Amplitudes and renormalization group techniques: A case study

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We explore the properties of a simple renormalizable shift-symmetric model with a higher-derivative kinetic energy and quartic-derivative coupling that can serve as a toy model for higher-derivative theories of gravity. The scattering amplitude behaves as in a normal effective field theory below the threshold for the production of ghosts, but has an unexpectedly soft behavior above the threshold. The physical running of the parameters is extracted from the two-point and four-point amplitudes. The results are compared to those obtained by other methods and are found to agree only in limiting cases. We draw several lessons that may also apply to gravity.

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I. INTRODUCTION

There are various ways of understanding the notion of "running coupling" and the associated renormalization group (RG), and they do not always give equivalent results. We can distinguish at least three different notions. The one that is of greatest significance in particle physics arises when one identifies the running coupling by a measurement of a physical amplitude, at a renormalization scale μ_R. We refer to this as "physical running." It is in this sense that one understands, for example, the running of the couplings of the standard model. The second notion is the dependence of the renormalization of the coupling on a scale μ that is introduced to fix dimensions, such as factors μ^{4-d} in dimensional regularization or $\log(\Lambda/\mu)$ if cutoff regularization is used. This is a less physical notion because such a scale is just an intermediate device that does not appear in physical observables. In particular, μ is conceptually different from the renormalization scale μ_R. However, it is often useful because it can be easier to calculate, and in the right circumstances (that we discuss in more detail later) it is a good proxy for the momentum dependence. Finally, there is a more general notion of "running," defined as the dependence of a coupling on some external mass or energy scale, typically a UV or IR cutoff, where one often considers the simultaneous running of many (even infinitely many) couplings, collected into some kind of potential or action. This originated with Wilson's RG [1] and its generalizations [2,3]. A more recent version of it applies to a scale-dependent one-particle-irreducible effective action, which we refer to here as the functional RG (FRG) [4,5]. These are much more broader definitions of RG, but in some particular cases they can be related to the other two. They have proven very useful in the understanding of critical phenomena and in other contexts; see Ref. [6] for a recent review.

One of the goals of this paper is to illustrate the relations between these notions in a very simple model consisting of a single scalar with the Lagrangian[1]

$$\mathcal{L} = -\frac{1}{2}Z_1(\partial\phi)^2 - \frac{1}{2}Z_2(\Box\phi)^2 - \frac{1}{4}g((\partial\phi)^2)^2. \quad (1)$$

It has a shift symmetry $\phi \to \phi + \text{const}$ and reflection symmetry $\phi \to -\phi$, and it is renormalizable despite the derivative interaction term. The higher-derivative interactions are similar to those that appear in gravitational theories. The inclusion of both two and four derivatives in the kinetic energies is typical of many applications of the FRG in which operators of different dimensions appear. It is also a crucial part of Lee-Wick theories [7] and quadratic gravity [8,9].

Superficially, this theory appears to be pathological. First, theories with higher-derivative kinetic energies contain extra degrees of freedom. This can be seen from the propagator in the full theory (with $Z_1 = 1$ and $Z_2 = 1/m^2$)

$$iD_F(q^2) = \frac{-i}{p^2 + \frac{p^4}{m^2}} = -i\left[\frac{1}{p^2} - \frac{1}{p^2 + m^2}\right]. \quad (2)$$

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[1]We use the metric signature $(-,+,+,+)$.

We see that besides a normal massless particle there is also a ghost with mass m. The signs in the original Lagrangian have been chosen to have the massive pole at timelike momentum, because the opposite sign would have the pole being tachyonic. Second, as we discuss, the theory is asymptotically free in the sense that the coupling runs logarithmically to zero in the UV limit, but only for negative coupling. In this way it is reminiscent of Symanzik's observation that ordinary $\lambda\Phi^4$ theory is asymptotically free for $\lambda < 0$ [10]. In spite of this, we can study the renormalization of this model and draw from it some useful lessons.

Besides being an interesting case study for several aspects of renormalization theory, our model is also of independent interest and has appeared recently in various different contexts. Without the higher-derivative kinetic term it is a textbook example of effective field theory (EFT), being the low-energy description of a $U(1)$-invariant linear sigma model in the Higgs phase [11,12]. With the higher-derivative kinetic term, it is the low-energy EFT for the higher-derivative version of the same model. As a conformal field theory, the higher-derivative model has been discussed in [13]. In the context of asymptotic safety, it has been presented as a type of matter interaction that would necessarily have to be present if gravity has a nontrivial fixed point [14,15]. It has also been treated by the full machinery of the FRG by two of the present authors [16], viewing it as a toy model for gravity. It was found to run from the higher-derivative free fixed point at high energy to the standard free fixed point at low energy. Finally, it has been studied recently by Tseytlin [17] and by Holdom [18], who found evidence that the model may be less pathological than would first appear.

We always assume that the field ϕ has mass dimension one, which is the natural choice when we interpret the two-derivative term as defining the propagator. Then, Z_2 and g have dimension of inverse mass to the power of 2 and 4, respectively. It is thus natural to reparametrize

$$\mathcal{L} = -\frac{Z_1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{Z_1}{2m^2}\Box\phi\Box\phi - \frac{Z_1^2 g}{4M^4}(\partial_\mu\phi\partial^\mu\phi)(\partial_\nu\phi\partial^\nu\phi), \tag{3}$$

where m and M are masses and g is dimensionless. Moreover, we have defined the coupling constant with an explicit factor of the mass M in order to make g dimensionless. The value of this somewhat redundant notation is that it facilitates the use of dimensional analysis by showing the mass factors explicitly. The notation is natural when one views this as the low-energy limit of the $U(1)$ linear sigma model, in which case the masses m and M are parametrically independent. Even though here we consider the theory as being potentially UV complete in itself, without the radial mode, we retain this notation. One can set $M = m$ without loss of generality.

In order to see this as a toy model for gravity, we recall that the action of quadratic gravity is schematically of the form $m_P^2 R + \frac{1}{\xi}C^2$ (where C is the Weyl tensor), so if we rescale the metric fluctuation by $m_P \sim 1/\sqrt{G}$, the action contains, among other terms,

$$(\partial h)^2 + \frac{1}{\xi m_P^2}(\Box h)^2 + \frac{1}{\xi m_P^4}(\partial h)^4.$$

Recalling that the mass of the ghost is $m = \xi m_P^2$, this becomes essentially the same as (3) with $Z_1 = 1$, $M = m$, and $g = \xi$.

Irrespective of the notation, it is important to keep in mind that the Lagrangian contains two mass scales: the mass of the ghost, m, and the scale $M/\sqrt[4]{g}$ at which tree-level unitarity is violated and above which one would appear to be in a strongly interacting regime, due to the E^4 derivative interaction. In this paper we always assume that $m < M/\sqrt[4]{g}$, in such a way that the massive ghosts can propagate and still be weakly interacting. Depending on the characteristic scale of the process, we thus have three energy regions which have different behavior:

(1) *Low energy:* This region is defined by energies that are small compared to the ghost mass m. The heavy ghost is not dynamically active and can be integrated out.
(2) *Intermediate energy:* This corresponds to energies above the mass m, but below the apparent strongly interacting regime. Here the heavy ghost is dynamically active. In the Appendix we also briefly comment on an intermediate case where $s \sim -u \gg m^2$ but $t \ll m^2$.
(3) *High energy:* This occurs when the energy is high enough that $gE^4/M^4 > 1$. At these energies, perturbation theory would seem to break down.

In this paper, we study the scattering amplitude of this theory in the first two regimes. At tree level it is given by

$$-\frac{1}{2}g(s^2 + t^2 + u^2). \tag{4}$$

At low energy, quantum corrections generate new effective interactions with six or eight derivatives. This is the expected behavior of a nonrenormalizable theory, treated with standard EFT methods. Somewhat unexpectedly, the higher-dimension operators cancel above the mass threshold for the production of ghosts, leaving us with a theory that looks renormalizable, with a logarithmically running coupling.

By comparing the loop corrections with the two- and four-point amplitudes, we identify the physical running (or lack of running) of the parameters. The results differ in general from those given by other methods, but agree in some limits.

We proceed as follows. We begin by describing the model in the absence of the higher-derivative kinetic term. It is useful to have this description because the full theory reduces to this EFT in the low-energy limit. In Secs. III and IV the calculation of the two-point function and the four-point scattering amplitude are presented. The final results are given in Eq. (16) for the two-point quantum correction, and in Eq. (29) for the full four-point correction. Section V is devoted to an analysis of the amplitude. In particular, we see how to match it at low energy to the previously obtained EFT results, and also consider the remarkable simplifications that occur in the high-energy limit [Eqs. (37) and (42)]. In Sec. VI we compare the physical beta functions derived from the amplitude to the beta functions obtained from the FRG and other definitions. Our main conclusions are summarized in Sec. VII.

