Stability of nonsingular cosmologies in Galileon models with torsion: A no-go theorem for eternal subluminality

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Generic models in Galileons or Horndeski theory do not have cosmological solutions that are free of instabilities and singularities in the entire time of evolution. We extend this no-go theorem to a spacetime with torsion. On this more general geometry the no-go argument now holds provided the additional hypothesis that the graviton is also subluminal throughout the entire evolution. Thus, critically different for Galileons' stability on a torsionful spacetime, an arguably unphysical although arbitrarily short (deep UV) phase occurring at an arbitrary time, when the speed of gravity (c_g) is slightly higher than luminal (c), and by at least an amount $c_g \geq \sqrt{2}c$, can lead to an all-time linearly stable and nonsingular cosmology. As a proof of principle we build a stable model for a cosmological bounce that is almost always subluminal, where the short-lived superluminal phase occurs before the bounce, and that transits to general relativity in the asymptotic past and future.

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Galileons is a well motivated modification of general relativity (GR) by a scalar, with higher derivatives in the action but with second order equations of motion [1]. The generalization is equivalent to Horndeski theory [2–4], and it has nonsingular solutions that generally suffer of gradient instabilities at some time in the entire evolution, up to some special cases [5–13]. Although these pathologies can happen away from the physically relevant phase, a conclusive resolution to this issue at all times in generic models seems unlikely because the no-go argument for stability also holds with very general extra matter [8,10,11]. We extend this no-go theorem to a spacetime with torsion in Sec. I and show that a torsionful geometry may support stable solutions in Galileons if there exists a superluminal phase. It can *formally* happen as a deep UV inconsistency at an arbitrary time; namely, it can be arbitrarily short and unrelated to the much longer physically relevant length scales, e.g., width of a bounce, but it casts doubt on Lorentz invariant UV completions [14–16]. In Sec. II we build a toy model for a bounce that is *always* stable.

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I. EXTENSION OF THE NO-GO THEOREM TO GALILEONS ON A SPACETIME WITH TORSION

We consider up to quartic generalized Galileons in the notation of [4], and we assume a spacetime with torsion:

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}(G_2 - G_3\tilde{\nabla}_\mu\tilde{\nabla}^\mu\phi + G_4\tilde{R} + G_{4,X}((\tilde{\nabla}_\mu\tilde{\nabla}^\mu\phi)^2 - (\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)\tilde{\nabla}^\nu\tilde{\nabla}^\mu\phi)), \quad (1)$$

where G_2, G_3, G_4 are arbitrary functions of ϕ and $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, g is the determinant of the metric with mostly $+$ signature, $G_{4,X} = \partial G_4/\partial X$, and $\tilde{R}$ and $\tilde{\nabla}$ denote the Ricci scalar and covariant derivative computed with a torsionful connection. The G_3 term is the simplest one to "feel" the torsion on the spacetime. For G_4 let us stress the specific order of contraction of Lorentz indices in the last term in (1). Indeed, second covariant derivatives with torsion do not commute on a scalar, and it was found in [17] that this choice is the only one that leads to a scalar with a wavelike dispersion relation, as in torsionless Galileons. Hence, (1) is the relevant choice for the question of how a different geometry can help the stability of the usual Galileon degrees of freedom.

A. Quadratic action for Galileons on a torsionful vs torsionless spacetime

We analyze the stability of the Friedmann-Lemaître-Robertson-Walker (FLRW) background against linearized perturbations. A straightforward computation shows that all independent components of torsion in (1) are nondynamical (See Appendix A 1 for a detailed derivation). The important aspect is that we can cast the quadratic action of Galileons with torsion in a form reminiscent of the usual Galileons without torsion [4]. Namely, from (1)

$$\mathcal{S}_\tau = \int \mathrm{d}\eta\, \mathrm{d}^3x a^4 \left[\frac{1}{2a^2}(\mathcal{G}_\tau(\dot{h}_{ij})^2 - \mathcal{F}_\tau(\partial_k h_{ij})^2)\right] \quad (2)$$

is the action for the graviton h_{ij} with speed $c_g^2 = \mathcal{F}_\tau/\mathcal{G}_\tau$, where η is conformal time. The vector sector is nondynamical. The scalar sector reads in the unitary gauge

$$\mathcal{S}_s = \int \mathrm{d}\eta\, \mathrm{d}^3x a^4 \bigg(-3\frac{\mathcal{G}_\tau}{a^2}\dot{\psi}^2 + \frac{\mathcal{F}_\tau}{a^2}(\partial_i\psi)^2 + 6\frac{\Theta}{a}\alpha\dot{\psi} + 2\frac{T}{a^2}\partial_i\alpha\partial_i\psi + 2\frac{\partial_i\partial_i B}{a^2}(a\Theta\alpha - \mathcal{G}_\tau\dot{\psi}) + \Sigma\alpha^2\bigg), \quad (3)$$

where ψ, α, and B are scalar perturbations and

$$\mathcal{G}_\tau = 2\frac{G_4^2}{G_4 + 2XG_{4,X}}, \quad \mathcal{F}_\tau = 2G_4, \quad T = \mathcal{F}_\tau(c_g^2 - 2), \quad (4)$$

$$\Theta = \frac{4\mathcal{G}_\tau^2\theta}{a\mathcal{F}_\tau^4}, \qquad \Sigma = \frac{2\mathcal{G}_\tau^3\sigma}{a\mathcal{F}_\tau^6}, \quad (5)$$

θ and σ (shown in the Appendix A 3) depend on two background fields: the scale factor of the FLRW metric $a(\eta)$ and the Galileon/Horndeski scalar, which in the context of linearized expressions we also denote as $\phi(\eta)$ and $X = \frac{\dot{\phi}^2}{2a^2}$. Let us also note that there is a nontrivial torsion background $x(\eta)$ expressed by the background equations in terms of $a(\eta), \phi(\eta)$. We show these details in Appendix A 1 for completeness.

