Emergent modified gravity: The perfect fluid and gravitational collapse

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Canonical gravity is a formulation of general relativity where an evolving space reproduces the usual spacetime theory, and, unlike other formulations, it requires fewer assumptions—in particular, in the relationship between the gravitational field and the metric field. This freedom in the assumptions is the key to emergent modified gravity, a canonical theory in which the spacetime metric is not fundamental, but rather it is an emergent object composed of the truly fundamental fields. This peculiar feature of emergent modified gravity compared to other approaches can be used to model new effects, such as those provided by quantum gravity in an effective description. We discuss how matter fields can be coupled to emergent modified gravity, we realize the coupling of the (isentropic) perfect fluid, and we explicitly obtain the Hamiltonian of spherically symmetric systems and identify the symmetries of the coupled system. We formulate the Oppenheimer-Snyder collapse model in canonical terms as an adaptation of the fluid frame to the canonical foliation, permitting us to easily extend the conditions of the model to emergent modified gravity and obtain an exact solution to the dust collapsing from spatial infinity, including some effects suggested by quantum gravity in spherical symmetry. In this solution, the collapsing dust forms a black hole, and then reaches a minimum radius inside the black hole. While the geometry on this minimum-radius surface is regular in the vacuum, it turns out to be singular in the presence of dust. However, the fact that the geometry is merely emergent in this picture, and all the fundamental fields that compose the phase space are regular on such a surface, allows us to continue the canonical solution past it in a meaningful way, obtaining the global structure for the interior of the star. Thus, the matter reaches the minimum-radius surface with vanishing velocity and finite positive acceleration, and it proceeds to emerge outward, now behaving as a white hole. This star-interior solution can then be complemented by the vacuum solution describing the star-exterior region by a continuous junction at the star's radius. This gluing process can be viewed as the imposition of boundary conditions, which is nonunique and does not follow from solving the equations of motion alone. This ambiguity in the gluing of the two regions can give rise to different physical outcomes of the collapse compatible with the canonical dynamics. We discuss two such phenomena: the formation of a wormhole characterized by two distinct asymptotic regions, and the transition from a black hole to a white hole characterized by a single asymptotic region.

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I. INTRODUCTION

General relativity (GR), in its canonical formulation, is a constrained gauge field theory where the gauge transformations, which are generated by the Hamiltonian constraint and the diffeomorphism constraint, are equivalent to spacetime coordinate transformations only *on shell*—that is, when the gauge generators vanish. The spacetime is foliated by spacelike hypersurfaces, and the field content is given by a set of spatial tensors on these hypersurfaces, with flow equations generated by the same constraints determining the evolution of these fields between adjacent hypersurfaces. These evolving spatial tensors can then reproduce the usual spacetime tensors of GR. The action of the gauge generators can also be understood geometrically: the Hamiltonian constraint $H[N]$, smeared by a scalar lapse function N, generates a normal, infinitesimal hypersurface deformation with length N, while the diffeomorphism constraint $\vec{H}[\vec{N}]$, smeared by the spatial shift vector $\vec{N}$, generates a tangential hypersurface deformation with length $\vec{N}$.

In ADM notation [1,2], the diffeomorphism constraint $\vec{H}[\vec{N}]$ and the Hamiltonian constraint have Poisson brackets

$$\{\vec{H}[\vec{N}_1],\vec{H}[\vec{N}_2]\} = -\vec{H}[\mathcal{L}_{\vec{N}_2}\vec{N}_1], \tag{1}$$

$$\{H[N],\vec{H}[\vec{N}]\} = -H[N^b\partial_b N], \tag{2}$$

$$\{H[N_1],H[N_2]\} = -\vec{H}[q^{ab}(N_2\partial_b N_1 - N_1\partial_b N_2)] \tag{3}$$

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that depend not only on $\vec{N}$ and N, but also on the inverse of the spatial metric q_{ab} on a spatial hypersurface. When a phase-space function, such as q^{ab}, appears in the argument of the constraint algebra, it is called a structure function, in contrast to the structure constants of usual Lie algebras. This constraint algebra, also known as the hypersurface deformation algebra, is central to canonical GR, and it lies at the heart of general covariance, the principle on which GR is built. The early results of [3], where the vacuum was considered with the spatial metric being the only configuration variable, state that the Hamiltonian constraint, at second-derivative order, is uniquely determined by the hypersurface deformation algebra (1)–(3), and given by that of GR, up to the choice of Newton's and the cosmological constants [3,4], implying that no modifications are allowed. A crucial loophole to this conclusion lies in the common assumption that the spatial metric q_{ab} must be a configuration variable. Physically, this is the assumption that the metric is gravity itself—that it is a fundamental field. This is the lesson we learned from GR: spacetime is a field itself, and it is the gravitational field. But the recent detailed study of [5] shows that this assumption is not necessary to obtain a field theory describing spacetime. A fully consistent spacetime theory can be obtained by instead assuming a set of fundamental fields composing the phase space, and considering nonclassical constraints that still respect the form of the hypersurface deformation algebra (1)–(3) up to the structure function q^{ab} differing from the classical one. This new structure function, composed by the fundamental variables of the phase space but not identical to any one of them, is interpreted as the inverse of the spatial metric. This is an emergent spatial metric, and when embedded into a four-dimensional manifold, it gives rise to an emergent spacetime that is not gravity, but is made of gravity. This is the theory of emergent modified gravity (EMG).

We learned two key lessons on the nature of spacetime in [5]: (1) the hypersurface deformation algebra can be used as a mechanism to obtain the spacetime metric in terms of the truly fundamental fields, and (2) the anomaly freedom of the constraint algebra does not imply general covariance. These two lessons gave birth to EMG. Lesson 1 tells us that spacetime is not gravity, but rather it is made of gravity, which can be used to weaken the assumptions that lead to the uniqueness results of [3], and it allows for modified gravity theories different from GR; we refer to the resulting spacetime metric as the *emergent spacetime*. Lesson 2 tells us that not all modifications allowed by the anomaly freedom of the constraint algebra are indeed covariant; rather, further *covariance conditions* must be demanded to obtain a covariant modified gravity theory.

An explicit realization of EMG theories have so far been obtained only in the spherically symmetric reduced model [5]. An earlier, special case of EMG in spherical symmetry was studied in [6,7] where holonomy corrections were introduced, motivated by loop quantum gravity (LQG) [8–10], and shown to have a nonsingular black hole solution. The global structure of such a solution is an interuniversal wormhole joining a black hole to a white hole through their interiors. Another application of spherically symmetric EMG is the covariant realization of modified Newtonian dynamics (MOND) as a solution to the dark matter puzzle [11–13]. Earlier attempts to obtain modified gravity theories in spherical symmetry include [14–16] modeling inverse-triad corrections and holonomy corrections both motivated by LQG; however, only the former corrections are special cases of the EMG Hamiltonian, while the holonomy corrections of these last three references were found not to be covariant in [5].

EMG differs from other theories of emergent gravity in that the former does not consider the gravitational field as emergent. Instead, in EMG the gravitational field is indeed fundamental—it is the spacetime that is emergent, such that the degrees of freedom of GR may be preserved in EMG. One of the better-known emergent gravity theories is entropic gravity [17]—which is in turn inspired by the holographic principle and the AdS/CFT correspondence [18–20]—where the gravitational force is a consequence of the temperature and change of entropy due to the change in the amount of information associated with the displacement of matter. A similar stance is taken in [21], which focuses on the thermodynamic nature of GR and argues that the Einstein equations are in some sense an equation of state derived from the proportionality of entropy and the horizon area together with the fundamental thermodynamic relation $\delta Q = T\mathrm{d}S$. It is, however, not clear what the degrees of freedom in entropic gravity and related theories really are, for instance, in the vacuum.

The explicit applications of EMG have so far been all in the vacuum—that is, where gravity is the only fundamental field. Matter coupling in EMG is the extension of the vacuum theory to that of a larger phase space including matter fields. Just as covariance conditions had to be placed on the emergent spacetime metric, covariance conditions have to be identified for the physical, possibly emergent manifestations of the introduced matter fields. Such detailed study would allow us to ensure that the matter coupling and possible modifications are indeed covariant. This paper aims to address some of these subtleties on matter coupling in EMG, and it will be focused on the perfect fluid.

Despite being the "simplest" form of a matter field, the perfect fluid couples in a simple, intuitive way only in the conventional geometric approach to general relativity through the stress-energy tensor, but its extension to a Hamiltonian or even Lagrangian formalism quickly becomes difficult. Dust was introduced into canonical general relativity in [22] as timelike dust and in [23] as null dust; the inclusion of pressure for a timelike perfect fluid was introduced in [24] starting from the Lagrangian formulation followed by the ADM decomposition to obtain its canonical counterpart. Because EMG has been formulated only in its canonical form, and no action principle is available for the regaining process of the emergent spacetime, the introduction of the perfect fluid in purely canonical terms with no reliance on Lagrangians is paramount. We do this explicitly in the present paper for both the timelike and null fluids by retaining the picture of the fluid as a collection of particles. In particular, unlike [24], where pressure $P(n, s)$ enters as a function of the particle number density n and entropy density s, our analysis here shows that pressure can arise, not as $P(n, s)$, but as a function of the phase-space momenta in a nontrivial way with no reliance on the introduction of entropy as an independent phase-space variable. The perfect fluid we derive here is therefore an example of an isentropic fluid in this context.

The main application of spherically symmetric EMG is to black holes and to gravitational collapse. In GR, once a star exceeds the Tolman-Oppenheimer-Volkoff (TOV) limit [25], its pressure is insufficient to stop the collapse of matter, and a black hole is formed, inevitably leading to the formation of a spacetime singularity at its center [26,27], which is commonly interpreted as a breakdown of GR as a valid description of spacetime in the high-curvature regions. On the other hand, the astrophysical observations indicate much of the black hole's exterior seems well described by GR, which predicts the formation of horizons and their well-known properties, implying that a black hole can grow by absorbing the collapsing matter, but it cannot shrink because nothing comes out of it.

One clue for the black hole's fate came with the discovery of Hawking radiation [28]. This radiation carries some of the black hole's energy away, and therefore it shrinks or evaporates until it exhausts its mass. Black hole evaporation then leads to the information loss paradox, because the Hawking radiation has a thermal spectrum and does not carry any additional information about the matter that formed the black hole: the information is lost, violating the unitarity of quantum mechanics [29–31]. Thus, pure evaporation of the black hole via Hawking radiation cannot be the full story unless we accept the information loss.

This paradox seems to be a pathological consequence of the assumptions for black hole evaporation: (1) Using classical gravity (specifically, GR), (2) neglecting backreaction, and (3) the Hawking-radiated matter is different from the collapsing matter [32]. A way out of the information loss paradox is thus to consider a deviation from these assumptions—for instance, by using a nonclassical gravity theory or introducing quantum gravity effects. Furthermore, quantum gravity effects are expected to play a significant role in the resolution of the classical singularities of black holes. Under the assumption that these divergences are indeed resolved, new paradigms to the black hole's fate can be provided. We will use EMG as the underlying theory replacing GR. Furthermore, as will be explained in more detail later, there is a specific parameter in the general EMG spherically symmetric Hamiltonian that can be interpreted as effective elements of quantum gravity—referring, in particular, to the holonomy variables used in LQG. While such a parameter belongs to a more general result of EMG, and we use LQG only as an interpretational tool, we may refer to it as a *holonomy* or *quantum* parameter. The effects of this parameter are the ones responsible for the nonsingular behavior of the black hole solution in [7] and will also play a central role in the new dynamics of the dust collapse solution we present here. Also, note that the general conclusions of singularity formation [27] can potentially be circumvented in EMG because some assumptions of GR, including the positive energy theorem [33], need not apply to the equations of EMG, since these are different from Einstein's equations.

In the present paper, we obtain an exact solution to the collapse of dust in spherically symmetric EMG with a nontrivial "holonomy" parameter with the expectation that it will reveal new important properties to the above puzzles. The resulting scenario is that of the matter falling in and producing a black hole. The radius of the star then reaches a minimum value and "bounces" back, emerging outward with the properties of a white hole. To complete the global structure of the spacetime, this solution must be continuously glued to the exterior region, which must be a solution to the vacuum equations, by the shared boundary given by the star's radius. The gluing process does not follow uniquely from solving the equations of motion, and different consistent gluings can lead to different physical phenomena. The two proposals for the outcome of the collapse that we will focus on are the formation of wormholes rather than simple black holes as the result of the collapse [34,35], and the transition from a collapsing black hole to an exploding white hole as a result of quantum gravity effects [32,36].

Within GR, traversable wormhole solutions can be obtained only by including matter with exotic properties such as negative energy density, but they are possible without exotic matter when deviations from GR are considered. This is the case of modified GR with an R^2 action [37], as well as the example given above of spherically symmetric EMG [7]. Furthermore, both examples show a wormhole solution in the vacuum.

