Static black hole in minimal Horndeski gravity with Maxwell and Yang-Mills fields and some aspects of its thermodynamics

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In this work we obtain a static spherically symmetric charged black hole solution in the framework of minimal Horndeski gravity with additional Maxwell and Yang-Mills fields. The obtained solution is examined; in particular, its asymptotics are studied. Thermodynamics of the black hole is investigated, namely, we use an effective surface gravity to derive black hole temperature. To obtain the first law of black hole thermodynamics, the Wald method is applied. We also use the extended thermodynamics approach, namely, it allows us to derive the Smarr relation, Gibbs free energy, and the thermal equation of state. The study of thermal values in the extended space shows rich phase behavior, in particular, the domain where the first-order phase transition takes place and the critical point with the second-order phase transition. We also study thermal behavior near the critical point, obtain critical exponents, and analyze the Ehrenfest equations at the critical point. Finally, we calculate the Prigogine-Defay ratio confirming the conclusion about the second-order phase transition at the critical point.

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I. INTRODUCTION

The recent decade has become a period of outstanding progress of observational astrophysics, first of all due to long-time expected detection of the gravitational waves, which required experimental setup of remarkably high accuracy [1]. In general, experimental observations show astonishing agreement with theoretical predictions made in the framework of general relativity (GR), which even nowadays is an exceptionally successful theory of gravity [2]. Notwithstanding its attractive features, there are some open issues that motivate people to look for alternative or more general approaches than Einsteinian theory of gravity that give answers to current puzzles. Among the most perplexing questions are the existence of singularities which, as it is proved, inevitably appear within general relativistic consideration, dark energy/dark matter issues, and consistent description of early stage evolution of the Universe.

To overcome the mentioned difficulties, various approaches were proposed and examined giving rise to different ways to modify the general relativistic setting of the problem. The key features, their advantages, and possible difficulties of the diverse approaches are given in thorough reviews [3–9]. Here we focus on scalar-tensor gravity theories, namely, the so-called Horndeski gravity [10,11], as one of the most promising approaches. We also point out that scalar-tensor theories of gravity may be considered as a conservative approach, since its formulation follows the way usually used in general relativity. We also point out that scalar-tensor theories have a rather long history, starting back from Brans-Dicke gravity established in the early 1960s [12]. The latter one also gained its second renaissance since the beginning of the new century, particularly because of its tight bonds with $F(R)$ gravity [8]. Strictly speaking, the Brans-Dicke theory is just a particular case of the general Horndeski gravity [11], but because of specific coupling between gravity and scalar sectors in Brans-Dicke-type theories and in Horndeski gravity they are often considered separately.

In his seminal paper [10], Horndeski proposed the most general four-dimensional scalar-tensor theory with the so-called derivative coupling between gravity and scalar fields, which gives rise to the second-order field equations. Horndeski gravity got its second revival when relations with the generalized Galileon model were established [13]. The Galileons first appeared in studies of the Dvali-Gabadadze-Poratti (DGP) model [14]; they got their name due to a

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specific shift symmetry, namely, $\varphi \rightarrow \varphi' = \varphi + b_\mu x^\mu + c$ (b_ν, c are constants). One of the most appealing features of Horndeski gravity related to the second order of the equations of motion is the absence of ghosts. On the other hand, the Cauchy problem is well posed in Horndeski gravity, making it an attractive model for various applications. Even though there is direct relation between the generalized Galileon theory and Horndeski gravity in four dimensions, higher-dimensional generalization of Horndeski gravity has not been obtained yet [11]. Since its relation to the DGP model and due to the fact that Horndeski theory terms in four-dimensional space-time can be derived via dimensional reduction [15,16], it can be claimed that Horndeski gravity, apart from its phenomenological origin, has some ties with string theory, at least in the low-energy limit of the latter. We also point out that Horndeski gravity can be generalized to become a multiscalar theory [17]; another interesting generalization is the so-called degenerate higher-order scalar-tensor theories [18], namely, the theories with higher-order equations of motion, but with some degeneracy conditions removing the Ostrogradsky instability. Horndeski gravity has numerous applications in cosmology; the most remarkable of them are pointed out in the review [11].

Black holes and other compact objects like neutron stars have attracted much attention since the second revival of Horndeski gravity [19–38]. Black hole solutions are important and useful toy models to study various effects, especially related to astrophysical black holes [39]. Gravity theories including general relativity usually have complicated structure; therefore, gaining some general results valid at least within a particular gravity theory might be a problem of immense difficulty, especially for the theories beyond general relativity. Therefore, black hole solutions are those objects that allow one to derive or test implications of theory and their study is a very important problem.

Black holes in Horndeski theory are known to have a nontrivial scalar field profile, particularly, the scalar field may be time dependent [11,23] or/and have singular behavior at the event horizon. The nontrivial profile of the scalar field significantly affects various properties of the black holes and usually requires careful study. Even though there a lot of black hole solutions in Horndeski gravity, not much attention is paid to the case where additional fields are taken into consideration [22,29,33,35]. It can be explained by the following two reasons: The first one is directly related to the cumbersome structure of the Horndeski theory giving rise to equations that are hardly tractable even under quite simple assumptions. The second reason, to our mind, is related to a rather general point of view that the main impact of the Horndeski gravity should take place on cosmological scales, whereas for the compact object, due to various screening mechanisms, they should mimic general relativistic black holes at least for a distant observer. But studies of black holes with additional material or gauge fields in Horndeski gravity allow one not only to reveal some specific features caused by the particular choice of the gravity model, but they, in principle, may give us a more general and broad view of some basic notions of black hole physics and show the range of their applicability to various gravity models.

In this paper a static black hole solution in a particular case of Horndeski theory with additional Maxwell and Yang-Mills field is considered. As far as we know, the interplay of Horndeski gravity and Yang-Mills field, even though both of them are taken probably in their simplest form, is studied for the first time. The Maxwell field in its standard form as well as for some of its nonlinear generalizations was considered in the case of Horndeski theory [22,29,33,35], whereas non-Abelian fields were examined mainly within general relativity [40–49] or more generally in Einstein-dilaton theory [42,50–52]. We also take into account the Maxwell field to examine interplay between the gauge fields in the framework of Horndeski gravity and as we will show there is an effective "coupling" between them which does not appear neither in general relativity, nor in a more general Einstein-dilaton theory [51,52]. We also pay considerable attention to the study of various aspects of thermodynamics for the obtained solution.

The work is organized as follows. In the following section, we obtain and study a static black hole solution in Horndeski gravity with additional Abelian and non-Abelian gauge fields. In the third section, we obtain and examine the black hole temperature. In the fourth section, we use the Wald approach to derive the first law of black hole thermodynamics, obtain other thermodynamics values such as entropy and heat capacity, and examine the latter one. In the fifth section, we use the extended thermodynamics approach to derive the extended first law and the Smarr relation. In the sixth section, we obtain the Gibbs free energy and study its behavior. Critical behavior in the extended approach is studied in the seventh section. Finally, in the last section, there are some conclusions and future prospects.

II. EQUATIONS OF MOTION FOR THE THEORY WITH NONMINIMAL DERIVATIVE COUPLING AND STATIC BLACK HOLE'S SOLUTION

General Horndeski gravity gives rise to complicated equations, which even for the geometries with high symmetry are difficult to handle; therefore, we consider one of its simplest particular cases, but which inherits a distinctive feature of the general Horndeski gravity, namely, its specific derivative coupling between gravity and additional scalar field. Similar to general Horndeski gravity, the equations of motion are of second order, making the theory free from Ostrogradski instability. We also take into account some gauge fields, namely, we consider both Abelian (electromagnetic) and non-Abelian ones, which are minimally coupled to gravity. The action for our system can be written in the form

$$S = \frac{1}{16\pi}\int d^{n+1}x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{2}(\alpha g^{\mu\nu} - \eta G^{\mu\nu})\partial_\mu\varphi\partial_\nu\varphi - \mathrm{Tr}(F^{(a)}_{\mu\nu}F^{(a)\mu\nu}) - \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right) + S_{\mathrm{GHY}}, \tag{1}$$

where $g_{\mu\nu}$ and g are the metric tensor and its determinant, respectively, R and $G_{\mu\nu}$ are the Ricci scalar and the Einstein tensor correspondingly, φ is the scalar field, α and η are coupling constants for it, and finally, $F^{(a)}_{\mu\nu}$ and $\mathcal{F}_{\mu\nu}$ are field strengths for non-Abelian and Abelian fields, respectively. We note that since there is no potential for the scalar field in the action (1), functions we obtain and analyze show their dependence on the ratio of the coupling parameters α/η; therefore, only one of them can be treated as a parameter that can be varied, but here we keep both in order to consider some limited cases. We also point out that the S_{GHY} term in the action (1) denotes the so-called boundary Gibbons-Hawking-York (GHY) term, which makes the variational problem well defined. For this theory with nonminimal derivative coupling the Gibbons-Hawking-York term can be written in the form

$$S_{\mathrm{GHY}} = \frac{1}{8\pi}\int d^n x\sqrt{|h|}\left(K + \frac{\eta}{4}[\nabla^\mu\varphi\nabla^\nu\varphi K_{\mu\nu} + (n^\mu n^\nu\nabla_\mu\varphi\nabla_\nu\varphi + (\nabla\varphi)^2)K]\right), \tag{2}$$

where h is the determinant of the boundary metric $h_{\mu\nu}$, $K_{\mu\nu}$ and K denote the extrinsic curvature tensor and its trace correspondingly, and finally n^μ is the vector normal to the boundary hypersurface.

We point out here that the field tensors for the gauge fields are defined in the standard way; namely, for the Yang-Mills field, we write

$$F^{(a)}_{\mu\nu} = \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu + \frac{1}{\bar{\sigma}}C^{(a)}_{(b)(c)}A^{(b)}_\mu A^{(c)}_\nu, \tag{3}$$

where $A^{(a)}_\mu$ is the Yang-Mills potential, $\bar{\sigma}$ is the coupling constant for the non-Abelian field, and $C^{(a)}_{(b)(c)}$ are the structure constants for the corresponding gauge group. In this work, the gauge group is chosen to be the special orthogonal one $SO(n)$.

The Maxwell field tensor is defined in the standard fashion,

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu, \tag{4}$$

and here $\mathcal{A}_\mu$ is the Maxwell field potential.

To obtain equations of motion for the system given by the action (1), the least action principle is used. For the gravitational part, we can write

$$\mathcal{E}_{\mu\nu} := G_{\mu\nu} + \Lambda g_{\mu\nu} - \left(\frac{1}{2}(\alpha T^{(1)}_{\mu\nu} + \eta T^{(2)}_{\mu\nu}) + T^{(3)}_{\mu\nu} + T^{(4)}_{\mu\nu}\right) = 0, \tag{5}$$

where we have

$$T^{(1)}_{\mu\nu} = \nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}\nabla^\lambda\varphi\nabla_\lambda\varphi, \tag{6}$$

$$T^{(2)}_{\mu\nu} = \frac{1}{2}\nabla_\mu\varphi\nabla_\nu\varphi R - 2\nabla^\lambda\varphi\nabla_\nu\varphi R_{\lambda\mu} + \frac{1}{2}\nabla^\lambda\varphi\nabla_\lambda\varphi G_{\mu\nu} - g_{\mu\nu}\left(-\frac{1}{2}\nabla_\lambda\nabla_\kappa\varphi\nabla^\lambda\nabla^\kappa\varphi + \frac{1}{2}(\nabla^2\varphi)^2 - R_{\lambda\kappa}\nabla^\lambda\varphi\nabla^\kappa\varphi\right) - \nabla_\mu\nabla^\lambda\varphi\nabla_\nu\nabla_\lambda\varphi + \nabla_\mu\nabla_\nu\varphi\nabla^2\varphi - R_{\lambda\mu\kappa\nu}\nabla^\lambda\varphi\nabla^\kappa\varphi, \tag{7}$$

$$T^{(3)}_{\mu\nu} = 2\mathrm{Tr}\left(F^{(a)}_{\mu\lambda}F^{(a)\lambda}_\nu\right) - \frac{g_{\mu\nu}}{2}\mathrm{Tr}\left(F^{(a)}_{\lambda\kappa}F^{(a)\lambda\kappa}\right), \tag{8}$$

$$T^{(4)}_{\mu\nu} = 2\mathcal{F}_{\mu\lambda}\mathcal{F}_\nu{}^\lambda - \frac{g_{\mu\nu}}{2}\mathcal{F}_{\lambda\kappa}\mathcal{F}^{\lambda\kappa}. \tag{9}$$

It is clear that on the right-hand side of Eq. (5) there are stress-energy tensors for the scalar and gauge fields given by the upper relations (6)–(9).

The least action principle also allows us to obtain equations of motion for the scalar and the gauge fields. For the scalar field φ, we arrive at the following equation:

$$\mathcal{E}_\varphi := (\alpha g_{\mu\nu} - \eta G_{\mu\nu})\nabla^\mu\nabla^\nu\varphi = 0. \tag{10}$$

For the Yang-Mills field, we obtain

$$\mathcal{E}_A{}^{(a)\nu} := \nabla_\mu(F^{(a)\mu\nu}) + \frac{1}{\bar{\sigma}}C^{(a)}_{(b)(c)}A^{(b)}_\mu F^{(c)\mu\nu} = 0. \tag{11}$$

Finally, for the Abelian gauge field, the standard Maxwell equations can be derived as

$$\mathcal{E}_{\mathcal{A}}{}^\nu := \nabla_\mu\mathcal{F}^{\mu\nu} = 0. \tag{12}$$

Here we are going to obtain a static black hole's solution, therefore we take the metric in the following form:

$$ds^2 = -U(r)dt^2 + W(r)dr^2 + r^2 d\Omega^2_{(n-1)}, \tag{13}$$

where $d\Omega^2_{(n-1)}$ represents the element of length for a unit $n-1$-dimensional hypersphere and the metric functions $U(r)$ and $W(r)$ will be obtained from the equations of motion. We also point out here that in the present work we assume that $n \geq 3$. It should be noted that a more general form for the static black hole metric was utilized in [53] with additional function $R(r)$ instead of r^2 in front of the angular part, but due to configuration of the gauge fields we consider here, we take the metric in the form (13).