Let us notice that despite the similarities between the quadratic actions in torsionless and torsionful Galileons, there is a crucial difference in (3) that helps the stability of the theory with torsion: namely, $\mathcal{G}_\tau \neq T$. This difference stems from the constraint equations imposed by the torsion perturbations.

Finally, let us bring (3) to a more useful form by using the equation for the Lagrange multiplier B ($\alpha = \frac{1}{a}\frac{\mathcal{G}_\tau}{\Theta}\dot{\psi}$) in $\mathcal{S}_s$. Thus, we obtain a single scalar mode,

$$\mathcal{S}_s = \int \mathrm{d}\eta\, \mathrm{d}^3x a^4 \left(\frac{1}{a^2}\mathcal{G}_\mathcal{S}\dot{\psi}^2 - \frac{1}{a^2}\mathcal{F}_\mathcal{S}(\partial_i\psi)^2\right), \quad (6)$$

where

$$\mathcal{G}_\mathcal{S} = 3\mathcal{G}_\tau + \frac{\mathcal{G}_\tau^2\Sigma}{\Theta^2}, \qquad \mathcal{F}_\mathcal{S} = \frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{a\mathcal{G}_\tau T}{\Theta}\right) - \mathcal{F}_\tau. \quad (7)$$

The no-go theorem follows a similar reasoning as in [5] in relation to wormholes, or as initially proved for a subclass of generalized Galileons in [6] and then extended to the full Horndeski action in [7]:

a. *No-go for nonsingular, all-time stable and subluminal solutions*: For Galileons on a spacetime with torsion (1) the following assumptions for a first order perturbative expansion about FLRW are mutually inconsistent:

(I) Nonsingular cosmology: namely, there is a lower bound on the scale factor $a(\eta) > b_1 > 0$.

(II) The graviton and the scalar mode are not ghosts, and they suffer no gradient instabilities: $\mathcal{G}_\tau > 0$, $\mathcal{F}_\tau > 0, \mathcal{F}_\mathcal{S} > 0, \mathcal{G}_\mathcal{S} > 0$.

(III) The graviton is always sub/luminal: $(c_g)^2 \leq 1$.

(IV) There is a lower bound $\mathcal{F}_\tau(\eta) > b_2 > 0$ as $\eta \to \pm\infty$ (No "strong gravity" at linear order [7,18]).

(V) Θ vanishes at most a finite number of times (to cover generic theories not defined by the equation $\Theta \equiv 0$ [12]).

The argument: It is key to notice that (I)–(III) imply

$$N =: \frac{a\mathcal{G}_\tau\mathcal{F}_\tau(c_g^2 - 2)}{\Theta} \neq 0, \quad (8)$$

because by (I–II) Θ is a regular (finite) function of H, ϕ.

Let us integrate the third inequality in (II). Using (7),

$$\Delta N = N_f - N_i > I(\eta_i, \eta_f), \qquad I(\eta_i, \eta_f) = \int_{\eta_i}^{\eta_f} \mathrm{d}\eta\, a^2\mathcal{F}_\tau, \quad (9)$$

where N_f and N_i are the values of N at some (conformal) times η_f and η_i respectively. Now, by (I), (II), and (IV)

(A) $\frac{dN}{d\eta} > a^2\mathcal{F}_\tau > b_1^2 b_2 > 0$,

(a) defining $I(\eta_i) := I(\eta_i, \eta_f)|_{\eta_f}$ and $I(\eta_f) := I(\eta_i, \eta_f)|_{\eta_i}$ we notice that they are *positive and growing* functions of η_i and η_f, for η_f and η_i fixed, respectively. $I(\eta_i)$ and $I(\eta_f)$ are differentiable and hence continuous because $a^2\mathcal{F}_\tau$ is continuous,

(b) N is monotonous increasing and hence, denoting with η_z any zero of Θ, then $N(\eta) \to \infty$ as $\eta \to \eta_z^-$ (η approaches η_z by the *left*) and $N(\eta) \to -\infty$ as $\eta \to \eta_z^+$,

(B) $\Delta N > 0$,

(C) $I(\eta_i)$ and $I(\eta_f)$ are not convergent as $\eta_i \to -\infty$, $\eta_f \to \infty$, respectively.

Now, by (V) N is finite almost everywhere; thus let us take a fixed value $-\infty < N_i < 0$ at some fixed time η_i. By (8) it follows that $N_f(\eta_f) < 0$ [Without lost of generality we can safely assume that there is no η_z (a zero of Θ) such that $\eta_i < \eta_z < \eta_f$[1]]. Then ΔN is also bounded by above as

[1]Indeed, if Θ has one zero η_z such that $\eta_i < \eta_z < \eta_f$, then by (A) N must be *positive* arbitrarily close *by the left of* η_z. But our fixed value $N_i(\eta_i) < 0$ means that $N(\eta)$ must have already vanished at some time η, with $\eta_i < \eta < \eta_z < \eta_f$ [because $N(\eta)$ is continuous for $\eta_i < \eta < \eta_z$], *thus already violating* (8). This clearly extends to any number of zeros. Thus, provided our starting point $-\infty < N_i(\eta_i) < 0$, we can exclude the case of any η_z in the time interval (η_i, η_f).