The black-to-white-hole transition proposal we will focus on is based on the work in [32,36], which can be described in a semiclassical treatment as follows: At the start of the collapse of matter, we may trust the classical theory, and the black hole is formed provided the critical mass is exceeded. At semiclassical regions characterized by higher curvature, the motion of matter and the gravitational field is modified due to quantum effects, possibly slowing down the collapse. At the highest-curvature regions, quantum effects are strong, and the semiclassical treatment might break down so that full quantum gravity may be needed for a detailed result. However, under the assumption that no physical divergences occur at the maximum-curvature regions, the collapsing matter will cross the would-be classical singularity surface and bounce, continuing its journey outwards. The matter that has crossed the would-be singularity is no longer collapsing, but expanding away from the black hole's center: this region then behaves as a white hole. The matter will then exit the horizon, and the black hole shrinks as it does so until the horizon disappears. In the full quantum gravity context, this process can be understood as a quantum transition from a black hole to a white hole with a transition amplitude associated to it that will depend on the quantum gravity model used. For an effort to compute this transition amplitude in the LQG approach, see Refs. [38–40]. Here, we will focus on the semiclassical treatment using spherically symmetric EMG as the theory providing the effective quantum gravity equations. Notice that this paradigm does not even require Hawking radiation, and the only major effects of its introduction would be to speed up the transition, and turn the collapsing and explosion phases asymmetric.

Previous results compatible with this paradigm include spherical models of null shells coupled to gravity where the classical solution collapsing into a black hole is connected to the (time-reversed) expanding solution emerging from a white hole via quantization [41] (see Ref. [42] for a canonical treatment of spherical null shells). The classical metric describing the exterior of such null shells was explicitly obtained in [36]. More recent work [43] studies the black-to-white-hole transition with the interior of the star described by an Oppenheimer-Snyder model modified by loop quantum cosmology (LQC) techniques [44]. While such a model can be useful to give us insights on what LQG may predict about this process, one cannot rely on the LQC equations being covariant, since homogeneity makes it impossible to address such a question. In this paper, we use the equations of EMG instead, which are covariant by construction, and several technical features of the resulting process differ significantly from those of [43].

In the last two examples, information falling into the black hole is not lost, because in the wormhole proposal it would simply emerge out in the next universe, while in the black-to-white transition proposal it would emerge out of the white hole after the transition. These two examples are therefore resolutions of the information loss paradox.

In this paper, we attempt to address all of the above issues in the context of EMG coupled to the perfect fluid. The organization of this paper is as follows: In Sec. II, we review canonical gravity and EMG, and we examine how the covariance conditions play a central role in defining the latter. In Sec. III, we briefly review how dust enters canonical gravity and identify the symmetries of the system associated with the dust that we will require to be preserved in EMG as additional conditions. In Sec. IV, we proceed to couple the perfect fluid to EMG by applying the covariance conditions in canonical form. We then obtain the explicit expression for the spherically symmetric Hamiltonian constraint for EMG in Sec. V. In Sec. VI, we formulate the Oppenheimer-Snyder model in canonical terms and then focus on the gravitational collapse of dust in EMG, obtaining an exact solution. Finally, in Sec. VII, we discuss the possible physical outcomes of the collapse compatible with this solution. We summarize the conclusions of this work in Sec. VIII.

II. EMERGENT MODIFIED GRAVITY

As usual in canonical theories, we assume that the spacetime, or the region of interest, is globally hyperbolic: $M = \Sigma \times \mathbb{R}$ with a three-dimensional "spatial" manifold Σ. Different choices of the embedding of Σ in M are parametrized by working with foliations of M into smooth families of spacelike hypersurfaces Σ_t, $t \in \mathbb{R}$, each of which is homeomorphic to Σ. For a given foliation, Σ can be embedded in M as a constant-time hypersurface: $\Sigma \cong \Sigma_{t_0} \cong (\Sigma_{t_0}, t_0) \hookrightarrow M$ for any fixed t_0.

Given a foliation into spacelike hypersurfaces Σ_t, a metric $g_{\mu\nu}$ on M defines the unit normal vector field n^μ on Σ_{t_0}, and it induces a unique spacelike metric $q_{ab}(t_0)$ on Σ_{t_0} by restricting the spacetime tensor $q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ to vector fields tangential to Σ_{t_0}, while $q_{\mu\nu} n^\nu = 0$. (We use greek letters for indices of spacetime tensors, and latin letters for indices of spatial tensors.) Time evolution then provides a family of spatial metrics, one for each hypersurface. Unambiguous evolution requires an additional structure that relates points between infinitesimally adjacent hypersurfaces, and this is provided by a time-evolution vector field

$$t^\mu = N n^\mu + N^a s_a^\mu \tag{4}$$

in spacetime, with the lapse function N and shift vector field N^a [2]. The three vector fields $s_a^\mu(t_0)$ inject $T\Sigma_{t_0}$ into TM such that $g_{\mu\nu} n^\mu s_a^\mu = 0$, playing the role of the spatial basis vectors on the spatial hypersurfaces. The lapse and shift describe, via (4), the frame of an observer in curved spacetime who measures the physical fields evolving on the hypersurfaces. The resulting spacetime metric or line element is then given by [2]

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + q_{ab}(\mathrm{d}x^a + N^a \mathrm{d}t)(\mathrm{d}x^b + N^b \mathrm{d}t). \tag{5}$$

Time-evolution and gauge transformations are described by the same flow via Poisson brackets, generated by the Hamiltonian constraint H and the diffeomorphism constraint H_a, differing only by their smearing function. Our notation here denotes H and H_a as the local versions of the constraints, and we use the square brackets to denote smearing by the function in the argument—i.e., $H[N] = \int \mathrm{d}x^3 H(x) N(x)$. Given a phase-space function $\mathcal{O}$, its infinitesimal gauge transformation is given by $\delta_\epsilon \mathcal{O} = \{\mathcal{O}, H[\epsilon^0, \epsilon^a]\}$, where $H[\epsilon^0, \epsilon^a] = H[\epsilon^0] + H_a[\epsilon^a]$, and ϵ^0 and ϵ^a are infinitesimal gauge parameters. The infinitesimal time evolution, on the other hand, is given by $\dot{\mathcal{O}} = \delta_t \mathcal{O} = \{\mathcal{O}, H[N, N^a]\}$—that is, lapse and shift play the role of gauge parameters in the time flow. In its role as the gauge flow generator, $H[\epsilon^0, \epsilon^a]$ must be a constraint: $H[\epsilon^0, \epsilon^a] = 0$ for all ϵ^0 and ϵ^a. Therefore, the dynamics is constrained too: $H[N, N^a] = 0$. We say that the physical solutions of the theory are "on shell,"—that is, the phase-space variables on each hypersurface that solve the equations of motion are such that the constraints vanish: $H[N] = 0$ and $H_a[N^a] = 0$.

The Poisson brackets do not directly provide the gauge transformations of N and N^a, because they do not have momenta; physically, this means that they do not evolve dynamically, but rather they specify the frame with respect to which evolution is defined. Instead, the gauge transformations of the lapse and shift are derived from the condition that the forms of the equations of motion of the phase space are gauge invariant. The gauge transformations obeying this condition are given by [45–47]

$$\delta_\epsilon N = \dot{\epsilon}^0 + \epsilon^a \partial_a N - N^a \partial_a \epsilon^0, \tag{6}$$

$$\delta_\epsilon N^a = \dot{\epsilon}^a + \epsilon^b \partial_b N^a - N^b \partial_b \epsilon^a + q^{ab}(\epsilon^0 \partial_b N - N \partial_b \epsilon^0). \tag{7}$$

We say that the spacetime (5) is covariant if and only if

$$\delta_\epsilon g_{\mu\nu}|_{\text{O.S.}} = \mathcal{L}_\xi g_{\mu\nu}|_{\text{O.S.}}, \tag{8}$$

where "O.S." means that we evaluate the expression on shell. The content of (8) is the condition that the canonical gauge transformations with the gauge parameters (ϵ^0, ϵ^a) reproduce diffeomorphisms of the spacetime metric with the 4-vector generator ξ^μ related by

$$\xi^\mu = \epsilon^0 n^\mu + \epsilon^a s^\mu_a = \xi^t t^\mu + \xi^a s^\mu_a, \tag{9}$$

$$\xi^t = \frac{\epsilon^0}{N}, \qquad \xi^a = \epsilon^a - \frac{\epsilon^0}{N} N^a. \tag{10}$$

The timelike components (tt and ta) of the spacetime-covariance condition (8) are automatically satisfied by the gauge transformations of the lapse and shift, (6) and (7), provided the covariance condition of the 3-metric, $\delta_\epsilon q_{ab}|_{\text{O.S.}} = \mathcal{L}_\xi q_{ab}|_{\text{O.S.}}$, is satisfied too. The latter, contrary to what is commonly stated, is not automatically satisfied just by virtue of the hypersurface deformation algebra; it can be simplified to the following series of conditions [5]:

$$\frac{\partial(\delta_{\epsilon^0} q^{ab})}{\partial(\partial_c \epsilon^0)}\bigg|_{\text{O.S.}} = \frac{\partial(\delta_{\epsilon^0} q^{ab})}{\partial(\partial_c \partial_d \epsilon^0)}\bigg|_{\text{O.S.}} = \cdots = 0, \tag{11}$$

where the series terminates on the highest-derivative order considered in the Hamiltonian constraint, which here we assume to be finite—that is, we assume a local theory.

If the phase space is composed of the spatial metric q_{ab} (the inverse of the structure function) as a configuration variable and its conjugate momenta p^{ab}, then the covariance condition (11) implies, from $\{q_{ab}, H[\epsilon^0]\} = \delta H[\epsilon^0]/\delta p^{ab}$, that the Hamiltonian constraint must not contain spatial derivatives of p^{ab}. If we use only up to second-order spatial derivatives of q_{ab}, the Hamiltonian constraint is uniquely determined by the hypersurface deformation algebra (1)–(3) up to the choice of Newton's constant and the cosmological constant [3,4]. It must therefore be the classical constraint of GR, and generally covariant modifications are ruled out under the stated conditions.

Here is where EMG differs from traditional canonical gravity. The assumption that the spatial metric q_{ab} is a configuration variable of the phase space is not necessary to obtain a field theory describing spacetime, and we may drop it altogether, except when trying to recover GR. Instead, we assume that the phase space is composed of certain fundamental fields different from the metric, and the metric is an emergent object to be regained by the following process. Leaving the diffeomorphism constraint unmodified, we allow for the Hamiltonian constraint to deviate from its classical expression, and we say it is modified. In vacuum, the only fundamental field is gravity. The Hamiltonian constraint in vacuum is then restricted to satisfy a hypersurface deformation algebra (1)–(3) where the structure function is allowed to be different (and not just related by a simple canonical transformation) from its classical expression. The inverse of the structure function obtained from such a procedure now plays the role of the *emergent* spatial metric, and it is used in (5) to define the *emergent* spacetime metric. The last step is to demand that this emergent metric and the modified constraint satisfy the covariance condition (11). This is EMG: a covariant theory of an emergent spacetime.

If one considers additional matter fields in the theory that present manifestations independently from the spacetime metric, then one has to make sure that such manifestations of the matter fields are covariant too. For example, if the matter field in consideration is described by some spacetime tensor f, then one has to apply the matter covariance condition on this field too:

$$\delta_\epsilon f|_{\text{O.S.}} = \mathcal{L}_\xi f|_{\text{O.S.}}. \tag{12}$$

Because the spacetime metric is emergent, it is allowed to depend not only on gravity, but also on the matter fields, as long as the anomaly freedom of the constraint algebra and all the covariance conditions are satisfied. Here, we will be interested in the perfect fluid by envisioning it as a collection of particles in the (emergent) spacetime; thus, it is fundamentally described by its 4-velocity u^μ (or covelocity u_μ). The matter covariance condition must then be applied to this quantity. For completeness, we will also place the covariance condition on the energy density, such that it transforms as a spacetime scalar.

III. CLASSICAL DUST IN CANONICAL GRAVITY

A. Pressureless dust

Unlike other matter fields, it is challenging to treat the perfect fluid—a model for collective particles—in both the Lagrangian and canonical formulations of general relativity due to the ambiguities related to the equation of state. This ambiguity stems from the pressure function not having a clear dependence on the generalized coordinates or the phase-space variables of the Lagrangian or canonical formulations, respectively. In [24], for instance, the pressure $P(n, s)$ is postulated as a function of the particle number density n and a new phase-space variable s they called entropy density. Our analysis here shows that pressure can be derived rather than postulated without introducing the entropy density s, and it is therefore an example of an isentropic perfect fluid in this context. Furthermore, as we will see below, the pressure function we derived here does not depend solely on the number density n, but rather on the “ratio” between the fluid’s radial and time momenta, as will become clear in Sec. IV. If one neglects the pressure, however, the fluid is called dust, and it is relatively easy to treat in both formulations, so this will be our starting point.

In the canonical formulation, the dust field is described by the coordinate fields $T(x)$ and $Z^i(x)$ (with $i = 1, 2, 3$) of its collective particles of rest mass μ as the configuration variables—note that these are different from the coordinates x of the manifold—and their respective conjugate momenta are denoted by $P^{(T)}(x)$ and $P_i^{(Z)}(x)$, representing the usual (densitized) energy and linear momentum (density) of the particles. The resulting Poisson brackets are therefore

$$\{T(\vec{x}), P^{(T)}(\vec{y})\} = \delta^3(\vec{x} - \vec{y}),$$
$$\{Z^i(\vec{x}), P_j^{(Z)}(\vec{y})\} = \delta^i_j \delta^3(\vec{x} - \vec{y}). \tag{13}$$

When coupled to gravity, the four canonical pairs $(T, P^{(T)})$, $(Z^i, P_i^{(Z)})$ add four degrees of freedom to the theory.

