Friedmann equations in the Codazzi parametrization of Cotton and extended theories of gravity and the dark sector

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The Friedmann equations of Cotton gravity provide a simple parametrization to reproduce, by tuning a single function, the Friedmann equations of several extensions of gravity, such as $f(R)$, modified Gauss-Bonnet $f(G)$, teleparallel $f(T)$, and more. It also includes the recently proposed conformal Killing gravity and mimetic gravity in Friedmann-Robertson-Walker space-times. The extensions generally have the form of a Codazzi tensor that may be associated to the dark sector. Fixing it by a suitable equation of state accomodates most of the postulated models that extend ΛCDM, as the Chevallier-Polarski-Lindler model.

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I. INTRODUCTION

In recent years there has been a flourishing of extended theories of gravity to address the problem of the dark sector. They modify the Einstein equations by adding a term H_{jk} to the energy-momentum tensor T_{jk} of matter:

$$R_{jk} - \frac{1}{2} g_{jk} R = \kappa (T_{jk} + H_{jk}). \tag{1}$$

The term originates from a new form of gravitational action or new particles.

Large-scale cosmology is staged in Friedmann-Robertson-Walker (FRW) space-times, where the Weyl tensor C_{jklm} is zero. This fact ushers Codazzi tensors.

If $\nabla_m C_{jkl}{}^m = 0$, then

$$R_{ij} - \frac{R}{2} g_{ij} = S_{ij} - g_{ij} S^k{}_k, \tag{2}$$

where the Schouten tensor $S_{ij} = R_{ij} - \frac{1}{6} R g_{ij}$ is a Codazzi tensor, i.e., $\nabla_i S_{jk} = \nabla_j S_{ik}$.

This means that $T_{jk} + H_{jk}$ has the same decomposition. The Codazzi condition ensures that $\nabla^k (T_{kl} + H_{kl}) = 0$.

Moreover, in a FRW space-time the sum must have the perfect fluid structure of the Einstein tensor.

The vast majority of extended models of gravity in FRW space-times specify this property for the radiation-matter sector, with conservation. This entails a Codazzi decomposition of the perfect fluid tensor. Then, necessarily, despite the often complex structure of the tensor H_{kl}, the dark sector is perfect fluid and conserved.

For these models:

$$H_{jk} = \mathcal{C}_{jk} - g_{jk} \mathcal{C}^p{}_p, \tag{3}$$

$$\nabla_i \mathcal{C}_{jk} = \nabla_j \mathcal{C}_{ik}. \tag{4}$$

The aim of this work is to uncover this common structure, albeit the different origins of the various cosmological models. We explicitly show this in plenty of well studied extended gravity models in FRW space-times: $f(R)$, Gauss-Bonnet $f(G)$, teleparallel $f(T)$, Einsteinian cubic $f(P)$, conformal Killing gravity, Lovelock.

An inclusive and simple model which they fit in, stems from Cotton gravity.

In 2021 Junpei Harada [1] introduced a modification of general relativity (GR) named "Cotton gravity" (CG), with field equations

$$C_{jkl} = \nabla_j T_{kl} - \nabla_k T_{jl} - \frac{1}{3} (g_{kl} \nabla_j T - g_{jl} \nabla_k T). \tag{5}$$

T_{kl} is the matter energy-momentum tensor with trace T and C_{jkl} is the Cotton tensor:

$$C_{jkl} \equiv \nabla_j \left(R_{kl} - \frac{R}{6} g_{kl} \right) - \nabla_k \left(R_{jl} - \frac{R}{6} g_{jl} \right), \tag{6}$$

where $C_{jkl} = -2 \nabla_m C_{jkl}{}^m$. The property $g^{kl} C_{jkl} = 0$ implies that $\nabla^p T_{jp} = 0$. Cotton gravity was devised so that any solution of GR is a solution of CG.

Soon after, Harada applied his theory to describe the rotation curves of 84 galaxies without assuming the presence of dark matter [2]. A wide class of spherically symmetric static vacuum solution was then obtained by Gogberashvili and Girvliani [3], with a long range

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modification of Newton's law. A static solution of Cotton gravity with electric and magnetic charges was obtained in [[4], Eq. (80)]

An important progress was made in [5] in showing that the equations of Cotton gravity are equivalent to the standard GR equations corrected by an arbitrary Codazzi tensor

$$R_{kl} - \frac{1}{2}Rg_{kl} = T_{kl} + \mathcal{C}_{kl} - g_{kl}\mathcal{C}^r{}_r,$$
$$\nabla_j\mathcal{C}_{kl} = \nabla_k\mathcal{C}_{jl}. \tag{7}$$

In the frame of FRW solutions, this is precisely the statement in Eq. (3). Cotton gravity exhibits the maximal freedom in specifying H_{kl}. We refer to Eq. (7) as the "Codazzi parametrization."

While Harada's equations have third order derivatives of the metric, the equivalent equations (7) are second order.

Sussman and Nájera [6] used (7) to produce FRW solutions of Cotton gravity. They posed a modified Friedmann equation with a scalar function $\mathcal{K}(t)$ and obtained the components of a perfect fluid Codazzi tensor for which Eq. (7) is satisfied. Very recently, they published a paper [7] with several nontrivial CG solutions that generalize the well known GR solutions: FLRW, Lemaitre-Tolman-Bondi, and Szekeres, as well as static perfect fluid spherically symmetric solutions (with application to galactic rotation curves) and nonstatic shear-free.

Also motivated by this result, we propose a general discussion based upon Theorem 2.1 in [5]: a FRW space-time always contains a perfect fluid Codazzi tensor that displays a freedom in its parameters.

In Sec. II we write the Friedmann equations for Cotton gravity in a FRW space-time.

In Sec. III we recognize that some of the most important extended theories of gravity have the following intriguing property: their Friedmann equations coincide with those of Cotton gravity by a suitable choice of the Codazzi tensor. We show this explicitly by providing the specific Codazzi tensor for $f(R)$ gravity, Gauss-Bonnet $f(G)$ gravity, $f(T)$ gravity, cubic Einsteinian and $f(P)$ gravity, Lovelock.

These findings are well corroborated by the generic gravity theory by Gürses and Heydarzade [8], whose very general form of gravitational action incorporates many extended gravity theories. They show that the field equations differ from the standard FRW ones by a perfect-fluid term.

In Sec. IV we show that the Codazzi parametrization of CG extends the recently introduced conformal Killing gravity [9–11], at least in FRW space-times.

In Sec. V we prove that the field equations of mimetic gravity become the Cotton equations if and only if the hosting space-time is generalized Robertson Walker, and FRW space-times are a special case.

In Sec. VI the dark sector is fixed by requesting an equation of state (EOS). It accommodates the best known redshift dependent models, such as the Chevallier-Polarski-Lindler model.