$|N_i| > \Delta N = |N_i| - |N_f| > 0$. Now, by (A), (C) $I(\eta_f)$ not only grows with η_f but is also unbounded from above; then there exists late enough in the evolution a critical time η_c such that if $\eta_f > \eta_c$, $I(\eta_f) > |N_i| > \Delta N$ for every fixed value N_i. This violates (9), and so we must have $N(\eta) > 0$. However, by a similar argument, fixing a value $\infty > N_f > 0$ at some η_f, necessarily $N_i(\eta_i) > 0$,[2] then $N_f > \Delta N = N_f - N_i > 0$, and by (A), (C) $I(\eta_i)$ is unbounded from above and there exists η_c early enough, such that if $\eta_i < \eta_c$, $I(\eta_i) > N_f > \Delta N$. Thus, eventually in the evolution (9) does not hold.

In fact, (III) can be relaxed to $c_g^2 < 2$, and the argument still holds. But, an almost everywhere subluminal graviton turning to $c_g^2 \geq 2$ during an arbitrarily short interval is enough to avoid this no-go argument, as we show below. Clearly, a minimal example needs $G_{4,X} \neq 0$.

II. EVERYWHERE STABLE BOUNCING COSMOLOGY IN GALILEONS WITH TORSION

A sufficient assumption to bypass the no-go theorem and obtain all-time stable solutions is, for instance, a period of *nonzero* width of superluminality of the graviton at some point in the evolution and by at least an amount $c_g \geq \sqrt{2}c$ ($c = 1$ in our units). Thus, in principle, the width τ_s of an arguably unphysical superluminal phase centered at a time η_s can be *arbitrarily short* and unrelated to the width τ_b and time of occurrence of the physically relevant bounce phase centered at a time η_b.

As a proof of principle let us show with a toy model that one can achieve stability of the FLRW bouncing background against linear order perturbations even when there is no relation between the timescales associated with the superluminal phase, necessary to avoid the no-go theorem, and the bounce phase. We assume that η_s, η_b are finite and without loss of generality $\eta_s < \eta_b = 0$. Furthermore, we demand for our model that the solution reduces in the asymptotic past and future to a solution that one could also obtain from conventional Einstein's gravity with a *luminal* graviton, minimally coupled to a massless scalar, and such that the torsion background is asymptotically vanishing. More precisely, we consider that $a(\eta)$ is positive and bounded from below, the bounce happening at the minimum $a(\eta_b)$, and for the latter asymptotics we require that the leading terms of the Lagrangian functions in (1) and the torsion background behave as follows as $\eta \to \pm\infty$:

$$G_2(\phi, X) \to \frac{1}{2a^2}\dot{\xi}^2, \qquad G_4(\phi, X) \to \frac{1}{2},$$
$$G_3(\phi, X) \to 0, \qquad x(\eta) \to 0, \tag{10}$$

[2] By a similar argument as in footnote 1, provided our starting point $\infty > N_f > 0$ we can exclude without lost of generality any η_z in the time interval (η_i, η_f).

where ξ is some invertible function of the Horndeski scalar ϕ, and we choose $M_{pl}^2/8\pi = 1$.

A. Construction of the model

a. *Procedure*: The following Ansatz for the Lagrangian functions has enough structure such that we can satisfy the asymptotic conditions (10), besides demanding the stability *at all times* (II), while simultaneously solving all the equations of motion:

$$G_2(\phi, X) = g_{20}(\phi) + g_{21}(\phi)X + g_{22}(\phi)X^2, \tag{11}$$

$$G_3(\phi, X) = g_{30}(\phi) + g_{31}(\phi)X, \tag{12}$$

$$G_4(\phi, X) = \frac{1}{2} + g_{40}(\phi) + g_{41}(\phi)X. \tag{13}$$

We reconstruct the seven unknown Lagrangian functions in (11)–(13), namely, g_{20}, g_{21}, g_{22}, g_{30}, g_{31}, g_{40}, and g_{41} stating first some solutions satisfying our requirements for $a(\eta)$ and the asymptotics (10), and then we work backward to find the Lagrangian functions whose dynamics correspond to the latter. We proceed as follows: without loss of generality we choose a model with a solution for the Hubble parameter, shown in Fig. 1, and the background of the Horndeski scalar field as

$$a = (\tau_b^2 + \eta^2)^{\frac{1}{4}}, \qquad H = \frac{\dot{a}}{a^2} = \frac{\eta}{2(\tau_b^2 + \eta^2)^{\frac{5}{4}}}, \qquad \phi = \eta, \tag{14}$$

such that our definition of bounce is satisfied. $\tau_b > 0$ fixes the maximum of H and the width of the bounce phase as the

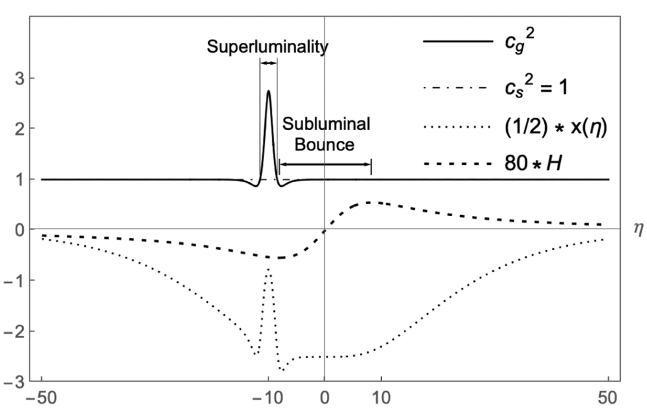

FIG. 1. Hubble parameter for a bounce at $\eta_b = 0$ with $\tau_b = 10$. Speed of sound for the scalar mode c_s^2. Speed of the graviton c_g^2 with *short* superluminality phase ($\tau_s \ll \tau_b$) happening at $\eta_s = -10$ before the bounce (For convenience displaying the graphs we choose here $\tau_b = 10\tau_s$). The graviton quickly becomes subluminal around η_s and approaches luminality from below in the past, and during the bounce phase and future. Torsion background $x(\eta)$ is exponentially vanishing in the asymptotic past and future.