For a static electrically charged solution, the gauge potential for the Maxwell (Abelian) field can be chosen in the form $\mathcal{A} = \mathcal{A}_0(r)dt$. From the Maxwell equations (12) we derive immediately that the electromagnetic field takes the form

$$\mathcal{F}_{rt} = \frac{q}{r^{n-1}}\sqrt{UW}. \tag{14}$$

It is known that the so-called Wu-Yang ansatz [40,42,45,46,50], being one of the simplest possible choices to satisfy the Yang-Mills equations (11), allowed us to derive various solutions in pure Yang-Mills theory and if gravity was taken into account it brought us to nontrivial black hole solutions. Therefore, the non-Abelian gauge potential takes the form as follows:

$$\mathbf{A}^{(a)} = \frac{\bar{q}}{r^2}C^{(a)}_{(i)(j)}x^i dx^j, \qquad r^2 = \sum_{j=1}^{n} x_j^2. \tag{15}$$

Here we point out that, in order to satisfy the equations of motion (11), we impose that $\bar{q} = \bar{\sigma}$ and, for simplicity, auxiliary Cartesian coordinates x_i were used and the indices a, i, j can take the following values: $1 \leq a \leq n(n-1)/2$, $2 \leq j+1 < i \leq n$. The relations between the coordinates x_i and the angular variables of a spherical coordinate system are standard,

$$\begin{aligned} &x_1 = r\cos\chi_{n-1}\sin\chi_{n-2}\ldots\sin\chi_1, \qquad &&x_2 = r\sin\chi_{n-1}\sin\chi_{n-2}\ldots\sin\chi_1,\\ &x_3 = r\cos\chi_{n-2}\sin\chi_{n-3}\ldots\sin\chi_1, &&x_4 = r\sin\chi_{n-2}\sin\chi_{n-3}\ldots\sin\chi_1,\\ &\ldots \\ &x_n = r\cos\chi_1, \end{aligned} \tag{16}$$

and the angular variables χ_i have typical ranges of variation; namely, for $1 \leq i \leq n-2$ we have $0 \leq \chi_i \leq \pi$ and $0 \leq \chi_{n-1} < 2\pi$. Using the angular variables we can also represent the length element for the unit sphere,

$$d\Omega^2_{n-1} = d\chi_1^2 + \sum_{j=2}^{n-1}\prod_{i=1}^{j-1}\sin^2\chi_i d\chi_j^2. \tag{17}$$

The gauge potential (15) can be rewritten in terms of angular variables, but its explicit form becomes more cumbersome. Using the relation (3) we are able to calculate the gauge field $F^{(a)}_{\mu\nu}$ and check that the equations of motion (11) are satisfied. Finally, we calculate the corresponding invariant for the Yang-Mills field,

$$\mathrm{Tr}(F^{(a)}_{\rho\sigma}F^{(a)\rho\sigma}) = (n-1)(n-2)\frac{\bar{q}^2}{r^4}. \tag{18}$$

Using the metric ansatz (13) and taking into account the gauge field tensors and their invariants we can write Eq. (5) in the following form:

$$\frac{(n-1)}{2rW}\left(\frac{W'}{W} + \frac{(n-2)}{r}(W-1)\right)\left(1 + \frac{3}{4}\eta\frac{(\varphi')^2}{W}\right) - \Lambda = \frac{\alpha}{4W}(\varphi')^2 + \frac{\eta}{2}\left(\frac{(n-1)(n-2)}{r^2W^2}\left(W - \frac{1}{2}\right)(\varphi')^2 + \frac{(n-1)}{rW^2}\varphi''\varphi'\right) + \frac{q^2}{r^{2(n-1)}} + \frac{(n-1)(n-2)\bar{q}^2}{2r^4}, \tag{19}$$

$$\frac{(n-1)}{2rW}\left(\frac{U'}{U} - \frac{(n-2)}{r}(W-1)\right)\left(1 + \frac{3}{4}\eta\frac{(\varphi')^2}{W}\right) + \Lambda = \frac{\alpha}{4W}(\varphi')^2 - \frac{\eta(n-1)(n-2)}{4r^2W}(\varphi')^2 - \frac{q^2}{r^{2(n-1)}} - \frac{(n-1)(n-2)\bar{q}^2}{2r^4}, \tag{20}$$

where the prime denotes the derivative with respect to r.

The equation for the scalar field (10) may be also be integrated at least once, and as a result we obtain

$$\sqrt{\frac{U}{W}}r^{n-1}\left[\alpha - \eta\frac{(n-1)}{2rW}\left(\frac{U'}{U} - \frac{(n-2)}{r}(W-1)\right)\right]\varphi' = C, \tag{21}$$

where C is an integration constant. The latter relation allows us to represent the derivative φ' as a function of $U(r)$ and $W(r)$ and of the derivative of the former one, but it can be hardly tractable in the general case. Further progress can be made if we impose $C = 0$ and this condition gives rise to a particular, but quite nontrivial, solution with a nontrivial scalar profile. The condition $C = 0$ is equivalent to the following constraint:

$$\alpha g_{rr} - \eta G_{rr} = 0. \tag{22}$$

Here we point out that the same condition (22) was used in our earlier works [33,35], as well as by other authors who studied black holes in Horndeski gravity [19–21].

Now Eqs. (19) and (20) can be solved together with the relation (22). As a result, we obtain

$$(\varphi')^2 = -\frac{4r^2 W}{2\alpha r^2 + \eta(n-1)(n-2)}\left(\Lambda + \frac{\alpha}{\eta} + q^2 r^{2(1-n)} + \frac{(n-1)(n-2)}{2}\bar{q}^2 r^{-4}\right); \tag{23}$$

$$UW = \frac{((\alpha - \Lambda\eta)r^2 + \eta(n-1)(n-2) - \eta q^2 r^{2(2-n)} - \eta(n-1)(n-2)\bar{q}^2 r^{-2}/2)^2}{(2\alpha r^2 + \eta(n-1)(n-2))^2}. \tag{24}$$

The square of the derivative φ' has to be positive outside of the black hole, which might be achieved if some conditions on the parameters α, η, Λ, q, and $\bar{q}$ are imposed. For instance, when both parameters α and η are positive, the cosmological constant Λ should be negative to provide positivity of the $(\varphi')^2$ in the outer domain. A similar conclusion is inferred if we impose $\alpha > 0$ and $\eta < 0$.

Finally, the metric function $U(r)$ can be written in the following form:

$$\begin{aligned} U(r) = {}& 1 - \frac{\mu}{r^{n-2}} - \frac{2\Lambda}{n(n-1)}r^2 - \frac{(n-2)}{(n-4)}\frac{\bar{q}^2}{r^2} + \frac{2q^2}{(n-1)(n-2)}r^{2(2-n)} + \frac{1}{2\alpha\eta(n-1)r^{n-2}} \\ & \times \left((\alpha + \Lambda\eta)^2 \int \frac{r^{n+1}}{r^2 + d^2}dr + \eta^2 q^4 \int \frac{r^{5-3n}}{r^2 + d^2}dr + 2\eta(\alpha + \Lambda\eta)q^2 \int \frac{r^{3-n}}{r^2 + d^2}dr + (n-1)(n-2)\right. \\ & \left. \times \eta\bar{q}^2\left((\alpha + \Lambda\eta)\int \frac{r^{n-3}}{r^2 + d^2}dr + \eta q^2 \int \frac{r^{-(n+1)}}{r^2 + d^2}dr + \frac{\eta}{4}(n-1)(n-2)\bar{q}^2 \int \frac{r^{n-7}}{r^2 + d^2}dr\right)\right), \end{aligned} \tag{25}$$

where $d^2 = \eta(n-1)(n-2)/2\alpha$. Even though all of the integrals in the upper relation can be calculated explicitly, we give here a more concise integral form for the metric function. There are also some peculiarities related to whether the dimension of space is odd or even. Details of the calculations of the integrals for the convenience of the reader are given in the Appendix. Taking into account these calculations, we give here the explicit form of the metric function $U(r)$ for odd n, while the corresponding expression for even n is given in the Appendix,

$$\begin{aligned} U(r) = {}& 1 - \frac{\mu}{r^{n-2}} - \frac{2\Lambda}{n(n-1)}r^2 - \frac{(n-2)}{(n-4)}\frac{\bar{q}^2}{r^2} + \frac{2q^2}{(n-1)(n-2)}r^{2(2-n)} + \frac{1}{2\alpha\eta(n-1)} \\ & \times \left[(\alpha + \Lambda\eta)^2\left(\sum_{j=0}^{\frac{n-1}{2}}(-1)^j d^{2j}\frac{r^{2(1-j)}}{n-2j} + (-1)^{\frac{n+1}{2}}\frac{d^n}{r^{n-2}}\arctan\left(\frac{r}{d}\right)\right) + 2\eta(\alpha + \Lambda\eta)q^2\right. \\ & \times \left(\sum_{j=0}^{\frac{n-5}{2}}\frac{(-1)^j r^{6-2n+2j}}{(4-n+2j)d^{2(j+1)}} + \frac{(-1)^{\frac{n-3}{2}}}{d^{n-2}r^{n-2}}\arctan\left(\frac{r}{d}\right)\right) + \eta^2 q^4\left(\sum_{j=0}^{\frac{3n-7}{2}}\frac{(-1)^j r^{2(4+j-2n)}}{(6+2j-3n)d^{2(j+1)}} + \frac{(-1)^{\frac{3n-5}{2}}}{d^{3n-4}r^{n-2}}\arctan\left(\frac{r}{d}\right)\right) \\ & + \eta(n-1)(n-2)\bar{q}^2 \times \left((\alpha + \Lambda\eta)\left(\sum_{j=0}^{\frac{n-5}{2}}\frac{(-1)^j d^{2j} r^{-2(1+j)}}{n-4-2j} + (-1)^{\frac{n-3}{2}}\frac{d^{n-4}}{r^{n-2}}\arctan\left(\frac{r}{d}\right)\right)\right. \\ & + \eta q^2\left(\sum_{j=0}^{\frac{n-1}{2}}\frac{(-1)^j r^{2(1+j-n)}}{(2j-n)d^{2(j+1)}} + \frac{(-1)^{\frac{n+1}{2}}}{d^{n+2}r^{n-2}}\arctan\left(\frac{r}{d}\right)\right) \\ & \left.\left. + \eta(n-1)(n-2)\frac{\bar{q}^2}{4}\left(\sum_{j=0}^{\frac{5-n}{2}}\frac{(-1)^j r^{2(j-2)}}{(2j+n-6)d^{2(j+1)}} + \frac{(-1)^{\frac{7-n}{2}}}{d^{8-n}r^{n-2}}\arctan\left(\frac{r}{d}\right)\right)\right)\right]. \end{aligned} \tag{26}$$

We note that in the upper relation its last sum is valid when $n < 7$. If $n > 7$, the last integral of the function (25) should be taken in the form (A1).

Because of special interest in the $n = 3$ case, here we give the explicit form for the metric function $U(r)$,

$$U(r) = 1 - \frac{\mu}{r} - \frac{\Lambda}{3}r^2 + \frac{q^2 + \bar{q}^2}{r^2} + \frac{1}{4\alpha\eta}\left[(\alpha + \Lambda\eta)^2\left(\frac{r^2}{3} - d^2\right) + \frac{\eta^2(q^2 + \bar{q}^2)^2}{d^2 r^2} \times \left(\frac{1}{d^2} - \frac{1}{3r^2}\right) + \left((\alpha + \Lambda\eta)^2 d + \frac{\eta(q^2 + \bar{q}^2)}{d^3}\right)^2 \frac{d}{r}\arctan\left(\frac{r}{d}\right)\right]. \tag{27}$$

We point out that for $n = 3$ both gauge fields equally contribute to the metric (27), while for higher dimensions ($n > 3$), the non-Abelian field in comparison with the Maxwell one gives slowly decaying terms if $r \to \infty$. Even though the explicit expressions for the metric functions (26) and (A5), as well as their particular cases (27) and (A6) correspondingly, are rather cumbersome, some important conclusions about their behavior can be derived relatively easily. First, for both types of parity of dimensions the behavior of the metric function $U(r)$ for large distances is asymptotically of anti–de Sitter (AdS) type if both coupling parameters α and η are positive (of the same sign); namely, we can write

$$U \simeq \frac{(\alpha - \Lambda\eta)^2}{2n(n-1)\alpha\eta} r^2 = \frac{\eta(\alpha/\eta - \Lambda)^2}{2n(n-1)\alpha} r^2. \tag{28}$$

Since the gauge fields give decaying terms in the outer far zone, it is natural that there is an AdS-type term that shows leading behavior if $r \to \infty$; similar results were obtained for nonlinear electromagnetic field [33,35].

If the radius r becomes very small ($r \to 0$) the metric function $U(r)$ shows singular behavior that is defined by the electromagnetic field part in the general case. Namely, when $r \to 0$ we can write

$$U(r) \simeq -\frac{q^4}{3(n-1)^2(n-2)^2} r^{4(2-n)}. \tag{29}$$

Therefore, the leading term for $r \to 0$ is defined by the Maxwell field, which is rather expectable. But the explicit form of the asymptotic (29) appears due to interplay of the gauge field contribution and Horndeski theory, although the asymptotic (29) does not depend on the coupling α and η. We point out that the Yang-Mills terms alone or the terms where effective coupling between the Maxwell and Yang-Mills fields are taken into account show less singular behavior in comparison with term (29) in the limit $r \to 0$ if $n > 3$. For $n = 3$, both gauge fields give rise to contributions of the same order, which is clearly seen from the explicit form of the metric function for this case (27). Namely, in this case we have

$$U(r) \simeq -\frac{(q^2 + \bar{q}^2)^2}{12r^4}. \tag{30}$$

One of the most important conclusions following from the asymptotic expressions (29) and (30), because of their negative signs, is that the singular behavior of the metric function $U(r)$ if $r \to 0$ is more similar to the Schwarzschild black hole than to the Reissner-Nordström one as it might be expected. That character of behavior of the metric function is also clearly reflected on the graph of the metric function $U(r)$ given by Fig. 1. The right graph of Fig. 1 also implies that, apart from the only event horizon, namely, the point where the function $U(r)$ crosses the horizontal axis, additional inner horizons may appear if one increases the electric charge q, but it also may occur if the parameter $\bar{q}$ goes up. A detailed analysis of this issue will be considered elsewhere. The other important conclusion, which is also directly related to the above mentioned features, is that for any charge q or $\bar{q}$ a naked singularity never occurs, as it usually takes place within general relativity if the charge of a black hole increases while other parameters of the black hole are held fixed.