Notation. $i, j, k, \ldots = 0, 1, 2, 3$, $\mu, \nu, \ldots = 1, 2, 3$. A dot operator $\dot{X} = u^k\nabla_k X$ is the time derivative in the comoving frame defined by $u^0 = 1$, $u^\mu = 0$. $X_{[ijk]}$ is the cyclic sum $X_{ijk} + X_{kij} + X_{jki}$.

II. FRIEDMANN EQUATIONS OF COTTON GRAVITY IN FRW SPACE-TIMES

Generalized Robertson Walker space-times (GRW) are Lorentzian manifolds that extend FRW space-times with the metric

$$ds^2 = -dt^2 + a(t)^2 g^\star_{\mu\nu}(\mathbf{x})dx^\mu dx^\nu, \tag{8}$$

where $g^\star_{\mu\nu}(\mathbf{x})$ is a positive definite metric and $a(t)$ is the scale factor. A covariant characterization is the existence of a vector field $u_k u^k = -1$ that is shear-free, vorticity-free, and acceleration-free, and the eigenvector of the Ricci tensor [12], i.e.,

$$\nabla_j u_k = H(g_{jk} + u_j u_k), \tag{9}$$

$$R_{ij}u^j = \xi u_i, \tag{10}$$

where $H = \dot{a}/a$ is Hubble's parameter, $\xi = 3(H^2 + \dot{H}) = 3\ddot{a}/a$. The condition (10) is equivalent to $\nabla_j H = -\dot{H}u_j$. Its divergence and the contracted Bianchi identity give

$$\dot{R} - 2\dot{\xi} = -2H(R - 4\xi) \tag{11}$$

whose solution is [13]

$$R = \frac{R^\star}{a^2} + 12H^2 + 6\dot{H} \tag{12}$$

where $R^\star$ is the spatial curvature. In $d = 4$ and whenever $C_{jklm}u^m = 0$ the GRW spacetime is a FRW space-time.

In a FRW space-time the natural form of the Codazzi tensor in Eq. (7) is perfect fluid. Λg_{kl} with a constant Λ, is trivially a Codazzi tensor.

This simple result is proven in [5] (Theorem 2.1).

Proposition 1. In a FRW space-time the tensor

$$\mathcal{C}_{kl} = \mathcal{A}u_k u_l + \mathcal{B}g_{kl} + \frac{\Lambda}{3}g_{kl} \tag{13}$$

is Codazzi provided that $\nabla_j\mathcal{A} = -\dot{\mathcal{A}}u_j$, $\nabla_j\mathcal{B} = -\dot{\mathcal{B}}u_j$,

$$\dot{\mathcal{B}} = -H\mathcal{A}. \tag{14}$$

Proof. The first two conditions mean that $\mathcal{A} = \mathcal{A}(t)$ and $\mathcal{B} = \mathcal{B}(t)$. Equation (14) requires $\dot{\mathcal{B}} \neq 0$. Next, with $\nabla_j\mathcal{A} = -\dot{\mathcal{A}}u_j$, $\nabla_j\mathcal{B} = -\dot{\mathcal{B}}u_j$, Eqs. (9) and (14) it is

$$\nabla_j\mathcal{C}_{kl} = -u_j u_k u_l(\dot{\mathcal{A}} + 2\dot{\mathcal{B}}) - \dot{\mathcal{B}}(u_l g_{jk} + u_k g_{jl} + u_j g_{kl}).$$

Therefore (13) is a Codazzi tensor for any choice of the scale factor. ■

It implies that any FRW space-time is a solution of Cotton gravity with (13), and leaves an interesting degree of freedom $\mathcal{B}$ in choosing the Codazzi tensor.

Equation (7) is written with the input (13), the general form of the Ricci tensor of a FRW space-time

$$R_{kl} = \frac{1}{3}(R - 4\xi)u_l u_k + \frac{1}{3}(R - \xi)g_{kl}$$

and the stress energy tensor $T_{kl} = (\mu + p)u_l u_k + p g_{kl}$ with energy density μ and pressure p of ordinary matter.

Contractions with $u^k u^l$ and g^{kl} and a simple rearrangement provide the Friedmann equations of Cotton gravity in a FRW space-time

$$\kappa\mu = \frac{R}{2} - \xi - 3\mathcal{B} - \Lambda, \tag{15}$$

$$\kappa p = -\frac{R}{6} - \frac{\xi}{3} + 3\mathcal{B} + \frac{\dot{\mathcal{B}}}{H} + \Lambda. \tag{16}$$

They are the standard Friedmann equations of GR augmented by the Codazzi terms. Such terms naturally correspond to the dark sector:

$$\begin{aligned} H_{kl} &= (\mathcal{A}u_k u_l + \mathcal{B}g_{kl}) - g_{kl}(4\mathcal{B} - \mathcal{A}), \\ &\equiv (\mu_D + p_D)u_k u_l + g_{kl}p_D. \end{aligned} \tag{17}$$

The function $\mathcal{B}(t)$ parametrizes the energy density and the pressure of the dark sector:

$$\mu_D = 3\mathcal{B}, \tag{18}$$

$$p_D = -3\mathcal{B} - \dot{\mathcal{B}}/H. \tag{19}$$

It implies the conservation law $\dot{\mu}_D = -3H(p_D + \mu_D)$ coming from $\nabla_k H^k{}_j = 0$ or the equivalent Codazzi condition for $\mathcal{A}u_k u_l + \mathcal{B}g_{kl}$ in FRW spacetimes.

III. REPRODUCING THE FRIEDMANN EQUATIONS OF EXTENDED THEORIES

We show that the Friedmann equations (15) and (16) of Cotton gravity may reproduce the Friedmann equations of other extended theories in absence of cosmological constant. With Eq. (12) and $\xi = 3(H^2 + \dot{H})$ we write them as

$$\kappa\mu = \frac{R^\star}{2a^2} + 3H^2 - 3\mathcal{B}, \tag{20}$$

$$\kappa p = -\frac{R^\star}{6a^2} - 3H^2 - 2\dot{H} + 3\mathcal{B} + \frac{\dot{\mathcal{B}}}{H}. \tag{21}$$

The comparison with the Friedmann equations of other theories selects the function $\mathcal{B}(t)$ that reproduces them.

In Ref. [8] Gürses and Heydarzade introduced the generic gravity theory. It is characterized by a very general form of gravitational action, with a scalar function $\mathcal{F}$ of the metric tensor, the Riemann tensor and its covariant derivatives at any order:

$$S = \int d^4x\sqrt{-g}\left[\frac{R - 2\Lambda}{\kappa} + \mathcal{F}(g, \mathrm{Riem}, \nabla\mathrm{Riem}, \nabla\nabla\mathrm{Riem}, \ldots)\right] + S_{\mathrm{mat}}. \tag{22}$$

The theory contains all modified theories of gravity based on curvature such as $f(R)$, $f(G)$, $f(P)$ theories.