II. EFFECTIVE FIELD THEORY AT LOW ENERGY

In generating an EFT one needs to know the low-energy degrees of freedom and the symmetries. The massless mode is the only one that is dynamical at low energy. The symmetry is the same as that of the full theory, which in this case consists of the shift-and-reflection symmetry. One then writes out a normal theory with only the massless particle, consistent with these symmetries. In general, this may have higher-derivative nonrenormalizable interactions. By this procedure we arrive at the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{g}{M^4}(\partial_\mu\phi\partial^\mu\phi)(\partial_\nu\phi\partial^\nu\phi) + \mathcal{L}_6 + \mathcal{L}_8 + \dots \tag{5}$$

Here $\mathcal{L}_6$ and $\mathcal{L}_8$ are Lagrangians with six and eight derivatives, which will be described more fully below. In principle, one might consider a notation where the coupling strength g differs from that of the original theory. However, we will see that the coupling in the effective theory is identified with the coupling of the full theory when the latter is renormalized at low energy.

At one loop, wave-function renormalization would arise from the tadpole diagram with two external legs, as shown in the two-point diagram of Fig. 1. However, this vanishes because the tadpole integral

$$\int \frac{d^d k}{(2\pi)^d}\frac{k_\mu k_\nu}{k^2} \tag{6}$$

is a scaleless integral which vanishes in dimensional regularization. This sets $Z_1 = 1$ in the EFT limit.

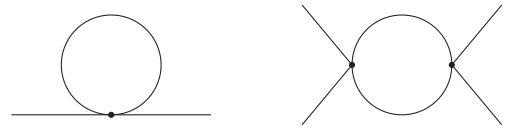

FIG. 1. Diagrams giving corrections to the two- and four-point functions.

For the scattering amplitude, the one-loop amplitude arises at order E^8 or, equivalently, it is described by a Lagrangian with eight derivatives. This can be seen dimensionally from the factor of g^2/M^8 which arises from two factors of the fundamental interaction. In dimensional regularization there are no other mass scales in the theory, and so the numerator factors arising from the one-loop amplitude must be powers of the external energies. There will be a divergence in this amplitude and the coefficients at order E^8 will need to be renormalized. Along with the renormalization will come the usual logs, and because this is a mass-independent renormalization, these must be factors of $\log s/\mu^2$ or similar logarithms. This tells us that the coefficients at order E^8 can be interpreted as "physically running" couplings. These logarithms will be finite and are predictions of the EFT.

In contrast, there will be no renormalization of the coefficients at order E^4 or E^6, as seen by the power counting described in the previous paragraph. This implies that no logarithms will be generated as well. The couplings at order E^4 and E^6 will not be running in the physical sense.

Let us see this in explicit detail. In $\phi + \phi \to \phi + \phi$ scattering, there are a limited number of kinematic invariants involved consistent with the crossing symmetry of the amplitude. This limits the number of effective Lagrangians involved. At dimensions six and eight, these can be taken to be

$$\mathcal{L}_6 = \frac{g_6}{4M^6}\partial_\mu\phi\partial^\mu\phi\Box\partial_\nu\phi\partial^\nu\phi + \frac{g_6'}{4M^6}\partial_\mu\phi\partial_\nu\phi\Box\partial^\mu\phi\partial^\nu\phi,$$
$$\mathcal{L}_8 = -\frac{g_8}{4M^8}\partial_\mu\phi\partial^\mu\phi\Box^2\partial_\nu\phi\partial^\nu\phi - \frac{g_8'}{4M^8}\partial_\mu\phi\partial_\nu\phi\Box^2\partial^\mu\phi\partial^\nu\phi. \tag{7}$$

The coupling constants in these Lagrangians cannot be predicted from EFT alone.

We have calculated the one-loop scattering amplitude in this theory. From the explicit calculation, the s channel gives

$$\frac{ig^2s^2(41s^2+t^2+u^2)}{1920\pi^2M^8\epsilon} - \frac{ig^2s^2\left(15(\log(\frac{-s}{4\pi\mu^2})+\gamma_E)(41s^2+t^2+u^2) - 1301s^2 - 46t^2 - 46u^2\right)}{28800\pi^2M^8} + O(\epsilon^1), \tag{8}$$

while the t and u channels can be found thanks to crossing symmetry. The t channel is given by the substitution $s \to t$ and $t \to s$, and u corresponds to the cyclic permutation $s \to u$, $t \to s$, $u \to t$. The total one-loop quantum correction to the four-point amplitude is

$$\frac{g^2\big(41(s^4+t^4+u^4)+2(s^2t^2+t^2u^2+u^2s^2)\big)}{1920\pi^2M^8\epsilon} - \frac{g^2}{28800\pi^2M^8}\left\{15\left[s^2(41s^2+t^2+u^2)\log\left(\frac{-s}{4\pi\mu^2}\right)\right.\right.$$
$$\left. + t^2(s^2+41t^2+u^2)\log\left(\frac{-t}{4\pi\mu^2}\right) + u^2(s^2+t^2+41u^2)\log\left(\frac{-u}{4\pi\mu^2}\right)\right]$$
$$\left. - (1301-615\gamma_E)(s^4+t^4+u^4) - 2(46-15\gamma)(s^2t^2+t^2u^2+u^2s^2)\right\} + O(\epsilon^1). \tag{9}$$

Because the field here is massless, the logarithms can only involve kinematic factors of s, t, and u.

The divergence in this expression can be absorbed into the renormalization of the dimension-eight coefficients in the effective Lagrangian. When renormalized at a scale $s = t = u = \mu_R^2$, the amplitude has the form

$$\mathcal{M} = -\frac{g}{2M^4}(s^2+t^2+u^2) + \frac{g_6}{2M^6}(s^3+t^3+u^3) + \frac{g_6'}{4M^6}(s^2t+s^2u+t^2u+t^2s+u^2s+u^2t)$$
$$-\frac{g_8(\mu_R)}{M^8}(s^4+t^4+u^4) - \frac{g_8'(\mu_R)}{2M^8}(s^2t^2+s^2u^2+t^2u^2)$$
$$-\frac{g^2}{1920\pi^2M^8}\left[41s^4\log\left(\frac{-s}{\mu_R^2}\right) + 41t^4\log\left(\frac{-t}{\mu_R^2}\right) + 41u^4\log\left(\frac{-u}{\mu_R^2}\right)\right.$$
$$\left. + s^2(t^2+u^2)\log\left(\frac{-s}{\mu_R^2}\right) + t^2(s^2+u^2)\log\left(\frac{-t}{\mu_R^2}\right) + u^2(t^2+s^2)\log\left(\frac{-u}{\mu_R^2}\right)\right]. \tag{10}$$

The values of $g_6, g_6', g_8(\mu_R)$, and $g_8'(\mu_R)$ are not predictions of the EFT and must be determined by either measurement or matching to the full theory. We will explicitly perform the matching below, using the amplitude of the full theory.

The "physical" beta functions of the various couplings can be read off from the amplitude. These are

$$\beta_g = 0,$$
$$\beta_{g_6} = 0,$$
$$\beta_{g_6'} = 0,$$
$$\beta_{g_8} = \frac{41g^2}{480\pi^2},$$
$$\beta_{g_8'} = \frac{g^2}{240\pi^2}. \tag{11}$$

These beta functions are predictions of the EFT.

The expected maximum limit of the EFT treatment of this matrix element occurs when

$$\frac{gE^4}{M^4} \sim 1, \tag{12}$$

where E^4 here represents any of the kinematic invariants $E^4 \sim s^2, t^2, u^2$. At these energies the interaction strength becomes large and the EFT treatment fails. All of the terms in the derivative expansion become relevant, with unknown coefficients. The actual limit of the EFT will be either when new degrees of freedom become dynamically active or at the energy implied by Eq. (12), whichever is lower.

The key elements of this section are that in the EFT treatment (1) the original coupling g is not renormalized and does not run in the physical sense, and (2) we need to renormalize the couplings of the eight-derivative Lagrangian, and these couplings are running couplings.

III. TWO-POINT FUNCTION

In this section, we continue to use the notation of Eq. (3). The Feynman rules are given by the propagator

$$-\frac{i}{p^2 + \frac{1}{m^2}p^4} \tag{13}$$

and the four-point vertex

$$-\frac{2ig^2}{M^4}\big[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_3)(p_4\cdot p_2)$$
$$+ (p_1\cdot p_4)(p_3\cdot p_2)\big]. \tag{14}$$

At one loop, the quantum corrections to the two-point function are given by the (tadpole) integral

$$-\frac{g}{Z_1 M^4}\mu^{4-d}\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+\frac{1}{m^2}q^4}\left[p^2q^2+2(p\cdot q)^2\right]. \quad (15)$$

In the absence of the four-derivative kinetic term (i.e., for $m\to\infty$), this is quartically divergent and is zero in dimensional regularization.

In the general case, setting $d=4-2\epsilon$, it is equal to

$$i\frac{3}{2}\frac{g}{Z_1}\left(\frac{m}{M}\right)^4 p^2\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}+\log 4\pi-\gamma-\log\frac{m^2}{\mu^2}+\frac{7}{6}+O(\epsilon)\right). \quad (16)$$

At one loop, only Z_1 receives quantum corrections, since there are no terms proportional to p^4. However, the μ dependence in (16) does not correspond to a logarithmic p dependence of the two-point function. This means that the μ dependence of the field renormalization can be reabsorbed once and for all without producing any large logs with the physical energy scale of the scattering process. For example, when setting $\mu=m$ the logarithm disappears altogether.

