

# Resonant phenomena in finite motions of test particles in oscillating dark matter configurations

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Nonlinear differential equations are derived that describe the time evolution of the test particle coordinates during finite motions in the gravitational field of oscillating dark matter. It is shown that in the weak field approximation, the radial oscillations of a test particle and oscillations in orbital motion are described by the Hill equation and the nonhomogeneous Hill equation, respectively. In the case of scalar dark matter with a logarithmic self-interaction, these equations are integrated numerically, and the solutions are compared with the corresponding solutions of the original nonlinear system to identify possible resonance effects.

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## I. INTRODUCTION

The nature of dark matter (DM) remains one of the main mysteries of modern cosmology. Currently, the most realistic hypothesis seems to be that DM consists of ultralight bosons, e.g., axions [1], with a mass in the  $10^{-23}$ – $10^{-21}$  eV range. New arguments in favor of this hypothesis are provided by very recent astronomical observations [2]. Coherent states of such ultralight particles are described by a classical scalar field, whose wave properties solve serious problems of the standard  $\Lambda$ CDM model on galactic and subgalactic scales [3–12]. In the early Universe, the primordial fluctuations of this field are stretched by inflation to form an almost homogeneous scalar background. This background oscillates near the minimum of the effective potential with a fundamental frequency  $\omega$  depending on both the mass  $m$  of the scalar field and the shape of the potential. However, these oscillations are unstable [13]. If the bosons of DM are not self-interacting, the corresponding free scalar field behaves on average as a dustlike matter [3]. In this case, due to Jeans instability, the scalar background breaks into diffuse lumps, which begin to collapse under the influence of self-gravity. On a scale comparable to the Jeans length, the collapse of a single lump stops due to the action of the so-called quantum pressure which appears from the growing gradient of the energy density of the scalar field. From the quantum mechanical point of view, this pressure is a manifestation of the Heisenberg uncertainty principle. The result of the established equilibrium is a “fuzzy” DM halo, in which the scalar field oscillates with the fundamental

frequency  $\omega \sim m$  [9]. In addition, as numerical simulations show, the central region of the formed halo, the core, undergoes quasinormal low-frequency oscillations, which are superimposed on the fundamental oscillations leading to their modulation [14,15].

If DM particles are self-interacting, another scenario for the formation of the localized DM configurations is realized. Self-interaction is described by additional higher-order terms in the effective potential. These terms can be either regular or singular. In the first case, they have a significant effect on the dynamics of the scalar field only at large oscillation amplitudes. In the second case, this is not necessary. An example of a singular potential will be considered in Sec. III of our paper. In both cases, the scalar field oscillations turn out to be unstable due to parametric resonance between the oscillating background and perturbations. As a result, the homogeneous scalar background breaks up into an ensemble of oscillating lumps, oscillons (pulsions). This mechanism works on both cosmological and astrophysical scales [16–23] (see Ref. [24] for a review). Under the action of gravity, after the completion of some relaxation processes, oscillons turn into long-lived self-gravitating oscillating objects with the size  $\sim m^{-1}$ , oscillatons (gravipulsions) [25–27], separated from the Hubble flow. This means that each individual oscillaton is the result of an established dynamic equilibrium between self-interaction, self-gravity, and quantum pressure and therefore must be described by the self-consistent system of Einstein-Klein-Gordon equations (see Ref. [28] for a recent review). Note that, within this system of equations, oscillatons can also arise from rather arbitrary localized initial conditions due to the gravitational cooling process [4,5,29].

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In any case, all the DM oscillations mentioned above lead to the corresponding oscillations of the gravitational field, which can be detected by their effect on the motion of photons and test particles. For example, as shown in [30], the gravitational time delay for a photon passing through an oscillating halo should cause small periodic fluctuations in the observed timing array of a pulsar located inside the halo, which can be detected in future pulsar timing experiments. In paper [31], it was proposed to use the laser interferometers for detecting the “gravitational waves” appearing due to the motion of the Earth through oscillating DM. The secular variations of the orbital period in the system of binary pulsars were discussed in [32,33] as a probe for ultralight oscillating DM. Also, in this context, in Refs. [34,35] the motion of test bodies in spherically symmetric time-periodic spacetimes induced by a nonself-interacting DM was numerically investigated. In paper [35] it was shown, in particular, that the orbital resonances may occur in the motion of stars in oscillating halos. In addition, in [35] it was demonstrated that spectroscopic emission lines from stars in such halos exhibit characteristic, periodic modulations due to variations in the gravitational frequency shift. These modulations were found analytically in Ref. [36].

Recently, in the above context, we studied the infinite motions of photons and test particles in the oscillating DM [37,38]. Namely, we considered deflection of their trajectories when passing through an oscillating spherically symmetric lump of DM. Using the geodesic method and the perturbative approach, we have found and calculated analytically periodic variations in the deflection angle of both light rays and trajectories of the test particles. In the present paper, we apply the same approach to study the finite motions of the test particles in oscillating DM, focusing on radial and orbital trajectories.