length of the domain where $\dot{H}(\eta) > 0$ around η_b. With this solution $X = 1/(2(\tau_b^2 + \eta^2)^{\frac{1}{2}})$. Now we solve the unknown Lagrangian functions in (11)–(13).

b. *Solving for g_{40} and g_{41} provided $\mathcal{G}_\tau > 0$, $\mathcal{F}_\tau > 0$ for all time, T vanishing at least once at some critical time, and G_4 asymptotics* (10): $\mathcal{G}_\tau$, $\mathcal{F}_\tau$ and the critical function T (4) depend only on G_4. Hence, we can solve for g_{40} and g_{41} from two algebraic equations in these variables:

$$\mathcal{F}_\tau(g_{40}, g_{41}) = 1, \tag{15}$$

$$T(g_{40}, g_{41}) = -1 - \frac{5}{4}\mathrm{Sech}\left(\frac{\eta - \eta_s}{\tau_s}\right) + 3\mathrm{Sech}\left(\frac{\eta - \eta_s}{\tau_s}\right)^2 \tag{16}$$

Equation (15) is a simple *choice* to realize the desired asymptotics of G_4 (16) and to obviously satisfy $\mathcal{F}_\tau > 0$. Equation (16) is an explicit *choice* to violate the subluminal graviton assumption at least during a short time $\tau_s \ll \tau_b$, which allows one to bypass the no-go theorem, as shown in Fig. 2. The solutions for g_{40} and g_{41} from the system of equations (15) and (16) are straightforward and everywhere nonsingular. Their graphs are shown in Fig. 4(b) in Appendix A 2. It can be readily verified that these solutions also imply $\mathcal{G}_\tau > 0$. They can be written at leading order as $\eta \to \pm\infty$ in the form

$$g_{40} = -g_{41}X = \frac{5}{8}e^{\mp\frac{(\eta-\eta_s)}{\tau_s}}. \tag{17}$$

c. *Solving for g_{30} and g_{31} provided $\mathcal{F}_\mathcal{S} > 0$, torsionless and G_3 asymptotics* (10): $\mathcal{F}_\mathcal{S}$ and the torsion background (x) depend on G_4, which is now fully fixed, and on G_3.

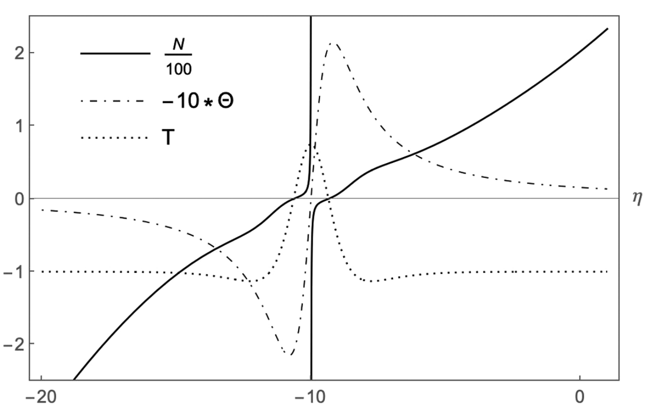

FIG. 2. Bypassing the no-go theorem. This choice for T (16) does not satisfy the all-time negativity condition, which critically means that the graviton is superluminal during a brief stage of evolution, around $\eta_s = -10$ as shown in Fig. 1, and that the function N in Eq. (8) vanishes. Hence, the no-go theorem does not hold, and we can build all-time stable solutions ($\tau_b = 10$, $\tau_s = 1, \eta_b = 0, \eta_s = -10$).

Hence, we can solve for g_{30} and g_{31} from two equations that are algebraic in these functions:

$$G_3(g_{30}, g_{31}) = \mathrm{Sech}\left(\frac{\eta}{\tau_b}\right), \qquad \Theta(g_{30}, g_{31}) = -H_s, \tag{18}$$

where

$$H_s = \frac{\eta - \eta_s S}{2(\tau_b^2(1-S) + \tau_s^2 S + (\eta - \eta_s S)^2)^{\frac{5}{4}}}, \tag{19}$$

and S is a step function which we specify below.