IV. THE SCATTERING AMPLITUDE

Let us now compute the corrections to the four-point function. Since we use the signature $-+++$, the Mandelstam variables are defined as

$$s=-(p_1+p_2)^2, \quad (17)$$

$$t=-(p_1+p_3)^2, \quad (18)$$

$$u=-(p_1+p_4)^2, \quad (19)$$

where all momenta are incoming. For this section we use the notation $Z_2=1/m^2$ and work in units with $M=1$, as this matches the previous work using the FRG.

In this case, one has to consider three Feynman diagrams that correspond to the s, t, and u channels. These are related by crossing symmetry. In the s channel, the integral one has to evaluate is

$$\frac{2g^2\mu^{4-d}}{Z_1^2M^8}\int\frac{d^dq}{(2\pi)^d}\frac{N}{(q^2+\frac{1}{m^2}q^4)((q+p)^2+\frac{1}{m^2}(q+p)^4)}, \quad (20)$$

where $p=p_1+p_2$ and the numerator is

$$N=\left[(p_1\cdot p_2)(q\cdot(q+p))+(p_1\cdot q)(p_2\cdot(q+p))+(q\cdot p_2)(p_1\cdot(q+p))\right]\times\left[(p_3\cdot p_4)(q\cdot(q+p))+(p_3\cdot q)(p_4\cdot(q+p))+(q\cdot p_4)(p_3\cdot(q+p))\right]. \quad (21)$$

The other channels only differ by permutations of the external momenta.

Using (2), the fourth-order propagators in the integral can be decomposed into a massless second-order propagator and a massive ghost propagator. This is equivalent to replacing the quartic propagators in the diagrams with either the massless or massive ones and summing over all possible combinations. In this way, for each channel the correction to the scattering amplitude becomes

$$\delta\mathcal{M}=\mathcal{M}_1-\mathcal{M}_2-\mathcal{M}_3+\mathcal{M}_4,$$

where $\mathcal{M}_1$ contains only the contributions of the massless particles, $\mathcal{M}_4$ contains that of the massive ghosts, and the other two mixed contributions with one massive and one massless propagator. In each partial amplitude we introduce a Feynman parameter, such that the denominators become (for the s channel)

$$\frac{1}{q^2(q+p)^2}=\int_0^1 dx\frac{1}{(q'^2+\Delta_1)^2}\quad\text{with}\quad \Delta_1=x(1-x)p^2, \quad (22)$$

$$\frac{1}{q^2[(q+p)^2+m^2]}=\int_0^1 dx\frac{1}{(q'^2+\Delta_2)^2}\quad\text{with}\quad \Delta_2=x(1-x)p^2+xm^2, \quad (23)$$

$$\frac{1}{(q^2+m^2)(q+p)^2}=\int_0^1 dx\frac{1}{(q'^2+\Delta_3)^2}\quad\text{with}\quad \Delta_3=(1-x)(xp^2+m^2), \quad (24)$$

$$\frac{1}{(q^2+m^2)[(q+p)^2+m^2]}=\int_0^1 dx\frac{1}{(q'^2+\Delta_4)^2}\quad\text{with}\quad \Delta_4=x(1-x)p^2+m^2, \quad (25)$$

and $q'=q+xp$. After some manipulations, the numerators become

$$N=N_0+N_1(q')^2+N_2(q')^4,$$

where

$$N_0 = x^2(1-x)^2 s^4,$$
$$N_1 = -\frac{1}{d}\left[(6+d)(x^2-x)+1\right]s^3,$$
$$N_2 = \frac{1}{4d(d+2)}\left[(d^2+6d+12)s^2+4(t^2+u^2)\right]. \tag{26}$$

Thus, the partial corrections to the amplitude become, in d dimensions,

$$\begin{aligned}\mathcal{M}_\ell &= 2g^2\frac{1}{Z_1^2}\int\frac{d^dq'}{(2\pi)^d}\int_0^1 dx\frac{N}{((q')^2+\Delta_\ell)^2}\\ &= \frac{1}{(4\pi)^{d/2}}\frac{2g^2}{Z_1^2}\int_0^1 dx\left[\Gamma\left(2-\frac{d}{2}\right)\Delta_\ell^{(d-4)/2}N_0+\frac{d}{2}\Gamma\left(1-\frac{d}{2}\right)\Delta_\ell^{(d-2)/2}N_1+\frac{d(d+2)}{4}\Gamma\left(-\frac{d}{2}\right)\Delta_\ell^{d/2}N_2\right].\end{aligned} \tag{27}$$

Finally, performing the x integration, without making any assumptions on the relative size of s, m, and M, we obtain for the s channel

$$\begin{aligned}&\frac{g^2m^4(13s^2+t^2+u^2)}{192\pi M^8\epsilon}-\frac{g^2}{5760\pi^2s^3M^8}\left\{-3s^5(41s^2+t^2+u^2)\log\left(-\frac{m^2}{s}\right)\right.\\ &-6m^4(-s+m^2)^3\left[(s^2+t^2+u^2)-2\frac{s}{m^2}(-9s^2+t^2+u^2)+\frac{s^2}{m^4}(41s^2+t^2+u^2)\right]\log\left(\frac{m^2}{m^2-s}\right)\\ &+s^2m^6\left[-2\frac{s}{m^2}(352s^2+37(t^2+u^2)-15\gamma_E(13s^2+t^2+u^2))\right.\\ &\left.-3(-31s^2+9(t^2+u^2))+6\frac{m^2}{s}(s^2+t^2+u^2)\right]\\ &+6s^{5/2}m^4\sqrt{4m^2-s}\left(16(6s^2+t^2+u^2)-8\frac{s}{m^2}(16s^2+t^2+u^2)+\frac{s^2}{m^4}(41s^2+t^2+u^2)\right)\operatorname{arccot}\sqrt{\frac{4m^2}{s}-1}\\ &\left.-30s^3m^4(13s^2+t^2+u^2)\log\left(\frac{4\pi\mu^2}{m^2}\right)\right\}.\end{aligned} \tag{28}$$

In this expression we can find a diverging contribution to the $(\partial\phi)^4$ operator, but again the scale parameter μ appears only in logs divided by the ghost mass m, and hence the most convenient choice is to set $\mu = m$ for each value of the kinematic variables s, t, and u. The scattering amplitude gains an imaginary part from both the on-shell loops of the massless modes thanks to $\log(-\frac{m^2}{s})$ and the on-shell ghosts in loops when $s > m^2$ in $\log(\frac{m^2}{m^2-s})$. The total quantum correction to the four-point amplitude is

$$\begin{aligned}&\frac{5g^2m^4(s^2+t^2+u^2)}{64\pi^2M^8\epsilon}+\frac{g^2}{5760\pi^2M^8}\left\{+\frac{m^4}{s^2}\left[-6m^4(s^2+t^2+u^2)+3sm^2\left(-31s^2+9(t^2+u^2)\right)\right.\right.\\ &\left.+2s^2\left((352-195\gamma_E)s^2-(15\gamma_E-37)(t^2+u^2)\right)\right]+6s^{-1/2}m^4\sqrt{4m^2-s}\left[16m^4(6s^2+t^2+u^2)\right.\\ &\left.-8sm^2(16s^2+t^2+u^2)+s^2(41s^2+t^2+u^2)\right]\operatorname{arccot}\sqrt{\frac{4m^2}{s}-1}+\frac{m^4}{t^2}\left[-6m^4(s^2+t^2+u^2)+3tm^2(-31t^2+9(s^2+u^2))\right.\\ &\left.+2t^2((352-195\gamma_E)t^2-(15\gamma_E-37)(s^2+u^2))\right]+6t^{-1/2}m^4\sqrt{4m^2-t}\left[16m^4(s^2+6t^2+u^2)\right.\\ &\left.-8tm^2(s^2+16t^2+u^2)+t^2(s^2+41t^2+u^2)\right]\operatorname{arccot}\sqrt{\frac{4m^2}{t}-1}\\ &+\frac{m^4}{u^2}\left[-6m^4(s^2+t^2+u^2)+3m^2\left(-31u^2+9(s^2+t^2)\right)+2u^2\left((352-195\gamma_E)u^2-(15\gamma_E-37)(s^2+t^2)\right)\right]\end{aligned}$$

$$
\begin{aligned}
&+ 6u^{-1/2}m^4\sqrt{4m^2-u}\left[16m^4(s^2+t^2+6u^2)-8um^2(s^2+t^2+16u^2)+u^2(s^2+t^2+41u^2)\right]\operatorname{arccot}\sqrt{\frac{4m^2}{u}-1}\\
&+ 3s^2(41s^2+t^2+u^2)\log\left(-\frac{m^2}{s}\right)+3t^2(s^2+41t^2+u^2)\log\left(-\frac{m^2}{t}\right)+3u^2(s^2+t^2+41u^2)\log\left(-\frac{m^2}{u}\right)\\
&+ \frac{6(u-m^2)^3}{u^3}\log\left(\frac{m^2}{m^2-u}\right)\left[m^4(s^2+t^2+u^2)-2um^2(s^2+t^2-9u^2)+u^2(s^2+t^2+41u^2)\right]\\
&+ \frac{6(t-m^2)^3}{t^3}\log\left(\frac{m^2}{m^2-t}\right)\left[m^4(s^2+t^2+u^2)-2tm^2(s^2-9t^2+u^2)+t^2(s^2+41t^2+u^2)\right]\\
&+ \frac{6(s-m^2)^3}{s^3}\log\left(\frac{m^2}{m^2-s}\right)\left[m^4(s^2+t^2+u^2)-2sm^2(-9s^2+t^2+u^2)+s^2(41s^2+t^2+u^2)\right]\\
&+ 450m^4(s^2+t^2+u^2)\log\left(\frac{4\pi\mu^2}{m^2}\right)\Bigg\}.
\end{aligned}
\tag{29}
$$