Our paper is organized as follows. In Sec. II, on the basis of geodesic equations, we derive exact equations that describe the time dependence of the radial and angular coordinates of a test particle making a finite motion inside an oscillating self-gravitating lump of DM. We show that in the weak field approximation these equations reduce to the homogeneous and nonhomogeneous Hill equations for radial and perturbed circular motions, respectively. In Sec. III we use these equations to describe the radial and perturbed circular motions in the case when the DM lump is formed by a real scalar field with a logarithmic self-interaction. Applying numerical integration, we compare solutions to Hill equations with solutions to exact equations to explore possible resonance effects. Discussion and concluding remarks can be found in Sec. IV.

## II. FINITE TRAJECTORIES OF TEST PARTICLES IN TIME-PERIODIC SPHERICALLY SYMMETRIC SPACETIMES

The motion of a particle in a gravitational field obeys the geodesic equation

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0, \quad (1)$$

where  $ds$  is the proper time interval. Consider a spherically symmetric oscillating localized lump of DM. The gravitational field inside the lump is described by the metric

$$ds^2 = B(t, r)dt^2 - A(t, r)dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \quad (2)$$

where  $A(t, r)$  and  $B(t, r)$  are time-periodic and tend to unity as  $r \rightarrow \infty$ . For the trajectories lying in the plane  $\vartheta = \pi/2$ , the geodesic equation reduces to the system

$$\frac{d}{ds} \ln \left( B \frac{dt}{ds} \right) = \frac{\dot{B}}{2B} \frac{dt}{ds} - \frac{\dot{A}}{2B} \left( \frac{dr}{ds} \right)^2 \left( \frac{dt}{ds} \right)^{-1}, \quad (3)$$

$$\frac{d^2 r}{ds^2} + \frac{B'}{2A} \left( \frac{dt}{ds} \right)^2 + \frac{\dot{A}}{A} \frac{dt}{ds} \frac{dr}{ds} + \frac{A'}{2A} \left( \frac{dr}{ds} \right)^2 = \frac{r}{A} \left( \frac{d\varphi}{ds} \right)^2, \quad (4)$$

$$\frac{d^2 \varphi}{ds^2} + \frac{2}{r} \frac{dr}{ds} \frac{d\varphi}{ds} = 0, \quad (5)$$

where  $(\dot{\phantom{x}}) = \partial/\partial t$ ,  $(\prime) = \partial/\partial r$ . From Eqs. (5) and (2) it follows that

$$\frac{d\varphi}{ds} = \frac{J}{r^2}, \quad (6)$$

$$B \left( \frac{dt}{ds} \right)^2 - A \left( \frac{dr}{ds} \right)^2 = 1 + \frac{J^2}{r^2}, \quad (7)$$

where  $J = \text{const}$  is the particle’s angular momentum.

Now suppose for a moment that the trajectory of the particle is known. Passing from variable  $s$  to variable  $t$  and denoting

$$Y(t) = B(t, r(t))(ds/dt)^{-1}, \quad (8)$$

from Eq. (7) we get

$$\left( \frac{dr}{dt} \right)^2 = \frac{B}{A} \left[ 1 - \frac{B}{Y^2} \left( 1 + \frac{J^2}{r^2} \right) \right]. \quad (9)$$

From Eq. (3) with (9), the equation for  $Y^2$  follows:

$$\frac{dY^2}{dt} = \left( \frac{\dot{B}}{B} - \frac{\dot{A}}{A} \right) Y^2 + \frac{\dot{A}B}{A} \left( 1 + \frac{J^2}{r^2} \right). \quad (10)$$

Given that  $d/ds = (Y/B)d/dt$  and using Eqs. (9) and (10), we find

$$\frac{d^2 r}{ds^2} = \frac{Y^2}{B^2} \left[ \frac{d^2 r}{dt^2} - \frac{\dot{B}}{2B} \frac{dr}{dt} - \frac{B'}{B} \left( \frac{dr}{dt} \right)^2 - \frac{\dot{A}}{2B} \left( \frac{dr}{dt} \right)^3 \right]. \quad (11)$$

Substituting (11) into Eq. (4) and taking into account (6), (8), and (9), we finally obtain

$$\frac{d^2 r}{dt^2} + \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{2B} \right) \frac{dr}{dt} + \left( \frac{\gamma(r)}{r} + \frac{A'}{2A} - \frac{B'}{B} \right) \left( \frac{dr}{dt} \right)^2 - \frac{\dot{A}}{2B} \left( \frac{dr}{dt} \right)^3 = \frac{\gamma(r)B}{rA} - \frac{B'}{2A}, \quad (12)$$

$$\left( \frac{d\varphi}{dt} \right)^2 = \frac{\gamma(r)}{r^2} \left[ B - A \left( \frac{dr}{dt} \right)^2 \right], \quad (13)$$