The *choice* of G_3 centered at the bounce $\eta_b = 0$ (18) vanishes exponentially fast in the asymptotic past and future, which satisfies (10). On the other hand, we choose an equation for Θ because it is the only free function in $\mathcal{F}_\mathcal{S}$. It has to satisfy the two remaining conditions: (i) it must be in accordance to the required asymptotics (10) and (ii) it must render $\mathcal{F}_\mathcal{S}$ positive everywhere. For (i) a close inspection of Θ in terms of the Ansatz (11)–(13) reveals that in order to recover a standard scalar minimally coupled to Einstein's gravity (10), then one needs $\Theta \overset{\eta\to\pm\infty}{\longrightarrow} -H$.[3] Hence, in Eq. (18) the *first requirement* for the step function S is that it suppresses the factors τ_s and η_s in H_s fast enough such that in the asymptotic past and future we recover asymptotics of our Hubble parameter (14) *at the necessary order* in η^4 that satisfies the asymptotics (10). On the other hand, (ii) can be easily satisfied with a step S that is nearly 1 in a domain of finite length $(\eta_s - \delta, \eta_b + \delta)$ where $\infty > \delta > 0$, for a large enough δ depending on the parameters $\tau_b, \tau_s, \eta_b, \eta_s$. As a proof of principle we choose

$$S = \mathrm{Sech}\left(\frac{\tau_s}{\tau_b}\frac{(\eta - \eta_s)}{\eta_s}\right), \tag{20}$$

which satisfies (i) and $\mathcal{F}_\mathcal{S} > 0$ everywhere. Suffice it to say for this toy model that this choice meets the requirements, as shown in Fig. 2, for instance for the parameters of the bounce $\eta_b = 0$, $\tau_b = 10$ and of the earlier and *shorter* superluminal phase $\eta_s = -10, \tau_s = \tau_b/10$.

The solutions for g_{30} and g_{31} obtained from (18) are everywhere regular, and their graphs are shown in Fig. 4(a). They can be written at leading order as $\eta \to \pm\infty$ in the form

$$g_{30} = -g_{31}X = \frac{3}{2}\frac{\eta_s}{\eta^2}e^{\mp\frac{\tau_s}{\tau_b}\frac{(\eta-\eta_s)}{|\eta_s|}}, \tag{21}$$

[3]Similar to the torsionless case, this choice introduces a well-known removable singularity in the unitary gauge known as γ crossing that can be seen to be harmless for the regularity of perturbations as in [19].

[4]In particular, it is not a trivial fact that one must choose the step function S such that the limit $\Theta = -H_s \to -H$ as $\eta \to \pm\infty$ is satisfied at *more* than leading order in η, in order to meet the required asymptotics of the Lagrangian functions (10) *only* at the leading order.

where we used $\eta_s < 0$, and the torsion background can be written at leading order as $\eta \to \pm\infty$, as

$$x = \mp\, \eta e^{\mp\frac{\eta}{\tau_b}}. \tag{22}$$

d. *Solving for g_{20}, g_{21}, and g_{22} provided $\mathcal{G}_s > 0$, always sub/luminal scalar, and the E-L equations for the background fields*: $\mathcal{G}_s$ and the Euler-Lagrange equations for the background fields $\mathcal{E}_{g_{00}} = 0$, $\mathcal{E}_{g_{ij}} = 0$ depend on G_3, G_4, which are fully fixed, and on G_2. Hence, we can solve for the Lagrangian functions $g_{20}(\phi)$, $g_{21}(\phi)$, and $g_{22}(\phi)$ from the following system of three equations, which again, is algebraic and linear in these functions:

$$\mathcal{G}_{\mathcal{S}} = \mathcal{F}_{\mathcal{S}}, \quad \mathcal{E}_{g_{00}} = 0, \quad \mathcal{E}_{g_{11}} = \mathcal{E}_{g_{22}} = \mathcal{E}_{g_{33}} = 0. \tag{23}$$

Because the Lagrangian functions $g_{30}(\phi)$ and $g_{31}(\phi)$ are such that $\mathcal{F}_{\mathcal{S}} > 0$, the choice of Eq. (23) is one possibility to simultaneously satisfy a nonghost scalar $\mathcal{G}_{\mathcal{S}} > 0$ and luminality for the scalar mode $c_s^2 = \mathcal{F}_{\mathcal{S}}/\mathcal{G}_{\mathcal{S}} = 1$. We choose luminality for no other reason than simplicity, although a subluminal choice is safer and better suited in many other cases. The unique solutions for $g_{20}(\phi)$, $g_{21}(\phi)$, and $g_{22}(\phi)$ obtained from (23) are nonsingular everywhere, and their graphs are shown in Fig. 3. They can be written at leading order as $\eta \to \pm\infty$ as

$$g_{20} = -\frac{\tau_b^2}{2}(\pm\eta)^{-5}, \qquad g_{21}X = \frac{3}{4}(\pm\eta)^{-3}, \tag{24}$$

$$g_{22}X^2 = \mp\frac{3}{4}(\pm\eta)^{-3}\frac{\tau_s}{\tau_b}e^{\mp\frac{\tau_s}{\tau_b}\frac{(\eta-\eta_s)}{|\eta_s|}}. \tag{25}$$

Let us note that because G_2, G_3, G_4 are such that the E-L equations (23) and their derivatives are satisfied, then the bouncing solution (14) is the correct one for the model with the Lagrangian functions that we just solved. Furthermore, the remaining E-L equation for the background scalar ($\mathcal{E}_\phi = 0$) is implied by the others because of gauge invariance, as can be readily verified, which certifies that the solution $\phi = \eta$ in (14) is also the correct one for the model with the Lagrangian functions just built.