The arccot functions can be rewritten as logs, using

$$
\operatorname{arccot}\sqrt{x-1}=\frac{i}{2}\left(\log\left(1-\frac{i}{\sqrt{x-1}}\right)-\log\left(1+\frac{i}{\sqrt{x-1}}\right)\right). \tag{30}
$$

A. The $Z_1 = 0$ case

It will be instructive to consider the case when there is no two-derivative kinetic term. Clearly, in this case we cannot assume the canonical normalization $Z_1 = 1$. In fact, we want to consider the limit when Z_1, m, and M all go to zero at the same rate. Defining the dimensionless field φ and the coupling γ by

$$
\frac{Z_1}{m^2}\phi^2=\varphi^2, \qquad \frac{Z_1^2}{M^4}\phi^4=\gamma\varphi^4,
$$

the action (3) becomes

$$
\mathcal{L}=\frac{1}{2}m^2\partial_\mu\varphi\partial^\mu\varphi-\frac{1}{2}\Box\varphi\Box\varphi-\frac{\gamma}{4}(\partial_\mu\varphi\partial^\mu\varphi)(\partial_\nu\varphi\partial^\nu\varphi), \tag{31}
$$

where the field is now canonically normalized with respect to the four-derivative kinetic term. Now we can simply set $m = 0$.

The calculation of the amplitude follows the steps of the general case but is much simpler. The s channel brings the following quantum correction:

$$
\frac{\gamma^2(13s^2+t^2+u^2)}{192\pi^2\epsilon}+\frac{\gamma^2(3(13s^2+t^2+u^2)(\log(\frac{4\pi\mu^2}{-s})-\gamma_E)+32s^2+5(t^2+u^2))}{576\pi^2}+O(\epsilon^1). \tag{32}
$$

Defining the renormalized coupling at the scale $s = t = u = \mu_R^2$ by the formula

$$
\gamma(\mu_R)=\gamma-\frac{\gamma^2}{16\pi^2}\left[\frac{5}{2}\left(\frac{1}{\epsilon}+\log\left(\frac{4\pi\mu^2}{\mu_R^2}\right)-\gamma_E\right)+\frac{7}{3}\right], \tag{33}
$$

(where the couplings on the rhs are the bare ones) and exploiting crossing symmetry, we obtain the complete four-point amplitude

$$
-\frac{\gamma(\mu_R)}{2}(s^2+t^2+u^2)+\frac{\gamma^2}{192\pi^2}\left[\log\left(\frac{\mu_R^2}{-s}\right)(13s^2+t^2+u^2)+\log\left(\frac{\mu_R^2}{-t}\right)(s^2+13t^2+u^2)+\log\left(\frac{\mu_R^2}{-u}\right)(s^2+t^2+13u^2)\right]+O(\epsilon^1). \tag{34}
$$

This agrees with [17].

In this case, the μ_R dependence is always associated with the dependence on the kinematic variables s, t, and u. The physical beta function is

$$
\mu_R\frac{\partial\gamma}{\partial\mu_R}=\frac{5\gamma^2}{16\pi^2}. \tag{35}
$$

V. UNDERSTANDING THE GENERAL AMPLITUDE

We study the scattering amplitude in this theory and identify the physical running (or lack of running) of the parameters. The results differ from those given by usual methods. The amplitude calculation is also an instructive example of EFT when treated at low energy. Finally, we identify a novel (as far as we know) phenomenon of the disappearance of certain operators as one increases the energy.

Here we discuss the general case where we start with Z_1 and Z_2 in principle different from zero. For this section, we revert to the notation of Eq. (3), where $Z_2 = 1/m^2$ and g is rescaled by a factor of M^4.

At one loop there is no renormalization of $Z_2 = 1/m^2$, as the one-loop tadpole diagram only has two factors of the external momentum. We have seen in (16) that the one-loop contribution to Z_1 is independent of the momentum. Therefore, we can renormalize to $Z_1 = 1$, and this result will be valid for all energies. From this we see that Z_1 is not a running parameter in the amplitude analysis and therefore

$$\beta_{Z_1} = 0 \tag{36}$$

for all energies. For the rest of this section we set $Z_1 = 1$.

A. Low energy

The full result simplifies in the low-energy limit. The logarithms involving mass factors can be Taylor expanded in the momentum, so that the only logarithms remaining are of the form $\log -s$, $\log -t$, and $\log -u$.

For $Z_2 s \ll Z_1$, we find that the quantum correction is given by

$$\begin{aligned}&\frac{5g^2m^4(s^2+t^2+u^2)}{64\pi^2M^8\epsilon} - \frac{g^2}{11520\pi^2M^8}\left\{-900m^4(s^2+t^2+u^2)\log\left(\frac{4\pi\mu^2}{m^2}\right) + 30(30\gamma_E - 11)m^4(s^2+t^2+u^2)\right.\\ &+ 6\left[s^2(41s^2+t^2+u^2)\log\left(\frac{-s}{m^2}\right) + t^2(s^2+41t^2+u^2)\log\left(\frac{-t}{m^2}\right) + u^2(s^2+t^2+41u^2)\log\left(\frac{-u}{m^2}\right)\right]\\ &\left.- 3(79(s^4+t^4+u^4) + 6(s^2t^2+t^2u^2+u^2s^2)) - 760m^2(s^3+t^3+u^3)\right\}.\end{aligned} \tag{37}$$

One can see that the logarithm, which is proportional to the original interaction, i.e., $s^2+t^2+u^2$, involves $\log(\mu^2/m^2)$ and is independent of the kinematic variables. This means that we can define a renormalized value of the coupling g by collecting all of the factors that multiply the invariant $s^2+t^2+u^2$ and identifying it with the coupling measured at low energy using the fundamental interaction. Then, we find

$$g(\mu) = g_B - \frac{5g^2m^4}{32\pi^2M^4}\left[\frac{1}{\epsilon} - \gamma_E - \log\left(\frac{4\pi\mu^2}{m^2}\right) + \frac{11}{30}\right]. \tag{38}$$

Here g_B is the original unrenormalized coupling. Now, if we define the beta function by the usual recipe of deriving with respect to μ, we find

$$\beta_g^\mu = \mu\frac{\partial g(\mu)}{\partial\mu} = \frac{5g^2m^4}{16\pi^2M^4}. \tag{39}$$

However, g does not depend on the energy so the physical beta function is

$$\beta_g = 0 \tag{40}$$

in the low-energy region.

The remainder of the amplitude involves powers of energy at order $E^6 \sim s^3, s^2t, \ldots$ and at order $E^8 \sim s^4, s^2t^2$. Those of order E^6 do not involve any logarithms, while there are logarithms at order E^4. A bit of inspection shows that the amplitude is exactly that of the EFT given in Eq. (10), with the identifications

$$\begin{aligned} g &= g,\\ g_6 &= -\frac{53g^2m^2}{384\pi^2M^2},\\ g_6' &= -\frac{7g^2m^2}{516\pi^2M^2},\\ g_8(\mu_R) &= \frac{79g^2}{1920\pi^2} + \frac{41g^2}{960\pi^2}\log\frac{\mu_R^2}{m^2},\\ g_8'(\mu_R) &= \frac{3g^2}{320\pi^2} + \frac{g^2}{480\pi^2}\log\frac{\mu_R^2}{m^2}. \end{aligned} \tag{41}$$

Whereas in the EFT by itself these parameters were unknown, here we see that they are predicted by the full theory. This procedure is referred to as *matching* the EFT to the full theory.

We see that in this region the heavy ghost is not dynamically active and the one-loop calculation amounts to integrating it out of the full theory to one-loop order. The

result is described by an EFT, with specific values of the coupling. This is an instructive example of EFT reasoning.