We also briefly examine the particular case if $\alpha = 0$, namely, when only derivative coupling between gravity and the scalar field part is considered. The particular solution for $\alpha = 0$ is substantially more simple than the general one examined above. Namely, for the squared derivative $(\varphi')^2$ and the product of the metric functions, we obtain

$$(\varphi')^2 = -\frac{4r^2 W}{\eta(n-1)(n-2)} \times \left(\Lambda + q^2 r^{2(1-n)} + \frac{1}{2}(n-1)(n-2)\bar{q}^2 r^{-4}\right), \tag{31}$$

$$UW = \left(1 - \frac{\Lambda r^2}{(n-1)(n-2)} - \frac{q^2}{(n-1)(n-2)} r^{2(2-n)} - \frac{\bar{q}^2}{2r^2}\right)^2. \tag{32}$$

The metric function $U(r)$ can be written in the form

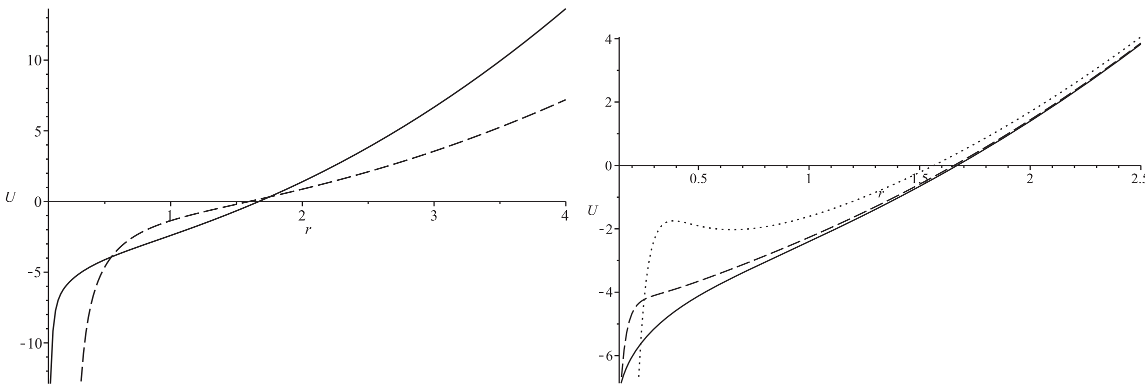

FIG. 1. Metric functions $U(r)$ for various dimensions n (left) and different values of the electric charge q (right). For both graphs we have $\Lambda = -2$, $\alpha = 0.2$, $\eta = 0.4$, $\bar{q} = 0.2$. Left: we have $q = 0.2$ and solid and dashed lines correspond to $n = 3$ and $n = 4$, respectively. Right: we have taken $n = 3$ and solid, dashed, and dotted curves correspond to $q = 0.2$, $q = 0.4$, and $q = 0.8$, respectively.

$$U(r) = 1 - \frac{\mu}{r^{n-2}} - \frac{2\Lambda}{n(n-1)} r^2 - \frac{(n-2)}{(n-4)} \frac{\bar{q}^2}{r^2} + \frac{2q^2}{(n-1)(n-2)} r^{2(2-n)} + \frac{\Lambda^2}{(n-1)^2(n^2-4)} r^4 - \frac{q^4}{3(n-1)^2(n-2)^2} r^{4(2-n)} - \frac{2\Lambda q^2}{(n-1)^2(n-2)(n-4)} r^{2(3-n)} + \frac{\Lambda\bar{q}^2}{(n-1)(n-2)} - \frac{q^2\bar{q}^2}{n(n-1)} r^{2(1-n)} + \frac{(n-2)}{4(n-6)} \frac{\bar{q}^4}{r^4}. \quad (33)$$

It should be emphasized that in (33) we impose $n \neq 4$ and $n \neq 6$. If, for instance, $n = 4$ the fourth term in the upper row and the third term in the middle row take an additional logarithmic factor ($\sim \ln r$) and if $n = 6$ this factor appears in the last term in the bottom row, but for both cases it does not change drastically the qualitative behavior of the metric function $U(r)$. We would like to note that for the particular case $\alpha = 0$ neither the product UW nor the function $U(r)$ depend on the parameter η. We point out that, if $r \to \infty$, the leading term of the metric function is of the order $\sim \Lambda^2 r^4$, and it is suppressed if $\alpha \neq 0$; since this term is always positive, it gives rise to the conclusion that there is no cosmological horizon for any sign of the cosmological constant. If $r \to 0$, the leading term of the metric (33) is the same as for the general case, namely, (29) and to some extent it is expectable since for small distances the metric is mainly defined by the leading electromagnetic field term. We also note that the product $UW \to \infty$ if $r \to \infty$ and it becomes singular if $r \to 0$, but this singular behavior, which also takes place if $\alpha \neq 0$ allows us to moderate singularities for the invariants of the Riemann tensor in comparison with standard general relativity solutions [33].

III. BLACK HOLE TEMPERATURE

One of the basic notions of black hole thermodynamics is temperature. The definition of the temperature is based on the geometrical notion of surface gravity, which can be applied not only to black holes within general relativity, but also to more general gravitational frameworks [54–56], including Horndeski gravity [57]. The surface gravity κ is defined as follows:

$$\kappa^2 = -\frac{1}{2} \nabla_a \chi_b \nabla^a \chi^b, \quad (34)$$

where χ^μ is a Killing vector, which is null on the event horizon. Since in our work the static configuration (13) is considered, the time translation vector $\chi^\mu = \partial/\partial t$ satisfies the mentioned condition. In the framework of general relativity and in various other approaches to gravity, the temperature is defined to be proportional to the surface gravity, namely,

$$T_{\rm BH} = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \frac{U'(r_+)}{\sqrt{U(r_+)W(r_+)}}, \quad (35)$$

where r_+ denotes the event horizon of the black hole. Having calculated the derivative $U'(r_+)$ and after simple algebra we write the temperature in the form

$$T_{\rm BH} = \frac{1}{4\pi(n-1)r_+} \left(\left(\frac{\alpha}{\eta} - \Lambda \right) r_+^2 + (n-1)(n-2) - \frac{q^2}{r_+^{2(n-2)}} - \frac{(n-1)(n-2)}{2r_+^2} \bar{q}^2 \right). \quad (36)$$

Surface gravity has clear geometric meaning and as it is mentioned above it is widely applicable, including Horndeski theory [57], but even for the latter theory there are some subtleties. The authors [57] consider a particular case of general Horndeski gravity similar to that considered here, but they also make several assumptions that single out a particular class of solutions that can be easily reduced to some general relativistic ones if the Horndeski coupling parameter η is turned off. It is also supposed that the scalar field shares the Killing symmetry and since no peculiarities of the scalar field are pointed out, we assume that it is supposed to be regular, in particular, at the event horizon. But in our case, due to the constraint (22), the first of the assumptions may be violated, in addition, the derivative of the scalar field has singular behavior at the horizon; therefore, the conclusions made in [57] cannot be applied directly to our solution. It was argued [58] that in Horndeski theory instead of the standard surface gravity its "effective" counterpart can be introduced and it can be explained by the fact that, in general, the speed of gravitons may differ from the speed of light [39,59]; namely, these speeds differ if the Lagrangian for the gravitational perturbation contains the Weyl tensor (the so-called Weyl criterion), which usually takes place in the Horndeski case [59]. Consequently, the effective or modified surface gravity gives rise to a modified relation for the black hole temperature [58], which can be written in the form

$$T=\frac{\kappa}{2\pi}\left(1+\frac{\eta}{4}\frac{(\varphi')^2}{W}\right)\bigg|_{r_+}=T_{\rm BH}\left(1+\frac{\eta}{4}\frac{(\varphi')^2}{W}\right)\bigg|_{r_+} = \sqrt{U(r_+)W(r_+)}T_{\rm BH}. \quad (37)$$

For the particular case of the solution given by the metric (13) with corresponding functions $U(r)$ and $W(r)$, we obtain

$$T=\frac{\eta}{8\pi(n-1)\alpha r_+(r_+^2+d^2)}\left(\left(\frac{\alpha}{\eta}-\Lambda\right)r_+^2+(n-1)(n-2) -\frac{q^2}{r_+^{2(n-2)}}-\frac{(n-1)(n-2)}{2r_+^2}\bar{q}^2\right)^2. \quad (38)$$

We point out that in the limit $\eta\to 0$ both the temperature (38) and its cousin (36) become singular, confirming the fact that our solution, from which both these expressions are derived, does not meet the criteria imposed in [57]. We also note that a similar conclusion about modification of the standard relation for temperature was recently made for a black hole in shift-symmetric Einstein-scalar Gauss-Bonnet theory [60]. Even though the temperature (38) is given by a relatively simple expression, not all its peculiarities can be seen easily, but nevertheless its key features can be described. First of all, due to the square over the main parentheses, an effective coupling between the terms of different origin appear; namely, there is a coupling between both gauge fields given by the term proportional to $q^2\bar{q}^2$, but we can also claim a coupling between the gauge and scalar fields reflected by the terms where coupling parameters are multiplied by q^2 or $\bar{q}^2$. To sum it up, the coupling we mention here is just a consequence of the coupling caused by Horndeski gravity and which appears in the metric functions $U(r)$ and $W(r)$.

Using the relation (38) we can easily analyze asymptotic behavior of the temperature. For instance, for large r_+ ($r_+\to\infty$) the temperature T (38) shows de Sitterian or anti–de Sitterian character depending on signs of the parameters α and η, namely, $T\sim(\alpha-\Lambda\eta)^2r_+/(2(n-1)\alpha\eta)$, but we pay more attention to the former one; the de Sitterian case will be examined elsewhere. For very small r_+ ($r_+\to 0$), the temperature is mainly defined by the gauge field terms, and if $n>3$ the leading term is related to the Maxwell field and is of the form $T\sim q^4/((n-1)^2(n-2))r_+^{7-4n}$. What is curious here, being caused by the nonminimal coupling, is this leading term does not have any dependence on the parameter η nor on the parameter α. If $n=3$ both gauge field terms give equal contribution, due to their symmetry even in the metric (27) and consequently it is reflected in temperature.

The analysis of the temperature as a function of the horizon radius for its intermediate values is not trivial since the contribution of various terms may be comparable to what affects the behavior of the temperature. The terms in the relation (38) have opposite signs, and the temperature might be a nonmonotonous function of the horizon radius r_+. In order to understand the dependence $T=T(r_+)$ better, we give some plots of this function for various values of parameters. Figure 2 shows this dependence if the cosmological constant Λ (the left graph) and the parameter of nonminimal coupling η (the right graph) are varied. The general features of both graphs are very similar; the function $T=T(r_+)$ has a specific "narrow" and "deep" minimum, and this minimum is not affected considerably by variation of either Λ or η. We conclude that it is mainly defined by the gauge field terms (it is given below). If the cosmological constant rises in absolute value, the temperature T also rises for large r_+ and it tends to be more monotonous in the range of intermediate values of r_+. The mentioned feature is also known for a Reissner-Nordström-AdS black hole and it causes nontrivial critical behavior within an extended thermodynamics approach; the latter will be considered in the following sections. Comparing both graphs of Fig. 2, we also conclude that variation of the cosmological constant Λ leads to more substantial change of the temperature for intermediate and relatively large values of the horizon radius r_+ than the variation of the coupling constant η and this result is expected because of the way those parameters contribute to the expression (38).

Figure 3 shows the influence of variation of the electric charge q on the temperature T. As it is pointed out above, since the terms caused by the gauge field become principal

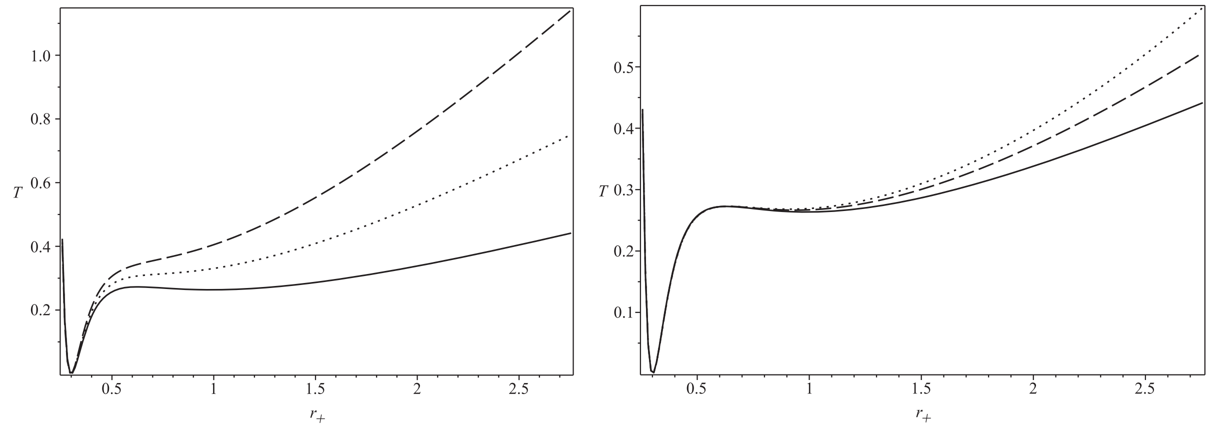

FIG. 2. Black hole's temperature T as a function of horizon radius r_+ for some values of the cosmological constant Λ (left) and the coupling parameter η (right). For both cases, from bottom to top corresponds to the increase in absolute value of the parameter we vary, whereas all other parameters are held fixed; namely, for both graphs we have taken $n = 4, \alpha = 0.1, q = \bar{q} = 0.2$. Left: we take $\eta = 0.2$ and $L_1 = -2$, $\Lambda_2 = -3$ and $\Lambda_3 = -4$. Right: we take $\Lambda = -2$ and $\eta_1 = 0.2$, $\eta_2 = 0.4$ and $\eta_3 = 0.8$.

ones for small radii of horizon r_+, it gives rise to the shift of the global minimum to the right if the charge q goes up. We also point out that the narrow domain close to the global minimum changes considerably, namely, it widens if the charge q increases. The other important consequence of this variation is the fact that the domain right to the global minimum also changes substantially; namely, its nonmonotonicity becomes less notable. We can conclude that further increase of the charge gives rise to its disappearance and it also affects the critical behavior of the black hole. Because of the same sign and inverse proportionality to the horizon radius r_+, a variation of the non-Abelian charge $\bar{q}$ gives qualitatively to the same changes in behavior of the temperature T, but due to different r_+ dependences in the general case, those changes might be substantial for intermediate values of r_+. Just for the particular case $n = 3$, both gauge fields give equal contribution.

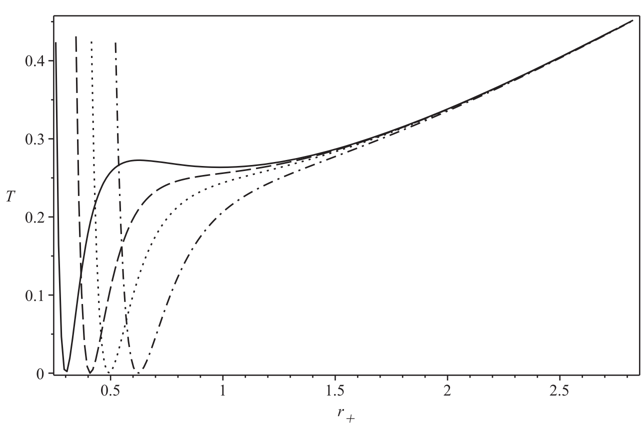

FIG. 3. Black hole's temperature T as a function of horizon radius r_+ for some values of the electric charge q if all the other parameters are held fixed. The solid, dashed, dotted, and dash-dotted curves correspond to $q_1 = 0.2$, $q_2 = 0.4$, $q_3 = 0.6$, and $q_4 = 1$, respectively. The fixed parameters are as follows: $n = 4$, $\alpha = 0.1$, $\eta = 0.2$, $\Lambda = -2$, and $\bar{q} = 0.2$.