The configuration variables define a 1-form du, describing the covelocity field of the dust particles with components $u_\mu \equiv -\partial_\mu T - W_i \partial_\mu Z^i$, where $W_i = P_i^{(Z)}/P^{(T)}$ is the (internal) 3-velocity. The covelocity field satisfies the normalization $g^{\mu\nu} u_\mu u_\nu = -s$, where one picks $s = 1$ for timelike dust and $s = 0$ for null dust.

The classical diffeomorphism and Hamiltonian constraints contributions of the dust are [22,23]

$$H_a^{\rm matter} = P^{(T)} \partial_a T + P_i^{(Z)} \partial_a Z^i, \tag{14}$$

$$H_s^{\rm dust} = \sqrt{s (P^{(T)})^2 + q^{ab} H_a^{\rm matter} H_b^{\rm matter}}, \tag{15}$$

respectively, where q^{ab} is the structure function of the constraint algebra (3). Energy quantities as observed in the Eulerian frame adapted to the foliation (observers with 4-velocity n^μ) can be obtained directly from the constraints. The Eulerian energy density $\rho_s^{(\rm E)}$ is given by

$$\sqrt{\det q} \rho_s^{(\rm E)} \equiv \frac{\delta H_s^{\rm dust}[N]}{\delta N} = H_s^{\rm dust} = P^{(T)} \sqrt{s + q^{ab} u_a u_b}, \tag{16}$$

and the Eulerian dust current $J_a^{(\rm E)}$ by

$$\sqrt{\det q} J_a^{(\rm E)} \equiv -\frac{\delta \vec{H}^{\rm matter}[\vec{N}]}{\delta N^a} = -H_a^{\rm matter} = P^{(T)} u_a, \tag{17}$$

which can be extended to the 4-current $\sqrt{\det q} J_\mu^{(\rm E)} = P^{(T)} u_\mu$, where the normal component yields the mass density of the dust. Using the particle rest mass μ, we can then obtain the Eulerian particle number density,

$$\mu n_s^{(\rm E)} \equiv -n^\nu J_\nu = \frac{P^{(T)}}{\sqrt{\det q}}. \tag{18}$$

This result confirms our interpretation of $P^{(T)}$ as the (densitized) mass density. The Eulerian spatial stress tensor is given by

$$\sqrt{\det q} S_{ab}^{(\rm E)} \equiv \frac{2}{N} \frac{\delta H_s^{\rm dust}[N]}{\delta q^{ab}} = \frac{(P^{(T)})^2}{H_s^{\rm dust}} u_a u_b, \tag{19}$$

and the Eulerian pressure by

$$P^{(\rm E)} \equiv \frac{q^{ab} S_{ab}}{3} = \frac{1}{3} \frac{(P^{(T)})^2}{H_s^{\rm dust}} q^{ab} u_a u_b. \tag{20}$$

The dust has a relative velocity with respect to Eulerian observers, such that it is boosted with respect to them by the Lorentz factor

$$\gamma^{(\rm E)} = -n^\mu u_\mu = \frac{\sqrt{s (P^{(T)})^2 + q^{ab} H_a^{\rm matter} H_b^{\rm matter}}}{P^{(T)}}$$
$$= \sqrt{s + q^{ab} u_a u_b}, \tag{21}$$

where we have used the equations of motion for $\dot{T}$ and $\dot{Z}^i$. We can use this boost factor to obtain the energy density in the dust frame:

$$\rho_s^{\rm dust} = (\gamma^{(\rm E)})^{-2} \rho_s^{(\rm E)} = \frac{1}{\sqrt{\det q}} \frac{(P^{(T)})^2}{H_s^{\rm dust}}. \tag{22}$$

B. Classical symmetries

The dust constraints (14) and (15) have the following important symmetry generators:

The phase-space function

$$Q_0[\alpha] = \int \mathrm{d}^3x\, \alpha P^{(T)} = \int \mathrm{d}^3x\, \alpha \sqrt{\det q}\mu n_s^{(\mathrm{E})}, \qquad (23)$$

where α is a constant, and we used (18) to write the second equality, commutes with the dust constraints: $\{Q_0[\alpha], H^{\mathrm{dust}}[N]\} = \{Q_0[\alpha], \vec{H}^{\mathrm{matter}}[\vec{N}]\} = 0$. This implies that $\dot{Q}_0[\alpha] = 0$, which in turn implies that (23) is a conserved global charge. Taking $\alpha = 1$, we identify this charge as the total mass of the collective particles in the Eulerian frame, implying the conservation of Eulerian mass and particle number.

A second symmetry generator of the dust constraints is given by the three global charges

$$Q_i[\beta^i] = \int \mathrm{d}^3x\, \beta^i P_i^{(Z)} = \int \mathrm{d}^3x \sqrt{\det q}\beta^i W_i \mu n_s^{(\mathrm{E})}, \qquad (24)$$

where $\beta^i = (\beta^1, \beta^2, \beta^3)$ are constants. Choosing a unit internal vector $\beta^j = \hat{\beta}^j$, we identify this quantity as the total linear mass-flux component in the direction $\hat{\beta}^j$.

The third symmetry generator of the dust constraints is related to its SO(3) global symmetry, corresponding to the internal rotation of the dust variables. In particular, this infinitesimal transformation takes the form

$$Z^i \to Z^i + \epsilon^i{}_{jk}\theta^j Z^k, \qquad (25)$$

$$P_i^{(Z)} \to P_i^{(Z)} + \epsilon_{ij}{}^k \theta^j P_k^{(Z)}, \qquad (26)$$

where $\theta^i = (\theta^1, \theta^2, \theta^3)$ are constant parameters denoting the angle of rotation along an internal axis with direction $\hat{\theta}^i$, the totally antisymmetric tensor ϵ_{ijk} (where $\epsilon_{123} = 1$) is the Lie algebra generator $\tau_i \in \mathfrak{so}(3)$ in the defining representation, and we raise (and lower) internal indices with the Kronecker delta δ^{ij} (and δ_{ij}). This transformation is generated by the phase-space function

$$\begin{aligned} G_j[\theta^j] &= \int \mathrm{d}^3x\, \theta^j \epsilon_{jk}{}^i Z^k P_i^{(Z)} \\ &= \int \mathrm{d}^3x \sqrt{\det q}(\theta^j \epsilon_{jk}{}^i Z^k \mu W_i n_s^{(\mathrm{E})}) \\ &= \int \mathrm{d}^3x \sqrt{\det q}(\vec{\theta} \cdot \vec{Z} \times (\vec{W}\mu n_s^{(\mathrm{E})})). \end{aligned} \qquad (27)$$

Choosing a unit internal vector $\theta^j = \hat{\theta}^j$, we identify this quantity as the total angular mass-flux component in the direction $\hat{\theta}^j$.

While the dust constraints (14) and (15) contain even more symmetries, the three global charges (23), (24), and (27) are basic conserved quantities that must hold in the generalization to the perfect fluid (at least for pressure functions that arise from conservative interactions between the fluid particles) and, furthermore, in EMG too, since we want to retain the picture of the perfect fluid as a collection of particles.

In the following, we will rederive the dust Hamiltonian constraint H_s^{dust} from its basic properties such as normalization and covariance conditions, and we will then generalize it to that of the perfect fluid, which includes the pressure in the form of an equation of state even in the context of EMG. We will refer to this more general constraint contribution as H_s^{matter}. The steps of this procedure are the following: We demand that the full (modified) constraints $H = H^{\mathrm{grav}} + H_s^{\mathrm{matter}}$ and $H_a = H_a^{\mathrm{grav}} + H_a^{\mathrm{matter}}$ form an anomaly-free hypersurface deformation algebra with (emergent) structure function q^{ab}; we demand that the (emergent) spacetime metric $g_{\mu\nu}$ be covariant, as well as the covelocity of the fluid u_μ, for which a further covariance condition will be formulated in the next section; we demand that the fluid's velocity be normalized; and, finally, we demand that the phase-space functions (23), (24), and (27) remain as symmetry generators in the modified theory, thus preserving the conserved quantities and their physical meaning. It is easier to consistently realize all of these demands in reverse order due to their increasing difficulty, and this is the approach we will take in the following sections. After imposing these conditions, we will solve them exactly, starting with the most general constraint ansatz with arbitrary dependence on the fluid's phase-space variables and the first-order derivatives of its configuration variables, while also allowing the structure function and the gravitational constraint contribution to depend on the fluid's configuration variables. Lastly, we will place a covariance condition on the energy density ρ_s such that it transforms as a spacetime scalar. We will also show that the pressure P will appear as an emergent property of the fluid, and its covariant transformation as a scalar will be implied by those of $g_{\mu\nu}$, u_μ, and ρ_s. We will do all of this in purely canonical grounds so that an underlying action is not necessary for the existence of the covariant Hamiltonian and, furthermore, so that the results hold for EMG too.

IV. THE PERFECT FLUID COUPLING

Throughout the first half of this section, we assume that the spacetime-covariance condition is already satisfied, because it simplifies the analysis of the other conditions. There is no loss of generality in making this assumption, because one must still implement it when deriving the explicit Hamiltonian constraint. We return to this condition in Sec. IV D.

A. Symmetry conditions

We start by defining our Hamiltonian constraint ansatz in such a way that we can split the constraint in the form $H = H^{\mathrm{grav}} + H_s^{\mathrm{matter}}$, where they have the dependence

$$\begin{aligned} &H^{\mathrm{grav}}(T, Z^i), \\ &H_s^{\mathrm{matter}}(T, Z^i, P^{(T)}, P_i^{(Z)}, \partial_a T, \partial_a Z^i), \end{aligned} \tag{28}$$

where we have suppressed the possible dependence on the gravitational variables to ease the notation. We also assume that in the vacuum limit defined by $T, Z^i, P^{(T)}, P_i^{(Z)} \to 0$, we obtain $H \to H^{\mathrm{grav}}$, with the latter depending only on the gravitational variables—therefore, H^{grav} has a more complicated dependence on the gravitational variables than H_s^{matter} does.

We now demand that the constraint ansatz (28) commute with the symmetry generators (23), (24), and (27). Due to the common complexity of H^{grav}, and in view that it depends on the gravitational variables much more heavily than H_s^{matter} does, we will assume that the symmetry generators (23), (24), and (27) must commute with H^{grav} and H_s^{matter} independently.

We obtain

$$\{Q_0[\alpha], H[\epsilon^0]\} = -\int \mathrm{d}^3x\, \alpha\epsilon^0 \frac{\partial H}{\partial T}, \tag{29}$$

$$\{Q_i[\beta^i], H[\epsilon^0]\} = -\int \mathrm{d}^3x\, \beta^i\epsilon^0 \frac{\partial H}{\partial Z^i}, \tag{30}$$

where we have integrated out the first-order derivative term and neglected boundary terms. Thus, $\{Q_0[\alpha], H^{\mathrm{grav}}[\epsilon^0]\} = 0$ and $\{Q_i[\beta^i], H_s^{\mathrm{matter}}[\epsilon^0]\} = 0$, for arbitrary α, β^i, and ϵ^0, reduce the phase-space dependence of the constraint ansatz (28) to

$$H_s^{\mathrm{matter}}(P^{(T)}, P_i^{(Z)}, \partial_a T, \partial_a Z^i), \tag{31}$$

with H^{grav} now completely independent of the fluid's variables. Therefore, H^{grav} is identical to the vacuum Hamiltonian constraint.

Similarly, the implementation of the symmetry generator (27) further reduces the phase-space dependence of the constraint ansatz to

$$H_s^{\mathrm{matter}}(P^{(T)}, \partial_a T, P_i^{(Z)}\partial_a Z^i, (\vec{P}^{(Z)})^2, (\partial\vec{Z})^2), \tag{32}$$

where $(\vec{P}^{(Z)})^2 = \delta^{ij} P_i^{(Z)} P_j^{(Z)}$ and $(\partial\vec{Z})^2 = \delta_{ij}\partial_a Z^i \partial_b Z^j$.

B. Normalization condition

The ADM decomposition of the normalization $g^{\alpha\beta}u_\alpha u_\beta = -s$, where we use the emergent inverse metric for this expression, can be rewritten as

$$u_t = N^b u_b - N\sqrt{s + q^{ab}u_a u_b}, \tag{33}$$

where the sign of the square root was chosen to preserve a negative (that is, future-pointing) u_t even in the case $N^b = 0$. Here, q^{ab} is the structure function of the resulting hypersurface deformation algebra—that is, the emergent one.

Using the expressions

$$u_t = -\dot{T} - \frac{P_i^{(Z)}}{P^{(T)}}\dot{Z}^i, \tag{34}$$

$$u_a = -H_a^{\mathrm{matter}}/P^{(T)}, \tag{35}$$

the normalization places a restriction on the equations of motion of the dust: Taking Hamilton's equations of motion $\dot{T} = \{T, H[N, \vec{N}]\}$—and similarly for Z^i—with unmodified diffeomorphism constraint, the normalization expression (33) can be written as

$$\begin{aligned} P^{(T)}\{T, H[N]\} &+ P_i^{(Z)}\{Z^i, H[N]\} \\ &= N\sqrt{s(P^{(T)})^2 + q^{ab}H_a^{\mathrm{matter}}H_b^{\mathrm{matter}}} \\ &=: NH_s^{\mathrm{dust}}, \end{aligned} \tag{36}$$

where H_s^{dust} differs from the classical expression (15), in that the classical structure function q^{ab} may be replaced by the emergent one. We shall use H_s^{dust} in this context in the following.