B. Above the mass threshold

Assume that all of the kinematic invariants are greater than m^2 in magnitude, i.e., $(s, |t|, |u|) \gg m^2$. If we use the definition (38) of the renormalized coupling defined below the mass threshold, the amplitude is finite and can be written in the form

$$\begin{aligned}\mathcal{M} = &-\frac{g}{2M^4}\left[1-\frac{17gm^4}{192\pi^2M^4}\right](s^2+t^2+u^2)\\ &-\frac{g^2m^4}{192\pi^2M^8}\left[\log\left(\frac{-s}{m^2}\right)(13s^2+t^2+u^2)\right.\\ &+\log\left(\frac{-t}{m^2}\right)(s^2+13t^2+u^2)\\ &\left.+\log\left(\frac{-u}{m^2}\right)(s^2+t^2+13u^2)\right]. \end{aligned} \tag{42}$$

We can instead define the coupling at the (off-shell) renormalization point $s = t = u = \mu_R^2$ by making the finite renormalization

$$\bar{g}(\mu_R) = g + \frac{5g^2m^4}{32\pi^2M^4}\left[\log\left(\frac{\mu_R^2}{m^2}\right) - \frac{17}{30}\right], \tag{43}$$

in which case the amplitude becomes

$$\begin{aligned}\mathcal{M} = &-\frac{\bar{g}(\mu_R)}{2M^4}(s^2+t^2+u^2)\\ &-\frac{\bar{g}^2m^4}{192\pi^2M^8}\left[\log\left(\frac{-s}{\mu_R^2}\right)(13s^2+t^2+u^2)\right.\\ &+\log\left(\frac{-t}{\mu_R^2}\right)(s^2+13t^2+u^2)\\ &\left.+\log\left(\frac{-u}{\mu_R^2}\right)(s^2+t^2+13u^2)\right], \end{aligned} \tag{44}$$

which agrees with the one calculated in the limit $Z_1 = 0$ [Eq. (34)]. This is understandable because at high energy the quartic terms in the propagator dominate the quadratic terms, and simply ignoring the quadratic terms yields the correct result.

There are a couple of striking observations that can be made from this result. The first is that all of the terms of order E^8 and E^6 have disappeared from the result. Because the general amplitude of Eq. (29) has many such terms, this requires special cancellations, which we will discuss below. The second is that here we *can* define a running coupling that removes the potentially large logarithms of the form $\log s/m^2$. We consider the (off-shell) renormalization point $s = t = u = \mu_R^2$.

For $(s, |t|, |u|) \gg m^2$ this captures an important part of the quantum correction. There are still logarithms left over, but they are not large. This corresponds to a beta function

$$\beta_{\bar{g}} = \frac{5\bar{g}^2m^4}{16\pi^2M^4}. \tag{45}$$

The factor of $17/30$ in Eq. (43) amounts to an optional threshold correction matching the amplitude above and below the threshold.

The disappearance of the E^8 and E^6 terms appears initially surprising. There are many such terms with factors such as $s^4, t^4, \ldots$ and $s^3, t^3, \ldots$ in the general result. However, we can begin to see that there are cancellations by looking at the logarithmic-type terms that arise at the highest order, E^8. We recall the result of Passarino and Veltman that all one-loop diagrams can be expressed in terms of factors of the scalar tadpole, bubble, triangle, and box diagrams. Here only the tadpoles and bubbles contribute. The tadpoles do not depend on the external momenta and do not give kinematic logs. These kinematic factors inside the logarithms come from the scalar bubble diagrams, which have the form

$$I_2(m_1, m_2, q^2) = \frac{1}{16\pi^2}\left[\frac{1}{\epsilon} + \gamma - \log 4\pi - \int_0^1 dx \log\left(\frac{xm_1^2 + (1-x)m_2^2 - q^2x(1-x)}{\mu^2}\right)\right]. \tag{46}$$

The logarithmic integral has the form

$$\begin{aligned}\int_0^1 dx \log\left(\frac{xm_1^2+(1-x)m_2^2-q^2x(1-x)}{\mu^2}\right) &= \log\left(-\frac{q^2}{\mu^2}\right) - 2, \quad (m_1 = m_2 = 0)\\ &= \log\frac{m^2}{\mu^2} + \left(1-\frac{m^2}{q^2}\right)\log\left(1-\frac{q^2}{m^2}\right) - 2, \quad (m_1 = 0, m_2 = m)\\ &= \log\frac{m^2}{\mu^2} + \sqrt{1-\frac{4m^2}{q2}}\log\left(\frac{\sqrt{1-4m^2/q^2}+1}{\sqrt{1-4m^2/q^2}-1}\right) - 2, \quad (m_1 = m_2 = m). \end{aligned} \tag{47}$$

The reader can see these logarithmic factors in the general amplitude. Moreover, one can see that there are common factors preceding these logs and the result involves the combination

$$I_2(0,0,q^2) - 2I_2(0,m,q^2) + I_2(m,m,q^2). \quad (48)$$

The divergences and the factor of $\log\mu^2$ cancel with this combination, leaving a finite result, as observed. In the low-energy region, the latter two components of this expression go to constants, and only $I_2(0,0,q^2)$ gives kinematic logarithms. This leads to the log dependence found in the low-energy/EFT limit in Eq. (10). However, at high energy each of the components involves equal factors of $\log(-q^2)$, and the leading energy dependence of this combination will cancel. This leads to the vanishing of the terms of order E^8 at high energy. It requires detailed work to verify that the remaining terms of order E^6 also cancel, but the general idea is the same. The amplitude that starts out containing orders E^4, E^6, E^8 at low energy ends up at order E^4 only at high energy. To the best of our knowledge, this vanishing of high-order energy dependence that was required within the EFT is a novel effect not previously described in the literature.

C. The very-high-energy region

The results of the preceding section hold for all energy scales such that the mass can be ignored. However, in the Introduction we distinguished an intermediate-energy scale from a high-energy scale. In the high-energy region a puzzling situation now presents itself. From Eq. (35) or Eq. (45) one sees that for $g < 0$ the coupling is asymptotically free (in agreement with earlier calculations [17]). However, it appears that a focus on the coupling constant is insufficient. Even if the coupling constant is running logarithmically to an asymptotically free fixed point, the amplitude itself is blowing up with energy.

At high enough energy, the one-loop scattering amplitude will become greater than unity. This occurs when the kinematic invariants s, t, $u \sim E^2$ are of order

$$\frac{g(E)E^4}{M^4} \sim 1. \quad (49)$$

A logarithmic decrease in the coupling does not offset the power-law growth. This behavior puts the notion of asymptotic freedom in question. We leave a more detailed discussion to a separate publication.

VI. DIFFERENT DEFINITIONS OF RUNNING PARAMETERS

Here we return to our introductory point that there are different flavors of RG techniques. In turn, we address the three that we highlighted.

A. Physical beta functions

Part of the renormalization program is the measurement process. Because we do not know the bare couplings, we need to measure the parameters. Within a given scheme, this involves measurements at a renormalization scale μ_R. When the physical amplitude depends on the energy in a particular way, then measurements at different renormalization scales will lead to different values of the coupling. What we refer to as the physical beta function describes how the coupling changes with μ_R.

The important point here is that this procedure studies the physical amplitudes and their dependence on the energy. We have used this method in the preceding analysis. In our work, the renormalization scheme involved identifies the couplings at the symmetric point $s = t = u = \mu_R^2$.

B. Using divergences or $\log\mu$ to define running couplings

Rather than calculate physical amplitudes, we often just look at the renormalization constants required for renormalizing the parameters. In dimensional regularization, these depend on $\log\mu^2$, where $\mu^{(4-d)}$ is the parameter introduced to keep the dimensions of Feynman integrals constant. The dependence on $\log\mu$ comes along with the $1/\epsilon$ of the renormalization constant and hence $\log\mu^2$ appears in the renormalization in the same way every time that the coupling appears. Then, beta functions can be calculated from $\mu\frac{\partial}{\partial\mu}$. In regularization schemes with a cutoff, beta functions can be found using $\Lambda\frac{\partial}{\partial\Lambda}$.

In mass-independent renormalization schemes, this procedure will identify the physical running. Because there are no other dimensionful factors around, the factors of $\log\mu$ will always be accompanied by factors of $\log E^2$, as in $\log s/\mu^2, \log t/\mu^2\ldots$ when applied in an amplitude.

However, there are a couple of ways that this could go wrong when there are factors of masses around. The logarithm could involve $\log m^2/\mu^2$ or $\log m^2/\Lambda^2$. In this case, the $\log\mu^2$ or $\log\Lambda^2$ dependence does not correlate with any energy dependence in the physical amplitude. It is just a constant and disappears when the renormalized coupling is defined.

A somewhat more unusual case where this method fails is in running parameters that are not associated with $1/\epsilon, \log\Lambda$, or $\log\mu$. An example of this is seen in our scattering amplitude. The low-energy analysis using the EFT shows that the couplings at order E^8 are running couplings at one loop; see Eq. (10). This occurs for the usual reason, with the EFT involving mass-independent renormalization, and the quantum corrections at order E^8 involving $1/\epsilon - \log E^2/\mu^2$. However, in the full theory these terms do not involve any renormalization or factors of $\log\mu$, as can be seen in Eq. (41), as they are finite predictions. Yet, since the amplitudes match the EFT

analysis exactly, g_8 and g_8' are physically running parameters.

C. One-loop beta functions from the FRG

In [16] the Lagrangian was parametrized as in (1) and the running of Z_1, Z_2, and g was calculated using the full FRG. The coupling then depends on a scale k that has the meaning of an IR cutoff. This calculation goes beyond the one-loop approximation, because the couplings on the rhs of the FRG equation are treated as running couplings. This kind of "RG improvement" amounts to a resummation of infinitely many diagrams. In order to compare with the amplitude calculation, we have to downgrade those results to the one-loop approximation. This is easily achieved by neglecting the RG improvement.