The authors prove a theorem for generic gravity in FLRW cosmology ([8], Theorem 5): the field equations always take the form $G_{kl} = \kappa T_{kl} + H_{kl}$, where $H_{kl} = \mathbb{A}g_{kl} + \mathbb{B}u_k u_l$ accounts for the contribution of all aforementioned higher order terms. The explicit expressions for $\mathbb{A}$ and $\mathbb{B}$ was given for the Einstein-Lovelock and for generalized Einstein-Gauss-Bonnet theories. In [14] the explicit analysis is extended to quadratic gravity.

In this general setting, we note the following:

Lemma 2. In a FLRW space-time if $H_{kl} = \mathbb{A}g_{kl} + \mathbb{B}u_k u_l$ is divergence-free then it is

$$H_{kl} = \mathcal{C}_{kl} - g_{kl}\mathcal{C}_p^p$$

with $\mathcal{C}_{kl}$ being a Codazzi tensor.

Proof. The divergence-free condition is $3H\mathbb{B} = \dot{\mathbb{A}} - \dot{\mathbb{B}}$. Let $\mathcal{A} = \mathbb{B}$ and $\mathcal{B} = \frac{1}{3}(\mathbb{B} - \mathbb{A})$, then $\dot{\mathcal{B}} = -H\mathcal{A}$. By Proposition 1 the tensor $\mathcal{C}_{kl} \equiv \mathcal{A}u_k u_l + \mathcal{B}g_{kl}$ satisfies the Codazzi condition. ■

Thus we may state that in FLRW cosmology Cotton gravity is equivalent to any generic gravity theory.

A. $f(R)$ gravity

Perhaps it is the best known extended theory of gravity. It was introduced by Buchdahl in 1970 [15] and gained popularity with the works on cosmic inflation by Starobinsky [16]. Recently $f(R)$ theories are possible candidates to explain the observed cosmic acceleration.

Investigations to explain both dark energy and inflation were pursued in the papers by Cognola *et al.* [17], Nojiri and Odintsov [18,19]. Capozziello considered $f(R)$ to discuss the issue of quintessence [20]. For general reviews on $f(R)$ see [21–23].

The action of $f(R)$ gravity is

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S^{(m)},$$

where $S^{(m)}$ is the matter term. With $f_R = df/dR$, the field equations are [23]

$$f_R R_{kl} - \frac{f}{2} g_{kl} - (\nabla_k \nabla_l - g_{kl}\Box) f_R = \kappa T_{kl}. \tag{23}$$

They can be rewritten in the form (1). In [24] it was proven that in a FRW space-time the resulting term H_{jk} is a perfect fluid tensor.

For the spatially flat ($R^\star = 0$) FRW space-time the Friedmann equations of $f(R)$ gravity are [[23], Eqs. (75) and (76)]:

$$\kappa\mu = 3 f_R H^2 - \frac{1}{2}(R f_R - f) + 3H\dot{R} f_{RR}, \tag{24}$$

$$(3H^2 + 2\dot{H}) f_R = -\left[\kappa p + \dot{R}^2 f_{RRR} + 2H\dot{R} f_{RR} + \ddot{R} f_{RR} + \frac{1}{2}(f - R f_R)\right]. \tag{25}$$

In comparing (24) with (20) we identify

$$\mathcal{B} = H^2(1 - f_R) + \frac{1}{6}(R f_R - f) - H\dot{R} f_{RR}. \tag{26}$$

In computing $\dot{\mathcal{B}}$ we note that $\dot{f}(R) = f_R(R)\dot{R}$, $\dot{f}_R(R) = f_{RR}\dot{R}$, $\dot{f}_{RR} = f_{RRR}\dot{R}$, so that

$$\dot{\mathcal{B}} = 2H\dot{H}(1 - f_R) + f_{RR}\left(\frac{R}{6} - \dot{H} - H^2\right)\dot{R} - H\ddot{R} f_{RR} - H\dot{R}^2 f_{RRR}.$$

The restriction $R^\star = 0$ in (12) gives $R = 12H^2 + 6\dot{H}$. We obtain $-\mathcal{A}$:

$$\frac{\dot{\mathcal{B}}}{H} = 2\dot{H}(1 - f_R) + (H\dot{R} - \ddot{R}) f_{RR} - \dot{R}^2 f_{RRR}. \tag{27}$$

Using now (21) we obtain (25).

Proposition 3. The Friedmann equations of Cotton gravity with the perfect fluid Codazzi tensor (13) are the Friedmann equations of $f(R)$ gravity with the choice (26).

B. $f(G)$ gravity

A second well-known extended theory that tries to solve the problem of dark energy is the Gauss-Bonnet gravity, alias $f(G)$ gravity [25–27]:

$$S = \int d^4x \sqrt{-g}\left[\frac{R}{2\kappa} + f(G)\right] + S^{(m)},$$

where $G = R^2 - 4R_{kl}R^{kl} + R_{jklm}R^{jklm}$ is the Gauss-Bonnet invariant.

The field equations may be written in the form $R_{kl} - \frac{1}{2} R g_{kl} = \kappa(T_{kl} + H_{kl})$ with the following divergence-free tensor H_{kl} [8,28]:

$$\begin{aligned} H_{kl} = {} & \frac{1}{2} g_{kl} f - 2 f_G (R R_{kl} - 4 R_{kq} R_l^q + 2 R_k{}^{pqr} R_{lpqr}) \\ & - 4 f_G R_k{}^{pq}{}_l R_{pq} + 2R(\nabla_k \nabla_l f_G - g_{kl}\Box f_G) \\ & - 4(R_l{}^p \nabla_p \nabla_k f_G + R_k{}^p \nabla_p \nabla_l f_G) + 4(\Box f_G) R_{kl} \\ & + 4(R^{pq} g_{kl} - R_k{}^{pq}{}_l) \nabla_p \nabla_q f_G, \end{aligned} \tag{28}$$

where $f_G = df/dG$. In a FRW space-time it is a perfect fluid tensor. For the spatially flat case, the Gauss-Bonnet invariant is $G = 24(\dot{H}H^2 + H^4)$ and the Friedmann equations of $f(G)$ gravity are expressible as [Eq. (5) in [29]]

$$\kappa\mu = 3H^2 - \kappa(G f_G - f - 24H^3 \dot{G} f_{GG}), \tag{29}$$

$$\kappa p = -3H^2 - 2\dot{H} + \kappa(G f_G - f) - 16\kappa H(H + \dot{H}) f_G - 8\kappa H^2 \ddot{f}_G. \tag{30}$$

With $\dot{f} = f_G \dot{G}$ we obtain $\dot{f}_G = f_{GG}\dot{G}$ and the first equation rewrites as $\kappa\mu = 3H^2 - \kappa(G f_G - f - 24H^3 \dot{f}_G)$.