where

$$\gamma(r) = \frac{J^2/r^2}{1 + J^2/r^2}, \quad 0 \leq \gamma < 1. \quad (14)$$

For Eqs. (12) and (13) we choose the initial conditions

$$r(0) = r_0, \quad (dr/dt)_{t=0} = 0, \quad \varphi(0) = 0. \quad (15)$$

The Eqs. (12)–(15) completely determine the time dependence of the coordinates of the moving particle. For radial motions we set  $\gamma = 0$ . For orbital motions, the requirement for the absence of a constant component on the right-hand side of Eq. (12) leads to the condition

$$\gamma(r_0) = \frac{r_0 \overline{B'(t, r_0)/A(t, r_0)}}{2 \overline{B(t, r_0)/A(t, r_0)}}, \quad (16)$$

where the bar means the average over the period of oscillations of the gravitational field. This condition gives the relation between  $r_0$  and the angular momentum  $J$  at which, in the case of a static mass distribution, the particle moves along a circular orbit of radius  $r_0$ . Of course, this is only possible if the right-hand side of Eq. (16) is positive and less than one. In this case, for a given  $J$ , the value of  $r_0$  is found by solving Eq. (16). Alternatively, one can choose  $r_0$  and calculate  $\gamma_0 = \gamma(r_0)$  using Eq. (16). This gives  $J^2 = r_0^2 \gamma_0 / (1 - \gamma_0)$  for the squared momentum, and hence  $\gamma(r)$  involved in Eqs. (12)–(14) can be written as

$$\gamma(r) = \left[ 1 + \frac{1 - \gamma_0}{\gamma_0} \left( \frac{r}{r_0} \right)^2 \right]^{-1}. \quad (17)$$

### A. Weak field approximation

Now let us assume that the oscillating lump of DM has a low density, so that its gravitational field is weak everywhere on the particle trajectory. It follows that

$$A = 1 - 2\psi + O(\varkappa^2), \quad B = 1 + 2\chi + O(\varkappa^2), \quad (18)$$

where  $\psi(t, r)$  and  $\chi(t, r)$  are small time-periodic functions of order  $\varkappa$ ,  $\varkappa \ll 1$  being a dimensionless small parameter proportional to the gravitational constant  $G$ . With finite

motions of particles in such a gravitational field, the particle velocities are small. Taking this into account and using (18), from Eq. (12) we obtain

$$\frac{d^2 r}{dt^2} - (2\dot{\psi}(t, r) + \dot{\chi}(t, r)) \frac{dr}{dt} = \frac{\gamma(r)}{r} - \chi'(t, r). \quad (19)$$

When deriving this equation, we neglected the terms of the orders  $\varkappa^2/R_g$ ,  $(\varkappa^2/T_g)dr/dt$ ,  $(\varkappa/R_g)(dr/dt)^2$ , and  $(\varkappa/T_g)(dr/dt)^3$ , where  $R_g$  is the characteristic radius of the gravitating lump,  $T_g$  is the oscillation period of the gravitational field. In the same approximation, Eq. (13) becomes

$$\left( \frac{d\varphi}{dt} \right)^2 = \frac{\gamma(r)}{r^2} [1 + 2\chi(t, r)], \quad (20)$$

and condition (16) takes the form

$$\gamma(r_0) = r_0 \bar{\chi}'(r_0), \quad (21)$$

where we denote  $\bar{\chi}(r_0) = \overline{\chi(t, r_0)}$ .

Note that Eq. (19) is generally nonlinear. Nevertheless, there are two types of finite motions for which one can restrict oneself to a linear analysis. Consider, at first, small radial oscillations near the center. Setting  $\gamma = 0$ , we expand  $\psi(t, r)$  and  $\chi(t, r)$  at  $r = 0$ , taking into account that  $\psi'(t, 0) = \chi'(t, 0) = 0$  due to the assumed smoothness of the gravitational field. Neglecting the terms of the orders  $(\varkappa/T_g)(r/R_g)^2 dr/dt$  and  $\varkappa r^2/R_g^3$ , we obtain

$$\frac{d^2 r}{dt^2} - (2\dot{\psi}(t, 0) + \dot{\chi}(t, 0)) \frac{dr}{dt} + \chi''(t, 0)r = 0. \quad (22)$$

Substitution

$$r(t) = u(t) \exp \left( \psi(t, 0) + \frac{1}{2} \chi(t, 0) \right) \quad (23)$$

transforms Eq. (22) into the equation

$$\frac{d^2 u}{dt^2} + \left( \ddot{\psi}(t, 0) + \frac{1}{2} \ddot{\chi}(t, 0) + \chi''(t, 0) \right) u = 0, \quad (24)$$

where in the brackets the terms of the order  $\varkappa^2/T_g^2$  were neglected. The remaining terms are  $T_g$ -periodic. Equations of this type are usually called Hill equations. They have numerous applications in physics and technology (see, e.g., [39]). The properties of solutions to the Hill equations depend significantly on the parameters involved in the equation. In the parameter space, there are regions in which solutions grow exponentially with time (resonant zones) and regions in which solutions are bounded (nonresonant zones). The Floquet theory of the Hill equation is presented, e.g., in the book [40].