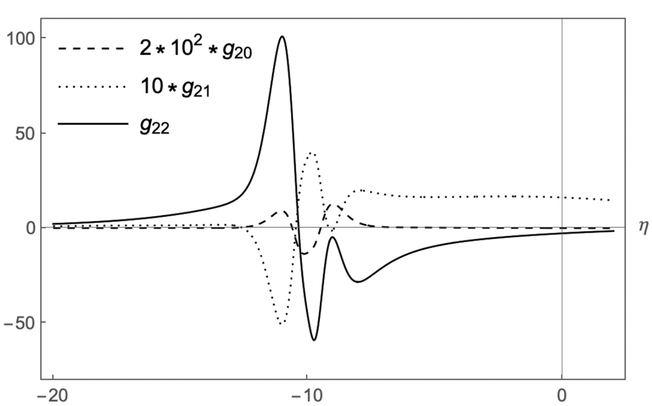

FIG. 3. Everywhere regular Lagrangian functions g_{20}, g_{21}, and g_{22}.

B. Asymptotic Lagrangian: Recovering GR

The leading order expressions of the Lagrangian functions as $\eta \to \pm\infty$ (17), (21), (24), and (25) in the Ansatz (11)–(13) tell that to the leading order the only nonvanishing Lagrangian functions are $G_4 = \frac{1}{2}$ and G_2. Hence, with the solution $\phi = \eta$ and the leading order expression $X = 1/(2\eta)$, considering only the leading contributions to G_2, namely $g_{21}X$, we can identify at leading order the corresponding action to (1) in the asymptotic past and future

$$S^\infty = \frac{1}{2}\int d^4x\sqrt{-g}(R - \partial_\mu\xi\partial^\mu\xi) \tag{26}$$

for a massless scalar field $\xi = \sqrt{\frac{3}{2}}\ln(\phi)$ minimally coupled to Einstein's gravity and with vanishing torsion background (22). Indeed, one can check that the field and Friedmann equations derived from (26) are satisfied by the leading order contributions as $\eta \to \pm\infty$ of the solutions that we started with in (14): namely, $a = \eta^{\frac{1}{2}}$, $H = \frac{1}{2}\eta^{-\frac{3}{2}}$, $\phi = \eta$, $\xi = \sqrt{\frac{3}{2}}\ln(\eta)$ solve

$$\ddot{\xi} + 2aH\dot{\xi} = 0, \qquad \dot{\xi}^2 - 6a^2H^2 = 0. \tag{27}$$

III. CONCLUSION

We first extended the no-go argument of [6,7] to Galileons on a spacetime with torsion (Horndeski-Cartan) (1). We showed that in generic models it is not possible to obtain a nonsingular FLRW cosmology that is always free of gradient instabilities against the scalar perturbation and an eternally sub/luminal graviton.

Then, we highlighted that unlike in the torsionless theory, where instabilities happen with certainty at some time in the entire evolution [6,7], a spacetime with torsion can support all-time *linearly* stable nonsingular solutions in Galileons if there exists at an arbitrary time a superluminal phase for the graviton and by at least an amount $c_g \geq \sqrt{2}c$. This unphysical phase could formally happen as a deep UV inconsistency, namely, arbitrarily short and unrelated to the physically relevant length scales that are pertinent to these models, such as time and much longer width of a bounce. Besides, this pathology in the classical solutions may still be informative raising the question about the possibility of Lorentz invariant UV completions [14,16] and whether causal paradoxes arise [1,13,15]. Finally, we showed a bouncing cosmology that is always stable, where a short superluminal phase happens before the bounce, and that transits to Einstein's gravity coupled to a massless scalar and with vanishing torsion in the asymptotic past and future.

At least in what concerns the stability and speed of solutions, this shows that the Horndeski-Cartan theory is fundamentally different from Horndeski on a torsionless geometry, in contrast to, e.g., the equivalence of Einstein-Cartan and GR.

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APPENDIX

1. Derivation of the quadratic action

To derive expressions (2) and (3) we follow the notation and detailed procedure in [17]. Briefly, we consider the perturbed metric $\mathrm{d}s^2 = (\eta_{\mu\nu} + \delta g_{\mu\nu})\mathrm{d}x^\mu \mathrm{d}x^\nu$ where $\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = a^2(\eta)(-\mathrm{d}\eta^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j)$ is a spatially flat FLRW background metric, η is conformal time, and we denote spatial indices with Latin letters such as $i = 1, 2, 3$ and spacetime indices with greek letters, such as $\mu = 0, 1, 2, 3$. The metric perturbation is written as

$$\delta g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = a^2(\eta)(-2\alpha\mathrm{d}\eta^2 + 2(\partial_i B + S_i)\mathrm{d}\eta\mathrm{d}x^i + (-2\psi\delta_{ij} + 2\partial_i\partial_j E + \partial_i F_j + \partial_j F_i + 2h_{ij})\mathrm{d}x^i\mathrm{d}x^j), \tag{A1}$$

with α, B, ψ, E scalar perturbations, S_i, F_i transverse vector perturbations, and h_{ij}, a symmetric, traceless, and transverse tensor perturbation.

On the other hand we consider the perturbed Galileon field $(\phi(x))$ as $\phi(\eta) + \Pi(\eta, \vec{x})$ in the linearized expressions, where in this context $\phi(\eta)$ is the background field.

To write explicit torsion in (1), we denote a torsionful and metric compatible covariant derivative on any vector V^μ as $\tilde{\nabla}_\mu V^\nu = \partial_\mu V^\nu + \tilde{\Gamma}^\nu_{\mu\lambda}V^\lambda$, where the nonsymmetric torsionful connection can be expressed in terms of the usual GR Christoffel connection $\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$, as $\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - K^\rho{}_{\mu\nu}$ (Namely, we introduce torsion in the second order formalism.). $K^\rho{}_{\mu\nu}$ is named contortion tensor, and with our convention of torsionful covariant derivatives it is antisymmetric in the first and third indices, $K_{\mu\nu\sigma} = -K_{\sigma\nu\mu}$, such that it has 24 independent components.