Comparison with (20) gives the following identification:

$$\mathcal{B} = \kappa(G f_G - f - 24H^3 \dot{f}_G). \tag{31}$$

After straightforward calculations we infer

$$-\mathcal{A} = \frac{\dot{\mathcal{B}}}{H} = 8\kappa[\dot{f}_G(H^3 - 2H\dot{H}) - H^2 \ddot{f}_G]. \tag{32}$$

Thus from (21) we obtain Eq. (30).

Proposition 4. The Friedmann equations of Cotton gravity with the perfect fluid Codazzi tensor (13) are the Friedmann equations of Gauss-Bonnet $f(G)$ gravity with the choice (31).

C. $f(T)$ gravity

In the framework of gravity theories with torsion, the "teleparallel equivalent of general relativity" is the best known one. It is widely discussed in [30] and briefly reviewed in [31]. It is based on the action [[31] Eq. (2.5)]:

$$S = \frac{1}{2\kappa}\int d^4x \sqrt{-g}[T + f(T)] + S^{(m)}.$$

The field equations are Eq. (263) in [30] or Eq. (2.6) in [31]. For the spatially flat FRW space-time, the Friedmann equations of $f(T)$ gravity are expressible as [Eqs. (2.9) and (2.10) in [31] or Eqs. (267) and (268) in [30]]:

$$H^2 = \frac{\kappa}{3}\mu - \frac{f}{6} - 2 f_T H^2, \tag{33}$$

$$\dot{H} = -\frac{1}{2}\frac{\kappa(p + \mu)}{1 + f_T - 12H^2 f_{TT}}. \tag{34}$$

The torsion scalar is $T = -6H^2$ [Eq. (269) in [30]], and $f_T = df/dT$. Comparing with Eq. (20) we identify

$$\mathcal{B} = -\frac{1}{6} f(T) - 2 f_T(T) H^2. \tag{35}$$

Note that $\dot{f}(T) = f_T \dot{T} = -12 H\dot{H} f_T$, $\dot{f}_T = f_{TT}\dot{T} = -12H\dot{H} f_{TT}$. Thus $\dot{\mathcal{B}} = -2H\dot{H} f_T + 24H^3\dot{H} f_{TT}$ and

$$-\mathcal{A} = \frac{\dot{\mathcal{B}}}{H} = -2\dot{H} f_T(T) + 24H^2\dot{H} f_{TT}(T). \tag{36}$$

Summing the Friedmann equations (20) and (21) of Cotton gravity we get

$$\frac{\kappa}{2}(p+\mu) = -\dot{H} + \frac{\dot{\mathcal{B}}}{2H}. \tag{37}$$

Inserting (36) in (37) gives Eq. (34).

Proposition 5. The Friedmann equations of Cotton gravity with the perfect fluid Codazzi tensor (13) are the Friedmann equations of $f(T)$ gravity with the choice (35).

D. Einsteinian cubic and $f(P)$ gravity

In [32] an extended theory is proposed, based on an invariant P constructed with cubic contractions of the Riemann tensor. The theory was subjected to three constraints: (1) the spectrum should be identical to that of GR (whence the name); (2) it is neither topological nor trivial in $d = 4$; (3) it is independent of the dimension.

The action is

$$S = \int d^4x\sqrt{-g}\left[\frac{R-2\Lambda}{2\kappa} + P\right] + S^{(m)},$$

$$\begin{aligned} P = &-\beta_1 R_j{}^{pq}{}_k R_p{}^{rs}{}_q R^j{}_{rs}{}^k + \beta_2 R_{jk}{}^{rs} R_{rs}{}^{pq} R_{pq}{}^{jk} \\ &+ \beta_3 R^j{}_k R_{pqrj} R^{pqrk} + \beta_4 R_{pqrs} R^{pqrs} + \beta_5 R_{jkpq} R^{kp} R^{jq} \\ &+ \beta_6 R^p_k R^j_p R^k_j + \beta_7 R R_{pq} R^{pq} + \beta_8 R^3. \end{aligned} \tag{38}$$

The aforementioned constraints impose three linear relations among the coefficients β_i.

In [33] the cosmological applications of Einsteinian cubic gravity at early and late times were investigated. In [34] the viability of the theoretical model is analyzed, by considering observational features such as cosmic chronometers data, baryon acoustic oscillations, and supernovae.

The field equation may be written in the form $R_{kl} - \frac{1}{2} R g_{kl} + \Lambda g_{kl} = \kappa(T_{kl} + H_{kl})$, where H_{kl} is an involved symmetric tensor containing contractions of the Riemann and the Ricci tensor.

The Friedmann equations are Eqs. (11) and (12) in [33]:

$$3H^2 = \kappa(\mu + 6\alpha\tilde{\beta} H^6) + \Lambda, \tag{39}$$

$$3H^2 + 2\dot{H} = -\kappa[p - 6\alpha\tilde{\beta} H^4 (H^2 + 2\dot{H})] + \Lambda, \tag{40}$$

with $\tilde{\beta} = -\beta_1 + 4\beta_2 + 2\beta_3 + 8\beta_4$. Comparison with (20) yields

$$\mathcal{B} = 6\kappa\tilde{\beta} H^6. \tag{41}$$

Then $-\mathcal{A} = \dot{\mathcal{B}}/H = 12\kappa\tilde{\beta} H^4 \dot{H}$. From Eq. (21) we get Eq. (40).

Proposition 6. The Friedmann equations of Cotton gravity with the perfect fluid Codazzi tensor (13) are the Friedmann equations of cubic Einsteinian gravity with the choice (41).

In the same paper [33] the authors proposed the $f(P)$ extension of Einsteinian cubic gravity:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa} + f(P)\right] + S^{(m)}.$$

The field equations are still of the type $R_{kl} - \frac{1}{2} R g_{kl} = \kappa(T_{kl} + \tilde{H}_{kl})$, where $\tilde{H}_{kl}$ is quite more involved. The Friedmann equations are Eqs. (26) and (27) in [33]:

$$3H^2 = \kappa\mu - \kappa f - 18\kappa\tilde{\beta} H^4 (H\partial_t - H^2 - \dot{H}) f_P, \tag{42}$$

$$\begin{aligned} 3H^2 + 2\dot{H} = &-\kappa P - \kappa f - 6\kappa\alpha\tilde{\beta} H^3 \\ &\times [H\partial_t^2 + 2(H^2 + 2\dot{H})\partial_t - 3H^3 - 5H\dot{H}] f_P, \end{aligned} \tag{43}$$

where $f_P = df/dP$ and $P = 6\tilde{\beta} H^4 (H^2 + 2\dot{H})$.