Now consider the motion along a trajectory close to circular. In this case we set

$$r(t) = r_0(1 + \eta(t)), \quad (25)$$

where  $\eta(t)$  is small. Making expansions in Eq. (19) at  $r = r_0$  and taking into account (21), we obtain

$$\begin{aligned} \frac{d^2\eta}{dt^2} - (2\dot{\psi}(t, r_0) + \dot{\chi}(t, r_0)) \frac{d\eta}{dt} \\ + \left( \chi''(t, r_0) + \frac{3\gamma(r_0)}{r_0^2} \right) \eta = -\frac{1}{r_0} \tilde{\chi}'(t, r_0), \end{aligned} \quad (26)$$

where  $\tilde{\chi}(t, r_0) = \chi(t, r_0) - \bar{\chi}(r_0)$  and the terms of the orders  $(r_0/R_g)(\kappa\eta/T_g)d\eta/dt$ ,  $\kappa^2\eta/R_g^2$ ,  $\kappa\eta^2/(r_0R_g)$ , and  $r_0\kappa\eta^2/R_g^3$  were neglected. Finally, substitution

$$\eta(t) = u(t) \exp\left(\psi(t, r_0) + \frac{1}{2}\chi(t, r_0)\right) \quad (27)$$

results in the nonhomogeneous Hill equation

$$\begin{aligned} \frac{d^2u}{dt^2} + \left( \ddot{\psi}(t, r_0) + \frac{1}{2}\ddot{\chi}(t, r_0) + \chi''(t, r_0) + \frac{3\gamma(r_0)}{r_0^2} \right) u \\ = -\frac{1}{r_0} \tilde{\chi}'(t, r_0), \end{aligned} \quad (28)$$

where we neglected the terms of the orders  $\kappa^2/T_g^2$  and  $\kappa^2/(r_0R_g)$  in the brackets and on the right-hand side, respectively.

The nonhomogeneous Hill equation has also been studied in the literature (see, e.g., [41] and references therein), but not in as much detail as its homogeneous counterpart. For our analysis, it is only important that the periodic forcing term on the right-hand side of Eq. (28) does not affect the location of the boundaries of the resonant zones [42,43].

### III. MOTIONS OF THE TEST PARTICLE IN A TIME-PERIODIC SPHERICALLY SYMMETRIC SCALAR FIELD

As examples, we consider radial and orbital motions of the test particle in the self-gravitating real scalar field with the potential

$$U(\phi) = \frac{m^2}{2} \phi^2 \left( 1 - \ln \frac{\phi^2}{\sigma^2} \right), \quad (29)$$

where  $\sigma$  is the characteristic magnitude of the field,  $m$  is the mass (in units  $\hbar = c = 1$ ). This potential is singular: its second derivative tends to infinity as  $\phi$  passes through zero. Originally, such potentials were considered in quantum field theory in connection with dilatation covariance of

relativistic field equations [44]. Later in Refs. [45,46] it was shown that the requirements of separability of noninteracting quantum subsystems and the validity of Planck's relation  $E = \hbar\omega$  for stationary states uniquely determine the logarithmic potential (29) (with a complex wave function) in the Schrödinger-Pauli and Klein-Gordon-type equations. Also, when taking into account quantum corrections, such potentials naturally appear in the inflationary cosmology [47–49], as well as in some supersymmetric extensions of the Standard Model (flat direction potentials in the gravity mediated supersymmetric breaking scenario) [50,51]. It is remarkable that in the Minkowski spacetime the potential (29) admits a whole family of exact solutions of the Klein-Gordon equation in the form  $\phi = a(t)w(\mathbf{r})$ , describing multidimensional localized time-periodic configurations of a real scalar field, the pulsions (oscillons) [52,53]. Moreover, it turned out that potential (29) is the only one that allows such solutions to exist [54]. It was also shown that pulsions can arise due to the fragmentation of a homogeneous scalar background oscillating around the local minimum of the potential (29) [17]. Stability analysis of the real pulsions have shown that, despite the absence of a global charge, there are values of oscillation amplitudes at which they are long-lived objects that retain their periodicity for a long time [17,55,56].

The above mentioned unique properties of the logarithmic potential motivate us to use this field model to test our approach. To do this, we need to take into account the effects of self-gravity. The corresponding solution of the Einstein–Klein–Gordon system was found in Ref. [27] by the Krylov-Bogoliubov method. This solution describes a self-gravitating field lump of an almost Gaussian shape that pulsates in time. In the weak field approximation, the corresponding metric functions  $A(t, r)$  and  $B(t, r)$  can be written as (18), where

$$\psi(t, r) = \frac{\kappa}{2} \left[ a^2 \rho^2 + V_{\max} \left( 1 - \frac{\sqrt{\pi} \operatorname{erf} \rho}{2\rho} e^{\rho^2} \right) \right] e^{3-\rho^2}, \quad (30)$$

$$\chi(t, r) = -\frac{\kappa}{2} \left[ a^2 \ln a^2 + V_{\max} \left( 1 + \frac{\sqrt{\pi} \operatorname{erf} \rho}{2\rho} e^{\rho^2} \right) \right] e^{3-\rho^2}, \quad (31)$$