Assuming that the field has dimension of mass, one arrives at the following beta functions:

$$k\partial_k Z_1 = -\frac{Z_1 + 2k^2 Z_2}{16\pi^2 (Z_1 + k^2 Z_2)^2} g k^4, \tag{50}$$

$$k\partial_k Z_2 = 0, \tag{51}$$

$$k\partial_k g = \frac{5(Z_1 + 2k^2 Z_2)}{32\pi^2 (Z_1 + k^2 Z_2)^3} g^2 k^4. \tag{52}$$

With dimensionless field the numerical coefficients in the beta functions remain the same, but of course the dimension of the couplings is different and so are the powers of k on the rhs. These one-loop beta functions differ from the full ones of [16], but the qualitative features of the RG flow remain the same.

In order to study the RG flow one has to make the couplings dimensionless, multiplying them by powers of k. One then obtains different results depending on the dimension of the field. When the field has dimension one, one finds the low-energy Gaussian fixed point corresponding to the free theory with a two-derivative propagator, as well as a bunch of other nontrivial fixed points. When the field is dimensionless, one finds the high-energy Gaussian fixed point corresponding to the free theory with a four-derivative propagator, as well as a bunch of other nontrivial fixed points. The two descriptions can be seen as two charts on a manifold, related by a well-defined coordinate transformation. In each chart, one of the free fixed points sits at the origin, while the other lies asymptotically at infinity.

The nice feature that was observed in [16] is that the running of Z_1 can be absorbed into a redefinition of the field, and then the dimension of the field changes continuously from zero near the UV fixed point to one near the IR fixed point.

D. Comparisons

We are now ready to compare the physical running with the results of the FRG and the μ running.

We begin by comparing the physical running of g to the μ running. At low energy, below the mass m, the amplitude does not give a physical running for g:

$$\beta_g = 0, \qquad E \ll m. \tag{53}$$

This is in disagreement with the $\log\mu$ derivative approach, which predicts a logarithmic running; see Eq. (39). This β function comes out from the $\log m^2/\mu^2$ in (38). Here μ is an unphysical parameter that disappears from all physical reactions after renormalization. The apparent running of the coupling g arises from taking the negative logarithmic derivative of the correction (38) with respect to μ. This often is appropriate in other settings because in mass-independent renormalization schemes the logarithmic factor is $\log q^2/\mu^2$ (where q is some kinematic energy factor) so that taking the derivative with respect to μ reveals the dependence of the amplitude on the kinematic variables $\log q^2$. However, here there is no dependence on any kinematic variable. If we perform renormalization at any kinematic scale below the mass threshold, it remains that value as long as the ghosts stay frozen. Of the two definitions of running given in dimensional regularization, the physical one matches the results from the EFT in Sec. II, where we did not observe any quantum corrections to the coupling g.

A well-known example can illustrate this point. In QED, the vacuum polarization correction involving a top-quark loop yields the correction

$$\Pi(q^2) = \frac{\alpha}{3\pi}\left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log\frac{m_t^2}{\mu^2} + \frac{q^2}{5m_t^2} + \ldots\right] \tag{54}$$

at low energy. However, despite the dependence on $\log\mu$, this does not imply that the top-quark loop contributes to the running of the electric charge at low energy. The top quark makes a contribution to the running of α only at energies above m_t, where the logarithmic factor involves $\log q^2$ instead of $\log m_t^2$.

There is agreement in the running of g in the high-energy region for energies above the mass m. Here the beta function describing the running coupling in the amplitude is

$$\beta_g = \frac{5g^2 m^4}{16\pi^2 M^4}, \qquad E \gg m. \tag{55}$$

This indeed agrees with Eq. (39), and does not change if we add a finite piece as in (43).

Again, we can see the relevant physics in the simpler case of the QED vacuum polarization. While the low-energy form of the function is given in Eq. (54), the high-energy version of this is

$$\Pi(q^2) = \frac{\alpha}{3\pi}\left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log\frac{m_t^2}{\mu^2} - \log\frac{-q^2}{m_t^2} + \ldots\right]. \tag{56}$$

Asymptotically the mass cancels out, and we could have performed the renormalization using a mass-independent scheme. However, since we previously chose to renormalize the electric charge at low energy, absorbing the $\log m_t^2/\mu^2$ factor into the coupling, the latter logarithm is potentially a large logarithm and should be resummed in a running coupling constant. The top-quark contribution to the electric charge is constant at low energy and runs at high energy.

This is the same behavior as is revealed in the running of g. We can see this relatively simply in the calculation. The loop integral in this model is proportional to

$$I_{\mu\nu\alpha\beta} = m^4 \int \frac{d^d p}{(2\pi)^d} \frac{p_\mu p_\nu (p-q)_\alpha (p-q)_\beta}{[m^2 p^2 - p^4][(m^2(p-q)^2 - (p-q)^4]} = F(q^2)(\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}) + \text{order } q \text{ terms}. \tag{57}$$

The only divergence appears in the first term F. We can simply evaluate this divergence by taking the trace of this integral,

$$\eta^{\mu\nu}\eta^{\alpha\beta} I_{\mu\nu\alpha\beta} = m^4 \int \frac{d^d p}{(2\pi)^d} \frac{1}{[m^2 - p^2][(m^2 - (p-q)^2]} = m^4 I_2(m, m, q) = d(d+2)F + \ldots, \tag{58}$$

where I_2 is given in Eq. (46). This is just the scalar bubble diagram, which along with the divergence carries the $\log m^2/\mu^2$ factor at low energy and $\log q^2/\mu^2$ at high energy, just as we have seen above. Other logarithms are possible in the amplitude, but this one is tied to the renormalization of the coupling at low energy and the running at high energy. The FRG and dimensional regularization of the running agree because it is uniquely the bubble diagram that determines the divergent factor in this calculation.

We now compare the physical running of g to the FRG results. At low energy the FRG running is a power law:

$$\beta_g = \frac{5(Z_1 + 2k^2/m^2)}{32\pi^2 (Z_1 + k^2/m^2)^3} \frac{g^2 k^4}{M^4} \to \frac{5 g^2 k^4}{32\pi^2 M^4}, \quad \text{for } k \ll m. \tag{59}$$

This beta function rapidly runs to zero at lower energies, asymptoting to the constant value of g found in the amplitude calculation.

On the other hand, at high energy the FRG gives

$$\beta_g = \frac{5(Z_1 + 2k^2/m^2)}{32\pi^2 (Z_1 + k^2/m^2)^3} \frac{g^2 k^4}{M^4} \to \frac{5 g^2 m^4}{16\pi^2 M^4}, \quad k \gg m, \tag{60}$$

which agrees with both the physical running and the μ running.

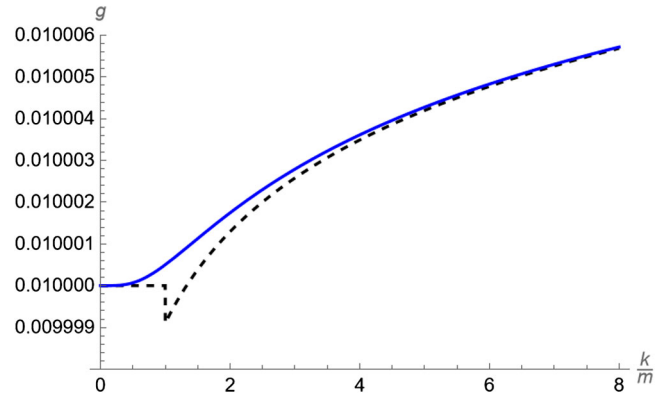

FIG. 2. Running coupling calculated from the FRG (blue continuous curve) and the one obtained by matching the low- and high-energy physical running (black dashed). They have been calculated here for the same low-energy limit $g = 0.01$.

Between these two limits, the behavior of the amplitude is more complicated, and if one tries to define a running coupling as in (43), namely, isolating the coefficient of $s^2 + t^2 + u^2$ in the full amplitude (29) and setting $s = t = u = \mu_R^2$, it turns out to be impossible to unambiguously identify it. Hence, the definition of a physical running is only meaningful in the asymptotic regions, where different power laws are clearly separated. The standard, conventional way of joining them is to assume that g does not run all the way up to the mass m and to match this to the high-energy logarithmic behavior (44) via (43). This is shown by the black dashed line in Fig. 2.

We note that, whereas the beta function becomes universal (scheme independent) at high energy, the relation between the low-energy value of the coupling and its high-energy behavior is not. By choosing a different constant in the brackets in (43) we can change the offset between the low- and high-energy parts of the curve in Fig. 2 and shift up or down the part of the curve above the threshold $k/m = 1$.

The same effect can also be obtained by using a different renormalization point. In the UV, the nonpolynomial dependence on the kinematical variables of the terms of the amplitude proportional to s^2, t^2, and u^2 is given by $\log(-s/\mu)$, $\log(-t/\mu)$ or $\log(-u/\mu)$. Thus, if we choose to renormalize at $s = at = bu = \mu_R^2$ where a and b are fixed constants, the coupling gets shifted by $-\log(ab)$, at the price of having $\log(\frac{-ta}{\mu_R^2})$ and $\log(\frac{-ub}{\mu_R^2})$ in Eq. (44). In this work we have used the symmetric point $a = b = 1$ and the definition (43), because these best capture the behavior of the amplitude and are best suited for comparison to the FRG, but we stress that these are arbitrary choices.