It is simple to identify

$$\mathcal{B} = -\kappa f - 18\kappa\tilde{\beta} H^4 (H\dot{f}_P - H^2 f_P - \dot{H} f_P). \tag{44}$$

Now $\dot{f}(P) = f_P \dot{P} = 18\tilde{\beta} H^3 (4H^2\dot{H} + 4\dot{H}^2 + H\ddot{H}) f_P$. After tedious but straightforward calculations it is

$$\frac{\dot{\mathcal{B}}}{H} = 6\kappa\tilde{\beta} H^3 [2H\dot{H} f_P - \dot{f}_P (4\dot{H} - H^2) - H\ddot{f}_P]. \tag{45}$$

Thus from (21) we get (43).

Proposition 7. The Friedmann equations of Cotton gravity with the perfect fluid Codazzi tensor (13) are the Friedmann equations of $f(P)$ gravity with the choice (44).

E. Regularized cubic Lovelock gravity

In Sec. 3 of [35] the authors focused on the cubic Lovelock gravity in a four-dimensional FRW space-time. They obtained the following Friedmann equations

$$\kappa\mu = 3J^2(1 + \alpha J^2 + \beta J^4), \tag{46}$$

$$\kappa(p+\mu) = -2\left(\dot{H} - \frac{R^\star}{6a^2}\right)(1 + 2\alpha J^2 + 3\beta J^4), \tag{47}$$

where $J^2 = H^2 + \frac{R^\star}{6a^2}$. We dropped their cosmological constant and the stress energy tensor is multiplied by a factor of 2 to match our notation. We state the following:

Lemma 8. If in (21) we put $-\mathcal{B} = F(J^2)$, where F is a smooth arbitrary function of $J^2 = H^2 + \frac{R^\star}{6a^2}$, then

$$\frac{\kappa}{2}(p+\mu) = -\left(\dot{H} - \frac{R^\star}{6a^2}\right)[1 + F_J(J^2)], \tag{48}$$

where $F_J = dF/dJ^2$.

Proof. From (21) we have $\frac{\kappa}{2}(p+\mu) = \frac{R^\star}{6a^2} - \dot{H} + \frac{\dot{\mathcal{B}}}{2H}$. If $-\mathcal{B} = F(J^2)$, then $\dot{\mathcal{B}} = -2F_{J^2}J\dot{J}$. On the other hand $J\dot{J} = H(\dot{H} - \frac{R^\star}{6a^2})$ so that $\frac{\dot{\mathcal{B}}}{2H} = -(\dot{H} - \frac{R^\star}{6a^2})F_{J^2}$ and the Lemma is proven. ∎

Choose $F_J(J^2) = \alpha J^2 + \beta J^4$ and (46), (47) are recovered.

F. Sussman Nájera model in Cotton gravity

In [6,7] the authors introduced the following modified Friedmann equation:

$$H^2 = \frac{\kappa}{3}\mu - \frac{R^\star}{6a^2} - \frac{\gamma}{a^2}\mathcal{K}(t), \tag{49}$$

with an arbitrary dimensionless function $\mathcal{K}(t)$ and a constant γ. Then they computed the components of the Codazzi tensor that solves Cotton gravity. Comparison of (49) with (20) gives $\mathcal{B}$ whence $\mathcal{A}$ is computed:

$$\mathcal{B} = -\frac{\gamma\mathcal{K}(t)}{a^2}, \tag{50}$$

$$\mathcal{A} = -\frac{\dot{\mathcal{B}}}{H} = \frac{\gamma\dot{\mathcal{K}}(t)}{a^2 H} - \frac{2\gamma\mathcal{K}(t)}{a^2}. \tag{51}$$

The expression of the Cotton tensor

$$\mathcal{C}_{kl} = \left[\frac{\gamma\dot{\mathcal{K}}(t)}{a^2 H} - \frac{2\gamma\mathcal{K}(t)}{a^2}\right]u_k u_l - \frac{\gamma\mathcal{K}(t)}{a^2}g_{kl} \tag{52}$$

compares with the components in Eq. (19) evaluated in [7].

IV. COMPARISON WITH CONFORMAL KILLING GRAVITY

After Cotton gravity, Harada introduced a new theory of gravity to explain the present accelerated phase of the Universe without explicit introduction of dark energy [9]:

$$\nabla_{[j}R_{kl]} - \frac{1}{3}\nabla_{[j}R g_{kl]} = \nabla_{[j}T_{kl]} - \frac{1}{6}\nabla_{[j}T g_{kl]}. \tag{53}$$

The equations are manifestly of third order in the derivatives of the metric tensor.

Shortly after in [11] we introduced a parametrization of the theory by showing that (53) is equivalent to the Einstein's equation in which the stress-energy tensor is augmented by a divergence-free conformal Killing tensor:

$$R_{kl} - \frac{1}{2}R g_{kl} = T_{kl} + K_{kl}, \tag{54}$$

$$\nabla_{[j}K_{kl]} = \frac{1}{6}\nabla_{[j}K g_{kl]}, \tag{55}$$

where $K = g^{pq}K_{pq}$. We named this theory conformal Killing gravity.

The second equation defines a divergence-free conformal Killing tensor. They are deeply investigated in differential geometry and in physics [36–39].

We proved existence of a conformal Killing tensor in any FRW space-time, obtaining two modified Friedmann equations that allow for the presence of a dark sector. When applied to a simple toy model, this theory reveals a phantom dark fluid with EOS parameter $w = -5/3$ [11].

In a second paper [10] Harada developed an interesting cosmological analysis confirming in general that the dark energy predicted by the conformal Killing gravity has the same EOS parameter.

Here we investigate the connections between Cotton and conformal Killing gravity.

To this end, consider a generic space-time endowed with a (0, 2) symmetric tensor satisfying the relation

$$\nabla_j K_{kl} = a_j g_{kl} - b_k g_{jl} - b_l g_{jk}. \tag{56}$$

We call such tensors Sinyukov-like (see [39,40]). If K_{kl} is the Ricci tensor, then we recover the Sinyukov manifolds, investigated for example in [41].

Here we consider divergence-free Sinyukov-like tensors:

$$\nabla_j K_{kl} = 5\frac{\nabla_j K}{18}g_{kl} - \frac{\nabla_k K}{18}g_{jl} - \frac{\nabla_l K}{18}g_{jk}, \tag{57}$$

where $K = K^p{}_p$. They satisfy the condition (55) that defines divergence-free conformal Killing tensors [38].

A space-time with a Sinyukov-like divergence-free tensor is a solution of conformal Killing gravity (54).