$\tau = mt$ ,  $\rho = mr$ ,  $\kappa = 4\pi G\sigma^2 \ll 1$  ( $G$  is the gravitational constant). The function  $a(\theta(\tau))$  oscillates in the range  $-a_{\max} \leq a(\theta) \leq a_{\max}$  ( $0 < a_{\max} < 1$ ) in the local minimum of the potential  $V(a)$ ,

$$a_{\theta\theta} = -dV/da, \quad V(a) = (a^2/2)(1 - \ln a^2), \quad (32)$$

where  $V_{\max} = V(a_{\max})$ ,  $d\theta/d\tau = 1 + \kappa\Omega + O(\kappa^2)$ , and the constant  $\kappa\Omega$  is the frequency correction due to gravitational effects (see Ref. [27] for details). The period (in  $\theta$ ) of these oscillations can be approximated by

$$T \approx 2\pi(1 - \ln a_{\max}^2)^{-1/2} \quad (a_{\max}^2 \ll 1), \quad (33)$$

$$T \approx 2\sqrt{2} \ln(1 - a_{\max}^2)^{-1} \quad (1 - a_{\max}^2 \ll 1). \quad (34)$$

The energy density of the field lump we are considering is concentrated on the characteristic scale  $r \sim R_g = m^{-1}$  and decays as

$$T_0^0 \sim m^2 \sigma^2 a^2(\theta) \rho^2 e^{3-\rho^2} \quad (\rho = r/R_g \gg 1). \quad (35)$$

As seen from Eqs. (30) and (31), at large distances from the lump the gravitational field turns into the static Schwarzschild field, in accordance with the Birkhoff theorem (see, e.g., [57]). However, inside the lump the gravitational field oscillates with the period  $T_g = [2m(1 + \chi\Omega)]^{-1} T$  (with respect to  $t$ ). Assuming no direct interaction with the scalar field, let us consider the radial and orbital motions of a test particle in this gravitational field.

### A. Radial motion

As has been shown, small radial oscillations of a test particle are generally described by Eqs. (23) and (24). Considering Eq. (24), we can put  $d/dt \approx md/d\theta$  with the accepted accuracy, and  $d/dr = md/d\rho$ . Using Eqs. (30)–(32) we obtain

$$\rho = v \exp\left(-\frac{\chi}{4} e^3 (a^2 \ln a^2 + 2V_{\max})\right), \quad (36)$$

$$\begin{aligned} \frac{d^2 v}{d\theta^2} + \chi e^3 \left[ \frac{3}{2} a^2 - a^2 \ln a^2 \left( \frac{1}{2} + \ln a^2 \right) \right. \\ \left. - V_{\max} \left( \frac{5}{3} + \ln a^2 \right) \right] v = 0. \end{aligned} \quad (37)$$

Equation (37) is the singular Hill equation because the expression in the square brackets is periodic in  $\theta$  and tends to infinity as  $a(\theta)$  passes through zero. For this equation, Fig. 1 shows the parametric resonance zones on the  $\chi - a_{\max}^2$  plane.

We solved Eqs. (32), (36), and (37) numerically in both resonant and nonresonant zones using initial conditions

$$a(0) = a_{\max}, \quad a_\theta(0) = 0, \quad \rho(0) = \rho_0, \quad \rho_\theta(0) = 0, \quad (38)$$

$$v(0) = \rho_0 \exp\left(\frac{\chi}{4} e^3 a_{\max}^2\right), \quad v_\theta(0) = 0. \quad (39)$$

For comparison, using the same initial conditions (38), we numerically solved the exact equation (12) with  $\gamma = 0$  and metric functions  $A(t, r)$  and  $B(t, r)$  given by formulas (18), (30), and (31). The results are shown in Fig. 2.

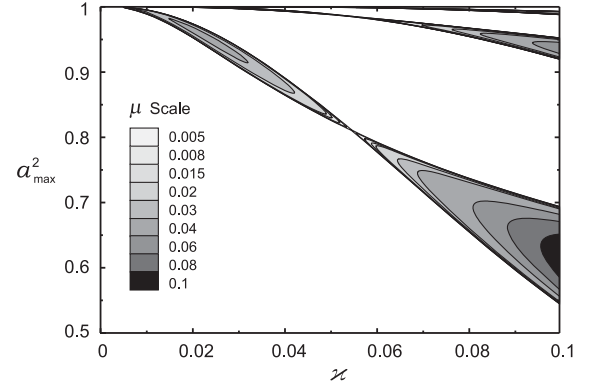


FIG. 1. Resonance zones and Floquet exponent  $\mu$  for Eq. (37).