The perturbed contortion tensor $K_{\mu\nu\sigma} = {}^0K_{\mu\nu\sigma}(\eta) + \delta K_{\mu\nu\sigma}(\eta, \vec{x})$ has only two nonvanishing background contributions on an isotropic and homogeneous spacetime, namely, ${}^0K_{0jk} = x(\eta)\delta_{jk}$, and ${}^0K_{ijk} = y(\eta)\epsilon_{ijk}$. For the spacetime dependent perturbation $\delta K_{\mu\nu\sigma}(\eta, \vec{x})$ the 24 independent components can be written in terms of irreducible components under small rotation group as eight scalars denoted as $C^{(n)}$ with $n = 1, \ldots, 8$, six (two-component) transverse vectors and two (two-component) traceless, symmetric, transverse tensors $T^{(1)}_{ij}, T^{(2)}_{ij}$. An explicit decomposition is given, for instance, in section $II.B$ in [17].

The four background fields ϕ, a, x, y obey five equations of which only four are independent (due to gauge redundancy), which we denote as $\mathcal{E}_f = \partial\mathcal{L}/\partial f = 0$ for f one of the following: $\phi, g_{00}, g_{ij}, {}^0K_{0jk}, {}^0K_{ijk}$. In particular, $\mathcal{E}_{{}^0K_{ijk}} = -2\varepsilon_{ijk}G_4 y/a^6 = 0$ implies that $y(\eta) \equiv 0$ and $\mathcal{E}_{{}^0K_{0ij}} = 0$ solves $x(\eta)$ in terms of a, ϕ

$$x(\eta) = -\frac{a^3\mathcal{G}_\tau(8HXG_{4,X} + a\dot{\phi}(G_3 - 2G_{4,\phi}))}{8G_4^2}. \tag{A2}$$

The quadratic action for the three tensor perturbations $h_{ij}, T^{(1)}_{ij}, T^{(2)}_{ij}$ is obtained as usual and implies $T^{(2)}_{ij} \equiv 0$ and

$$T^{(1)}_{ij} = \frac{2a^2XG_{4,X}}{G_4 + 2XG_{4,X}}\dot{h}_{ij} - 2xh_{ij}. \tag{A3}$$

Using these equations back in the quadratic action gives (2). Notice the difference in $\mathcal{G}_\tau$ between the graviton h_{ij} in Galileons in a torsionful and a torsionless spacetime due to the nontrivial $T^{(1)}_{ij}$.

Similarly, the part of the quadratic action relevant to the thirteen scalar perturbations $\Pi, \alpha, B, \psi, E, C^{(n)}$ (with $n = 1, \ldots, 8$) is written as (3) after integrating out all of the eight nondynamical torsion perturbations $C^{(n)}$. This is simpler in the unitary gauge, where $\Pi = 0$ and $E = 0$, because one can recognize (See [17] for the theory with $c = 0$.) that there are five Lagrange multipliers $C^{(1)}, C^{(5)}, C^{(7)}, C^{(2)}, B$. The constraint equations imposed by the first three Lagrange multipliers imply the vanishing of $C^{(6)}, C^{(4)}, C^{(8)}$ respectively, and with the equation for $C^{(2)}$ one can express the only nontrivial torsion scalar as

$$C^{(3)} = -\frac{2a^2XG_{4,X}}{G_4 + 2XG_{4,X}}\dot{\psi} + 2x\psi - \frac{a^3(2G_4H + \Theta) + a^2\dot{\phi}G_{4,\phi}}{2G_4}\alpha. \tag{A4}$$

Using (A4) in the quadratic action gives (3), where critically $\mathcal{G}_\tau \neq T$ as opposed to Galileons without torsion.

2. Lagrangian functions

The Lagrangian functions g_{20}, g_{21}, g_{22}, g_{30}, g_{31}, g_{40}, and g_{41} have an exact solution. We show below the graphs of these functions around the bounce at $\eta_b = 0$ with width $\tau_b = 10$ and at the short superluminality phase at $\eta_s = -10$ with width $\tau_s \ll \tau_b$ (we choose here $\tau_b = 10\tau_s$ for convenience displaying the graphs).

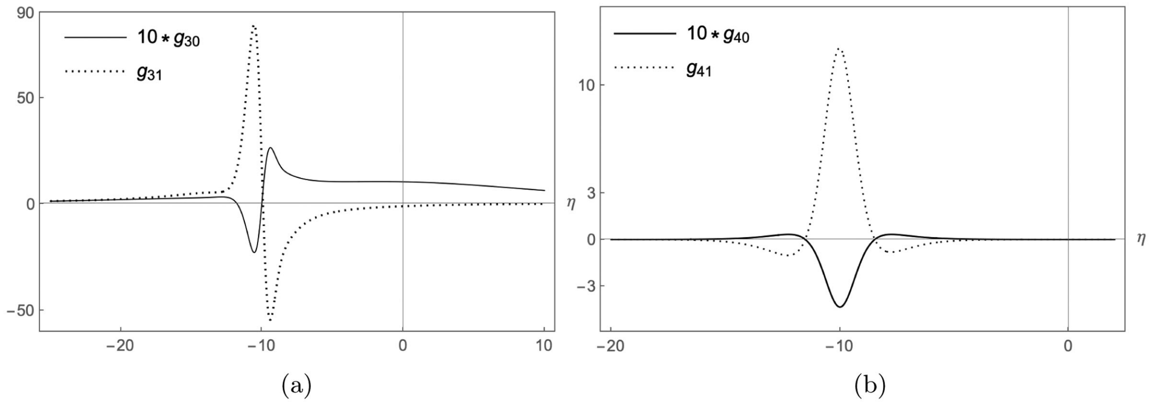

FIG. 4. Everywhere regular Lagrangian functions. (a) g_{30} and g_{31}. (b) g_{40} and g_{41}.