On the other hand (57) implies the Codazzi condition

$$\nabla_j\left[K_{kl} - \frac{K}{3}g_{kl}\right] = \nabla_k\left[K_{jl} - \frac{K}{3}g_{jl}\right]. \tag{58}$$

Then $\mathcal{C}_{kl} = K_{kl} - \frac{1}{3}K g_{kl}$ is a Codazzi tensor with $\mathcal{C}_r^r = -\frac{1}{3}K$. From $R_{kl} - \frac{1}{2}R g_{kl} = T_{kl} + K_{kl}$ we recover the paradigm (7) with the same stress-energy tensor.

Proposition 9. A space-time with a divergence-free Sinyukov-like tensor (57) is a solution both of conformal Killing gravity (54) and of Cotton gravity (7), with the same stress-energy tensor.

We show that, rather surprisingly, any FRW space-time is equipped with a Sinyukov-like tensor.

We recall that a vector Z_j is a conformal Killing vector [38] (CKV for short) if the following condition holds:

$$\nabla_j Z_i + \nabla_i Z_j = 2\psi g_{ij}, \tag{59}$$

where the scalar function ψ is called conformal factor.

Let $Z_j = Fu_j$ with $u_j u^j = -1$ and F a scalar function. The following result holds:

In a GRW space-time Fu_j is a CKV if and only if $\dot{F} = HF = \psi$ and $\nabla_i F = -u_i \dot{F}$; i.e., F depends only on time. ([42] Theorem 2.1, [43] Theorem 1)

In this case, since $H = \dot{a}/a$, we obtain $F(t) = ka(t)$ for some constant k. According to Rani *et al.* [38], the CKV originates a conformal Killing tensor

$$K_{ij} = F^2 u_i u_j + F_1 g_{ij} \tag{60}$$

for arbitrary scalar function F_1. Let us choose F_1 in order that $0 = \nabla_p K^p{}_j$. A simple evaluation using (9) shows that $\nabla_i F_1 = -5F\dot{F}u_i$. Then F_1 depends only on time, and $\dot{F}_1 = 5F\dot{F}$. An integration gives $F_1 = \frac{5}{2}F^2 - \Lambda$ being Λ a constant. Now

$$K_{jk} = F^2\left(u_j u_k + \frac{5}{2}g_{jk}\right) - \Lambda g_{jk}. \tag{61}$$

Next evaluate $\nabla_i K_{jk} = HF^2(-5u_i g_{jk} + g_{ij}u_k + g_{ik}u_j)$. Contraction with g^{jk}: $\nabla_i K = -18HF^2 u_i$. It turns out that K_{jk} satisfies (57); i.e., it is divergence-free and Sinyukov-like. ■

The associated Codazzi tensor $\mathcal{C}_{kl} = K_{kl} - \frac{K}{3}g_{kl}$ is

$$\mathcal{C}_{ij} = F^2\left(u_i u_j - \frac{1}{2}g_{ij}\right) + \frac{\Lambda}{3}g_{ij}. \tag{62}$$

We have proven the following:

Proposition 10. Any GRW space time, and thus any FRW space-time, is a solution of both Cotton and conformal Killing gravity with the same stress-energy tensor.

The Codazzi tensor (62) is not as general as (13), since it is fixed up to a constant. In fact the condition (14) is more general than $\dot{F} = HF$. In a FRW space-time Cotton gravity is more general than conformal Killing gravity.

The conformal Killing tensor (61) is used in [11] to obtain the Friedmann equations of conformal Killing gravity. The eigenvalue equation $K_{ij}u^i = \lambda u_j$ gives $\lambda = \frac{3}{2}F^2 - \Lambda$ and thus $K = 6\lambda + 2\Lambda$. We rewrite the tensor as

$$K_{ij} = \frac{2\lambda + 2\Lambda}{3}u_i u_j + \frac{5\lambda + 2\Lambda}{3}g_{ij}. \tag{63}$$

Note that $2\lambda = 3F^2 - 2\Lambda = 3k^2 a^2(t) - 2\Lambda$.

V. COMPARISON WITH MIMETIC GRAVITY

In 2013, Chamseddine and Mukhanov [44–46] proposed a modification of GR where the conformal degree of freedom is distinguished. This is done by parametrizing the physical metric tensor g_{kl} in terms of an auxiliary metric $\tilde{g}_{kl}$ and a scalar field ϕ, called mimetic field:

$$g_{kl}(\tilde{g}, \phi) = -(\tilde{g}^{pq}\nabla_p\phi\nabla_q\phi)\tilde{g}_{kl}, \tag{64}$$

where $\tilde{g}^{pq} \equiv (\tilde{g}^{-1})_{pq}$. Then $g^{kl} = -(\tilde{g}^{pq}\nabla_p\phi\nabla_q\phi)^{-1}\tilde{g}^{kl}$. The compatibility condition follows:

$$g^{kl}\nabla_k\phi\nabla_l\phi = -1. \tag{65}$$

A conformal transformation of the auxiliary metric $\tilde{g}'_{kl} = \Omega^2\tilde{g}_{kl}$ leaves the physical metric invariant. Mimetic gravity may be viewed as a conformal extension of Einstein theory, which is locally Weyl invariant: this fact was pointed out by Barvinsky [47].

The gravitational action depends upon the auxiliary metric and the mimetic field. Alternatively, it depends on the physical metric but with the constraint (65):

$$S = \int d^4x\sqrt{-g}[R + \zeta(g^{pq}\nabla_p\phi\nabla_q\phi + 1) - V(\phi)] + S^{(m)}. \tag{66}$$

$V(\phi)$ is a potential and ζ is a Lagrange multiplier. For a thorough review of mimetic gravity see [48].

The first field equation is obtained by minimizing with respect to the metric:

$$R_{kl} - \frac{1}{2}Rg_{kl} = T_{kl} + 2\zeta\nabla_k\phi\nabla_l\phi + g_{kl}V(\phi). \tag{67}$$

It has the form of an extended theory with dark sector explicitly represented by the mimetic field (whence the name of “mimetic dark matter” in the literature).

The trace and the constraint give $2\zeta = R + T + 4V$. The covariant divergence of (67) is

$$2[\nabla^k\zeta\nabla_k\phi + \zeta\nabla^k\nabla_k\phi]\nabla_l\phi + \nabla_l V = 0, \tag{68}$$

where we used $\nabla_k R^k{}_l - \frac{1}{2}\nabla_l R = 0$, $\nabla^k T_{kl} = 0$ and $\nabla_j(\nabla^p\phi\nabla_p\phi) = 0$.