IV. WALD PROCEDURE, CONSERVED QUANTITIES, AND THE FIRST LAW OF BLACK HOLE THERMODYNAMICS

The Wald approach is a consistent method to derive the first law of black hole mechanics (thermodynamics). Being a generalization of the standard Noether procedure to obtain conserved quantities, it allows us to obtain the latter one for general diffeomorphism invariant theories and it was successfully applied to various gravity theories. Moreover, the approach was generalized for the theories with internal gauge degrees of freedom [61]. Gauge invariant derivation of the zeroth and the first laws of black holes was also made recently [62]. To briefly describe the procedure, we write the variation of the Lagrangian for the system (1),

$$\delta\mathcal{L} = \sqrt{-g}\Big(\mathcal{E}_{\mu\nu}\delta g^{\mu\nu} + \mathcal{E}_{\varphi}\delta\varphi + \mathrm{Tr}\big(\mathcal{E}_{A}{}^{(a)\mu}\delta A^{(a)}_{\mu}\big) + \mathcal{E}_{\mathcal{A}}{}^{\mu}\delta\mathcal{A}_{\mu}\Big) + \sqrt{-g}\nabla_{\mu}\mathcal{J}^{\mu}, \tag{39}$$

and here $\mathcal{E}_{\mu\nu}$, $\mathcal{E}_{\varphi}$, $\mathcal{E}_{A}{}^{(a)\mu}$, and $\mathcal{A}^{\mu}$ are the left-hand sides of the equations of motion (5), (10)–(12), respectively, for the dynamical fields we consider. The last term in the upper variation is the so-called boundary term, which is transformed into a hypersurface integral enclosing the chosen volume. $\mathcal{J}^{\mu}$ is the surface current which can be given as a sum of corresponding dynamical fields currents, namely,

$$\mathcal{J}^{\mu} = \mathcal{J}^{\mu}_{g} + \mathcal{J}^{\mu}_{\varphi} + \mathcal{J}^{\mu}_{A} + \mathcal{J}^{\mu}_{\mathcal{A}}, \tag{40}$$

where the respective components are defined as follows:

$$\mathcal{J}^{\mu}_{g} = 2\frac{\partial\mathcal{L}}{\partial R_{\kappa\lambda\mu\nu}}\nabla_{\lambda}(\delta g_{\kappa\nu}) - 2\nabla_{\lambda}\left(\frac{\partial\mathcal{L}}{\partial R_{\kappa\mu\lambda\nu}}\right)\delta g_{\kappa\nu}, \tag{41}$$

$$\mathcal{J}^{\mu}_{\varphi}=\frac{\partial\mathcal{L}}{\partial(\varphi_{\mu})}\delta\varphi,\qquad \mathcal{J}^{\mu}_{A}=-4\mathrm{Tr}\left(F^{(a)\mu\lambda}\delta A^{(a)}_{\lambda}\right),$$
$$\mathcal{J}^{\mu}_{\mathcal{A}}=-4\mathcal{F}^{\mu\lambda}\delta\mathcal{A}_{\lambda}. \tag{42}$$

If the equations of motion are satisfied, the only contribution to the variation of the Lagrangian (39) and respectively to the action is given by the hypersurface term. Having the current $\mathcal{J}^{\mu}$ (40) we can construct the corresponding current from $J_{(1)}=\mathcal{J}_{\mu}dx^{\mu}=g_{\lambda\mu}\mathcal{J}^{\lambda}dx^{\mu}$ and then we define its Hodge dual, which is essential in the Wald approach,

$$\Theta(\psi,\delta\psi)=*J_{(1)}(\psi,\delta\psi), \tag{43}$$

where ψ is used to denote all the dynamical fields and $\delta\psi$ are their variations. The diffeomorphism is generated by a vector field ξ^{μ}; therefore, the variation of dynamical fields can be written in the form

$$\delta_{\xi}\psi=\mathcal{L}_{\xi}\psi, \tag{44}$$

where $\mathcal{L}_{\xi}$ is the corresponding Lie derivative, generated by the vector ξ^{μ}. The variation of the Lagrangian of the system can be written also as the corresponding Lie derivative, namely,

$$\delta_{\xi}*L=\mathcal{L}_{\xi}*L=d(i_{\xi}*\mathcal{L}). \tag{45}$$

Here we point out that, since the Lagrangian in our case is defined as a scalar function, i.e., 0-form, in the latter relation the Hodge dual of the Lagrangian is used. We note that to derive the second equality in the upper relation the so-called Cartan magic formula is used. Rewriting the formula for the variation of the Lagrangian (39) in terms of forms and taking into account the relations (43) and (45) as well as the notation (44), we obtain

$$d(i_{\xi}*L)=\mathcal{E}_{\psi}\mathcal{L}_{\xi}\psi+d\Theta(\psi,\mathcal{L}_{\xi}\psi)$$
$$\Rightarrow d(\Theta(\psi,\mathcal{L}_{\xi}\psi)-i_{\xi}*L)=-\mathcal{E}_{\psi}\mathcal{L}_{\xi}\psi, \tag{46}$$

where $\mathcal{E}_{\psi}$ correspond to the equations of motion for the dynamical fields. If the equations of motion are satisfied, the right-hand side of the latter relation will be equal to zero. Now we introduce a Noether current n-form,

$$J_{\xi}=\Theta(\psi,\delta\psi)-i_{\xi}*L, \tag{47}$$

which is obviously closed on shell. Moreover, it implies that this form is exact on shell, namely,

$$J_{\xi}=dQ_{\xi}. \tag{48}$$

The integral over a closed $n-1$-dimensional hypersurface Σ_{n-1} is the so-called Noether charge related to the vector field ξ^{μ} which generates the diffeomorphism. Then following the Wald approach, the space of the solutions of the equations of motion is defined to be the phase space of the theory, and variation of the dynamical fields $\delta_{\xi}\psi$ taken on shell is a phase space vector flow generated by the vector ξ^{μ}. This flow can be generated by a Hamiltonian $\mathcal{H}_{\xi}$, which is related to a symplectic form defined on a Cauchy hypersurface Σ; namely, for its on shell variation we write

$$\delta\mathcal{H}_{\xi}=\int_{\Sigma}\Omega(\psi,\delta\psi,\mathcal{L}_{\xi}\psi)=\int_{\Sigma}(\delta\Theta(\psi,\mathcal{L}_{\xi}\psi)-\mathcal{L}_{\xi}\Theta(\psi,\delta\psi)). \tag{49}$$

Using the definition of the Noether current (47) and Cartan magic formula for the Lie derivative, we can rewrite the latter relation as follows:

$$\delta\mathcal{H}_{\xi}=\int_{\Sigma}(\delta J_{\xi}+\delta(i_{\xi}*L)-i_{\xi}d\Theta-d(i_{\xi}\Theta))$$
$$=\int_{\Sigma}(\delta(dQ_{\xi})-d(i_{\xi}\Theta))=\int_{\partial\Sigma}(\delta Q_{\xi}-i_{\xi}\Theta). \tag{50}$$

We note that in the second equality we have used the on shell condition, which allows us to remove the second and the third terms in the first integral. In the second integral, we use the definition of the Noether charge and the fact that the exterior derivative and the variation for the Noether charge Q_{ξ} commute, allowing us to derive the last equality and the integral over the boundary $\partial\Sigma$. If ξ^{μ} is supposed to be a generator of a symmetry, then $\mathcal{L}_{\xi}\phi=0$ and, consequently, $\delta\mathcal{H}_{\xi}=0$. If the hypersurface Σ has two boundaries, what actually takes place for black holes, namely the infinity and the event horizon, therefore from upper relation we obtain

$$\delta\mathcal{H}_{r_+}\equiv\int_{\partial\Sigma_+}(\delta Q_{\xi}-i_{\xi}\Theta)=\int_{\infty}(\delta Q_{\xi}-i_{\xi}\Theta)\equiv\delta\mathcal{H}_{\infty}, \tag{51}$$

where $\partial\Sigma_+$ is the event horizon hypersurface. The written relation allows us to derive the first law of black hole thermodynamics.

Before derivation of the first law of black hole thermodynamics, we give an explicit relation for the components of the Noether charge, namely, we write

$$Q_{\lambda_1\ldots\lambda_{n-1}}=\varepsilon_{\lambda_1\ldots\lambda_{n-1}\mu\nu}\left(\frac{\partial\mathcal{L}}{\partial R_{\kappa\lambda\mu\nu}}\nabla_{\lambda}\xi_{\kappa}-2\xi_{[\kappa}\nabla_{\lambda]}\left(\frac{\partial\mathcal{L}}{\partial R_{\kappa\lambda\mu\nu}}\right)\right.$$
$$\left.-2\mathrm{Tr}\left(F^{(a)\mu\nu}A^{(a)}_{\lambda}\right)\xi^{\lambda}-2\mathcal{F}^{\mu\nu}\mathcal{A}_{\lambda}\xi^{\lambda}\right). \tag{52}$$

Using the upper relation as well as the relation for the Hodge dual of the surface current (43), we can calculate the differences of variations that are given under the integrals in the relation (51). Similarly, as in the previous section, the time translation vector ξ^{μ} can be chosen for corresponding calculations. It is a Killing vector and it is null on the event horizon. For more clarity, we split the calculations of the difference of the variations in two parts, namely, for the gravity part together with nonminimally coupled scalar field and for the gauge fields. The gravity part together with the scalar field contribution gives rise to the following relation:

$$(\delta Q_\xi - i_\xi\Theta)_{gs} = -(n-1)r^{n-2}\delta U\hat{\Omega}_{n-1}, \tag{53}$$

where δU is the variation of the metric function U and $\hat{\Omega}_{n-1}$ is the surface $n-1$-form. The total variation for nonminimally coupled theory excluding gauge field contribution depends on the variation of the metric function δU only. We point out that a similar result is derived in pure Einsteinian theory, for instance, for the Schwarzschild solution. The gauge fields give an independent contribution and it takes the form

$$(\delta Q_\xi - i_\xi\Theta)_{gf} = \frac{2r^{n-1}}{\sqrt{UW}}\mathcal{A}_0\left(\left(\frac{\delta U}{U}+\frac{\delta W}{W}\right)\mathcal{A}_0'-2\delta\mathcal{A}_0'\right)\hat{\Omega}_{n-1}, \tag{54}$$

where $\mathcal{A}_0$ is the time component of the electromagnetic field potential and $\mathcal{A}_0' = \mathcal{F}_{rt}$ is its radial derivative (electric field). We would like to stress that the Yang-Mills field does not give any contribution to the difference of variations due to the fact that the constant $\bar{q}$ associated with Yang-Mills coupling is held fixed. The total variation is the sum of both of the above written variations,

$$(\delta Q_\xi - i_\xi\Theta)_{\text{tot}} = r^{n-2}\left(-(n-1)\delta U + \frac{2r}{\sqrt{UW}}\mathcal{A}_0 \times\left(\left(\frac{\delta U}{U}+\frac{\delta W}{W}\right)\mathcal{A}_0'-2\delta\mathcal{A}_0'\right)\right)\hat{\Omega}_{n-1}. \tag{55}$$

For convenience we assume that the electric potential is equal to zero at the event horizon $\mathcal{A}_0|_{r_+} = 0$. Taking this condition into account and performing integration over an $n-1$-dimensional hypersphere of the radius r_+, we obtain the explicit relation for the variation of the Hamiltonian $\mathcal{H}_{r_+}$ at the horizon,

$$\delta\mathcal{H}_{r_+} = (n-1)\omega_{n-1}r_+^{n-2}U'(r_+)\delta r_+, \tag{56}$$

where $\omega_{n-1} = 2\pi^{(n-1)/2}/\Gamma((n-1)/2)$ is the surface of a unit $n-1$-dimensional hypersphere. Variation of the Hamiltonian $\delta\mathcal{H}_\infty$ takes the form as follows:

$$\delta\mathcal{H}_\infty = (n-1)\omega_{n-1}\delta\mu - 4\omega_{n-1}\mathcal{A}_0\delta q. \tag{57}$$

Since, as pointed out above, the variation of the Hamiltonian at the horizon and at the infinity are equal, therefore we obtain

$$(n-1)\omega_{n-1}r_+^{n-2}U'(r_+)\delta r_+ = (n-1)\omega_{n-1}\delta\mu - 4\omega_{n-1}\mathcal{A}_0\delta q. \tag{58}$$

Finally, to derive the first law of black hole thermodynamics, it is necessary to find the relations between the variations of observable entities such us mass or charge of the black hole and corresponding variations in the given above relation.

The electric charge is defined in the standard way, namely, we use the Gauss law and obtain

$$Q_e = \frac{1}{4\pi}\int_{\Sigma_\infty} *F = \frac{\omega_{n-1}}{4\pi}q. \tag{59}$$

The electric potential measured at the infinity with respect to the horizon is defined as follows:

$$\Phi_e = \mathcal{A}_\mu\xi^\mu|_\infty - \mathcal{A}_\mu\xi^\mu|_{r_+} = \mathcal{A}_0. \tag{60}$$

We point out that the time translation vector $\xi^\mu = \partial/\partial t$ is used here to calculate the electric potential. The black hole's mass can be defined as

$$M = \frac{(n-1)\omega_{n-1}}{16\pi}\mu. \tag{61}$$

Variation of the mass (61) together with the relations (59) and (60) allow us to rewrite the right-hand side of Eq. (58) in the form of a typical thermodynamic relation. On the left-hand side of that relation, we can use the relation for the temperature (38) in order to avoid introducing additional scalar charges and its corresponding conjugate value if the physical meaning of both these values is not clarified. Then the entropy of the black hole can be defined in a typical manner, namely,

$$S = \frac{\omega_{n-1}}{4}r_+^{n-1}. \tag{62}$$

Therefore, the entropy is equal to a quarter of the black hole horizon area, similar to as it takes place in general relativity (GR). Finally, the first law of black hole thermodynamics can be written in the form

$$\delta M = T\delta S + \Phi_e\delta Q_e. \tag{63}$$

The obtained relation is completely of the same form as for the Reissner-Nordström black hole in the framework of GR, even though the explicit relation for the temperature (38) differs from its general relativistic cousin. The fact that the thermodynamic relations like the first law are the same in different theories may be an additional confirmation of universality of black hole thermodynamics, which at least for some cases are insensitive to the underlying theories that allow one to obtain corresponding thermodynamic relations.

