbol{z}} (-1)^{\#v(\boldsymbol{z})} [g(w_1, \boldsymbol{z}) - g(v_1, \boldsymbol{z})]$$

$$= \sum_{\boldsymbol{z}} (-1)^{\#v(\boldsymbol{z})} g(w_1, \boldsymbol{z}) - \sum_{\boldsymbol{z}} (-1)^{\#v(\boldsymbol{z})} g(v_1, \boldsymbol{z}), \tag{B25}$$

where $\boldsymbol{z} = (z_2, \ldots, z_n) \in \mathbb{R}^{n-1}$, and $z_i = v_i$ or w_i for each $i = 2, \ldots, n$, with $\#v(\boldsymbol{z})$ being as before the number of components z_i of $\boldsymbol{z}$ that are equal to v_i. If we define

$$g_1(w) := \sum_{\boldsymbol{z}} (-1)^{\#v(\boldsymbol{z})} g(w, \boldsymbol{z}), \tag{B26}$$

then g_1 is continuously differentiable, and by the mean value theorem there exists $c_1 \in (v_1, w_1)$ such that

$$\Delta_K^{(n)}(g) = g_1'(c_1)(w_1 - v_1)$$

$$= (w_1 - v_1) \sum_{\boldsymbol{z}} (-1)^{\#v(\boldsymbol{z})} \frac{\partial g}{\partial z_1}(c_1, \boldsymbol{z}). \tag{B27}$$

If $n = 1$, we stop here. Otherwise, we can define $K_1 := [v_2, w_2] \times \cdots \times [v_n, w_n]$, and realize from Eq. (B27) that

$$\Delta_K^{(n)}(g) = (w_1 - v_1) \Delta_{K_1}^{(n-1)} \left(\frac{\partial g}{\partial z_1}(c_1, \cdot) \right). \tag{B28}$$

Since g is of class $\mathcal{C}^n(R)$, the partial derivative is of class $\mathcal{C}^{n-1}(R)$, and in particular so is its restriction to K_1. We can then repeat the previous argument with the partial derivative instead of g, concluding after this second step that there exists $c_2 \in (v_2, w_2)$ such that

$$\Delta_K^{(n)}(g) = (w_1 - v_1)(w_2 - v_2) \sum_{\boldsymbol{z}} (-1)^{\#v(\boldsymbol{z})} \frac{\partial^2 g}{\partial z_1 \partial z_2}(c_1, c_2, \boldsymbol{z}), \tag{B29}$$

where now $\boldsymbol{z} \in \mathbb{R}^{n-2}$ runs only over the last $n - 2$ components (if any). In general, after n iterations we find that there exists $\boldsymbol{c}$ in the interior of K such that

$$\Delta_K^{(n)}(g) = \left[\prod_{i=1}^n (w_i - v_i) \right] \frac{\partial^n g}{\partial z_1 \cdots \partial z_n}(\boldsymbol{c})$$

$$= \mathrm{Vol}(K) \frac{\partial^n g}{\partial z_1 \cdots \partial z_n}(\boldsymbol{c}). \tag{B30}$$

Now, the variation of g in R is given by

$$\mathrm{Var}_R(g) = \sup_P \sum_j |\Delta_{R_j}^{(n)} g|, \tag{B31}$$

where the supremum is taken over all possible rectangular partitions of R. By Eq. (B30), given an arbitrary partition P of R,

$$\sum_j |\Delta_{R_j}^{(n)} g| = \sum_j \left| \frac{\partial^n g}{\partial z_1 \cdots \partial z_n}(\boldsymbol{c}_j) \right| \mathrm{Vol}(R_j), \tag{B32}$$

where each $\boldsymbol{c}_j$ belongs to the interior of R_j. This corresponds to the Riemann sum of $\partial^n g / \partial z_1 \cdots \partial z_n$ associated with the partition P with tags $\boldsymbol{c}_j$. Since $\partial^n g / \partial z_1 \cdots \partial z_n$ is continuous, in particular it is Riemann integrable, and therefore given any $\epsilon > 0$, there exists $\delta > 0$ such that if $\|P\| < \delta$, then

$$\left| \int_R d\mathbf{z} \left| \frac{\partial^n g}{\partial z_1 \cdots \partial z_n}(\mathbf{z}) \right| - \sum_j \left| \frac{\partial^n g}{\partial z_1 \cdots \partial z_n}(\mathbf{c}_j) \right| \mathrm{Vol}(R_j) \right| < \epsilon.$$
(B33)

In particular,

$$\int_R d\mathbf{z} \left| \frac{\partial^n g}{\partial z_1 \cdots \partial z_n}(\mathbf{z}) \right| - \epsilon < \sum_j \left| \frac{\partial^n g}{\partial z_1 \cdots \partial z_n}(\mathbf{c}_j) \right| \mathrm{Vol}(R_j)$$
$$\leq \mathrm{Var}_R(g).$$
(B34)

This is true for all $\epsilon > 0$, and thus

$$\int_R d\mathbf{z} \left| \frac{\partial^n g}{\partial z_1 \cdots \partial z_n}(\mathbf{z}) \right| \leq \mathrm{Var}_R(g).$$
(B35)

Conversely, for any partition P,

$$\sum_j |\Delta_{R_j}^{(n)} g| = \sum_j \left| \int_{R_j} d\mathbf{z} \frac{\partial^n g}{\partial z_1 \cdots \partial z_n}(\mathbf{z}) \right|$$
$$\leq \sum_j \int_{R_j} d\mathbf{z} \left| \frac{\partial^n g}{\partial z_1 \cdots \partial z_n}(\mathbf{z}) \right|$$
$$= \int_R d\mathbf{z} \left| \frac{\partial^n g}{\partial z_1 \cdots \partial z_n}(\mathbf{z}) \right|.$$
(B36)

The supremum taken over all possible partitions still verifies the inequality, i.e,

$$\mathrm{Var}_R(g) \leq \int_R d\mathbf{z} \left| \frac{\partial^n g}{\partial z_1 \cdots \partial z_n}(\mathbf{z}) \right|.$$
(B37)

Combining Eqs. (B35) and (B37) gives us the desired result. ∎

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