### B. Orbital motion

In this case, the particle trajectory, close to circular, is generally described by Eqs. (25), (27), (28), (20), and (21). Using Eqs. (30)–(32), with the accepted accuracy, we obtain

$$\eta = u \exp S(a, \rho_0), \quad (40)$$

$$\begin{aligned} \frac{d^2 u}{d\theta^2} + \left\{ \frac{\gamma_0}{\rho_0^2} + \frac{\chi}{2} [2V_{\max}(1 - \ln a^2) + (3 - 2\rho_0^2)a^2 \right. \\ \left. - a^2 \ln a^2 (1 + 2 \ln a^2) + 4\overline{a^2 \ln a^2}] e^{3-\rho_0^2} \right\} u \\ = -\chi a^2 \widetilde{\ln a^2} e^{3-\rho_0^2}, \end{aligned} \quad (41)$$

where

$$\begin{aligned} S(a, \rho) = \frac{\chi}{4} e^{3-\rho^2} \\ \times \left[ 2a^2 \rho^2 - a^2 \ln a^2 + V_{\max} \left( 1 - \frac{3\sqrt{\pi} \text{erf} \rho}{2\rho} e^{\rho^2} \right) \right], \end{aligned} \quad (42)$$

$$\begin{aligned} \gamma_0 = \chi \rho_0^2 e^{3-\rho_0^2} \\ \times \left\{ \overline{a^2 \ln a^2} + V_{\max} \left[ 1 - \frac{1}{2\rho_0^2} \left( 1 - \frac{\sqrt{\pi} \text{erf} \rho_0}{2\rho_0} e^{\rho_0^2} \right) \right] \right\}, \end{aligned} \quad (43)$$

and  $\overline{a^2 \ln a^2} = a^2 \ln a^2 - \overline{a^2 \ln a^2}$ . Eq. (41) is nonhomogeneous singular Hill equation. The resonance zones of this equation are depicted in Fig. 3 on the  $\chi - a_{\max}^2$  plane with fixed  $\rho_0$ . We solved Eqs. (32), (40), and (41) numerically using the initial conditions

$$a(0) = a_{\max}, \quad a_\theta(0) = 0, \quad \eta(0) = 0, \quad \eta_\theta(0) = 0, \quad (44)$$

$$u(0) = 0, \quad u_\theta(0) = 0. \quad (45)$$

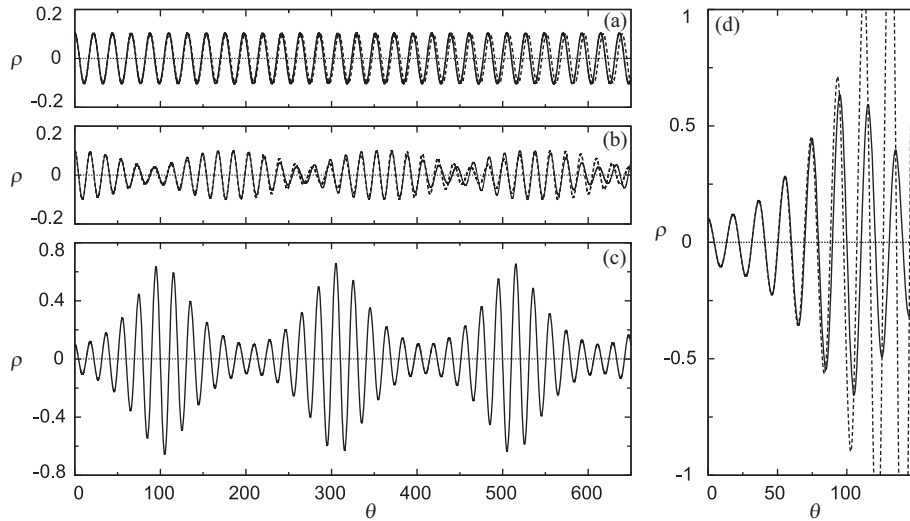


FIG. 2. Solutions of the equations (32), (36) and (37) far from the resonance zone (a), in the vicinity of the resonance zone (b), and at the center of the resonance zone (c),  $\rho_0 = 0.1$ ,  $\chi = 0.01$ ,  $a_{\max} = 0.9$ (a), 0.995(b), 0.997(c). The results are shown in dashed lines. The corresponding solutions of the nonlinear Eq. (12) for the same parameters are shown in solid lines. Panel (d) depicts the comparison of the initial fragment of (c) with the exponentially growing solution obtained in the linear approximation.

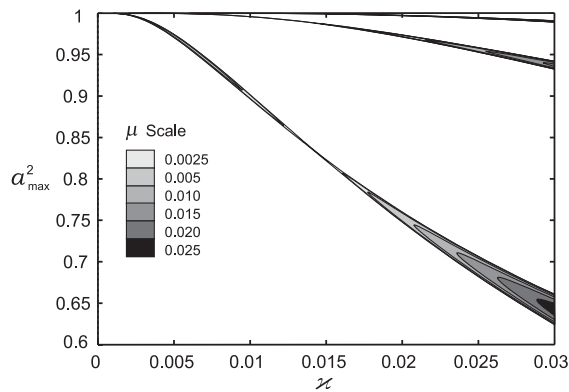


FIG. 3. Resonance zones and Floquet exponent  $\mu$  for Eq. (41) with  $\rho_0 = 0.3$ .