3. Coefficients in the final action for the scalar perturbations

The coefficients θ and σ for the quadratic action (3) are

$$\begin{aligned}\theta = &-2aG_4H(G_4{}^2 + 4(2G_{4,X}{}^2 - G_{4,XX}G_4)X^2) - (G_{4,\phi}(G_4{}^2 + 4G_{4,X}{}^2X^2 - 2G_4X(G_{4,X} + 2G_{4,XX}X)) \\ &+ G_4X(-(G_{3,X} - 2G_{4,\phi X})(G_4 + 2G_{4,X}X) + G_3(3G_{4,X} + 2G_{4,XX}X)))\dot{\phi},\end{aligned} \tag{A5}$$

$$\begin{aligned}\sigma = a(&-24H^2(G_4{}^4 + 16G_{4,X}{}^4X^4 + 8G_{4,X}{}^2G_4X^3(G_{4,X} - 2G_{4,XX}X) + 2G_4{}^2X^2(15G_{4,X}{}^2 + 8G_{4,XX}{}^2X^2 \\ &+ 4G_{4,X}X(G_{4,XX} - G_{4,XXX}X)) - G_4{}^3X(G_{4,X} + 4X(4G_{4,XX} + G_{4,XXX}X))) \\ &+ X(-3G_3{}^2(G_4{}^2 + 2X^2(5G_{4,X}{}^2 + 8G_{4,XX}{}^2X^2 + 4G_{4,X}X(G_{4,XX} - G_{4,XXX}X)) - G_4X(11G_{4,X} + 4X(5G_{4,XX} \\ &+ G_{4,XXX}X))) + 6G_3(-X(G_4 + 2G_{4,X}X)(2(G_{3,XX} - 2G_{4,\phi XX})X(G_4 + 2G_{4,X}X) \\ &+ G_{3,X}(5G_4 - 2X(G_{4,X} + 4G_{4,XX}X)) + 2G_{4,\phi X}(-5G_4 + 2X(G_{4,X} + 4G_{4,XX}X))) \\ &+ 2G_{4,\phi}(G_4{}^2 + 2X^2(5G_{4,X}{}^2 + 8G_{4,XX}{}^2X^2 + 4G_{4,X}X(G_{4,XX} - G_{4,XXX}X)) \\ &- G_4X(11G_{4,X} + 4X(5G_{4,XX} + G_{4,XXX}X)))) + 4((G_4 + 2G_{4,X}X)^2(G_{2,X}(G_4 + 2G_{4,X}X) \\ &- 2G_{3,\phi}(G_4 + 2G_{4,X}X) + X(-3(G_{3,X} - 2G_{4,\phi X})^2X + 2(G_{2,XX} - G_{3,\phi X})(G_4 + 2G_{4,X}X))) \\ &+ 3G_{4,\phi}X(G_4 + 2G_{4,X}X)(2(G_{3,XX} - 2G_{4,\phi XX})X(G_4 + 2G_{4,X}X) + G_{3,X}(5G_4 - 2X(G_{4,X} + 4G_{4,XX}X)) \\ &+ 2G_{4,\phi X}(-5G_4 + 2X(G_{4,X} + 4G_{4,XX}X))) - 3G_{4,\phi}{}^2(G_4{}^2 + 2X^2(5G_{4,X}{}^2 + 8G_{4,XX}{}^2X^2 \\ &+ 4G_{4,X}X(G_{4,XX} - G_{4,XXX}X)) - G_4X(11G_{4,X} + 4X(5G_{4,XX} + G_{4,XXX}X)))))) \\ &+ 24H(X((G_4 + 2G_{4,X}X)((2G_{3,X} - 5G_{4,\phi X})G_4{}^2 + G_4(-(G_{3,X} + 2G_{4,\phi X})G_{4,X} + (G_{3,XX} - 2G_{4,\phi XX})G_4)X \\ &+ 2((G_{3,X} - 4G_{4,\phi X})G_{4,X}{}^2 + ((G_{3,XX} - 2G_{4,\phi XX})G_{4,X} - 2(G_{3,X} - 2G_{4,\phi X})G_{4,XX})G_4)X^2) \\ &+ G_3(-6G_{4,X}G_4{}^2 + 3G_4(G_{4,X}{}^2 - 3G_{4,XX}G_4)X - 2(3G_{4,X}{}^3 - 2G_{4,X}G_{4,XX}G_4 + G_{4,XXX}G_4{}^2)X^2 \\ &- 4(G_{4,X}{}^2G_{4,XX} - 2G_{4,XX}{}^2G_4 + G_{4,X}G_{4,XXX}G_4)X^3)) + G_{4,\phi}(-G_4{}^3 + 4G_{4,X}{}^2X^3(G_{4,X} + 2G_{4,XX}X) \\ &- 2G_4X^2(9G_{4,X}{}^2 + 8G_{4,XX}{}^2X^2 + 4G_{4,X}X(G_{4,XX} - G_{4,XXX}X)) + 2G_4{}^2X(3G_{4,X} \\ &+ X(9G_{4,XX} + 2G_{4,XXX}X))))\dot{\phi}.\end{aligned} \tag{A6}$$

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