We would also like to stress that, even from a naive thermodynamic point of view, the temperature T (38) satisfies a simple consistency relation, which follows directly from the first law (63), namely, $\frac{\partial T}{\partial Q_e} = \frac{\partial \Phi_e}{\partial S}$, whereas the temperature T (36) does not. To obtain the consistency relation for the temperature (36), an additional scalar charge was introduced [29], which was used in earlier paper [33,35], but its physical meaning is not clear. Moreover, in the framework of the standard thermodynamics, there are only two variable macroscopic parameters of the black hole, namely, its mass or, directly related to it, the radius of the event horizon r_+ and the electric charge q (or Q_e). Any additional independent thermodynamic variable should be related to an independent macroscopic parameter (integration constant), but there are not any more independent macroscopic values in the standard framework. Thus, the "scalar charge" considered in the earlier paper was introduced just to have consistent thermodynamics relations, but its physical meaning remains obscure.

Heat capacity or specific heat is an important notion to analyze thermal stability, particularly as it is widely used in black hole thermodynamics. Thermally stable systems are characterized by positive specific heat and, if the specific heat turns out to be negative, the system tends to decay. To obtain the heat capacity, we use the standard definition for the latter and write

$$C_Q = T\left(\frac{\partial S}{\partial T}\right)_Q = T\frac{\partial S}{\partial r_+}\left(\frac{\partial r_+}{\partial T}\right)_Q = \frac{(n-1)\omega_{n-1}}{4} r_+^{n-2}\left(\left(\frac{\alpha}{\eta}-\Lambda\right)r_+^2 + (n-1)(n-2) - \frac{q^2}{r_+^{2(n-2)}} - \frac{(n-1)(n-2)\bar{q}^2}{2r_+^2}\right)\left[-\frac{3r_+^2+d^2}{r_+(r_+^2+d^2)}\left(\left(\frac{\alpha}{\eta}-\Lambda\right)r_+^2 + (n-1)(n-2) - \frac{q^2}{r_+^{2(n-2)}} - \frac{(n-1)(n-2)\bar{q}^2}{2r_+^2}\right) + 4\left(\left(\frac{\alpha}{\eta}-\Lambda\right)r_+ + \frac{(n-2)q^2}{r_+^{2n-3}} + \frac{(n-1)(n-2)\bar{q}^2}{2r_+^3}\right)\right]^{-1}. \tag{64}$$

The obtained relation (64) has a relatively more cumbersome structure in comparison to the expression for the temperature (38), but since the derivative of the temperature T over the horizon radius r_+ makes contribution to the heat capacity, some important conclusions about the behavior of the latter can be derived immediately knowing the peculiar features of the temperature. Namely, since the temperature may in general have three extrema points, it means that the heat capacity as a function of r_+ may have three discontinuity points, separating stable and unstable domains. We point out here that since for relatively large r_+ the temperature shows rising character for any variation of black hole parameters, at least in the observed domain, therefore we can conclude that the specific heat C_Q is positive and the black hole is thermally stable. For smaller radii of the horizon, the sign of C_Q and consequent conclusion about thermal stability or instability substantially depend on the chosen values of black hole parameters and the parameters of the Lagrangian. To make the behavior of the function $C_Q = C_Q(r_+)$ more transparent, we give corresponding graphs, showing its behavior near discontinuity points and how it is affected by variations of certain parameters, namely, its electric charge q and the cosmological constant Λ.

Figure 4 shows the rightmost discontinuity point for two values of the electric charge. As it was noted above, the heat capacity C_Q to the right of the discontinuity point is positive and it goes up if the horizon radius r_+ increases. This feature is typical for most types of black holes with AdS asymptotic. To the left of the asymptotes, the heat capacity becomes negative, therefore this range of r_+ is a domain of instability. We also point out that for smaller radius r_+ there is a second discontinuity point that is reflected by very fast decrease of the heat capacity C_Q if the radius of the horizon goes down. We also conclude that discontinuity points become closer if the charge q goes up, and further increase of the charge gives rise to merging of the singularity points and consecutive shrinkage of the unstable domain, at least for the considered range of the

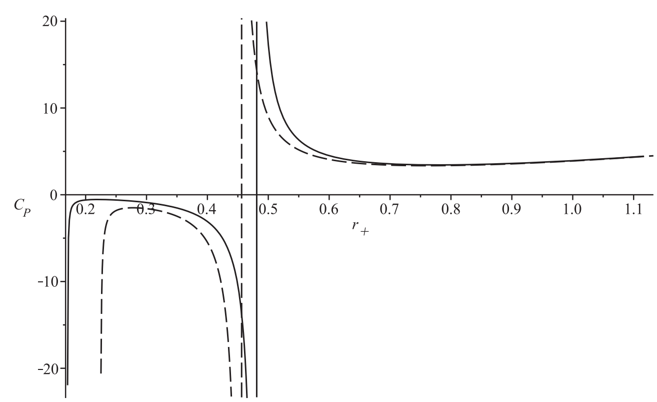

FIG. 4. Heat capacity C_Q as a function of horizon radius r_+ for two values of the electric charge q whenever the other parameters are held fixed. The solid and dashed curves correspond to $q = 0.1$ and $q = 0.125$, respectively. The fixed parameters are as follows: $n = 3$, $\alpha = 0.2$, $\eta = 0.3$, $\Lambda = -3$, and $\bar{q} = 0.01$.

parameters. A similar conclusion can be made if the absolute value of Λ goes up. Then the peculiarity of the heat capacity diminishes, what is shown in Fig. 5; namely, the height of the peak drops down and finally vanishes if the absolute value of the cosmological constant Λ rises.

We also point out that the heat capacity C_Q (64) within the extended thermodynamics approach can be treated as the heat capacity under constant pressure C_P, and the pressure is introduced below. It is valid since all the parameters are held fixed in the relation (64).

V. EXTENDED THERMODYNAMICS

The so-called extended thermodynamics has attracted considerable attention for more than a decade [63–68]. Even though some basic assumptions for the extended thermodynamics are still disputed, this approach gives rise to wider thermodynamic phase space, allowing us to describe richer thermodynamics and establish at least formal, but deeper ties with the thermodynamics of various systems usually considered in condensed matter physics. In particular, it establishes profound relations between phase transition phenomena of condensed matter systems and phase transitions (transformations) in black hole physics. The key assumption of the extended approach is the fact that the cosmological constant is considered to be a thermodynamic value. Namely, the cosmological constant Λ was identified with thermodynamic pressure,

$$P = -\frac{\Lambda}{8\pi}. \tag{65}$$

It should be pointed out that there is some analogy to ideal fluid, where a corresponding term related to the thermodynamic pressure goes along with the metric tensor in the energy-momentum tensor of the fluid, but it will not be discussed in the current work. The introduced thermodynamic pressure (65) gives rise to the consequence that the black hole mass should be identified now with the enthalpy $M = H$ [66], but not with the internal energy as it was in the standard thermodynamics. Having the pressure P (65) the corresponding conjugate thermodynamic volume V can be defined as follows:

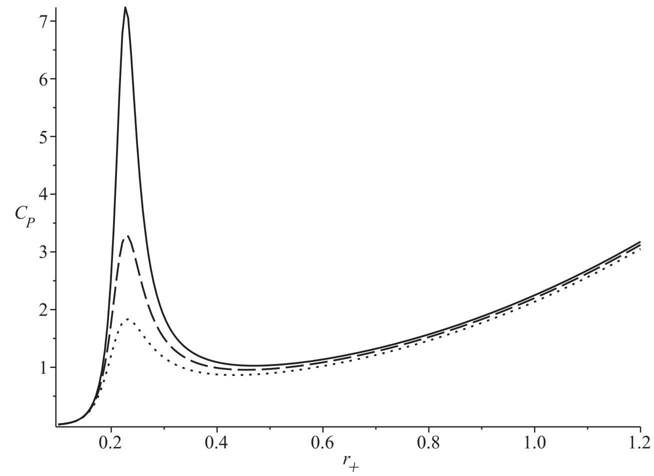

FIG. 5. Dissolution of the peak for the heat capacity C_Q for large absolute values of Λ. From higher to lower, the peaks correspond to the increase of the absolute value of the cosmological constant.

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,Q_e}. \tag{66}$$

The explicit relation for the thermodynamic volume depends on the parity of dimension n, similar to as it is for the metric function $U(r)$ (25). Namely, for odd n the explicit expression for the thermodynamic volume V is as follows:

$$\begin{aligned} V = \omega_{n-1}\Bigg(\frac{r_+^n}{n} - \frac{\eta}{4\alpha}\Bigg(2\left(\frac{\alpha}{\eta}+\Lambda\right)\Bigg[\sum_{j=0}^{\frac{n-1}{2}}(-1)^j d^{2j}\frac{r_+^{n-2j}}{n-2j} + (-1)^{\frac{n+1}{2}}d^n \arctan\left(\frac{r_+}{d}\right)\Bigg] + 2q^2 \\ \times \Bigg[\sum_{j=0}^{\frac{n-5}{2}}(-1)^j d^{2j}\frac{r_+^{4+2j-n}}{(4+2j-n)d^{2(j+1)}} + \frac{(-1)^{\frac{n-3}{2}}}{d^{n-2}}\arctan\left(\frac{r_+}{d}\right)\Bigg] + (n-1)(n-2)\bar{q}^2 \\ \times \Bigg[\sum_{j=0}^{\frac{n-5}{2}}(-1)^j d^{2j}\frac{r_+^{n-2j-4}}{n-2j-4} + (-1)^{\frac{n-3}{2}}d^{n-4}\arctan\left(\frac{r_+}{d}\right)\Bigg]\Bigg)\Bigg). \end{aligned} \tag{67}$$

The explicit expression for the thermodynamic volume can be written similarly for even n. The obtained relation (67) is in agreement with a respective relation obtained in [33] for corresponding limits in both cases. Since $n = 3$ is of special interest, we also write the thermodynamic volume for this case,

$$V = 4\pi\left(\frac{r_+^3}{3} - \frac{\eta}{2\alpha}\left[\left(\frac{\alpha}{\eta}+\Lambda\right)\left(\frac{1}{3}r_+^3 - d^2 r_+\right) + \left(d^3\left(\frac{\alpha}{\eta}+\Lambda\right) + \frac{1}{d}(q^2+\bar{q}^2)\right)\arctan\left(\frac{r_+}{d}\right)\right]\right). \tag{68}$$

To derive the Smarr relation for the black hole, we introduce an additional intensive thermodynamic variable, which in some sense is similar to the pressure (65) introduced above. The new variable and its conjugate are defined as follows:

$$\Pi = \frac{\alpha}{8\pi\eta}, \qquad \Psi = \left(\frac{\partial M}{\partial \Pi}\right)_{S,Q_e,P}. \tag{69}$$

Taking corresponding derivatives, we write the explicit relation for the extensive conjugate value Ψ. Namely, for odd n ($n < 7$) we obtain

$$\begin{aligned}\Psi = \omega_{n-1}\Bigg(&\frac{\eta}{2\alpha}\Bigg[\left(\frac{\alpha}{\eta}+L\right)\left(\sum_{j=0}^{\frac{n-1}{2}}(-1)^j d^{2j}\frac{r_+^{n-2j}}{n-2j}+(-1)^{\frac{n-1}{2}}d^n\arctan\left(\frac{r_+}{d}\right)\right)+(n-1)(n-2)\frac{\bar{q}^2}{2}\\
&\times\left(\sum_{j=0}^{\frac{n-5}{2}}(-1)^j d^{2j}\frac{r_+^{n-2j-4}}{n-2j-4}+(-1)^{\frac{n-3}{2}}d^{n-4}\arctan\left(\frac{r_+}{d}\right)\right)+q^2\left(\sum_{j=0}^{\frac{n-5}{2}}(-1)^j\frac{r_+^{4+2j-n}}{(4+2j-n)d^{2(j+1)}}\right.\\
&\left.+\frac{(-1)^{\frac{n-3}{2}}}{d^{n-2}}\arctan\left(\frac{r_+}{d}\right)\right)\Bigg]+\frac{\eta^2 r_+^{n-2}}{8\alpha^2(r_+^2+d^2)}\left(\left(\frac{\alpha}{\eta}+\Lambda\right)r_+^2+\frac{q^2}{r_+^{2(n-2)}}+\frac{(n-1)(n-2)}{2}\frac{\bar{q}^2}{r_+^2}\right)^2-\frac{\eta^2}{8\alpha^2}\\
&\times\Bigg[(n+2)\left(\frac{\alpha}{\eta}+\Lambda\right)^2\left(\sum_{j=0}^{\frac{n-1}{2}}(-1)^j d^{2j}\frac{r_+^{n-2j}}{n-2j}+(-1)^{\frac{n-1}{2}}d^n\arctan\left(\frac{r_+}{d}\right)\right)+2(4-n)\left(\frac{\alpha}{\eta}+\Lambda\right)q^2\\
&\times\left(\sum_{j=0}^{\frac{n-5}{2}}(-1)^j\frac{r_+^{4+2j-n}}{(4+2j-n)d^{2(j+1)}}+\frac{(-1)^{\frac{n-3}{2}}}{d^{n-2}}\arctan\left(\frac{r_+}{d}\right)\right)+3(2-n)q^4\left(\sum_{j=0}^{\frac{3n-7}{2}}(-1)^j\frac{r_+^{6+2j-3n}}{(6+2j-3n)d^{2(j+1)}}\right.\\
&\left.+\frac{(-1)^{\frac{3n-5}{2}}}{d^{3n-4}}\arctan\left(\frac{r_+}{d}\right)\right)+(n-1)(n-2)^2\left(\frac{\alpha}{\eta}+\Lambda\right)\bar{q}^2\left(\sum_{j=0}^{\frac{n-5}{2}}(-1)^j d^{2j}\frac{r_+^{n-2j-4}}{n-2j-4}+(-1)^{\frac{n-3}{2}}d^{n-4}\right.\\
&\left.\times\arctan\left(\frac{r_+}{d}\right)\right)-n(n-1)(n-2)q^2\bar{q}^2\left(\sum_{j=0}^{\frac{n-1}{2}}(-1)^j\frac{r_+^{2j-n}}{(2j-n)d^{2(j+1)}}+\frac{(-1)^{\frac{n+1}{2}}}{d^{n+2}}\arctan\left(\frac{r_+}{d}\right)\right)\\
&+\frac{1}{4}(n-1)^2(n-2)^2(n-6)\bar{q}^4\left(\sum_{j=0}^{\frac{5-n}{2}}(-1)^j\frac{r_+^{n+2j-6}}{(n+2j-6)d^{2(j+1)}}+\frac{(-1)^{\frac{7-n}{2}}}{d^{8-n}}\arctan\left(\frac{r_+}{d}\right)\right)\Bigg]\Bigg).\end{aligned} \tag{70}$$