In on-shell configurations, choosing the parameters a and b is equivalent to fixing the scattering angle. This angle should be held fixed along the running from the IR to the UV regime; otherwise, one could observe different beta functions and consequently different runnings of the

coupling. If one allows the scattering angle to also depend on s, the amplitude is no longer described by the universal running coupling, as demonstrated by the example of peripheral scattering in the Appendix.

The FRG gives a continuous interpolation for the running of the quartic coupling and can separately account for the six- and eight-derivative couplings that will inevitably be generated.

In order to compare the RG trajectory of the coupling in the FRG with the trajectory of the physical coupling, we have to make an identification of the argument of the former, which is an arbitrary cutoff scale k, with the argument of the latter, which at the symmetric point is $\sqrt{s}$. If we just put $k^2 = s$ and we adjust the initial conditions so that the two trajectories have the same IR limit $g(0)$, then in the UV limit they differ by a small offset. This can be fixed by choosing $k = \sqrt{s}/\xi$, where $\xi = e^{25/40-17/60} \approx 1.4$. This is illustrated again in Fig. 2.

The other running parameter within the FRG is Z_1. In the notation of this section, the general expression was

$$\beta_{Z_1} = -\frac{Z_1 + 2k^2/m^2}{16\pi^2(Z_1 + k^2/m^2)^2}\frac{gk^4}{M^4}. \tag{61}$$

This is in disagreement with the amplitude calculation, for which Z_1 does not run at all energies,

$$\beta_{Z_1} = 0. \tag{62}$$

If we had defined the running of Z_1 not by the dependence on energy or renormalization scale, but by the dependence of the counterterm on the unphysical parameter μ that appears in dimensional regularization, we would have identified

$$\beta^{\mu}_{Z_1} = \frac{3}{16\pi^2}\frac{gm^4}{M^4}. \tag{63}$$

We noted that the asymptotic form at large k of the FRG result was

$$\beta_{Z_1} = \frac{gm^2k^2}{M^4} + \frac{3}{16\pi^2}\frac{gm^4}{M^4} + \dots \tag{64}$$

which, if one disregards the power-law running, would agree on the logarithmic running. So in this case the difference between schemes is not limited to a particular kinematical domain and Z_1 should not be considered a physically running coupling at all.

The one-loop correction to the kinetic energy term was found to be

$$\frac{3gm^4}{16\pi^2M^4}p^2\left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log\frac{m^2}{\mu^2} + \frac{7}{6}\right]. \tag{65}$$

The portion to focus on is again the $\log m^2/\mu^2$ and what is going on is very similar to the low-energy regime of g. If we perform wave-function renormalization at any kinematic scale, setting $Z_1 = 1$, it remains that value at any other scale. Taking the derivative with respect to μ does not give us physical information in this case.

The other issue is that of power-law corrections found within the FRG. In the case of Z_1 this is a less significant issue than for g, since Z_1 is a redundant coupling and is not associated directly with any scattering process. Another way to say this is that in a two-point function the only invariant scale is p^2, which on shell is just equal to the pole mass m^2. Nevertheless, we can interpret the difference between the beta functions computed here and those coming from the FRG as follows. In this paper we have perturbed around a generic free theory containing both kinetic terms, which is not a fixed point in general: only the theories with $Z_1 = 0$ or $Z_2 = 0$ are fixed points. There is a trivial running with p that goes from one to the other, since the quartic term dominates in the UV and the quadratic one dominates in the IR, but the dimension of the field remains fixed and does not enter in any of our conclusions. However, the canonical dimension of the field at a free fixed point is fixed: it is one at the two-derivative fixed point and zero at the four-derivative one. In the FRG this is correctly taken into account. Within the context of the FRG, the power running of Z_1 with the scale k is necessary to correctly interpolate between the low-energy and high-energy Gaussian fixed points.

VII. DISCUSSION

We have explored the running of couplings in a simple model with higher-derivative interactions and kinetic energy. The scattering amplitude reveals what we are calling the "physical" running, as it describes the running parameters seen in physical processes. This differs from some other definitions of running couplings using different methods, and we have used explicit calculations to illustrate these differences.

Some of the lessons from this work can be summarized as follows.

(1) Physical running couplings can only be defined far from mass thresholds, and there are different patterns of running above and below the threshold. In our case, the coupling g does not run below the threshold and runs logarithmically above it. EFT is useful in understanding the low-energy region.

(2) Power-law running is not seen in the physical amplitudes. Instead, in the EFT regime, the effects that depend on higher powers of the kinematic invariants are organized as higher-order operators in an effective Lagrangian. These higher-order operators disappear altogether above the mass threshold (operator "melting").

(3) Alternate methods of defining running couplings using $\Lambda\frac{\partial}{\partial\Lambda}$, $k\frac{\partial}{\partial k}$, or $\mu\frac{\partial}{\partial\mu}$ (where Λ, k, and μ refer to UV cutoffs, IR cutoffs, and the dimensional regularization auxiliary scale) sometimes yield running behavior which is not seen in physical processes. In our case, this is found in the coupling Z_1. The culprit is factors of $\log m^2/\Lambda^2$, etc., which do not involve any of the kinematic invariants and hence do not change with the energy scale of the physical reaction.

The last point calls into question the utility of these alternate methods if the results are not reflected in physical amplitudes. In the case of the μ running, terms like $\log(\mu^2/m^2)$ in the amplitude do not correspond to physical running. Sometimes such terms arise below the mass threshold and are replaced by genuine running $\log(\mu^2/-q^2)$ above threshold, as we have seen in the example of the top-quark contribution to the running of the electromagnetic coupling. However, there may be other examples of beta functions where this is not properly accounted for. This seems to be the case, for example, in the nonlinear sigma models [19,20] and quadratic gravity [21,22] whose beta functions need to be recalculated. We will report these results separately in joint work with G. Menezes.

The FRG deserves a separate discussion. One way of viewing it is as a method to compute the effective action, i.e., as an alternative to the path integral. One starts from a given form of the scale-dependent effective action, presumed valid at some UV scale Λ, and then includes the effect of quantum fluctuations with momenta between Λ and a lower scale k by integrating the FRG from Λ to k. When one integrates down to $k=0$ the full effective action is obtained, and it contains all of the information about all scattering amplitudes. For an explicit example of such a calculation in the context of ϕ^4 theory, we refer to [23]. Used in this way, k is by itself an unphysical variable. If one were to calculate at some nonzero value of $k=k_*$ one should then add in the quantum corrections that come from the region of $k=0$ up to that value of $k=k_*$, which were excluded by the cutoff. The physical amplitudes would be independent of the choice to work at any nonzero value of k_*. This is the origin of the renormalization group equations: describing how the coupling must change with the cutoff in order to hold the physical properties fixed.

The second way to use the FRG is based on giving a physical meaning to k (a so-called "cutoff identification"). It rests on the ability of the FRG to account for decoupling phenomena, which is one of its main strengths (see, e.g., [24]). If some physical variable F, with dimension of mass, is the only mass scale of the system, and if it enters in loop calculations in the same way as a mass or an IR cutoff, then the F dependence of the effective action (at $k=0$) will be the same as the k dependence of the running effective action. This is because for $k<F$ the decoupling theorem implies that the effective action becomes independent of k. We refer to [25] for a clear exposition of this logic.

In our calculations, the identification of the cutoff with external momenta sometimes works and sometimes does not. Consider first the corrections to the two-point function. If one had calculated the tadpole diagram with cutoff regularization rather than dimensional regularization, the tadpole integral is quadratically divergent and one would find a result that has the form

$$\frac{3gm^4}{16\pi^2M^4}\,p^2\left[c\Lambda^2-\log\frac{m^2}{\Lambda^2}+\ldots\right], \tag{66}$$

where the constant c would depend on how the cutoff is implemented. In this case, Λ is a UV cutoff so that quantum corrections below Λ are included. However, in this case we know that, treated as a regularization scheme, both the power-law and logarithmic dependence on Λ disappear after renormalization and we again have $Z_1=1$. So in this case, tracing the cutoff dependence of the counterterm does not say anything about the two-point function. In the FRG calculation the tadpole gives rise to quadratic running of Z_1. However, the external momentum does not enter in any way in the tadpole integral and so this not a case where one can trade the external momentum dependence for k dependence.

In the running of the coupling g we see both outcomes. At low energy the power-law running found in the FRG is not observed in the amplitude. It is an aspect of the threshold behavior, interpolating between a constant in the IR limit and the logarithmic behavior at the high energy. The threshold behavior of the couplings in FRG is not universal, and in any case there is no definition of physical running to compare with in that regime.

At high energy, the dependence on $\log k^2$ correctly mirrors the dependence of the amplitude on $\log E^2$ and gives the correct beta function. This is due to the fact that for fixed ratios t/s and u/s, and in the limit when $m/s\to 0$, the amplitude depends only on a single mass scale s, which enters in the denominators of the loop integrals in a way that is reminiscent of an IR cutoff. Thus, in this regime, the k dependence of the running coupling correctly reflects the s dependence of the amplitude. At energies close to m, the amplitude becomes a complicated function of s and m and no RG calculation exactly reproduces the amplitude.