Using the constraint ansatz compatible with the symmetry conditions given by (32), the Eq. (36), which must be satisfied for arbitrary N, simplifies into the condition

$$P^{(T)}\frac{\mathrm{d}H}{\mathrm{d}P^{(T)}} + P_i^{(Z)}\frac{\mathrm{d}H}{\mathrm{d}P_i^{(Z)}} = H_s^{\mathrm{dust}}, \tag{37}$$

where "d" stands for total derivative. The solution to the normalization condition then reduces the phase-space dependence of the ansatz (32) to

$$H_s^{\mathrm{matter}} = H_s^{\mathrm{dust}} - \bar{f}((P^{(Z)})^2/(P^{(T)})^2, \partial_a T, (\partial\vec{Z})^2) \tag{38}$$

for some undetermined function $\bar{f}$.

C. Matter covariance conditions I

We now impose the covariance condition on the fluid's covelocity field:

$$\delta_\epsilon u_\mu|_{\mathrm{O.S.}} = \mathcal{L}_\xi u_\mu|_{\mathrm{O.S.}}. \tag{39}$$

To this end, we perform the ADM decomposition of its infinitesimal coordinate transformation:

$$\mathcal{L}_\xi u_\mu = \frac{\epsilon^0}{N}\dot{u}_\mu + \left(\epsilon^a - \frac{\epsilon^0}{N}N^a\right)\partial_a u_\mu + u_t\partial_\mu\left(\frac{\epsilon^0}{N}\right) + u_a\partial_\mu\left(\epsilon^a - \frac{\epsilon^0}{N}N^a\right). \tag{40}$$

Explicitly, the components are

$$\begin{aligned}\mathcal{L}_\xi u_a &= \frac{\epsilon^0}{N}\left(\dot{u}_a - \frac{u_t - N^b u_b}{N}\partial_a N - (N^b\partial_b u_a + u_b\partial_a N^b)\right) + \frac{u_t - N^b u_b}{N}\partial_a\epsilon^0 + \epsilon^b\partial_b u_a + u_b\partial_a\epsilon^b \\ &= \frac{\epsilon^0}{N}\left(\dot{u}_a + \sqrt{s + q^{ab}u_a u_b}\partial_a N - (N^b\partial_b u_a + u_b\partial_a N^b)\right) - \sqrt{s + q^{ab}u_a u_b}\partial_a\epsilon^0 + \epsilon^b\partial_b u_a + u_b\partial_a\epsilon^b, \end{aligned}\tag{41}$$

$$\begin{aligned}\mathcal{L}_\xi u_t &= \frac{\epsilon^0}{N}\partial_t(u_t - u_a N^a) + \left(\epsilon^a - \frac{\epsilon^0}{N}N^a\right)\partial_a u_t + \frac{u_t - N^a u_a}{N}\left(\dot{\epsilon}^0 - \frac{\epsilon^0}{N}\dot{N}\right) + u_a\dot{\epsilon}^a + \frac{\epsilon^0}{N}N^a\dot{u}_a \\ &= -\epsilon^0\partial_t\sqrt{s + q^{ab}u_a u_b} + \left(\epsilon^a - \frac{\epsilon^0}{N}N^a\right)\partial_a u_t - \sqrt{s + q^{ab}u_a u_b}\dot{\epsilon}^0 + u_a\dot{\epsilon}^a, \end{aligned}\tag{42}$$

where we use the normalization expression (33) to simplify these components into their second lines.

Also, the following calculation will be useful:

$$\begin{aligned}\{u_a, H[N]\} &= -\partial_a(\{T, H[N]\}) - \frac{P_i^{(Z)}}{P^{(T)}}\partial_a(\{Z^i, H[N]\}) - \left\{\frac{P_i^{(Z)}}{P^{(T)}}, H[N]\right\}\partial_a Z^i \\ &= -\partial_a\left(\{T, H[N]\} + \frac{P_i^{(Z)}}{P^{(T)}}\{Z^i, H[N]\}\right) + \{Z^i, H[N]\}\partial_a\left(\frac{P_i^{(Z)}}{P^{(T)}}\right) - \left\{\frac{P_i^{(Z)}}{P^{(T)}}, H[N]\right\}\partial_a Z^i \\ &= -\partial_a\left(\frac{N H_s^{\rm dust}}{P^{(T)}}\right) + \{Z^i, H[N]\}\partial_a\left(\frac{P_i^{(Z)}}{P^{(T)}}\right) - \left\{\frac{P_i^{(Z)}}{P^{(T)}}, H[N]\right\}\partial_a Z^i, \end{aligned}\tag{43}$$

where we use the normalization condition (36) in the last line.

We will now focus on the spatial component of the covariance condition (41). Using Hamilton's equations of motion $\dot{u}_a = \{u_a, H[N] + H_b[N^b]\}$, (41) becomes

$$\frac{1}{\epsilon^0}\left(\frac{P^{(T)}}{H_s^{\rm dust}}\{u_a, H[\epsilon^0]\} + \partial_a\epsilon^0\right)\bigg|_{\rm O.S.} = \frac{1}{N}\left(\frac{P^{(T)}}{H_s^{\rm dust}}\{u_a, H[N]\} + \partial_a N\right)\bigg|_{\rm O.S.}. \tag{44}$$

Using (43), it can be rewritten into

$$\frac{1}{\epsilon^0}(\{Z^i, H[\epsilon^0]\}\partial_a W_i - \{W_i, H[\epsilon^0]\}\partial_a Z^i)|_{\rm O.S.} = \frac{1}{N}(\{Z^i, H[N]\}\partial_a W_i - \{W_i, H[N]\}\partial_a Z^i)|_{\rm O.S.}.$$

Upon substitution of the ansatz (32), this condition implies, for arbitrary ϵ^0 and N, the equation

$$P^{(T)}\frac{{\rm d}H^{\rm matter}}{{\rm d}(\partial_b Z^i)} - P_i^{(Z)}\frac{{\rm d}H^{\rm matter}}{{\rm d}(\partial_b T)}\bigg|_{\rm O.S.} = 0, \tag{45}$$

where "d" stands for total derivative. The solution to this equation implies that the phase-space dependence on the configuration variables' derivatives of the constraint ansatz is of the form $H_s^{\rm matter}(P^{(T)}\partial_a T + P_i^{(Z)}\partial_a Z^i)$. Consistency between this solution to the covariance condition and the solution to the normalization condition (38) yields

$$H^{\rm matter} = H_s^{\rm dust} - \bar{f}((P^{(Z)})^2/(P^{(T)})^2). \tag{46}$$

We now focus on the time component of the covariance condition $\delta_\epsilon u_t|_{\rm O.S.} = \mathcal{L}_\xi u_t|_{\rm O.S.}$, which, using (6), (7), (33), (42), and (44), becomes

$$\frac{1}{\epsilon^0}\left\{\sqrt{s + q^{ab}u_b u_c}, H[\epsilon^0]\right\} - u_a q^{ab}\frac{\partial_b\epsilon^0}{\epsilon^0}\bigg|_{\rm O.S.} = \frac{1}{N}\left\{\sqrt{s + q^{ab}u_b u_c}, H[N]\right\} - u_a q^{ab}\frac{\partial_b N}{N}\bigg|_{\rm O.S.}. \tag{47}$$

Using the spacetime-covariance condition (11), this further simplifies to

$$\frac{u_a q^{ab}}{\epsilon^0}\left(\frac{P^{(T)}}{H_s^{\rm dust}}\{u_a, H[\epsilon^0]\} - \partial_b\epsilon^0\right)\Big|_{\rm O.S.} = \frac{u_a q^{ab}}{N}\left(\frac{P^{(T)}}{H_s^{\rm dust}}\{u_a, H[N]\} - \partial_b N\right)\Big|_{\rm O.S.}. \quad (48)$$

If the spatial covariance condition (44) is satisifed, then (48) is automatically satisfied too; thus, it does not imply an independent equation.

D. Anomaly freedom and spacetime covariance

Note that the undetermined function $\bar{f}$ in (46) can still depend on the structure function, which we have suppressed so far for notational ease. This dependence can be fully addressed by the requirement of anomaly freedom.

Anomaly freedom is realized if the Hamiltonian constraint satisfies the brackets (1)–(3). The bracket (1) is trivially satisfied because we have assumed the classical form of the diffeomorphism constraint. For the bracket (2) to be satisfied, the Hamiltonian constraint must be a density-weight-one function. We assume this is the case for the gravitational contribution. On the other hand, imposing that the matter contribution of the form (46) is a density-weight-one function fixes its dependence on the structure function,

$$\bar{f} = \sqrt{\det q}\, P((P^{(Z)})^2/(P^{(T)})^2), \quad (49)$$

where P is an undetermined function of $\delta^{ij}W_iW_j$, and it may also depend on the gravitational variables as long as it remains a density-weight-zero. We will discuss the physical meaning of the function P below and conclude that it in fact plays the role of pressure. For now, we simply note that P is a function of $W^2 = (P^{(Z)})^2/(P^{(T)})^2$, the magnitude of the internal velocity of the fluid's particles. This is precisely one of the basic ways to understand pressure, as an effect of particle flow.

We now focus on the spacetime-covariance condition (11) and the last bracket (3), which is the most complicated one. We first note that since $H^{\rm grav}$ is independent of the fluid's variables as found in the previous conditions, it follows that the gravitational contribution must already satisfy $\{H^{\rm grav}[N_1], H^{\rm grav}[N_2]\} = -\vec{H}^{\rm grav}[q^{ab}(N_2\partial_b N_1 - N_1\partial_b N_2)]$ by itself, and this in turn implies that q^{ab} is independent of the fluid's variables. Since the covariance condition for the structure function (11) involves the full constraint $H = H^{\rm grav} + H_s^{\rm matter}$, but both q^{ab} and $H^{\rm grav}$ are independent of the fluid's variables, it follows that Eq. (11) must hold if one replaces H with either of the two contributions independently. We assume that the gravitational contribution already satisfies this covariance condition. On the other hand, the fluid's contribution to this condition becomes

$$\frac{\partial\{q^{ab}, P[\sqrt{\det q}\,\epsilon^0]\}_{\rm grav}}{\partial(\partial_c\epsilon^0)}\Big|_{\rm O.S.} = \frac{\partial\{q^{ab}, P[\sqrt{\det q}\,\epsilon^0]\}_{\rm grav}}{\partial(\partial_c\partial_d\epsilon^0)}\Big|_{\rm O.S.} = \cdots = 0, \quad (50)$$

where we use the subscript "grav" to restrict the action of the Poisson brackets to the gravitational variables.

Coming back to the bracket (3), a direct evaluation shows that

$$\{H_s^{\rm matter}[N_1], H_s^{\rm matter}[N_2]\}_{\rm matter} = -\vec{H}^{\rm matter}[q^{ab}(N_2\partial_b N_1 - N_1\partial_b N_2)], \quad (51)$$

where the subscript "matter" restricts the action of the Poisson brackets to the matter variables. Furthermore, using the spacetime-covariance condition, we find that $\{H^{\rm grav}[N_1], H_s^{\rm matter}[N_2]\} = -\{H^{\rm grav}[N_1], P[\sqrt{\det q}N_2]\}_{\rm grav}$. Therefore, for the bracket (3) to hold for the full Hamiltonian constraint, we obtain the following condition on the modified pressure function:

$$\{P[\sqrt{\det q}N_1], H[N_2]\} - \{P[\sqrt{\det q}N_2], H[N_1]\} = \{P[\sqrt{\det q}N_2], P[\sqrt{\det q}N_1]\}. \quad (52)$$

The two conditions (50) and (52) are nontrivial, and there may be room for the pressure function to receive modifications—note that the action of the Poisson brackets on the structure function in (52) always vanishes because the spacetime-covariance condition requires that its bracket with $H^{\rm grav}$ and P not generate spatial derivatives of N_1 and N_2, while the nonderivative terms cancel due to the antisymmetry of the bracket.