For the dimensions $n \geq 7$ there is a different contribution in the bottom line of the above relation. It follows from the corresponding term in the metric function $U(r)$. The explicit expression for Ψ if n is even can be derived similarly. We also write the thermodynamic function Ψ for the $n = 3$ case,

$$\begin{aligned}\Psi = 4\pi\Bigg(&\frac{\eta}{4\alpha d}\left(1-\frac{\eta\Lambda}{\alpha}\right)(q^2+\bar{q}^2)\arctan\left(\frac{r_+}{d}\right)-\frac{1}{8}\left(1+\frac{\eta\Lambda}{\alpha}\right)\left(1+5\frac{\eta\Lambda}{\alpha}\right)\left(\frac{r_+^3}{3}-d^2 r_+ + d^3\arctan\left(\frac{r_+}{d}\right)\right)\\
&-\frac{3\eta^2}{8\alpha^2 d^2}(q^2+\bar{q}^2)^2\left(\frac{1}{3r_+^3}-\frac{1}{d^2 r_+}-\frac{1}{d^3}\arctan\left(\frac{r_+}{d}\right)\right)+\frac{\eta^2 r_+}{8\alpha^2(r_+^2+d^2)}\left(\left(\frac{\alpha}{\eta}+\Lambda\right)r_+^2+\frac{q^2+\bar{q}^2}{r_+^2}\right)^2\Bigg).\end{aligned} \tag{71}$$

Since a non-Abelian field is also included into the action, it gives a contribution into the metric function $U(r)$ and all the derived quantities, so we assume that the non-Abelian parameter $\bar{q}$ can be varied as well. We introduce a non-Abelian charge similarly as it was defined, for instance, in [50],

$$Q_n = \frac{1}{4\pi\sqrt{(n-1)(n-2)}}\int_{\Sigma_{n-1}} d^{n-1}\chi J(\chi)\sqrt{\mathrm{Tr}(F^{(a)}_{\mu\nu}F^{(a)\mu\nu})} = \frac{\omega_{n-1}}{4\pi}\bar{q}. \tag{72}$$

The integral in upper relation is taken over a sphere enclosing the black hole and $J(\chi)$ denotes the Jacobian for the chosen spherical coordinates. The Yang-Mills charge Q_n (magnetic) now can be considered as a thermodynamic value similar to the electric charge of the Maxwell field. Therefore, a thermodynamic conjugate value to the charge Q_n can be introduced,

$$U = \left(\frac{\partial M}{\partial Q_n}\right)_{S,Q_e,P,\Pi}. \quad (73)$$

We do not give the explicit expression for the potential U, but it can be obtained easily. Having introduced additional thermodynamic variables such as P, Π, Q_n and their thermodynamic conjugates, we are able to write the so-called extended first law, which takes the form

$$\delta M = T\delta S + \Phi_e \delta Q_e + V\delta P + \Psi\delta\Pi + U\delta Q_n. \quad (74)$$

Taking into account the pairs of conjugate variables, we also write the Smarr relation,

$$(n-2)M = (n-1)TS - 2VP - 2\Psi\Pi + (2-n)\Phi_e Q_e + UQ_n. \quad (75)$$

If the non-Abelian field is set to zero ($\bar{q}=0$) the obtained relation is reduced to the corresponding equation derived for the electrically charged black hole in Horndeski gravity [33]. If we compare with the general relativistic case, the Smarr relation (75) and the generalized first law (74) gain only one additional term caused by the thermodynamic variable Π and its conjugate value Ψ. The latter two relations may be considered as an additional argument in favor of universality of black hole thermodynamics, which allows us to write the fundamental thermodynamic relations that take the same or at least very similar form for various underlying theories of gravity.

VI. GIBBS FREE ENERGY

If a thermodynamic system undergoes phase transitions, the Gibbs free energy is more convenient than the enthalpy identified with the black hole's mass M. The Gibbs free energy is defined as follows:

$$G = M - T_{\rm BH}S. \quad (76)$$

The explicit relation for the Gibbs free energy for odd n ($n<7$) takes the form

$$\begin{aligned} G = \frac{\omega_{n-1}}{16\pi}\Bigg(& r_+^{n-2} + \frac{2\Lambda}{n(n-1)}r_+^n + \frac{2(2n-3)}{(n-1)(n-2)}q^2 r_+^{2-n} - \frac{3(n-2)}{n-4}\bar{q}^2 r_+^{n-4} - \frac{\eta r_+^{n-2}}{2\alpha(n-1)(r^2+d^2)}\Bigg(\left(\frac{\alpha}{\eta}+\Lambda\right) \\ & \times r_+^2 + \frac{q^2}{r_+^{2(n-2)}} + \frac{(n-1)(n-2)\bar{q}^2}{2r_+^2}\Bigg)^2 + \frac{\eta}{2\alpha}\Bigg[\left(\frac{\alpha}{\eta}+\Lambda\right)^2\left(\sum_{j=0}^{\frac{n-1}{2}}(-1)^j d^{2j}\frac{r_+^{n-2j}}{n-2j} + (-1)^{\frac{n-1}{2}}d^n \arctan\left(\frac{r_+}{d}\right)\right) \\ & + 2\left(\frac{\alpha}{\eta}+\Lambda\right)q^2\left(\sum_{j=0}^{\frac{n-5}{2}}(-1)^j\frac{r_+^{4+2j-n}}{(4+2j-n)d^{2(j+1)}} + \frac{(-1)^{\frac{n-3}{2}}}{d^{n-2}}\arctan\left(\frac{r_+}{d}\right)\right) + q^4\left(\sum_{j=0}^{\frac{3n-7}{2}}(-1)^j\frac{r_+^{6+2j-3n}}{(6+2j-3n)d^{2(j+1)}}\right. \\ & \left. + \frac{(-1)^{\frac{3n-5}{2}}}{d^{3n-4}}\arctan\left(\frac{r_+}{d}\right)\right) + (n-1)(n-2)\bar{q}^2\Bigg(\left(\frac{\alpha}{\eta}+\Lambda\right)\left(\sum_{j=0}^{\frac{n-5}{2}}(-1)^j d^{2j}\frac{r_+^{n-2j-4}}{n-2j-4}\right. \\ & \left. + (-1)^{\frac{n-3}{2}}d^{n-4}\arctan\left(\frac{r_+}{d}\right)\right) + q^2\left(\sum_{j=0}^{\frac{n-1}{2}}(-1)^j\frac{r_+^{2j-n}}{(2j-n)d^{2(j+1)}} + \frac{(-1)^{\frac{n+1}{2}}}{d^{n+2}}\arctan\left(\frac{r_+}{d}\right)\right) \\ & + \frac{(n-1)(n-2)}{4}\bar{q}^2\left(\sum_{j=0}^{\frac{5-n}{2}}(-1)^j\frac{r_+^{n+2j-6}}{(n+2j-6)d^{2(j+1)}} + \frac{(-1)^{\frac{7-n}{2}}}{d^{8-n}}\arctan\left(\frac{r_+}{d}\right)\right)\Bigg)\Bigg]\Bigg). \end{aligned} \quad (77)$$

Similar to above, we give the explicit relation for $n=3$ because of a special interest in this case,

$$\begin{aligned} G = \frac{1}{4}\Bigg(& r_+ + \frac{\Lambda}{3}r_+^3 + 3\frac{q^2+\bar{q}^2}{r_+} + \frac{\eta}{2\alpha}\Bigg[\left(\frac{\alpha}{\eta}+\Lambda\right)^2\left(\frac{r_+^3}{3} - r_+ d^2\right) + \frac{(q^2+\bar{q}^2)^2}{r_+ d^2}\left(\frac{1}{d^2} - \frac{1}{3r_+^2}\right) \\ & + \frac{1}{d}\left(\left(\frac{\alpha}{\eta}+\Lambda\right)d^2 + \frac{q^2+\bar{q}^2}{d^2}\right)^2\arctan\left(\frac{r_+}{d}\right) - \frac{r_+}{2(r_+^2+d^2)}\left(\left(\frac{\alpha}{\eta}+\Lambda\right)r_+^2 + \frac{q^2+\bar{q}^2}{r_+^2}\right)^2\Bigg]\Bigg). \end{aligned} \quad (78)$$

Since the Gibbs free energy G (77) and its particular case (78) have rather intricate forms and their temperature dependences are given implicitly, it is difficult to analyze their behavior. To understand it better, we give a corresponding graph, which shows the dependence $G = G(T)$, while the pressure and all the other parameters are fixed. Namely, Fig. 6 shows that for smaller pressure P the Gibbs free energy has swallowtail behavior and it gives rise to the conclusion that there is a phase transition of the first order. From the qualitative point of view, the behavior of the Gibbs free energy is the same as for the

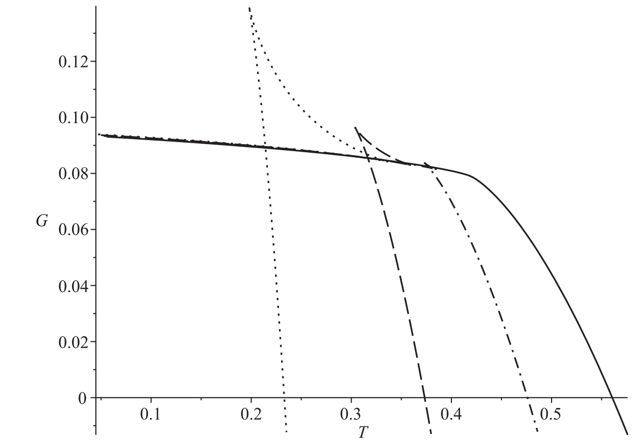

FIG. 6. Gibbs free energy G as a function of temperature T for various values of pressure P or cosmological constant Λ, while other parameters are held fixed. The dotted, dashed, dash-dotted, and solid lines correspond to $\Lambda = -1.5$, $\Lambda = -3.5$, $\Lambda = -5.5$, and $\Lambda = -7.5$, respectively. The fixed parameters are as follows: $n = 3$, $q = 0.1$, $\bar{q} = 0.1$, $\alpha = 0.2$, and $\eta = 0.3$.

Reissner-Nordström-AdS black hole in general relativity [66]. The Gibbs free energy in Horndeski gravity for a nonlinearly charged black hole was also examined in our earlier paper [33] and, again, from the qualitative point of view there is complete agreement between current and earlier results. If the pressure goes up, the swallowtail gradually diminishes and after reaching of a critical value it completely vanishes. The Gibbs free energy turns out to be a smooth function of the temperature T and it also means the disappearance of the phase transition. The critical point when the behavior of the Gibbs free energy becomes smooth is supposed to be a point of the second-order phase transition, which is usually takes place for a Van der Waals system or Reissner-Nordström-AdS black hole [66]. Because of the interest in the critical point and near critical behavior, some aspects of this issue will be examined in the following section. For better illustration of the swallowtail behavior and its gradual diminishing with increase of the pressure, we add the 3D figure for the Gibbs free energy (Fig. 7).

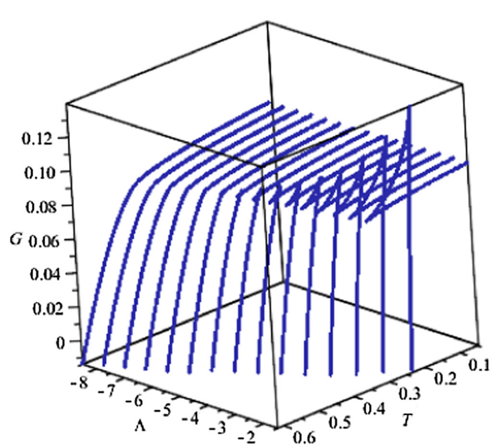

FIG. 7. Gibbs free energy G as a function of temperature T and pressure P (or the cosmological constant Λ).

VII. CRITICAL BEHAVIOR IN THE EXTENDED PHASE SPACE

Since additional thermodynamic variables are defined, we are able to extend corresponding thermodynamic phase space for the system and consequently derive and examine richer thermal behavior of the black hole. One of the key relations in thermodynamics of any conventional system is its thermal equation of state, which establishes a relation between its macroscopic values such as temperature T, pressure P, and volume V. Having defined the pressure P (65) and using the relation for the temperature (38), we can rewrite the latter relation in a form of the thermal equation of state, namely, we write

$$P = \frac{1}{8\pi}\left(\frac{q^2}{r_+^{2(n-1)}} + \frac{(n-1)(n-2)\bar{q}^2}{2r_+^4} - \frac{(n-1)(n-2)}{r_+^2} - \xi\right) \pm \frac{1}{4\pi r_+^2}\sqrt{2(n-1)\pi\xi r_+(r_+^2+d^2)T}, \tag{79}$$

where for convenience we denote $\xi = \alpha/\eta$, which is directly related to the above introduced thermodynamic value Π (69). We also point out that to obtain the expression (79) we extract the cosmological constant Λ from the relation (38), solving the corresponding quadratic equation for the parameter Λ; therefore, the sign $\pm$ appears in the upper relation. To have the pressure P positive in all the range of variation, we pick up the sign $+$ only and consider it in the following relations. We also point out that, instead of thermodynamic volume (66), we still keep the horizon radius r_+, partially because of the complexity of the relation (66) which does not allow us to express r_+ as an explicit function of V. On the other hand, it does not change or modify conclusions about the critical behavior that we are to derive. In addition, we remark that Eq. (79) being completely "geometrical" in nature can be rewritten in terms of "physical" variables in a similar fashion as it was done in [66,67], but such a redefinition of thermodynamic values does not affect any physical conclusions at all. We also point out that some hints about possible critical behavior of a nonlinearly charged back hole were obtained in our earlier paper [33], and a more detailed consideration of criticality issues were made in [69].