The model discussed in this paper reiterates several points made by one of us in the past [26–28], but there are also some new aspects. The disappearance of the higher-order operators of the low-energy EFT is expected when

the model is UV completed in a linear $U(1)$ sigma model, but surprisingly also happens when the four-derivative kinetic term becomes important. This offers a glimpse of how, in a derivatively coupled theory, one could transition from the low-energy EFT regime to an asymptotically free (and possibly asymptotically safe) regime. In principle, this could provide an alternative UV completion to the $U(1)$ linear sigma model, mentioned in the Introduction.

This kind of behavior may also be extended to gravitational theories. For example, it raises the possibility that at least some of these higher-order operators, such as those of order R^3, should not be used above certain thresholds, because the coefficients of the higher-order operators vanish. Our model seems to enter a strong-coupling regime at very high energy. This is because the powers of momentum of the interaction overwhelm the logarithmic decrease of the coupling. We have not discussed the physics of this regime in this paper, but we plan to return to it in the future.

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APPENDIX: PERIPHERAL SCATTERING

The peripheral scattering limit is that of large s and $t \sim 0$. Although this appendix is only peripheral to our main discussion (pun intended), it does have an interesting feature which we comment on in the discussion.

In this case, the s and u channels give similar contributions to the scattering amplitude, since on shell $u = -s$ and the Mandelstam variables appear only quadratically or in the log in the dominant terms in the high-energy limit:

$$\frac{7g^2m^4s^2}{48M^8\pi^2\epsilon} + \frac{g^2m^4}{576M^8\pi^2}\left\{(74-84\gamma_E)s^2 + 42s^2\left[\log\left(\frac{4\pi\mu^2}{s}\right) + \log\left(-\frac{4\pi\mu^2}{s}\right)\right]\right\} + O(\epsilon^1). \tag{A1}$$

On the other hand, the t channel is a bit more subtle, since the terms that are powers of t in the denominator could give some divergences. However, if we expand (28) with t and s exchanged at small t, all of the divergent terms are actually zero and we obtain

$$\begin{aligned}&\frac{g^2m^4s^2}{96\pi^2M^8\epsilon} + \frac{g^2m^4s^2\left[6\log(\frac{4\pi\mu^2}{m^2}) - (6\gamma-5)\right]}{576\pi^2M^8} + \frac{g^2m^2st}{96\pi^2M^8\epsilon} + \frac{g^2m^2st\left[7s + 12m^2\log(\frac{4\pi\mu^2}{m^2}) + 2m^2(-6\gamma+5)\right]}{1152\pi^2M^8}\\ &\quad + \frac{7g^2t^2m^4}{96\pi^2M^8\epsilon} + \frac{g^2t^2\left[-6s^2\log\left(-\frac{t}{m^2}\right) + 9s^2 + 35m^2s + 420m^4\log(\frac{4\pi\mu^2}{m^2}) - m^4(420\gamma+140)\right]}{5760\pi^2M^8}\\ &\quad + O(t^3).\end{aligned} \tag{A2}$$

Anyway, the first line is clearly dominant. Hence, the one-loop quantum corrections in peripheral scattering are

$$\frac{5g^2m^4s^2}{32\pi^2\epsilon M^8} + \frac{g^2m^4s^2}{576M^8\pi^2}\left\{(79-90\gamma_E) + 42\left[\log\left(\frac{4\pi\mu^2}{s}\right) + \log\left(-\frac{4\pi\mu^2}{s}\right)\right] + 6\log\left(\frac{4\pi\mu^2}{m^2}\right)\right\} + O(t). \tag{A3}$$

After renormalization, in the present notation the amplitude has the form

$$\mathcal{M} = \frac{gs^2}{M^4}\left[1 + \frac{7gm^4}{96\pi^2M^4}\left(\log\left(\frac{-s}{m^2}\right) + \log\left(\frac{s}{m^2}\right)\right) + \frac{79gm^4}{576\pi^2M^4}\right] + \mathcal{O}(st). \tag{A4}$$

For this process one can define a running coupling

$$\tilde{g}(\mu_R) = g + \frac{7g^2m^4}{48\pi^2M^4}\log\frac{\mu_R^2}{m^2} + \frac{79g^2m^4}{576\pi^2M^4}$$

when renormalizing at the scale $s = \mu_R^2$, where again the $79/576\pi^2$ factor is optional. This removes the potentially large logarithms, and carries the beta function

$$\beta_{\tilde{g}} = \frac{7g^2m^4}{24\pi^2M^4}.$$

It is interesting that one can define a physical beta function in this region, yet it is different from that found when all of the kinematic variables are large.

We can understand this in the following way. The universal beta functions that one calculates from perturbation theory are only universal as long as one considers processes that depend on a single momentum scale. This is the case, for example, for $2 \to 2$ scattering at a fixed angle: the ratios of the Mandelstam variables are fixed and the amplitude depends just on s. In the case of peripheral scattering we are changing the scattering angle together with the energy, and the amplitude is not a function of s alone. While there is no guarantee that a running coupling can be defined in this setting, it appears possible in the one-loop calculation.

[1] K. G. Wilson and J. B. Kogut, The renormalization group and the epsilon expansion, Phys. Rep. **12**, 75 (1974).
[2] F. J. Wegner and A. Houghton, Renormalization group equation for critical phenomena, Phys. Rev. A **8**, 401 (1973).
[3] J. Polchinski, Renormalization and effective Lagrangians, Nucl. Phys. **B231**, 269 (1984).
[4] C. Wetterich, Exact evolution equation for the average potential, Phys. Lett. B **301**, 90 (1993).
[5] T. R. Morris, The exact renormalization group and approximate solutions, Int. J. Mod. Phys. A **09**, 2411 (1994).
[6] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J. M. Pawlowski, M. Tissier, and N. Wschebor, The nonperturbative functional renormalization group and its applications, Phys. Rep. **910**, 1 (2021).
[7] T. D. Lee and G. C. Wick, Finite theory of quantum electrodynamics, Phys. Rev. D **2**, 1033 (1970).
[8] A. Salvio, Quadratic gravity, Front. Phys. **6**, 77 (2018).
[9] J. F. Donoghue and G. Menezes, On quadratic gravity, Nuovo Cimento C **45**, 26 (2022).
[10] K. Symanzik, A field theory with computable large-momenta behavior, Lett. Nuovo Cimento **6S2**, 77 (1973).
[11] C. P. Burgess, *Introduction to Effective Field Theory* (Cambridge University Press, Cambridge, England, 2020), ISBN 978-1-139-04804-0, 978-0-521-19547-8.
[12] J. F. Donoghue and L. Sorbo, *A Prelude to Quantum Field Theory* (Princeton University Press, Princeton, NJ, 2022), ISBN 978-0-691-22349-0, 978-0-691-22348-3, 978-0-691-22350-6.
[13] M. Safari, A. Stergiou, G. P. Vacca, and O. Zanusso, Scale and conformal invariance in higher derivative shift symmetric theories, J. High Energy Phys. 02 (2022) 034.
[14] G. P. de Brito, A. Eichhorn, and R. R. L. d. Santos, The weak-gravity bound and the need for spin in asymptotically safe matter-gravity models, J. High Energy Phys. 11 (2021) 110.
[15] C. Laporte, A. D. Pereira, F. Saueressig, and J. Wang, Scalar-tensor theories within asymptotic safety, J. High Energy Phys. 12 (2021) 001.
[16] D. Buccio and R. Percacci, Renormalization group flows between Gaussian fixed points, J. High Energy Phys. 10 (2022) 113.
[17] A. A. Tseytlin, Comments on 4-derivative scalar theory in 4 dimensions, Teor. Mat. Fiz. **217**, 649 (2023).
[18] B. Holdom, Running couplings and unitarity in a 4-derivative scalar field theory, Phys. Lett. B **843**, 138023 (2023).
[19] P. Hasenfratz, Four-dimensional asymptotically free nonlinear sigma models, Nucl. Phys. **B321**, 139 (1989).
[20] R. Percacci and O. Zanusso, One loop beta functions and fixed points in higher derivative sigma models, Phys. Rev. D **81**, 065012 (2010).
[21] E. S. Fradkin and A. A. Tseytlin, Renormalizable asymptotically free quantum theory of gravity, Phys. Lett. **104B**, 377 (1981); Nucl. Phys. **B201**, 469 (1982).
[22] I. G. Avramidi and A. O. Barvinsky, Asymptotic freedom in higher derivative quantum gravity, Phys. Lett. **159B**, 269 (1985).
[23] A. Codello, R. Percacci, L. Rachwał, and A. Tonero, Computing the effective action with the functional renormalization group, Eur. Phys. J. C **76**, 226 (2016).
[24] J. Berges, N. Tetradis, and C. Wetterich, Non-perturbative renormalization flow in quantum field theory and statistical physics, Phys. Rep. **363**, 223 (2002).
[25] M. Reuter and H. Weyer, Renormalization group improved gravitational actions: A Brans-Dicke approach, Phys. Rev. D **69**, 104022 (2004).
[26] M. M. Anber, J. F. Donoghue, and M. El-Houssieny, Running couplings and operator mixing in the gravitational corrections to coupling constants, Phys. Rev. D **83**, 124003 (2011).
[27] M. M. Anber and J. F. Donoghue, On the running of the gravitational constant, Phys. Rev. D **85**, 104016 (2012).
[28] J. F. Donoghue, A critique of the asymptotic safety program, Front. Phys. **8**, 56 (2020).