E. Pressure regained

Generally, in EMG, the metric no longer enters the Hamiltonian directly as a phase-space variable, and therefore, Einstein's equations no longer hold in the sense that a reconstructed Lagrangian is not necessarily minimized with respect to variations of the metric, and one does not recover Einstein's equations except in the classical limit. However, when the matter Hamiltonian resembles the classical one, then the same classical stress-energy components are obtained—up to the emergent metric. This is the case for the constraint (68) when the P function is independent of the gravitational variables (which we assume in this subsection to extract its physical meaning), and this will help us in interpreting the variables of the modified theory with the more familiar stress-energy components. The following canonical stress-energy components are the values that Eulerian observers would measure—that is, in the foliation basis, for which we will use the components

$$u_0 \equiv n^\mu u_\mu = -\frac{\sqrt{s(P^{(T)})^2 + q^{ab} H_a^{\rm matter} H_b^{\rm matter}}}{P^{(T)}}, \tag{53}$$

$$u_a = -\frac{H_a^{\rm matter}}{P^{(T)}}, \tag{54}$$

$$g_{00} \equiv n^\mu n^\nu g_{\mu\nu} = -1. \tag{55}$$

The energy density is

$$\rho^{(\rm E)} \equiv \frac{1}{\sqrt{\det q}} \frac{\delta H_s^{\rm matter}[N]}{\delta N} = \rho_s^{\rm dust} u_0 u_0 - P, \tag{56}$$

where the expression for $\rho_s^{\rm dust}$ is given by (22) by replacing the classical structure function with the emergent one. The mass current density is

$$J_a^{(\rm E)} \equiv \frac{1}{\sqrt{\det q}} \frac{\delta \vec{H}^{\rm matter}[\vec{N}]}{\delta N^a} = -\rho_s^{\rm dust} u_0 u_a, \tag{57}$$

and the spatial stress tensor is

$$S_{ab}^{(\rm E)} \equiv \frac{2}{N\sqrt{\det q}} \frac{\delta H_s^{\rm matter}[N]}{\delta q^{ab}} = \rho_s^{\rm dust} u_a u_b + q_{ab} P. \tag{58}$$

The pressure is

$$P^{(\rm E)} \equiv \frac{1}{3} q^{ab} S_{ab} = \frac{1}{3} \rho_s^{\rm dust} q^{ab} u_a u_b + P. \tag{59}$$

We now compare the canonical energy results to the respective components of the stress-energy tensor $T_{\mu\nu} = \rho_s u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu)$. The normal component is

$$T_{00} \equiv n^\mu n^\nu T_{\mu\nu} = (\rho_s + P) u_0 u_0 - P. \tag{60}$$

A comparison with the canonical energy density, equating $T_{00} = \rho_s^{(\rm E)}$, then establishes

$$\rho_s = \rho_s^{\rm dust} - P \tag{61}$$

and confirms P as the pressure function. This result shows that the energy has a kinetic contribution $\rho_s^{\rm dust}$ from the fluid particles and an interaction contribution P from the pressure. The normal-spatial components,

$$T_{0a} \equiv n^\mu s_a^\nu T_{\mu\nu} = -(\rho_s + P) u_0 u_a = -\rho_s^{\rm dust} u_0 u_a, \tag{62}$$

then agree with the Eulerian mass current upon equating $T_{0a} = J_a^{(\rm E)}$. Finally, the spatial components

$$T_{ab} = (\rho_s + P) u_a u_b + q_{ab} P = \rho_s^{\rm dust} u_a u_b + q_{ab} P \tag{63}$$

also agree with the canonical stress $T_{ab} = S_{ab}^{(\rm E)}$. The pressure and the freedom of the equation of state have been, thus, regained.

F. Matter covariance conditions II

As a final consistency check, we want to make sure that the energy density (61) is a spacetime scalar, which is equivalent to the statement that $\rho_s^{\rm dust}$ and P are spacetime scalars—that is, that their gauge transformations correspond to the infinitesimal coordinate transformation of a scalar field. This is the case if and only if the equation $\delta_\epsilon \rho_s^{\rm dust}|_{\rm O.S.} = \mathcal{L}_\xi \rho_s^{\rm dust}|_{\rm O.S.}$, and similarly for P, is satisfied. Performing the ADM decomposition of this equation, and using Hamilton's equations of motion, it can be simplified to

$$\frac{1}{\epsilon^0} \{\rho_s^{\rm dust}, H[\epsilon^0]\}|_{\rm O.S.} = \frac{1}{N} \{\rho_s^{\rm dust}, H[N]\}|_{\rm O.S.}. \tag{64}$$

Using the spacetime and matter covariance conditions, (11) and (44), and the fact that $\rho_s^{\rm dust}$ is independent of the gravitational variables, this can be written as

$$\frac{1}{\epsilon^0} \left(\{P^{(T)}, H_s^{\rm matter}[\epsilon^0]\} - \frac{P^{(T)}}{H_s^{\rm dust}} q^{ab} H_a^{\rm matter} \partial_b \epsilon^0 \right)\Big|_{\rm O.S.} = \frac{1}{N} \left(\{P^{(T)}, H_s^{\rm matter}[N]\} - \frac{P^{(T)}}{H_s^{\rm dust}} q^{ab} H_a^{\rm matter} \partial_b N \right)\Big|_{\rm O.S.}.$$

This equation is automatically satisfied by the constraint (68). Similarly, the covariance condition for the pressure P to transform as a spacetime scalar is obtained from (64) by replacing $\rho_s^{\rm dust}$ with P. This condition then reduces to

$$\frac{\partial(\{P, H[\sqrt{\det q}\epsilon^0]\})}{\partial(\partial_c \epsilon^0)}\Big|_{\rm O.S.} = \frac{\partial\{P, H[\sqrt{\det q}\epsilon^0]\}}{\partial(\partial_c \partial_d \epsilon^0)}\Big|_{\rm O.S.} = \cdots = 0, \tag{65}$$

where we have used (50). The anomaly-freedom condition (52) and the pressure-covariance condition (65) can be combined and respectively simplified to

$$\{P[\sqrt{\det q} N_2], P[\sqrt{\det q} N_1]\} = 0, \tag{66}$$

$$\frac{\partial\{P, H^{\rm grav}[\sqrt{\det q}\epsilon^0]\}}{\partial(\partial_c \epsilon^0)}\Big|_{\rm O.S.} = \frac{\partial\{P, H^{\rm grav}[\sqrt{\det q}\epsilon^0]\}}{\partial(\partial_c \partial_d \epsilon^0)}\Big|_{\rm O.S.} = \cdots = 0. \tag{67}$$

Therefore, the conditions for symmetry, normalization, anomaly freedom, and covariance of the 4-velocity imply the covariance of the energy density, but the pressure requires the additional condition (65) to transform as a spacetime scalar. When the pressure function is independent of the gravitational variables, this condition is automatically satisfied.

This completes the implementation of all the symmetry, anomaly-freedom, and covariance conditions on the fluid's Hamiltonian constraint contribution, which acquires the form

$$H_s^{\rm matter} = H_s^{\rm dust} - \sqrt{\det q}P, \tag{68}$$

where P is an undetermined function of $\delta^{ij}W_iW_j$ and the gravitational variables, such that it preserves its density-weight-zero and satisfies the conditions (50), (66), and (67). An evaluation of these conditions requires specifying the gravitational phase space and constraint contribution, which we cannot do in the general case, but we will be able to evaluate them in the spherically symmetric case.

The timelike case $s = 1$ of the Hamiltonian constraint for the perfect fluid (68) is the same constraint—up to the emergent metric and the pressure's dependence on the gravitational variables—obtained via the ADM decomposition of the action of the perfect fluid [24], but it does not require the introduction of entropy as an extra phase-space variable, and unlike in this reference, the P function we derived does not depend solely on the number density (18).

V. EMERGENT MODIFIED GRAVITY: SPHERICAL SYMMETRY

A. Vacuum

In the spherically symmetric theory in vacuum, the spacetime metric takes the general form

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + q_{xx}(\mathrm{d}x + N^x\mathrm{d}t)^2 + q_{\vartheta\vartheta}\mathrm{d}\Omega^2. \tag{69}$$

The classical spatial metric components can be written in terms of the classical radial and angular densitized triads E^x and E^φ, respectively: $q_{xx} = (E^\varphi)^2/E^x$ and $q_{\vartheta\vartheta} = E^x$.

The symplectic structure of the canonical theory is

$$\{K_x(x), E^x(y)\} = \{K_\varphi(x), E^\varphi(y)\} = \delta(x-y), \tag{70}$$

where K_x and K_φ are the radial and angular components of the extrinsic curvature. Within spherical symmetry, only the Hamiltonian constraint and the radial diffeomorphism constraint are nontrivial. Consequently, the hypersurface deformation algebra (1)–(3) simplifies to

$$\{H_x[N_1^x], H_x[N_2^x]\} = -H_x[\mathcal{L}_{N_2^x}N_1^x], \tag{71}$$

$$\{H[N], H_x[N^x]\} = -H[N^xN'], \tag{72}$$

$$\{H[N_1], H[N_2]\} = -H_x[q^{xx}(N_2N_1' - N_1N_2')], \tag{73}$$

where the prime denotes a derivative with respect to the radial coordinate x, and $q^{xx} = E^x/(E^\varphi)^2$ is simply the inverse of the classical spatial metric component q_{xx}. The algebra of (71)–(73) holds even in the presence of matter—when gauge fields are present, typically the third bracket receives a contribution from the respective Gauss constraint, but we shall neglect this in the following, since the perfect fluid is not a gauge field. We denote the constraints in vacuum by $H^{\rm grav}$ and $H_x^{\rm grav}$.

In emergent modified gravity, we keep $H_x^{\rm grav}$ in its classical form,

$$H_x^{\rm grav} = E^\varphi K_\varphi' - K_x(E^x)', \tag{74}$$

such that the bracket (71) is unmodified. However, we allow for the Hamiltonian constraint $H^{\rm grav}$ to be nonclassical. According to the regaining procedure of EMG, we demand that the modified $H^{\rm grav}$ be anomaly free, such that it reproduces the brackets (72) and (73) up to a modified q^{xx} such that its phase-space dependence differs from the classical one. In practice, we start with a general ansatz for $H^{\rm grav}$ as a function of the phase-space variables and their derivatives up to second order [5,6]. Imposing the anomaly freedom of the hypersurface deformation algebra [Eqs. (71)–(73)] restricts this ansatz severely by providing a set of differential equations it must satisfy, and it determines the expression for q^{xx}. The inverse of this modified structure function is then replaced in the line element (69). Unlike q^{xx}, the angular component $q^{\vartheta\vartheta}$ cannot be recovered from the hypersurface deformation algebra, because the underlying spherical symmetry trivializes the angular diffeomorphism constraints. Therefore, we simply take the classical value $q^{\vartheta\vartheta} = 1/E^x$ and use its inverse in (69). This determines the emergent metric.

Once the emergent metric has been identified by imposing anomaly freedom, we must impose the covariance condition (11), which in spherical symmetry reduces to the simpler conditions

$$\frac{\partial(\delta_{\epsilon^0}q^{xx})}{\partial(\epsilon^0)'}\bigg|_{\rm O.S.} = \frac{\partial(\delta_{\epsilon^0}q^{xx})}{\partial(\epsilon^0)''}\bigg|_{\rm O.S.} = \cdots = 0,$$
$$\frac{\partial(\delta_{\epsilon^0}q^{\vartheta\vartheta})}{\partial(\epsilon^0)'}\bigg|_{\rm O.S.} = \frac{\partial(\delta_{\epsilon^0}q^{\vartheta\vartheta})}{\partial(\epsilon^0)''}\bigg|_{\rm O.S.} = \cdots = 0. \tag{75}$$

The resulting equations of the anomaly-freedom and covariance conditions are restrictive enough that they can all be solved exactly in the vacuum for the most general constraint ansatz, where the phase-space dependence involves up to second-order derivatives; the most general vacuum Hamiltonian constraint in spherical symmetry is given by [5]

$$
\begin{aligned}
H^{\rm grav} = -\lambda_0 \frac{\sqrt{E^x}}{2} \Bigg[& E^\varphi \left(-\Lambda_0 + \frac{\alpha_0}{E^x} + \left(c_f \frac{\alpha_2}{E^x} + 2 \frac{\partial c_f}{\partial E^x} \right) \frac{\sin^2(\bar\lambda K_\varphi)}{\bar\lambda^2} \right. \\
& \left. + 2 \left(q \frac{\alpha_2}{E^x} + 2 \frac{\partial q}{\partial E^x} \right) \frac{\sin(2\bar\lambda K_\varphi)}{2\bar\lambda} \right) + 4 K_x \left(c_f \frac{\sin(2\bar\lambda K_\varphi)}{2\bar\lambda} + q \cos(2\bar\lambda K_\varphi) \right) \\
& + \frac{((E^x)')^2}{E^\varphi} \left(-\frac{\alpha_2}{4E^x} \cos^2(\bar\lambda K_\varphi) + \frac{K_x}{E^\varphi} \bar\lambda^2 \frac{\sin(2\bar\lambda K_\varphi)}{2\bar\lambda} \right) + \left(\frac{(E^x)'(E^\varphi)'}{(E^\varphi)^2} - \frac{(E^x)''}{E^\varphi} \right) \cos^2(\bar\lambda K_\varphi) \Bigg], \qquad (76)
\end{aligned}
$$

with the structure function

$$
q^{xx} = \left(\left(c_f + \left(\frac{\bar\lambda (E^x)'}{2E^\varphi} \right)^2 \right) \cos^2(\lambda K_\varphi) - 2q\bar\lambda^2 \frac{\sin(2\bar\lambda K_\varphi)}{2\bar\lambda} \right) \lambda_0^2 \frac{E^x}{(E^\varphi)^2}. \qquad (77)
$$

Furthermore, it can be shown that the constraint (76) has the weak observable

$$
\begin{aligned}
\mathcal{M} = d_0 + & \frac{d_2}{2} \left(\exp \int \mathrm{d}E^x \frac{\alpha_2}{2E^x} \right) \left(c_f \frac{\sin^2(\bar\lambda K_\varphi)}{\bar\lambda^2} + 2q \frac{\sin(2\bar\lambda K_\varphi)}{2\bar\lambda} - \cos^2(\bar\lambda K_\varphi) \left(\frac{(E^x)'}{2E^\varphi} \right)^2 \right) \\
& + \frac{d_2}{4} \int \mathrm{d}E^x \left(\left(c_{f0} + \frac{\alpha_0}{E^x} \right) \exp \int \mathrm{d}E^x \frac{\alpha_2}{2E^x} \right), \qquad (78)
\end{aligned}
$$

such that $\dot{\mathcal{M}} = \mathcal{D}_H H + \mathcal{D}_x H_x|_{\rm O.S.} = 0$, where $\mathcal{D}_H$ and $\mathcal{D}_x$ are functions of the phase space, meaning that $\mathcal{M}$ is a constant of the motion. The parameters $\bar\lambda$, d_0, and d_2 are constants, while the rest of the parameters are functions of E^x, all undetermined by the covariance conditions, representing allowed covariant modifications. The classical constraint and structure functions are recovered in the limit $\lambda_0, c_f, \alpha_0, \alpha_2 \to 1$ and $\bar\lambda, q \to 0$, and $\Lambda_0 \to -\Lambda$ if there is a cosmological constant.[1] In the classical limit, and further taking $d_0 \to 0$ and $d_2 \to 1$, the observable (78) becomes the classical mass function. The parameters c_f, q, λ_0, and $\bar\lambda$ are characteristic of EMG, because they appear directly in the structure function—hence, the emergent metric. On the other hand, the parameters Λ_0, α_0, and α_2 are modification functions allowed even for a nonemergent metric, and hence may be studied in the context of 2D dilaton gravity; these modifications have been treated before in the Hamiltonian formalism—for instance, in [14–16,49], and a Lagrangian approach with similar modifications can be found in [50].