Following the key assumption that the equation of state for black holes (79) is analogous to the Van der Waals equation of state, far reaching consequences can be derived. In particular, critical behavior can be studied and one of the most important issues here is a phase transition between the so-called large and small black holes. The central notion here is the so-called inflection point, defined as follows:

$$\left(\frac{\partial P}{\partial r_+}\right)_T = 0, \qquad \left(\frac{\partial^2 P}{\partial r_+^2}\right)_T = 0. \tag{80}$$

It is worth noting that if we use the volume V (66), to find the inflection point the derivatives with respect to the volume V should be equated to zero, but using the relation $\frac{\partial P}{\partial V} = \frac{\partial P}{\partial r_+}\frac{\partial r_+}{\partial V}$, and assuming that the derivative $\frac{\partial V}{\partial r_+} \neq 0$ since the volume is supposed to be a monotonous function of r_+, we again arrive at the relations (80). We also point out that other thermodynamic parameters we used in the extended description are held fixed. The relation for critical radius can be derived straightforwardly using the relations (80), namely, after simple calculations we write

$$-3(n-2) + \frac{(2n-1)q^2}{r_c^{2(n-2)}} + \frac{5(n-2)\bar{q}^2}{r_c^2} + \frac{3r_c^4 + 22d^2r_c^2 + 15d^4}{2(r_c^2+d^2)(r_c^2+3d^2)}\left(n-2-\frac{q^2}{r_c^{2(n-2)}} - \frac{(n-2)\bar{q}^2}{r_c^2}\right) = 0, \tag{81}$$

where r_c is the critical radius r_c. The critical temperature T_c and pressure P_c can be written as functions of the critical radius r_c,

$$T_c = \frac{2(n-1)(r_c^2+d^2)}{\pi\xi r_c(r_c^2+3d^2)^2}\left(n-2-\frac{q^2}{r_c^{2(n-2)}} - \frac{(n-2)\bar{q}^2}{r_c^2}\right)^2; \tag{82}$$

$$P_c = \frac{1}{8\pi}\left(\frac{q^2}{r_c^{2(n-1)}} + \frac{(n-1)(n-2)\bar{q}^2}{2r_c^4} - \frac{(n-1)(n-2)}{r_c^2} - \xi + \frac{4(n-1)(r_c^2+d^2)}{r_c^2(r_c^2+3d^2)}\left(n-2-\frac{q^2}{r_c^{2(n-2)}} - \frac{(n-2)\bar{q}^2}{r_c^2}\right)\right). \tag{83}$$

The equation for the critical horizon radius r_c (81) does not have an exact analytical solution for general dimension n and arbitrary chosen parameters q, $\bar{q}$, and d. Therefore, the critical values such as T_c and P_c cannot be given as explicit functions of the mentioned parameters of the black hole in the general case, as it takes for the Van der Waals gas or even simpler black hole solutions such as, for instance, the Reissner-Nordström-AdS one [66]. In general the critical values can be calculated numerically for arbitrary values of n, ξ and black hole charges q and $\bar{q}$. It should be pointed out that for some particular cases analytical solutions can be, in principle, obtained. Because of some interest in the analytical solution and taking into account the fact that analytical solutions often are easier to analyze, we note several particular cases where at least it is possible to derive an analytical solution for the critical radius r_c and, consequently, to other two critical values T_c and P_c. First of all, if $n = 3$ Eq. (81) takes the form

$$-3 + \frac{5(q^2+\bar{q}^2)}{r_c^2} + \frac{3r_c^4 + 22d^2r_c^2 + 15d^4}{2(r_c^2+d^2)(r_c^2+3d^2)}\left(1 - \frac{q^2+\bar{q}^2}{r_c^2}\right) = 0. \tag{84}$$

The latter equation can be rewritten in a form of a cubic equation for the square of the critical radius r_c^2. A similar equation can be written if the electric charge $q = 0$, but in this case for any n, the only difference with Eq. (84) is hidden in the parameter d, which is dimension dependent.

Another interesting particular case is $\alpha = 0$ and it is easy to verify that equation for the critical radius r_c (81) reduces to the form

$$\frac{(4n-7)q^2}{r_c^{2(n-2)}} + 5\frac{(n-2)\bar{q}^2}{r_c^2} + 2 - n = 0. \tag{85}$$

Corresponding relations for the critical temperature T_c and the pressure P_c can be rewritten as follows:

$$T_c = \frac{4(n-2)}{9\pi r_c}\left(1 - \frac{\bar{q}^2}{r_c^2} - \frac{q^2}{(n-2)r_c^{2(n-2)}}\right)^2, \tag{86}$$

$$P_c = \frac{(n-1)(n-2)}{24\pi r_c^2}\left(1 - \frac{5\bar{q}^2}{2r_c^2} - \frac{(4n-7)}{(n-1)(n-2)}\frac{q^2}{r_c^{2(n-2)}}\right). \tag{87}$$

If $n = 3$ Eq. (85) turns to be a quadratic one and the critical radius can be easily written as

$$r_c^2 = 5(q^2+\bar{q}^2). \tag{88}$$

Substituting the critical radius into the upper relations (86) and (87), we obtain corresponding critical values T_c and P_c. After the computation, we write the explicit expression for the so-called critical ratio,

$$\rho_c \equiv \frac{P_c r_c}{T_c} = \frac{75}{512}. \tag{89}$$

Thus, the critical ratio r_c as it is expected is a dimensionless number that does not depend on the parameters of the solution such as its charges q, $\bar{q}$, and this conclusion is in perfect agreement with the definition of the critical ratio for conventional systems as well as in within the extended phase space thermodynamics for black holes. On the other hand, it is known that for the standard Van der Waals system and the Reissner-Nordström-AdS black hole the critical ratio is $\rho_c = 3/8$ and, as we see in our case, it is considerably smaller. We also note that exact analytical solutions of Eq. (85) can be also derived for $n = 4$ and $n = 5$, where Eq. (85) for r_c^2 turns out to be quadratic and cubic, respectively, but here we do not give explicit relations for corresponding values.

Another important particular case of Eq. (85) is related to the situation when one of the charges is set to zero. Namely, if $q = 0$, then the square of the critical radius for any dimension is

$$r_c^2 = 5\bar{q}^2. \tag{90}$$

Using this result, we write the critical ratio ρ_c for this particular case,

$$\rho_c = \frac{75(n-1)}{1024}. \tag{91}$$

The obtained relation is in perfect agreement with the relation (89) if $n = 3$. Finally, we assume that $\bar{q} = 0$, then Eq. (85) immediately gives us

$$r_c^{2(n-2)} = \frac{(4n-7)}{(n-2)} q^2. \tag{92}$$

The latter expression gives rise to the following critical ratio:

$$\rho_c = \frac{3(4n-7)^2}{512(n-2)}. \tag{93}$$

Similar to the upper case, there is perfect agreement with the ratio ρ_c (89) if $n = 3$, but in contrast to the upper case its dimension dependence is different. The latter relation also shows that, for higher-dimensional cases, at least when n is not too high the critical ratio (93) is also smaller than the corresponding ratio for higher-dimensional generalization of the Reissner-Nordström-AdS black hole, which equals $\rho_c = (2n-3)/(4(n-1))$.

If a thermodynamic system undergoes a second-order phase transition, there are universal parameters, namely, the critical exponents that characterize behavior of certain thermodynamic values near the critical point and do not depend on the parameters of the system [70]. To obtain the critical exponents it is useful to introduce the so-called reduced variables, which show how close to the critical point the system is,

$$t = \frac{T}{T_c} - 1, \qquad \omega = \frac{r_+}{r_c} - 1. \tag{94}$$

Now the critical exponents $\bar{\alpha}$, β,γ, and δ are defined as follows:

$$C_V \sim |t|^{-\bar{\alpha}}, \qquad \Delta V_{ls} \sim |t|^{\beta}, \qquad \kappa_T \sim t^{\gamma}, \qquad P - P_c \sim |\omega|^{\delta}. \tag{95}$$

Here we point out that C_V is the heat capacity under constant volume, ΔV_{ls} is the volume difference for large and small phases, and κ_T is the isothermal compressibility. We also note that instead of the commonly used notation α for the first of the critical exponents we use $\bar{\alpha}$, because the symbol α is used to denote one of the coupling constants.

It follows from the definition of the entropy S (62) that the heat capacity under fixed volume exactly equals zero: $C_V = T(\partial S/\partial T)_V = 0$, therefore we immediately conclude that the critical exponent $\bar{\alpha} = 0$. To derive the other critical exponents, we rewrite the equation of state (79) near the critical point in the following form:

$$P = P_c + At + Bt\omega + C\omega^3 + Dt^2 + \cdots, \tag{96}$$

where

$$A = T_c \left(\frac{\partial P}{\partial T}\right)_{r_+}\bigg|_{r_c}, \qquad B = r_c T_c \left(\frac{\partial^2 P}{\partial T \partial r_+}\right)\bigg|_{r_c},$$
$$C = \frac{r_c^3}{6}\left(\frac{\partial^3 P}{\partial r_+^3}\right)_T\bigg|_{r_c}, \qquad D = \frac{T_c^2}{2}\left(\frac{\partial^2 P}{\partial T^2}\right)_{r_+}\bigg|_{r_c}. \tag{97}$$

The derivatives noted above can be either calculated numerically for a general case of solution or, for some particular cases, even analytical expressions can be derived. In any case, the following procedure is identical. Differentiating Eq. (96) and taking into account Maxwell's area law, we can write

$$\int_{\omega_l}^{\omega_s} \omega dP = \int_{\omega_l}^{\omega_s} (Bt + C\omega^3) d\omega = 0. \tag{98}$$

After integration we arrive at the relation

$$Bt(\omega_s^2 - \omega_l^2) + \frac{C}{2}(\omega_s^4 - \omega_l^4) = 0. \tag{99}$$

The obtained equation gives rise to a nontrivial solution $\omega_s = -\omega_l$. Since for both phases we have the same pressure and using the equation of state (96) we obtain

$$Bt(\omega_s - \omega_l) + C(\omega_s^3 - \omega_l^3) = 0. \tag{100}$$

Solving the latter equation for ω_s and taking into account the relation for ω_s and ω_l, finally we arrive at the following expression:

$$\omega_l \simeq \sqrt{-\frac{B}{C}t} = \sqrt{\frac{B}{C}\frac{(T_c - T)}{T_c}}. \tag{101}$$

Now we are able to write the expression for the volume difference ΔV_{ls} and extract the critical exponent from it,

$$\Delta V_{ls} \simeq V_c(\omega_l - \omega_s) = 2V_c\omega_l \sim |-t|^{1/2} \Rightarrow \beta = \frac{1}{2}. \tag{102}$$

Using the definition of the isothermal compressibility κ_T and the equation of state (96) we can derive the critical exponent γ,

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T \sim \frac{1}{Bt}, \Rightarrow \gamma = 1. \tag{103}$$

Finally, considering the critical isotherm, we obtain the critical exponent δ. Namely, from the equation of state (96) it follows that

$$P - P_c \sim C\omega^3, \Rightarrow \delta = 3. \tag{104}$$

All the critical exponents we have derived take the same value as their counterparts for the Reissner-Nordström-AdS black hole [66] and in the case of Horndeski gravity they were derived in the work [69], but for a different black hole solution. The same critical exponents were derived for various solutions in different frameworks, as it was mentioned in the reviewing paper [68]. Therefore, we can conclude that the critical behavior shows some universal features, at least for the vast number of black hole solutions in various independent frameworks.

We also note that in [69] the authors used a different equation of state identifying the thermodynamic pressure P not with the cosmological constant Λ, but relating the pressure with the ratio of the coupling constants α/η. In their case that definition of pressure was reasonable, since asymptotic behavior of the metric function $U(r)$ for infinitely large distances in their case is defined by the ratio α/η; in fact, that solution has an additional constraint, giving rise to the noted behavior. In our case we do not impose any specific constraints, thus asymptotic behavior at the infinity is equally defined by the cosmological constant Λ and the ratio α/η. Actually, we have an effective cosmological constant $\Lambda_{\mathrm{eff}} \sim \frac{\eta}{\alpha}(\frac{\alpha}{\eta} - \Lambda)^2$, whereas in [69] the effective cosmological constant is of the form $\Lambda_{\mathrm{eff}} \sim \frac{\alpha}{\eta}$. We also suppose that in our case the thermodynamic pressure can be defined to be proportional to the ratio $\frac{\alpha}{\eta}$, giving rise to a bit more cumbersome equation of state instead of Eq. (79). However, taking into account the results of the work [69], we do not think that it changes drastically the critical behavior or gives rise to other critical exponents.

Since here we focus on the analysis of the thermal behavior of the system at the critical point or in close vicinity of it, we also consider Ehrenfest's equations that were developed for the study of the phase transition of the second order, which is supposed to take place at the critical point. The Ehrenfest equations characterize discontinuity of such thermodynamic parameters as the heat capacity under constant pressure C_P, the isothermal compressibility κ_T, and the volume expansion coefficient $\tilde{\alpha}$, namely, we write

$$\left(\frac{\partial P}{\partial T}\right)_S = \frac{C_{P_2} - C_{P_1}}{VT(\tilde{\alpha}_2 - \tilde{\alpha}_1)} = \frac{\Delta C_P}{VT\Delta\tilde{\alpha}}, \tag{105}$$

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{\tilde{\alpha}_2 - \tilde{\alpha}_1}{\kappa_{T_2} - \kappa_{T_1}} = \frac{\Delta\tilde{\alpha}}{\Delta\kappa_T}. \tag{106}$$

We point out here that heat capacity C_P in the upper relation is given by relation (64), because the latter one was derived under the assumption that Λ was held fixed. The volume expansion coefficient $\tilde{\alpha}$ is defined as follows: $\tilde{\alpha} = 1/V(\partial V/\partial T)_P$. We show that mentioned thermodynamic quantities such as C_p, $\tilde{\alpha}$, and κ_T have infinite discontinuity at the critical point. Let us consider the isothermal compressibility,

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = -\frac{1}{V}\frac{\partial V}{\partial r_+}\left(\frac{\partial r_+}{\partial P}\right)_T. \tag{107}$$

Taking into account the first of the conditions (80), we conclude that at the critical point the derivative $(\partial r_+/\partial P)_T \to \infty$; therefore, there is an infinite gap for the isothermal compressibility κ_T at the critical point. The other two thermodynamic quantities also have an infinite gap at the critical point and it is enough to consider one of them, because the other one can be shown in exactly the same way. Let us consider again the heat capacity (64). It is clear that to show its discontinuity at the critical point we should show that the derivative $(\partial r_+/\partial T)_P$ has an infinite gap at the critical point, because both the temperature T and the derivative $\partial S/\partial r_+$ are continuous and take finite values at that point. To make the analysis more transparent, we write the derivative $(\partial T/\partial r_+)_P$ taken at the critical point r_c,

$$\left(\frac{\partial T}{\partial r_+}\right)_P\bigg|_c = \frac{r_c^2\chi(r_c)}{8(n-1)\pi\xi(r_c^2 + d^2)} \times \left(\frac{(r_c^2 + 3d^2)}{(r_c^2 + d^2)}\chi(r_c) + 2r_c\chi'(r_c)\right), \tag{108}$$

where we denote $\chi(r) = \xi - \Lambda + (n-1)(n-2)/r^2 - q^2/r^{2(n-1)} - (n-1)(n-2)\bar{q}^2/2r^4$ and $\chi'(r)$ is its derivative with respect to r. Now if we write the derivative

$(\partial P/\partial r_+)_T$ at the critical point r_c and use the expression for the critical temperature T_c (82), we obtain

$$\left(\frac{\partial P}{\partial r_+}\right)_T\bigg|_c = -\frac{1}{16\pi r_c}\left(\frac{(r_c^2+3d^2)}{(r_c^2+d^2)}\chi(r_c)+2r_c\chi'(r_c)\right)=0. \tag{109}$$

Where the last equality is nothing else but the condition (80), it follows that the expression in the parentheses in the upper relation equals zero. Since there is identical contribution in the relation (108), we conclude that the derivative $(\partial T/\partial r_+)_P$ equals zero at the critical point r_c and as a result the heat capacity C_P is discontinuous with infinite gap at this point.