Further, assuming that  $\gamma(r)$ ,  $A(t, r)$ , and  $B(t, r)$  are given by formulas (17), (18), (30), and (31), we numerically solved Eq. (12) to find  $\eta = (\rho - \rho_0)/\rho_0$  with the same initial conditions (44). The results can be compared in Fig. 4. In addition, we show the evolution of the orbital trajectory obtained by integrating Eq. (13).

#### IV. DISCUSSION

Thus, based on the geodesic equation, we have derived the nonlinear system determining the time dependence of the test particle coordinates in finite motions in the oscillating spherically symmetric spacetime. In the weak field approximation we have reduced the nonlinear equation (12) for the radial coordinate to linear equations, namely, to the Hill equation (24) for the radial oscillations and the nonhomogeneous Hill equation (28) for the

oscillations near circular orbit, respectively. Using these equations we studied resonance effects in radial and orbital motions of test particles inside oscillating spherically symmetric field lump in the scalar model with the logarithmic self-interaction (29). Equations (24) and (28) then take the form (37) and (41), respectively.

The resonant solutions of the Hill equation at large times  $\theta$  have the asymptotic form  $\sim F(\theta)e^{\mu\theta}$ , where  $F(\theta)$  is a  $T$ -periodic ( $T/2$ -periodic or  $T/2$ -antiperiodic) function, and the Floquet exponent  $\mu$  is positive in parametric resonance zones (see, e.g., [40]). In the case of radial motions, the resonance zones of Eq. (37) are depicted in Fig. 1 on the  $\chi - a_{\max}^2$  plane. We solved numerically Eq. (37) at different points of this plane and compared the solutions with the corresponding solutions of Eq. (12) in the domain of sufficiently small  $\chi$  and  $\rho_0$  for which our approximation is valid. We have found that, far from the resonance zones, the solutions are in good agreement with each other over a sufficiently large time interval (see Fig. 2(a) as an example). It can be seen that the oscillations are practically sinusoidal. As you approach (with fixed  $\chi$ ) the resonance zone, these oscillations become modulated, turning into beats [Fig. 2(b)]. In the resonance zone, the beats acquire a different character [Fig. 2(c)]. As can be seen from Fig. 2(d), the increase in the oscillation amplitude in each beat is well approximated by an exponent with a growth rate close to the Floquet exponent in the corresponding solution of the Hill equation. This suggests a resonant mechanism for the growth of oscillations. When the oscillation amplitude reaches sufficiently large values, the high-order nonlinear terms in Eq. (12) come into play, periodically suppressing the resonance and limiting the amplitude of the beats.