The parameter $\bar\lambda$ is special, as it is responsible for the nonsingular black hole solution obtained [7] that we previously discussed. Furthermore, it has an interpretation within LQG as a quantum parameter, as we now briefly explain. The starting point of the loop quantization is carried out not by quantizing the classical phase space directly, but instead by working in the holonomy-flux representation. In spherically symmetric LQG [10,51], the holonomies are given by

$$
h_e^x[K_x] = \exp\left(i \int_e \mathrm{d}x\, K_x \right), \qquad (79)
$$

$$
h_{v,\lambda}^\varphi[K_\varphi] = \exp\left(i \int_\lambda \mathrm{d}\theta\, K_\varphi \right) = \exp\left(i\lambda K_\varphi(v) \right), \qquad (80)
$$

where e stands for an arbitrary radial curve of finite coordinate length, while v stands for an arbitrary point in the radial line, and λ is the coordinate length of an arbitrary angular curve on the 2-sphere intersecting the point v. The explicit integration in the angular holonomy (80) is possible due to spherical symmetry, but the radial holonomy integration must remain formal. Similarly, the fluxes are given by direct integration of the densitized triads E^x and E^φ over finite, two-dimensional surfaces with normals in the radial and angular directions, respectively. For the following argument, we need only the expression of the holonomies.

Because the loop quantization is based on the holonomy-flux variables, the Hamiltonian constraint must be rewritten in terms of them, rather than the bare curvatures K_x and K_φ, leading to a modified constraint in the spirit of EMG. However, the radial holonomy (79) is essentially nonlocal, and it cannot be studied within EMG, which has only been formulated locally. On the other hand, the angular holonomies (80) are indeed local when restricting the state space to spherically symmetric states. Therefore, angular

[1] An equivalent constraint was recently obtained in [48] starting with a simpler ansatz, and hence it is a special case of the one in [5].

holonomy effects have a chance to appear in the local equations of EMG. Furthermore, since the angular holonomies can be integrated, they become simple complex exponentials of the angular curvature (80). Because the Hamiltonian constraint must be Hermitian as an operator in the quantum theory, or simply real prior to quantization, the holonomy modifications in the Hamiltonian constraint will appear as trigonometric functions of K_φ. This is indeed the case for the EMG constraint (76). Therefore, the parameter λ can be given the interpretation of an angular holonomy length within the LQG theory, and hence acquire the status of a quantum parameter.

The holonomy parameter in (80) may depend on the scale, here given by a dependence on E^x. While the parameter $\bar{\lambda}$ in the EMG constraint (76) is constant, a detailed analysis of canonical transformations done in [5] shows that a nonconstant λ does arise in EMG, but through a canonical transformation, it can be traded in for the constant $\bar{\lambda}$, provided that the other undetermined functions acquire nonclassical expressions in the case where λ is nonconstant. In the following sections, however, we will only consider a constant holonomy parameter for simplicity, and we drop the bar in $\bar{\lambda}$ and write this *constant* parameter as λ for notational ease. We will sometimes refer to the effects of λ as quantum effects due to its interpretation in LQG, but note that the appearance of $\lambda \neq 0$ is allowed by the covariance conditions regardless of LQG.

B. The perfect fluid

The angular components of the momentum variables of the perfect fluid would imply via (27) that it has an angular mass flux or angular momentum, which breaks the spherical symmetry. Therefore, these angular components must vanish, and the perfect fluid is left with only two nontrivial canonical pairs:

$$\{T(x), P_T(y)\} = \{X(x), P_X(y)\} = \delta(x-y). \quad (81)$$

The covariance condition for the pressure function (50) involving the angular component of the structure function $q^{\vartheta\vartheta} = 1/E^x$ immediately requires that P not depend on spatial derivatives of K_x. Taking the ansatz $P(E^x, K_\varphi, W^2)$, the anomaly-freedom condition (66) and the spacetime-covariance condition (50) are automatically satisfied, while the pressure-covariance condition (67) simplifies to

$$\left.\frac{\partial P}{\partial K_\varphi}\right|_{\text{O.S.}} = 0. \quad (82)$$

Therefore, the dependence of the modified pressure on the gravitational variables reduces to only E^x.

The Hamiltonian constraint contribution of the perfect fluid is given by[2]

$$H_s^{\text{matter}} = \sqrt{sP_T^2 + q^{xx}(P_T T' + P_X X')^2} - \sqrt{\det q}\, P\left(E^x, \frac{P_X^2}{P_T^2}\right), \quad (83)$$

and the diffeomorphism constraint contribution by

$$H_x^{\text{matter}} = P_T T' + P_X X', \quad (84)$$

where the determinant is understood to be integrated in the angular coordinates, $\sqrt{\det q} = 4\pi\sqrt{q_{xx}}E^x$, and q^{xx} is the emergent structure function (77).

The associated Eulerian energy density is

$$\rho_s^{(\text{E})} = \frac{1}{\sqrt{\det q}}\sqrt{sP_T^2 + q^{xx}(P_T T' + P_X X')^2}, \quad (85)$$

and the energy density in the dust frame is

$$\rho_s^{\text{dust}} = \frac{1}{\sqrt{\det q}}\frac{P_T^2}{\sqrt{sP_T^2 + q^{xx}(P_T T' + P_X X')^2}}. \quad (86)$$

The full Hamiltonian constraint in the presence of a perfect fluid is given by $H = H^{\text{grav}} + H_s^{\text{matter}}$, and the full diffeomorphism constraint by $H_x = H_x^{\text{grav}} + H_x^{\text{matter}}$.

C. Reflection symmetry surface

A particular characteristic of the constraints coupled to the perfect fluid that will be relevant for our discussion is the following: When taking the classical value $q = 0$ in (76), the full constraints $H = H^{\text{grav}} + H_s^{\text{matter}}$ and $H_x = H_x^{\text{grav}} + H_x^{\text{matter}}$ are symmetric under the time-reversal operation $K_\varphi \to -K_\varphi$, $K_x \to -K_x$, $T \to -T$, $P_X \to -P_X$. Furthermore, it possesses an additional reflection-symmetry surface at $K_\varphi = -\pi/(2\lambda)$ in the sense that, defining $K_\varphi = -\pi/(2\lambda) - \delta$, the Hamiltonian is symmetric under the operation $\delta \to -\delta$, $K_x \to -K_x$, $T \to -T$, $P_X \to -P_X$, which is almost identical to time reversal. The meaning of this symmetry is the following: Given a solution to the equations of motion in the region $\delta < 0$, the solution in region $\delta > 0$ will be a perfect reflection of the solution in $\delta < 0$, with the only difference being the flow of time. Thus, the solution in the region $\delta > 0$ will look like the

[2]In [48], it was suggested that the constraint contribution of timelike dust in spherical symmetry could be given by $H^{\text{dust}} = \sqrt{P_T^2 + q^{xx}P_T^2(T')^2}$ using the emergent structure function, but no underlying theory for the dust was presented. While our procedure shows that this version is indeed anomaly free and covariant, it overlooks the radial phase-space variables of the dust which are necessary for the rise of the (modified) pressure function of the perfect fluid.

time-reversed solution in the region $\delta < 0$. We will exploit this symmetry in obtaining the global structure of the collapse solution.

VI. GRAVITATIONAL COLLAPSE OF TIMELIKE DUST

A. Frame transformations and gauge fixing

Consider a coordinate transformation to a frame described by a one-form field $v_\mu \mathrm{d}x^\mu = v_t \mathrm{d}t + v_x \mathrm{d}x$ and normalized $g^{\mu\nu} v_\mu v_\nu = -s$, where we take $s = 1, 0, -1$ for timelike, null, and spacelike frames, respectively. Using the inverse metric in the ADM decomposition,

$$g^{\mu\nu} = q^{ab} s^\mu_a s^\nu_a - \frac{1}{N^2}(t^\mu - N^a s^\mu_a)(t^\nu - N^b s^\nu_b), \quad (87)$$

the normalization can be written as

$$-\frac{(v_t)^2}{N^2} + 2\frac{N^x}{N^2} v_t v_x + \left(q^{xx} - \frac{(N^x)^2}{N^2}\right)(v_x)^2 = -s, \quad (88)$$

which can be solved for v_t in terms of v_x or vice versa—for example,

$$v_t = \pm\sqrt{N^2(s + q^{xx}(v_x)^2)} + N^x v_x, \quad (89)$$

where the sign choice corresponds to either an ingoing or an outgoing frame. It will also be convenient to use the components of the velocity $v^\mu \equiv g^{\mu\nu} v_\nu$:

$$v^t = -\frac{v_t - N^x v_x}{N^2}, \qquad v^x = \frac{v_t - N^x v_x}{N^2} N^x. \quad (90)$$

The covelocity is a one-form, and thus defines a parameter γ along its integral curve via $\mathrm{d}\gamma \equiv v_\mu \mathrm{d}x^\mu$. If the curve is timelike (or null), we can replace the time coordinate in the metric for the parameter γ as the new time coordinate as follows. The infinitesimal time along the integral curve is

$$\mathrm{d}t = \frac{1}{v_t}\mathrm{d}\gamma - \frac{v_x}{v_t}\mathrm{d}x. \quad (91)$$

Substituting this into a metric of the general form (69) and rearranging terms, we obtain

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{-N^2 + q_{xx}(N^x)^2}{(v_t)^2}\mathrm{d}\gamma^2 + \frac{2N^2 q_{xx}}{(v_t)^2 v_x}\left(\frac{N^x}{N^2} v_t v_x + \left(q^{xx} - \frac{(N^x)^2}{N^2}\right)(v_x)^2\right)\mathrm{d}\gamma \mathrm{d}x \\
&\quad - \frac{N^2 q_{xx}}{(v_t)^2}\left(-\frac{(v_t)^2}{N^2} + 2\frac{N^x}{N^2} v_t v_x + \left(q^{xx} - \frac{(N^x)^2}{N^2}\right)(v_x)^2\right)\mathrm{d}x^2 \\
&= \frac{-N^2 + q_{xx}(N^x)^2}{(v_t)^2}\mathrm{d}\gamma^2 - \frac{2N^2 q_{xx}}{(v_t)^2 v_x}\left(s - \frac{(v_t)^2}{N^2} + \frac{N^x}{N^2} v_t v_x\right)\mathrm{d}\gamma \mathrm{d}x + s\frac{N^2 q_{xx}}{(v_t)^2}\mathrm{d}x^2, \quad (92)
\end{aligned}$$

where we use (88) to obtain the last line, and we have suppressed the angular part of the metric. If this frame is timelike, $s = 1$, the parameter γ is the proper time along the integral curve, and we denote it as $\gamma = -\tau$ (the sign is due to the Lorentzian signature); we can then redefine the metric variables compatible with the new coordinates:

$$\mathrm{d}s^2 =: -N^2_{(\tau)}\mathrm{d}\tau^2 + q^{(\tau)}_{xx}(\mathrm{d}x + N^x_{(\tau)}\mathrm{d}\tau)^2, \quad (93)$$

where

$$q^{(\tau)}_{xx} = \frac{N^2 q_{xx}}{(v_t)^2}, \qquad N^x_{(\tau)} = -v^x, \qquad N_{(\tau)} = 1. \quad (94)$$

Furthermore, in the new coordinates, the covelocity has the components $v_\mu \mathrm{d}x^\mu = -\mathrm{d}\tau$, and hence, the velocity components are

$$v^\tau_{(\mathrm{new})} = g^{\tau\mu} v_\mu = -g^{\tau\tau} = 1, \qquad v^x_{(\mathrm{new})} = g^{x\mu} v_\mu = -g^{x\tau} = -N^x_\tau. \quad (95)$$

Therefore, the frame of an observer characterized by its normalized timelike covector field v_μ will, in the canonical context, always be given by a unit lapse $N = 1$, while a residual gauge freedom rests on the undetermined shift, which is the observer's (negative) radial velocity $N^x = -v^x$. We will exploit this fact in the following sections to obtain the spacetime solutions associated with free-falling observers directly from the canonical equations of motion.