It is also established that there is a subtlety in the definition of the so-called phase transitions of the second order according to Ehrenfest's classification. More precisely, the character of the phase transition with discontinuous second derivatives as we have here is defined by the Prigogine-Defay ratio, which is introduced as follows:

$$\tilde{\Pi} = \frac{(\partial P/\partial T)_S}{(\partial P/\partial T)_V} = \frac{\Delta C_P \Delta\kappa_T}{VT(\Delta\tilde{\alpha})^2}, \tag{110}$$

where obviously the Prigogine-Defay ratio is calculated at the critical point. Taking into account corresponding relations for the thermodynamic values C_P, $\tilde{\alpha}$, and κ_T and substituting them into the upper relation, after simple transformations we obtain

$$\tilde{\Pi} = -\frac{(\partial S/\partial r_+)(\partial r_+/\partial P)_T}{(\partial V/\partial r_+)(\partial r_+/\partial T)_P}\bigg|_c. \tag{111}$$

Calculating derivatives $\partial S/\partial r_+$ and $\partial V/\partial r_+$ and taking into account the relations (108) and (109), we obtain

$$\tilde{\Pi} = 1. \tag{112}$$

Therefore, since the Prigogine-Defay ratio equals 1, the phase transition at the critical point is exactly of the second order. We point out that in contrast to the considered case for dilatonic black holes the Prigogine-Defay ratio is $\tilde{\Pi} < 1$ [51], giving rise to the conclusion about a glass-type phase transition for the latter case.

VIII. DISCUSSION

In this work, a static charged black hole solution is obtained in Horndeski gravity with linear Maxwell and Yang-Mills fields. Because of the chosen form of the field potentials for the gauge fields, namely, the Maxwell field is purely electric and the non-Abelian field is of magnetic character, the explicit relations for the metric function are derived in a closed form. We point out that due to the nature of the Horndeski gravity the explicit relations for the metric function $U(r)$ have some differences for even (A5) and odd (26) dimensions of space n. This is a specific feature of Horndeski gravity and similar differences occurred even for pure Horndeski gravity [32], but it also affects the terms related to the gauge field [33]. The other distinctive feature of the obtained solution is a specific effective coupling between the gauge fields that is reflected by the terms proportional to the product of charges q and $\bar{q}$ ($\sim q^2\bar{q}^2$) in the metric function $U(r)$ (25) and in the following explicit relations. We point out that effective coupling of similar characters never appears in the framework of general relativity or Einstein-dilaton theory [51,52], but it may appear in "higher-order" gravity theories, for instance, when Gauss-Bonnet or higher Lovelock terms are taken into account, but as far as we know this issue has not been studied yet. It would be interesting to consider this issue in those theories and compare both results. It should be noted that for $n = 3$ both Abelian and non-Abelian fields give identical contribution to the metric function (27). We also note that the obtained solution belongs to a class that does not have a limit when both coupling parameters α and η are switched off. This constraint is caused by the relation (22).

The intricate expression of the metric function $U(r)$ in its integral (25) or explicit (26) [or/and (A5)] forms turns a thorough analysis of the metric function into a difficult task. However, asymptotic cases can be analyzed relatively easily. First of all, it follows from (28) that asymptotic behavior at the infinity will be of AdS or dS types, depending on the signs of the coupling constants. We also point out that, in this case, instead of the bare cosmological constant Λ, there is an effective one defined by both the bare constant Λ and the ratio of the coupling parameters α/η, namely, $\Lambda_{\text{eff}} \sim \eta/\alpha(\alpha/\eta-\Lambda)^2$. It should be noted that imposing additional constraints on the metric functions $U(r)$ and $W(r)$, another effective cosmological constant Λ_{eff} can be obtained; namely, in [69] the effective cosmological constant was obtained to be proportional to the ratio of the coupling parameters $\Lambda_{\text{eff}} \sim \alpha/\eta$. Therefore, it is an interesting issue to examine various options of how the effective cosmological constant appears and what form it takes. The latter is also important from the point of view of the extended thermodynamics because it is directly related to the definition of the thermodynamic pressure P. In this work, we consider mainly the solution with AdS asymptotic, but as we have mentioned, our solution may have de Sitterian asymptotic depending on the signs of the parameters, but this solution has its own peculiarities and it needs additional careful study.

For very small distances $r \to 0$ the leading contribution into the metric function $U(r)$ is mainly defined by the gauge field. In our case for $n > 3$ the dominant contribution is given by the Maxwell field, whereas for $n = 3$ both gauge fields contribute equally. Because of a specific interplay between Horndeski gravity and gauge field terms, the dominant term for $r \to 0$ is always of negative sign, making the behavior of the metric function more similar to the Schwarzschild solution than to the Reissner-Nordström one. In addition, the leading term is always proportional to $\sim q^4$, whereas in general relativity the linear Maxwell field contribution is of the order of $\sim q^2$. The negative sign of the mentioned contribution for $r \to 0$ gives rise to the conclusion that for this particular solution in Horndeski gravity a naked singularity never exists as it may happen for a charged solution in general relativity, for instance, for the Reissner-Nordström solution. Figure 1 confirms the mentioned conclusion. Figure 1 also shows that increase of the charge (or even both charges) can give rise to the appearance of additional horizons, but it needs more careful examination and it will be considered elsewhere.

We also study thermal properties of the black hole. First, we calculate the black hole temperature. To obtain it we have used the concept of modified surface gravity, introduced in [58], where the authors argued that due to the difference between speeds of gravitons and photons the concept of surface gravity needs a revision. The modified surface gravity and corresponding black hole temperature allowed us to avoid introducing additional scalar charges that are ill defined, as it was done earlier [29,33], to maintain the first law of black hole thermodynamics. An additional benefit we obtain using the modified surface gravity concept is the fact that the entropy we introduce takes the same form as in general relativity. To obtain the first law we use the Wald approach [71,72]. We also point out that the concept of the effective surface gravity should be carefully analyzed as it is performed in general relativity. Both temperature T and entropy S allowed us to calculate heat capacity C_Q and examine it. Its examination shows that it might have singularity points and instability domains that disappear under certain conditions. These singularities give a hint about possible critical behavior of the black hole, which is also studied in the extended thermodynamics framework.

Finally, introducing the thermodynamic pressure P (65) we obtain the thermal equation of state (79). In addition to the pressure, we have also introduced the thermal quantity Π which has a similar nature to the pressure, as was pointed out in [69], but this issue should be carefully studied. The extended thermal phase space allowed us to derive the Smarr relation (75). We also obtained the Gibbs free energy. The study of the Gibbs free energy for relatively small pressure shows swallowtail behavior (see Figs. 6 and 7) and increase of the pressure gives rise to gradual diminishing of the swallowtail behavior with its following dissolution. The swallowtail character of the $G = G(T)$ function means that the system undergoes a phase transition of the first order for corresponding values of the pressure P and it disappears when the swallowtail vanishes with increase of the pressure. The study of the equation of state (79) gives rise to the critical radius r_c (or critical volume V_c) which is obtained for some particular cases and, consequently, for those cases we derived explicit relations for the critical ratios ρ_c. General relations for the critical values can be studied only numerically. Studying the thermal behavior near the critical point, we obtained the critical exponents $\bar{\alpha}$, β, γ, and δ. Their numerical values are the same as even for the Reissner-Nordström-AdS black hole [66] and for other black hole solutions in Horndeski gravity with different equations of state [69] which confirms the universal character of thermodynamic relations. We have also analyzed the Ehrenfest equations to study the behavior at the critical point and calculated the Prigogine-Defay ratio $\tilde{\Pi}$, which is shown to be equal to 1. We make the conclusion that at the critical point we have the second-order phase transition. It would also be interesting to study carefully the critical behavior if instead of the cosmological constant Λ the ratio α/η is used to define thermodynamic pressure. The other interesting and important issue is to study in more detail the domain where the first-order phase transition occurs, namely, to obtain and examine the Clausius-Clapeyron equation.

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APPENDIX: SOME INTEGRALS AND EXPLICIT FORM FOR THE METRIC FUNCTION

For the sake of convenience, we give some technicalities related to the explicit calculation of the metric function $U(r)$ (25). Here we collect all the necessary integrals and we point out that there are some subtleties for odd and even dimensions n, namely, we obtain

$$\int \frac{r^m}{r^2+d^2}dr = \sum_{j=0}^{(m-2)/2}(-1)^j d^{2j}\frac{r^{m-2j-1}}{m-2j-1} + (-1)^{\frac{m}{2}} d^{m-1}\arctan\left(\frac{r}{d}\right), \tag{A1}$$

if m is a positive even number. If m is a positive odd number, the latter integral takes the form

$$\int \frac{r^m}{r^2+d^2}dr = \sum_{j=0}^{(m-3)/2}(-1)^j d^{2j}\frac{r^{m-2j-1}}{m-2j-1} + (-1)^{\frac{m-1}{2}}\frac{d^{m-1}}{2}\ln\left(1+\frac{r^2}{d^2}\right), \tag{A2}$$

and if in the latter relation $m = 1$ there is just a logarithmic contribution. There are also integrals of the form

$$\int \frac{r^{-m}}{r^2+d^2}dr = \sum_{j=0}^{(m-2)/2}(-1)^j \frac{r^{1+2j-m}}{(1+2j-m)d^{2(j+1)}} + \frac{(-1)^{\frac{m}{2}}}{d^{m+1}}\arctan\left(\frac{r}{d}\right), \tag{A3}$$

if m is an even positive. While in the case of an odd integer the upper integral takes the following form:

$$\int \frac{r^{-m}}{r^2+d^2}dr = \sum_{j=0}^{(m-3)/2}(-1)^j \frac{r^{1+2j-m}}{(1+2j-m)d^{2(j+1)}} + \frac{(-1)^{\frac{m+1}{2}}}{2d^{m+1}}\ln\left(1+\frac{d^2}{r^2}\right). \tag{A4}$$

Having used the given integrals, we are able to write the explicit form of the metric function $U(r)$ (25). Particularly, for odd n the metric function is given in the main body of the paper (26), while for even n we write

$$\begin{aligned} U(r) = {} & 1-\frac{\mu}{r^{n-2}}-\frac{2\Lambda}{n(n-1)}r^2-\frac{(n-2)}{(n-4)}\frac{\bar{q}^2}{r^2}+\frac{2q^2}{(n-1)(n-2)}r^{2(2-n)}+\frac{1}{2\alpha\eta(n-1)} \\ & \times\left[(\alpha+\Lambda\eta)^2\left(\sum_{j=0}^{\frac{n-2}{2}}(-1)^j d^{2j}\frac{r^{2(1-j)}}{n-2j}+(-1)^{\frac{n}{2}}\frac{d^n}{2r^{n-2}}\ln\left(1+\frac{r^2}{d^2}\right)\right)\right. \\ & +2\eta(\alpha+\Lambda\eta)q^2\left(\sum_{j=0}^{\frac{n-6}{2}}\frac{(-1)^j r^{6-2n+2j}}{(4-n+2j)d^{2(j+1)}}+\frac{(-1)^{\frac{n-2}{2}}}{2(dr)^{n-2}}\ln\left(1+\frac{d^2}{r^2}\right)\right) \\ & +\eta^2q^4\left(\sum_{j=0}^{\frac{3n-8}{2}}\frac{(-1)^j r^{2(4+j-2n)}}{(6+2j-3n)d^{2(j+1)}}+\frac{(-1)^{\frac{3n-4}{2}}}{2d^{3n-4}r^{n-2}}\ln\left(1+\frac{d^2}{r^2}\right)\right)+\eta(n-1)(n-2)\bar{q}^2 \\ & \times\left((\alpha+\Lambda\eta)\left(\sum_{j=0}^{\frac{n-6}{2}}\frac{(-1)^j d^{2j}r^{-2(1+j)}}{n-4-2j}+(-1)^{\frac{n-2}{2}}\frac{d^{n-4}}{2r^{n-2}}\ln\left(1+\frac{r^2}{d^2}\right)\right)\right. \\ & +\eta q^2\left(\sum_{j=0}^{\frac{n-2}{2}}\frac{(-1)^j r^{2(1+j-n)}}{(2j-n)d^{2(j+1)}}+\frac{(-1)^{\frac{n+2}{2}}}{2d^{n+2}r^{n-2}}\ln\left(1+\frac{d^2}{r^2}\right)\right) \\ & \left.\left.+\eta(n-1)(n-2)\frac{\bar{q}^2}{4}\left(\sum_{j=0}^{\frac{n-10}{2}}\frac{(-1)^j d^{2j}r^{-2(j+3)}}{n-8-2j}+(-1)^{\frac{n-8}{2}}\frac{d^{n-8}}{2r^{n-2}}\ln\left(1+\frac{r^2}{d^2}\right)\right)\right)\right]. \end{aligned} \tag{A5}$$

We remark that the upper relation is written for the case $n > 7$. Similar to the odd n case, the exponent in the numerator of the last integral in (25) changes its sign if $n < 7$, therefore integral (A4) should be chosen instead of (A2).

For the case $n = 4$ the metric function $U(r)$ is as follows:

$$\begin{aligned} U(r) = {} & 1-\frac{\mu}{r^2}-\frac{\Lambda}{6}r^2+\frac{q^2}{3r^4}-2\frac{\bar{q}^2}{r^2}\ln\left(\frac{r}{d}\right)+\frac{1}{6\alpha\eta}\left[\frac{(\alpha+\Lambda\eta)^2}{2}\left(\frac{r^2}{2}-d^2\right)+\left((\alpha+\Lambda\eta)\frac{d^4}{2}+3\eta\bar{q}^2\right)\right. \\ & \times\frac{(\alpha+\Lambda\eta)}{r^2}\ln\left(1+\frac{r^2}{d^2}\right)+\frac{\eta^2q^4}{2d^2r^4}\left(-\frac{1}{3r^4}+\frac{1}{2(dr)^2}-\frac{1}{d^4}\right)+3\frac{\eta^2q^2\bar{q}^2}{d^2r^4}\left(\frac{1}{d^2}-\frac{1}{2r^2}\right) \\ & \left.-\frac{9\eta^2\bar{q}^4}{2d^2r^4}+\left(\frac{\eta^2}{2d^2}\left(\frac{q^2}{d^2}-3\bar{q}^2\right)^2-\eta(\alpha+\Lambda\eta)q^2\right)\frac{1}{d^2r^2}\ln\left(1+\frac{d^2}{r^2}\right)\right]. \end{aligned} \tag{A6}$$

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