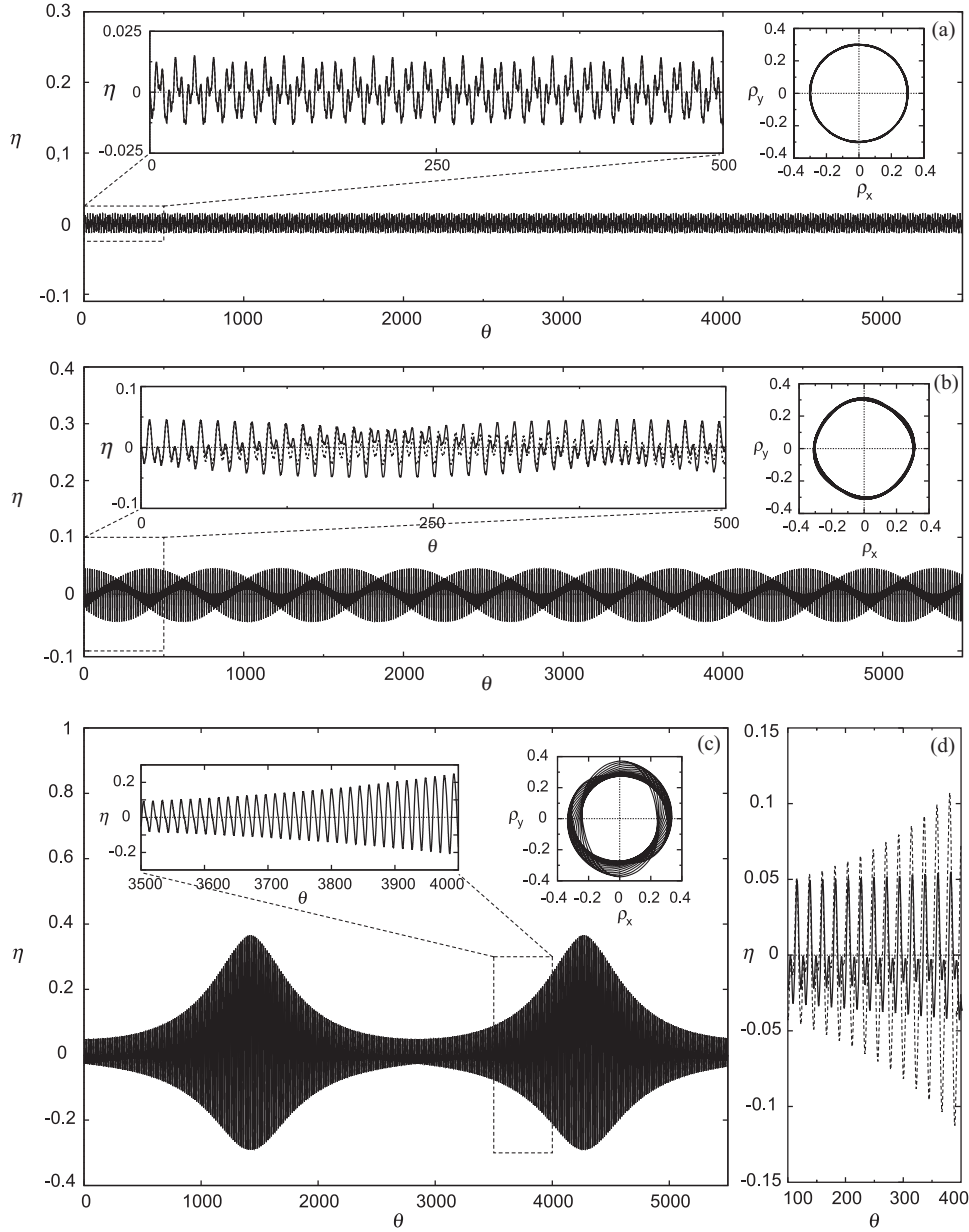


FIG. 4. Solutions of the equations (32), (40), and (41) far from the resonance zone (a), in the vicinity of the resonance zone (b), and at the center of the same zone (c). The results of integration are shown in dashed lines,  $\rho_0 = 0.3$ ,  $\varkappa = 0.005$ ,  $a_{\max} = 0.950(a)$ ,  $0.985(b)$ ,  $0.98625(c)$ . The corresponding solutions of the nonlinear system (12), (13) are shown in solid lines. Panel (d) presents the comparison of the initial fragment of (c) with the rapidly growing solution obtained in the linear approximation [see Eqs. (40) and (41)]. The nearly circular particle orbits shown in the right insets of panels (a), (b), and (c) correspond to the oscillation curves depicted in the left insets.

Now consider the orbital motions. The resonance zones of Eq. (41) are shown in Fig. 3 on the  $\varkappa - a_{\max}^2$  plane for a fixed  $\rho_0$ . Far from resonance zones, the solutions of Eqs. (40) and (41) are in excellent agreement with solutions of Eq. (12) (see Fig. 4(a), where the curves in the inset practically coincide). Near the resonant zones, the solutions gradually diverge with time [Fig. 4(b)]. Nevertheless, it is clear that the solutions of Eqs. (40) and (41) here gives an acceptable description of the beats that have appeared.

However, in the resonance zones, the solutions diverge significantly [see Figs. 4(c) and 4(d)]. First, we have found that the resonance zones of Eq. (12) are located within the resonance zones of Eq. (41), but they are much narrower. Secondly, the growth rate of resonant solutions of Eq. (12) is much less (more than three times at maximum) than the Floquet exponent for the corresponding solutions of Eq. (41). We believe that, as in the case of radial motions, the high-order nonlinear terms in Eq. (12) tame the

resonance, which results in limiting the growth of oscillations and appearance of the long-period beats [Fig. 4(c)]. As expected, the maximum values of the amplitude and period of the beats were observed at the center of the resonance zone of Eq. (12). In this regard, our results are similar to those obtained in Ref. [35] for the case of orbital motion of a test particle inside an oscillating star with uniform density. The role of nonlinear terms was also discussed in Ref. [58] in the study of parametric resonance in Bose-Einstein condensates.

Summing up, we can conclude that in the absence of resonance, the finite motions of a test particle are well described in the linear approximation by the Hill equations. In the resonance zones, the nonlinearities suppress the resonance and limit the growth of the oscillation amplitude. Here, the resonant effects manifest themselves only in a change in the shape of the beats and must be described within the framework of the original nonlinear system.

When applied to the motion of short-period stars near the center of the Galaxy, where the density of dark matter is presumably greatest, the resonance phenomena we are discussing can manifest themselves in the form of characteristic oscillations (with the specific beats) of the star's radial velocity and as slow variations of the orbital eccentricity with a beating period. Because these effects are small, very precise and long-term measurements of the orbital parameters and radial velocity are required to

estimate the deviation of the star's radial velocity from that expected for stable orbital motion.

As concerns the central star cluster, however, so far this deviation has been evaluated only for the most well-studied short-period star S0-2 in the search for its binarity [59]. In this research, the radial velocity variations from the S0-2 orbital model have been studied using data from several dozen of radial velocity measurements but no significant periodic signals have been detected at the current level of measurement accuracy. The expected uncertainties with Extremely Large Telescope will be lowered by one order of magnitude or so, but as it was estimated in [35], only reduction of this uncertainties by two orders of magnitude can reveal fluctuations caused by the above mentioned effects of axion DM instability.

We hope that upcoming long-term spectroscopic studies of stars in the center of the Galaxy using new generation telescopes will provide more accurate data and confirm the ultra-light axion nature of dark matter or impose new constraints on the models under consideration.

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