On the other hand, if the frame is null, $s = 0$, the parameter γ is an affine parameter, and we denote it as $\gamma = u$. In this case, the metric (92) becomes of the Eddington-Finkelstein form:

$$\mathrm{d}s^2 = -\frac{N^2 - q_{xx}(N^x)^2}{(v_t)^2}\mathrm{d}u^2 + 2\frac{\sqrt{N^2 q_{xx}}}{v_t}\mathrm{d}u\mathrm{d}x. \quad (96)$$

This cannot be expressed in the ADM decomposition, which is based on spatial foliations; hence, there is no gauge choice in the canonical formalism to reproduce these coordinates, but they are still useful to extend coordinate charts once the spacetime solution is known in other coordinates.

For completion, similar coordinate transformations can be done for spacelike and null frames that replace the spatial coordinate. In particular, starting from a metric of the general form (69), two null transformations described by the one-forms $\mathrm{d}u = v^{(1)}_\mu \mathrm{d}x^\mu$ and $\mathrm{d}v = v^{(2)}_\mu \mathrm{d}x^\mu$ replacing the time and spatial coordinates with the null coordinates u and v, respectively, renders the metric into a Kruskal-Szekeres form:

$$\mathrm{d}s^2 = -\frac{N^2 - q_{xx}(N^x)^2}{v^{(1)}_t v^{(2)}_t}\mathrm{d}u\mathrm{d}v. \tag{97}$$

The coordinates (u, v) can be more easily related to the original coordinates (t, x) by using (89) together with $v_t v^t + v_x v^x = 0$ to obtain

$$\frac{v^{(i)}_t}{v^{(i)}_x} = s_{(i)}\sqrt{N^2 q^{xx}} + N^x, \tag{98}$$

where $s_{(1)} = +1$ and $s_{(2)} = -1$. Then,

$$\begin{aligned} \mathrm{d}\gamma^{(i)} &= v^{(i)}_t \mathrm{d}t + v^{(i)}_x \mathrm{d}x \\ &= v^{(i)}_t\left(\mathrm{d}t + \left(s_{(i)}\sqrt{N^2 q^{xx}} + N^x\right)^{-1}\mathrm{d}x\right), \end{aligned} \tag{99}$$

where $\gamma^{(1)} = u$ and $\gamma^{(2)} = v$. If the spacetime in the original coordinates is static, then we can choose null geodesics which have constant $v^{(i)}_t$ (they are Killing-conserved quantities) and can be absorbed into u and v, and the expression (99) can be directly integrated.

B. The fluid frame and the Gullstrand-Painlevé gauge

We start by choosing a space foliation compatible with the fluid frame. This means that the covelocity of the fluid has the components $u_\mu = n_\mu = g_{\mu\nu}n^\nu$, or $u_t = -N$ and $u_x = 0$. The vanishing of the spatial component then implies that $H^{\text{matter}}_x = 0$. Furthermore, since the velocity of the fluid has been adapted to the foliation, the fluid variable T can parametrize time—that is, $\dot{T} = 1$. Using the equations of motion for this, combined with $H^{\text{matter}}_x = 0$, $u_t = -N$, and $u_X = 0$, implies the equations

$$N = 1, \qquad P_X = 0. \tag{100}$$

The unit lapse is consistent with our previous discussions, such that this gauge is indeed associated with the fluid frame. We will see later that the choice of adapting the space foliation to the fluid frame leads to the Oppenheimer-Snyder model of uniformly distributed dust in the classical case with vanishing pressure [26,52], but unlike the uniformity conditions, the adaptation of the foliation to the fluid frame can be easily extended to EMG. Finally, the on-shell conditions $H_x = 0$ and $H = 0$, respectively, simplify to

$$K_x = \frac{E^\varphi K'_\varphi}{(E^x)'}, \tag{101a}$$

$$H^{\text{grav}} = -P_T - \sqrt{\det q}P. \tag{101b}$$

We now define the Gullstrand-Painlevé (GP) gauge as

$$N = 1, \qquad E^x = x^2, \tag{102}$$

which is compatible with the fluid frame due to the unit lapse. This choice must satisfy the consistency equation $\dot{E}^x = 0$, which implies the value of the shift:

$$N^x = -\lambda_0 \frac{\sin(2\lambda K_\varphi)}{2\lambda}\left(c_f + \lambda^2\frac{x^2}{(E^\varphi)^2}\right). \tag{103}$$

Following our previous discussion on frame transformations involving unit lapse, we conclude that the shift (103) provides the negative observer's radial velocity, and since this is the fluid frame itself, it is the fluid velocity. The general equation of motion for the remaining fluid variable is

$$\dot{X} = N^x X'. \tag{104}$$

The remaining nontrivial equations are given by the equations of motion $\dot{K}_\varphi$ and $\dot{E}^\varphi$. We will solve these equations for two special cases below: the classical case, and the case with holonomy modifications.

C. Classical collapse

An exact solution of the classical collapse of dust is known [26,52]. We will reproduce this result with our canonical methods, because it will be useful as a guide to later solve the modified equations and to check the classical limit. To this end, we will take the full classical limit given by $c_f, \lambda_0, \alpha_i \to 1$, $\lambda, q \to 0$, with vanishing cosmological constant and pressure $\Lambda_0 = P = 0$.

1. Star exterior

The classical GP equations of motion for dust are

$$N^x = -K_\varphi, \tag{105a}$$

$$K_\varphi \frac{P_T}{E^\varphi} + \frac{1}{2}\partial_{t_{\text{GP}}}(K^2_\varphi) + \frac{x^2}{(E^\varphi)^2}\partial_{t_{\text{GP}}}\ln E^\varphi = 0, \tag{105b}$$

$$\dot{K}_\varphi = -\frac{1}{2x}\left[1 - \frac{x^2}{(E^\varphi)^2} + (xK^2_\varphi)'\right], \tag{105c}$$

$$\partial_{t_{\text{GP}}}\left(\frac{E^\varphi}{x}\right) = -K_\varphi\left(\frac{E^\varphi}{x}\right)'. \tag{105d}$$

As is well known, the exterior Schwarzschild metric in vacuum,

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{x}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{x}\right)^{-1}\mathrm{d}x^2 + x^2\mathrm{d}\Omega^2, \tag{106}$$

is the solution to the equations of motion in the Schwarzschild gauge given by

$$N^x = 0, \qquad E^x = x^2. \tag{107}$$

The (generalized) Gullstrand-Painlevé metric,

$$\mathrm{d}s^2 = -\mathrm{d}t_{\mathrm{GP}}^2 + \frac{1}{\epsilon^2}\left(\mathrm{d}x - s\sqrt{\frac{2M}{x} - \frac{2M}{R_0}}\mathrm{d}t_{\mathrm{GP}}\right)^2 + x^2\mathrm{d}\Omega^2, \tag{108}$$

with constant ε, $s = \pm 1$, and $x < R_0 = 2M/(1-\varepsilon^2)$, is obtained either by direct coordinate transformation of the Schwarzschild metric (106) associated with a frame as explained in Sec. VI A whose 4-velocity is that of timelike geodesics with Killing-conserved energy ε—that is, $v_t = -\varepsilon$ (i.e., the geodesics are at rest at $x = R_0$)—or by solving the equations of motion in the GP gauge (105) for vacuum—that is, for $P_T = 0$. The signs $s = -1$ and $s = +1$ are associated with ingoing or outgoing frames, respectively.

For future use, the classical solution to each phase-space variable in the GP gauge is

$$E^\varphi = \frac{x}{\varepsilon}, \tag{109}$$

$$K_\varphi = s\sqrt{\frac{2M}{x} - \frac{2M}{R_0}}, \tag{110}$$

$$K_x = -\frac{s}{2\varepsilon}\left(\frac{2M}{x} - \frac{2M}{R_0}\right)^{-1/2}\frac{M}{x^2}. \tag{111}$$

Substituting these expressions into the vacuum observable (78), with $d_0 = 1$ and d_2, yields $\mathcal{M} = M$, as expected.

2. Star interior

We will now obtain the solution for the interior of a star of dust. Let us assume that the dust at some initial time t_0 is at rest confined to the region $x < R_0$; i.e., the star has an initial radius R_0. The radius of the star at other times, $R(t)$, follows a geodesic motion, and because the interior metric must match the exterior one at $x = R(t)$, the radius $R(t)$ follows the geodesic equation of the exterior metric; this equation is easily obtained by recalling that in GP coordinates, the shift is the negative velocity of the frame $N^x = -\mathrm{d}x/\mathrm{d}t_{\mathrm{GP}}$:

$$\frac{\mathrm{d}R}{\mathrm{d}t_{\mathrm{GP}}} = s\sqrt{\frac{2M}{R} - \frac{2M}{R_0}}, \tag{112}$$

whose general solution is the inversion of

$$t_{\mathrm{GP}} - t_0 = \frac{sR}{\varepsilon^2 - 1}\sqrt{\frac{2M}{R} - \frac{2M}{R_0}} - \frac{sM}{(\varepsilon^2-1)^{3/2}}\ln\left(\frac{\sqrt{\frac{2M}{R_0}} + \sqrt{\frac{2M}{R} - \frac{2M}{R_0}}}{\sqrt{\frac{2M}{R_0}} - \sqrt{\frac{2M}{R} - \frac{2M}{R_0}}}\right), \tag{113}$$

and the constant ε is the Killing-conserved energy of the free-falling frame at rest at the radius R_0. The case $\varepsilon \to 1$, $s = -1$ corresponds to collapsing dust from rest at infinity such that

$$R(t_{\mathrm{GP}}) = \left(\frac{9M}{2}t_{\mathrm{GP}}^2\right)^{1/3}. \tag{114}$$

Substituting $R(t_{\mathrm{GP}})$ into the metric (108) then gives the boundary metric that joins the star's exterior and interior metrics. The latter is obtained by solving the GP gauge equations of motion (105). To this end, based on the vacuum solution, we take the ansatz

$$E^\varphi = \frac{x}{\varepsilon(t_{\mathrm{GP}}, x)}, \qquad K_\varphi = s\sqrt{\frac{2m(t_{\mathrm{GP}}, x)}{x} + \varepsilon(t_{\mathrm{GP}}, x)^2 - 1}, \tag{115}$$

with $s = \pm 1$. The intuition behind this ansatz is that at any given time t_{GP}, there will be a total mass $m(t_{\mathrm{GP}}, x)$ contained within the radius x, which will be experienced by the observers parametrized by ε, which is no longer a Killing-conserved quantity. Defining $\varepsilon^2 = 1 - E$ and using the ansatz (115), the last two equations of (105) become

$$\dot{m} = -K_\varphi m', \tag{116}$$

$$\dot{E} = -K_\varphi E'. \tag{117}$$

Using the method of separation of variables—i.e., assuming the forms $E = \lambda_E E_t(t_{\mathrm{GP}})E_x(x)$ and $m = \lambda_m m_t(t_{\mathrm{GP}})\times m_x(x)$ with λ_E and λ_m constants—and imposing the boundary conditions $m(t_{\mathrm{GP}}, x = R(t_{\mathrm{GP}})) = M$ and $E(t_0) = 1 - \varepsilon^2 = 2M/R_0$, one can solve these equations exactly, obtaining

$$m = M\frac{x^3}{R^3}, \tag{118}$$

$$E = \frac{2M}{R_0}\frac{x^2}{R^2} = \frac{2m}{x}\frac{R}{R_0}, \tag{119}$$

$$P_T = \frac{m'}{\varepsilon}, \tag{120}$$

with only (104), the equation for X, being difficult to solve exactly, except in the limit $\varepsilon \to 1$, $R_0 \to \infty$. In this limit, and imposing the boundary condition $X(t_{\rm GP}, x = R(t_{\rm GP})) = R(t_{\rm GP})$, the solution is given by

$$X = \left(\frac{(9Mt_{\rm GP}^2/2)^{1/2} + x^{3/2}}{2}\right)^{2/3}. \tag{121}$$

This solution describes a uniformly distributed dust with energy density $\rho^{\rm dust}(R)$ and therefore corresponds to the Oppenheimer-Snyder collapse model [26]: The associated energy density is

$$\rho^{\rm dust} = \frac{\varepsilon}{4\pi x^2} P_T = \frac{m}{4\pi x^3/3}, \tag{122}$$

and the global conserved charge (23) with $\alpha = \varepsilon$ gives the total mass

$$Q[\varepsilon] = \int {\rm d}x\, \varepsilon P_T = M. \tag{123}$$

The resulting metric for the star interior is given by [52]

$$ {\rm d}s^2 = -{\rm d}t_{\rm GP}^2 + \left(1 - \frac{2m}{x}\frac{R}{R_0}\right)^{-1}\left({\rm d}x - s\sqrt{\frac{2m}{x}}\sqrt{1-\frac{R}{R_0}}{\rm d}t_{\rm GP}\right)^2 + x^2 {\rm d}\Omega^2. \tag{124}$$

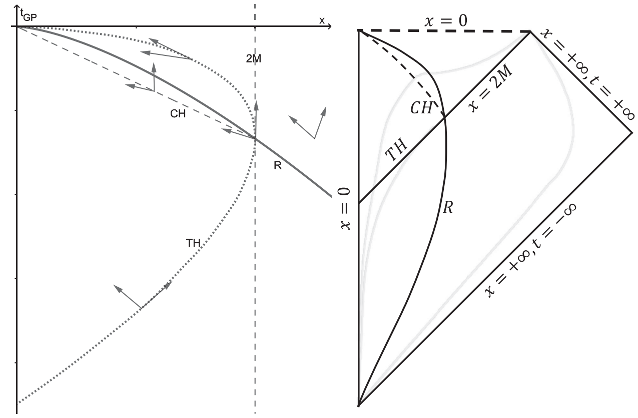
The physical singularity at $t_{\rm GP} \to 0$ persists, where $R \to 0$: all of the dust collapses into the coordinate point ($t_{\rm GP} \to 0$, $x \to 0$), and the usual spacelike singularity of the vacuum black hole appears from then on.