Variation of the action with respect to the mimetic field gives

$$2\nabla^p(\zeta\nabla_p\phi) = -\frac{\partial V}{\partial\phi}. \tag{69}$$

Since $g^{pq}\nabla_p\phi\nabla_q\phi = -1$ the vector field $u_k = -\nabla_k\phi$ is unit timelike and closed, i.e., $\nabla_j u_k = \nabla_k u_j$. Then it is vorticity-free and acceleration-free:

$$\nabla_j u_k = H(g_{jk} + u_j u_k) + \sigma_{jk}, \tag{70}$$

with σ_{jk} being the shear tensor. The corresponding metric is (see [44,46,49])

$$ds^2 = -dt^2 + g^\star_{\mu\nu}(\mathbf{x}, t)dx^\mu dx^\nu. \tag{71}$$

By fixing the hypersurfaces of constant time of (71) to be of constant ϕ, the solution of the constraint $g^{pq}\nabla_p\phi\nabla_q\phi = -1$ may be written (see [44] and references therein or [46]):

$$\phi = \pm t + \text{const.} \tag{72}$$

Thereby choosing $\phi = t$ and using $u_k = -\nabla_k\phi$, it is $u_0 = -1$, $u_\mu = 0$.

We then conclude that the general metric for mimetic gravity is (71). In this context $V = V(t)$, while in general ζ is a function of $\mathbf{x}$ and t.

The field equations take the form

$$R_{kl} - \frac{1}{2}Rg_{kl} = T_{kl} + 2\zeta u_k u_l + Vg_{kl}, \tag{73}$$

and (68) becomes $\nabla_l V = -2(\dot\zeta + 3H\zeta)u_l$. Transvecting it with u^l gives the interesting relation

$$\dot V = 2\dot\zeta + 6H\zeta, \tag{74}$$

where we used $\nabla_p u^p = 3H$ derived from (70).

Now note that (73) may be rewritten as in Cotton gravity $R_{kl} - \frac{1}{2}Rg_{kl} = T_{kl} + \mathcal{C}_{kl} - g_{kl}\mathcal{C}^r_r$, with

$$\mathcal{C}_{kl} = 2\zeta u_k u_l + \frac{1}{3}g_{kl}(2\zeta - V). \tag{75}$$

It is a perfect fluid tensor with $\mathcal{A} = 2\zeta$ and $3\mathcal{B} = 2\zeta - V$ but in general it is not Codazzi. Nevertheless, in view of (74), it is always

$$-\frac{\dot{\mathcal{B}}}{H} = \frac{1}{3H}(\dot V - 2\dot\zeta) = 2\zeta = \mathcal{A}.$$

We report Theorem 2.1 in [5] restricted to the case of vanishing acceleration:

The perfect fluid tensor $\mathcal{C}_{kl} = \mathcal{A}u_k u_l + \mathcal{B}g_{kl}$ is Codazzi if and only if (1) $\nabla_j u_k = H(g_{jk} + u_j u_k)$, (2) $\nabla_j H = -\dot H u_j$, (3) $\nabla_j\mathcal{A} = -\dot{\mathcal{A}}u_j$ and $\nabla_j\mathcal{B} = -\dot{\mathcal{B}}u_j$, and (4) $H = -\dot{\mathcal{B}}/\mathcal{A}$. This can be rephrased as follows:

Proposition 11. The field equation (67) of mimetic gravity is the field equation of Cotton gravity if and only if the space-time is GRW, $V = V(t)$ and $\zeta = \zeta(t)$. In particular, in a FRW space-time the field equations (67) are the Cotton equations.

Cotton gravity can include other versions of mimetic gravity. As noted in [45] (see also the review [35]) in order to have viable cosmological perturbations the action (66) has to include higher derivative terms. For example it is possible to add $\frac{1}{2}\gamma(\Box\phi)^2$ being γ a constant. The new field equations are [Eq. 110 in [35]]:

$$\begin{aligned}R_{kl} - \frac{1}{2}Rg_{kl} &= T_{kl} + g_{kl}[V(\phi) + \gamma\nabla_p\chi\nabla^p\phi] \\ &\quad + 2\zeta\nabla_k\phi\nabla_l\phi - \gamma[\nabla_k\phi\nabla_l\chi + \nabla_k\chi\nabla_l\phi],\end{aligned} \tag{76}$$

where $\Box\phi = \chi$. The background is a FRW space-time. Since $u_k = -\nabla_k\phi$ we get $3H = -\Box\phi$. Moreover, recalling that $\nabla_k H = -\dot H u_k$, it is $\nabla_k\chi = -3\nabla_k H = 3\dot H u_k$ and the previous equation rewrites as

$$R_{kl} - \frac{1}{2}Rg_{kl} = T_{kl} + 2(\zeta + 3\gamma\dot H)u_k u_l + g_{kl}[V(\phi) + 3\gamma\dot H]. \tag{77}$$

Thus we recognize

$$\mathcal{C}_{kl} = 2(\zeta + 3\gamma\dot H)u_k u_l + \frac{1}{3}g_{kl}(2\zeta - V + 3\gamma\dot H). \tag{78}$$

This is again perfect fluid, with $\mathcal{A} = 2(\zeta + 3\gamma\dot H)$ and $3\mathcal{B} = 2\zeta - V + 3\gamma\dot H$ being in this context $V = V(t)$ and $\zeta = \zeta(t)$. The covariant divergence of (77) gives the conservation law

$$2[3H\zeta + 9\gamma H\dot H + \dot\zeta + 3\gamma\ddot H]u_l + \nabla_l V + 3\gamma\nabla_l\dot H = 0. \tag{79}$$

Transvecting this with u^l gives

$$\gamma\ddot H = 2H(\zeta + 3\gamma\dot H) - \frac{2\dot\zeta}{3} + \frac{\dot V}{3}. \tag{80}$$

Thus $3\dot{\mathcal{B}} = 2\dot\zeta - \dot V + 3\gamma\ddot H$ and using (80) it is $\frac{\dot{\mathcal{B}}}{H} = 2(\zeta + 3\gamma\dot H) = -\mathcal{A}$ and (78) is a Codazzi tensor.

We have proven the following:

Proposition 12. In a FRW space-time the field equations (77) are the Cotton equations.

A Lagrangian containing also a term proportional to $(\nabla_k\nabla_l\phi)^2$ was investigated by Casalino *et al.* [50]. Its viability was tested in the light of the multi messenger detection of the gravitational wave event GW170817 and its optical counterpart [51]. As a result, the coefficient multiplying this term was shown to be $< 10^{-15}$; thus the term should be suppressed.

In closing, we recall that Nojiri and Odintsov [52] introduced mimetic $f(R)$ gravity.

VI. FIXING THE DARK SECTOR

From the above discussion it is clear that the dark sector is described by the Codazzi terms and emerges from geometry. In Cotton gravity the term $\mathcal{B}$ remains unfixed, so that further restrictions are needed.

We make a standard cosmological analysis by supposing that the content of energy in T_{jk} is from radiation (r) and matter (m): $\mu = \mu_r + \mu_m$, where

$$\mu_r = \frac{\mu_{r,0}}{(a/a_0)^4}, \qquad \mu_m = \frac{\mu_{m,0}}{(a/a_0)^3}.$$

By setting $\frac{8\pi G}{3H_0^2} = 1/\mu_c$, $\Omega_{r,0} = \mu_{r,0}/\mu_c$, $\Omega_{m,0} = \mu_{m,0}/\mu_c$, $\Omega_{k,0} = -\frac{R^\star}{6H_0^2 a_0^2}$, and $\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$, we get

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{(a/a_0)^4} + \frac{\Omega_{m,0}}{(a/a_0)^3} + \frac{\Omega_{k,0}}{(a/a_0)^2} + \Omega_\Lambda + \frac{\mathcal{B}}{H_0^2}.$$

In terms of redshift $1 + z = a_0/a$ the equation becomes

$$\frac{H^2}{H_0^2} = \Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_\Lambda + \frac{\mathcal{B}(z)}{H_0^2}. \tag{81}$$

If $\mathcal{B} = 0$, then the standard ΛCDM model is recovered.

It is quite remarkable that we only need to assume the presence of matter and radiation, while the theory provides the term that can be interpreted as a dark sector. As Harada argued, the dark sector appears as a purely geometric effect due to the presence of the Codazzi tensor.

Let us write the condition $\dot{\mathcal{B}} = -H\mathcal{A}$ as a function of the redshift. With $\dot{z} = -(1+z)H$ it is $\dot{\mathcal{B}} = \frac{d\mathcal{B}}{dz}\dot{z} = -\frac{d\mathcal{B}}{dz}(1+z)H$. The condition becomes

$$\mathcal{A} = \frac{d\mathcal{B}}{dz}(1+z). \tag{82}$$

In this representation $\mathcal{A}$ does not depend on the Hubble parameter.

Now recall Eqs. (18) and (19): $\kappa\mu_D = 3\mathcal{B}$ and $\kappa p_D = -3\mathcal{B} - \dot{\mathcal{B}}/H$. Suppose that an EOS $p_D = w(z)\mu_D$ is valid, where the parameter w may be redshift dependent.

In general the dark sector is characterized by $w < -1/3$. The regime $-1 < w < -1/3$ is usually called "quintessence," while the one with $w < -1$ is called "phantom." The consequences of a phantom energy in the Universe were pointed out in the seminal paper [53].

The EOS and (82) imply the equation

$$3\mathcal{B}(1 + w(z)) = (1+z)\frac{d\mathcal{B}}{dz},$$

with solution

$$\mathcal{B}(z) = \mathcal{B}_0 \exp\left[3\int_0^z \frac{1+w(z')}{1+z'}dz'\right]. \tag{83}$$

Inserting this in (81) we have

$$\frac{H^2}{H_0^2} = \Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_\Lambda + \frac{\mathcal{B}_0}{H_0^2}\exp\left[3\int_0^z \frac{1+w(z')}{1+z'}dz'\right]. \tag{84}$$

This is substantially Eq. (14) in [54] with the difference that here Λ is not dynamical. We also note the balance

$$1 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{k,0} + \Omega_\Lambda + \Omega_{D,0}, \tag{85}$$

where $\Omega_{D,0} = \mathcal{B}_0/H_0^2$ is the present-time dark energy density.

This analysis generalizes the considerations in [10,11]. In particular, if $w(z) = w$ is constant, we get the wCDM model, which generalizes the ΛCDM model with $w \neq -1$, and is reviewed in [54]:

$$\mathcal{B}(z) = \mathcal{B}_0(1+z)^{3(1+w)}, \qquad \mathcal{A}(z) = 3(1+w)\mathcal{B}(z).$$

Reversing to cosmic time we get

$$\mathcal{B}(t) = \frac{\mathcal{B}_0}{(a(t)/a_0)^{3(1+w)}}, \tag{86}$$

$$\mathcal{A}(t) = 3(1+w)\mathcal{B}(t). \tag{87}$$

In the case $w = -5/3$ we recover the phantom term typical of conformal Killing gravity discovered in [9–11]. The Codazzi tensor becomes

$$\mathcal{C}_{kl} = \frac{\mathcal{B}_0}{(a/a_0)^{3(1+w)}}[3(1+w)u_k u_l + g_{kl}]. \tag{88}$$

There are many redshift-dependent models that parametrize the shape of dark energy: they were used for example in [54] to test deviations from the ΛCDM model. More recently they were discussed on the base of JWST results [55]. The same parametrizations can be used to fix $\mathcal{B}$ and $\mathcal{A}$ using (83). We recall some of them here.

A. Chevallier-Polarski-Linder (CPL) model

It is one of the most used redshift-dependent parametrization and was introduced in [56,57]. It supposes that

$$w(z) = w_0 + w_a\frac{z}{1+z},$$

where w_0 is the present time dark energy EOS parameter and the correction describes its evolution. It features a good behavior at high z and it is linear at low z (see [55,58] for details). From (83) we obtain

$$\mathcal{B}(z) = \mathcal{B}_0(1+z)^{3(1+w_0+w_a)} \exp\left(-\frac{3w_a z}{1+z}\right), \qquad (89)$$

$$\mathcal{A}(z) = 3\mathcal{B}(z)\left(1 + w_0 + \frac{w_a z}{1+z}\right). \qquad (90)$$

The CPL model has a counterpart in the Codazzi parametrization of Cotton gravity.

In [58] the authors observe that the recent data from JWST reveal a very large number of massive galaxies at high redshift. This fact poses challenges to the standard ΛCDM model. Based on the CPL model and testing with the new datasets, they propose a scenario in which the dark sector consists of a negative cosmological constant. A similar model was considered in [59].

B. Jassal-Bagla-Padmanabhan model

In Ref. [60] Jassal *et al.* introduce the following expression for the EOS parameter, claiming that it solves some issues present in the CPL model (see [55] and references therein):

$$w(z) = w_0 + w_a \frac{z}{(1+z)^2}.$$

From (83) we easily obtain

$$\mathcal{B}(z) = \mathcal{B}_0(1+z)^{3(1+w_0)} \exp\left[\frac{3}{2}\frac{w_a z^2}{(1+z)^2}\right], \qquad (91)$$

$$\mathcal{A}(z) = 3\mathcal{B}(z)\left[1 + w_0 + \frac{w_a z}{(1+z)^2}\right]. \qquad (92)$$

Also this model has a counterpart in the Codazzi parametrization of Cotton gravity, without explicit introduction of dark energy.

VII. CONCLUSIONS

Cotton gravity offers a simple setting to reproduce the Friedmann equations of well-known extended theories. In all cases the dark sector arising from geometry is described by a Codazzi tensor with the proper choice of a single function. We also showed that the recently proposed conformal Killing gravity is absorbed in Cotton gravity at least for cosmological FRW space-times; this is also true for mimetic gravity. The dark sector may be fixed requesting an EOS: this can accommodate in a unified description the best known redshift dependent models.

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