If the star falls from infinity, then $R_0 \to \infty$, and $R(t_{\rm GP})$ is given by (114); the metric (124) in this case can be rewritten as

$$ {\rm d}s^2 = -{\rm d}t_{\rm GP}^2 + ({\rm d}x - K_\varphi {\rm d}t_{\rm GP})^2 + x^2 {\rm d}\Omega^2, \tag{125}$$

where the curvature is simply $K_\varphi = -\sqrt{2m/x}$ with range $K_\varphi \in [0, -\infty)$—corresponding to $t_{\rm GP} < 0$. However, the range $K_\varphi \in (\infty, 0]$—corresponding to $t_{\rm GP} > 0$—is a solution too, describing the time-reversed process: an exploding star with the dust coming out of the singularity.

The spacetime diagram of the collapse in GP coordinates is shown in Fig. 1.



(a) (b)

FIG. 1. Classical collapse of a dust star with infinite initial radius $R_0 \to \infty$ in (a) GP coordinates and (b) its conformal diagram. Some light cones have been represented with arrows in (a) for reference on the causal structure. The solid line labeled by R is the star's radius. It crosses its own horizon at the time $t_{\rm GP} = -4M/3$. The causal horizon, labeled as CH, is defined by the vanishing velocity of outgoing null rays in GP coordinates, and beyond which it changes sign, becoming ingoing null rays, and constant x curves become spacelike. This surface is given by the solution to $2m = x$, or $t_{\rm GP} = -2x/3$. The trapping horizon, labeled as TH, is the surface beyond which light cannot escape the event horizon $x = 2M$ in vacuum. The trapping horizon is obtained by solving the outgoing null geodesic that intersects the star's radius at the event horizon. Unlike in the vacuum solution, in the presence of dust, the trapping and causal horizons are not identical, and neither of them is a constant x curve. The faint gray lines in (b) denote constant x curves.

D. Holonomy model collapse

Having solved the classical system, we are now in the position to solve the modified equations with finite holonomy parameter λ. For simplicity, we take the limit $c_f, \alpha_0, \alpha_2 \to 1$ and consider constant λ_0 as an aid to rescale the resulting metric. We also take vanishing cosmological constant and pressure, $\Lambda_0 = P = 0$.

1. Star exterior

Similarly to the classical system, the vacuum equations of motion in the Schwarzschild gauge,

$$N^x = 0, \qquad E^x = x^2, \tag{126}$$

can be solved exactly, yielding the emergent metric [7]

$$ {\rm d}s^2 = -\left(1 - \frac{2M}{x}\right)\frac{{\rm d}t^2}{\mu^2\lambda_0^2} + \left(1 + \lambda^2\left(1 - \frac{2M}{x}\right)\right)^{-1} \times \left(1 - \frac{2M}{x}\right)^{-1}\frac{{\rm d}x^2}{\lambda_0^2} + x^2 {\rm d}\Omega^2, \tag{127}$$

where μ is a constant of the equations of motion that, together with λ_0, may rescale the metric, and M is a constant that can be related to the classical mass, as we will show. Demanding asymptotic flatness fixes $\mu = 1/\lambda_0$ and $\lambda_0 = \sqrt{1+\lambda^2}$. Unlike the classical Schwarzschild metric, this has a second coordinate singularity at

$$x_\lambda = \frac{2M\lambda^2}{1+\lambda^2}, \tag{128}$$

which is not a geometric singularity, but a minimum radius placed at the maximum-curvature surface.

The spacetime metric in the GP coordinates can again be obtained either by direct coordinate transformation of (127) using a free-falling timelike frame or by solving the vacuum equations of motion in the GP gauge (102); the resulting phase-space variables in this gauge are given by

$$N^x = -s\sqrt{\frac{2M}{x} - \frac{2M}{R_0}}\sqrt{1-\frac{x_\lambda}{x}}, \tag{129}$$

$$E^\varphi = \frac{x}{\varepsilon}, \tag{130}$$

$$\frac{\sin^2(\lambda K_\varphi)}{\lambda^2} = \frac{2M/x + \varepsilon^2 - 1}{1+\lambda^2\varepsilon^2}, \tag{131}$$

$$K_x = \frac{K_\varphi'}{2\varepsilon}. \tag{132}$$

Plugging these expressions into the weak observable (78), with $d_0 = 0$ and $d_2 = 1$, yields $\mathcal{M} = M$, reinforcing the interpretation of M as that of the mass even in the modified theory. The emergent GP metric is

$$\mathrm{d}s^2 = -\mathrm{d}t_{\mathrm{GP}}^2 + \frac{1}{\varepsilon^2}\left(1-\frac{x_\lambda}{x}\right)^{-1}\left(\mathrm{d}x - s\sqrt{\frac{2M}{x} - \frac{2M}{R_0}}\sqrt{1-\frac{x_\lambda}{x}}\mathrm{d}t_{\mathrm{GP}}\right)^2 + x^2\mathrm{d}\Omega^2, \tag{133}$$

where the constant ε is the Killing-conserved energy of the free-falling frame, $s = \mathrm{sgn}(\sin(2\lambda K_\varphi))$, and $R_0 = 2M/(1-\varepsilon^2)$.

The Schwarzschild time t and the GP time t_{GP} are related by the coordinate transformation

$$\mathrm{d}t_{\mathrm{GP}} = \varepsilon\mathrm{d}t - s\sqrt{\frac{2M}{x} - \frac{2M}{R_0}}\left(1-\frac{2M}{x}\right)^{-1}\sqrt{1-\frac{x_\lambda}{x}}\mathrm{d}x \tag{134}$$

or

$$t_{\mathrm{GP}} = \varepsilon t + s\sqrt{\frac{2M}{R_0}}\left(-\sqrt{R_0-x}\sqrt{x-x_\lambda} + 2\sqrt{R_0-2M}\sqrt{2M-x_\lambda}\,\mathrm{arctanh}\left(\frac{\sqrt{R_0-2M}\sqrt{x-x_\lambda}}{\sqrt{R_0-x}\sqrt{2M-x_\lambda}}\right)\right.$$
$$\left. - (R_0 - 4M + x_\lambda)\arctan\left(\frac{\sqrt{x-x_\lambda}}{\sqrt{R_0-x}}\right)\right), \tag{135}$$

where the integration constant has been chosen such that $t_{\mathrm{GP}} = 0$ coincides with $t = 0$ at $x = x_\lambda$.

Using the fact that the shift in the GP gauge is the negative radial velocity of the observers, $N^x = -\mathrm{d}x/\mathrm{d}t_{\mathrm{GP}}$, the radial coordinate $x = R$ of the timelike geodesics then follows the equation

$$\frac{\mathrm{d}R}{\mathrm{d}t_{\mathrm{GP}}} = s\sqrt{\frac{2M}{R} - \frac{2M}{R_0}}\sqrt{1-\frac{x_\lambda}{R}}, \tag{136}$$

and we see that $s = -1$ corresponds to an infalling frame; the acceleration is

$$\frac{\mathrm{d}^2R}{\mathrm{d}t_{\mathrm{GP}}^2} = \frac{M}{R^2}\left(-\left(\frac{2M}{R} - \frac{2M}{R_0}\right)^{-1} + \frac{\lambda^2}{1+\lambda^2}\left(1-\frac{x_\lambda}{R}\right)^{-1}\right)\left(\frac{\mathrm{d}R}{\mathrm{d}t_{\mathrm{GP}}}\right)^2. \tag{137}$$

The velocity equation (136) has the general solution

$$t_{\mathrm{GP}} - t_0 = \frac{s}{\sqrt{2M}}\left(R\sqrt{\frac{R_0}{R}}\sqrt{\frac{R_0}{R}-1}\sqrt{1-\frac{x_\lambda}{R}} + R_0\left(1-\frac{x_\lambda}{R_0}\right)\left(\sqrt{\frac{R_0}{R}}\arctan\left(\sqrt{\frac{1-x_\lambda/x}{R_0/R-1}}\right) - \frac{\pi}{2}\right)\right), \tag{138}$$

so that the observer is at rest at the radial coordinate $x = R_0$ at time t_0. The geodesic then reaches $R = x_\lambda$ at time

$$t_{\rm GP} = t_0 + \frac{s}{\sqrt{2M}}\frac{\pi}{2}(R_0 + x_\lambda), \tag{139}$$

and at this minimum radius the velocity (136) vanishes, while the acceleration (137) is finite and positive:

$$\frac{{\rm d}^2 R}{{\rm d}t_{\rm GP}^2}\xrightarrow[R\to x_\lambda]{}\frac{1}{2x_\lambda}\left(\frac{2M}{x_\lambda} - \frac{2M}{R_0}\right). \tag{140}$$

If the observer falls from infinity, $R_0 \to \infty$ at $t_{\rm GP} \to -\infty$, the time coordinates are instead related by

$$t_{\rm GP} = t - s\sqrt{2M}\left(2\sqrt{x - x_\lambda} - \sqrt{2M - x_\lambda}\ln\left(\frac{\sqrt{2M - x_\lambda} + \sqrt{x - x_\lambda}}{\sqrt{2M - x_\lambda} - \sqrt{x - x_\lambda}}\right)\right), \tag{141}$$

where again the integration constant has been chosen such that $t_{\rm GP} = 0$ coincides with $t = 0$ at $x = x_\lambda$. The expression for the geodesic, which we relabel as $R(t_{\rm GP})$, is instead the solution to

$$t_{\rm GP} = \frac{2s}{3}R\sqrt{\frac{R}{2M}}\sqrt{1 - \frac{x_\lambda}{R}}\left(1 + 2\frac{x_\lambda}{R}\right), \tag{142}$$

where the integration constant has been chosen such that $t_{\rm GP} = 0$ at $R = x_\lambda$. The explicit inversion of (142) is given by

$$R(t_{\rm GP}) = -x_\lambda + \left(\frac{9M}{4}t_{\rm GP}^2 + x_\lambda^3 + \sqrt{\left(\frac{9M}{4}t_{\rm GP}^2\right)^2 + \frac{9M}{2}t_{\rm GP}^2 x_\lambda^3}\right)^{1/3} + x_\lambda^2\left(\frac{9M}{4}t_{\rm GP}^2 + x_\lambda^3 + \sqrt{\left(\frac{9M}{4}t_{\rm GP}^2\right)^2 + \frac{9M}{2}t_{\rm GP}^2 x_\lambda^3}\right)^{-1/3}. \tag{143}$$

This geodesic coordinate reaches the minimum radius $x_\lambda = 2M\lambda^2/(1+\lambda^2)$ at $t_{\rm GP} = 0$ (and $t = 0$) with vanishing velocity $\dot{R}(x = x_\lambda) = 0$, and finite, positive acceleration

$$\frac{{\rm d}^2 R}{{\rm d}t_{\rm GP}^2}\xrightarrow[R\to x_\lambda]{}\frac{M}{x_\lambda^2}. \tag{144}$$

2. Star interior

We now focus on the star-interior region in the GP gauge (102). Based on the classical solution for the star interior and on the vacuum solution in the holonomy model, we take the ansatz

$$E^\varphi = \frac{x}{\varepsilon(t_{\rm GP}, x)},$$
$$\frac{\sin(\lambda K_\varphi)}{\lambda} = \frac{s}{\sqrt{1+\lambda^2\varepsilon^2}}\sqrt{\frac{2m(t_{\rm GP}, x)}{x} + E}, \tag{145}$$

where $s = \pm 1$ is the sign of $\sin(\lambda K_\varphi)$, and we define $\varepsilon^2 =: 1 - E$. Using this ansatz, the relevant equations of motion in the GP gauge become

$$N^x = -\frac{\sin(2\lambda K_\varphi)}{2\lambda}\frac{1+\lambda^2\varepsilon^2}{1+\lambda^2}, \tag{146}$$

$$\frac{\varepsilon}{x}P_T + \dot{K}_\varphi + \frac{\lambda}{2}\cot(\lambda K_\varphi)\frac{\dot{E}}{1+\lambda^2\varepsilon^2} = 0, \tag{147}$$

$$\frac{\dot{m}}{1+\lambda^2\varepsilon^2} = -\frac{\sin(2\lambda K_\varphi)}{2\lambda}\frac{m'}{\sqrt{1+\lambda^2}}, \tag{148}$$

$$\frac{\dot{E}}{1+\lambda^2\varepsilon^2} = -\frac{\sin(2\lambda K_\varphi)}{2\lambda}\frac{E'}{\sqrt{1+\lambda^2}}. \tag{149}$$

Unfortunately, these equations are too complicated to obtain an exact general solution. However, we can solve exactly for the special case of the dust collapsing from rest at infinity. Thus, restricting ourselves to the case $E \to 0$, $R_0 \to \infty$, and defining $m(t_{\rm GP}, x) = \frac{x}{2}m_1(x/t_{\rm GP})$, Eq. (149) is trivially satisfied, while Eq. (148) becomes

$$z^2\frac{\partial m_1}{\partial z} = s_2\sqrt{m_1}\sqrt{1 - \frac{\lambda^2 m_1}{1+\lambda^2}}\left(m_1 + z\frac{\partial m_1}{\partial z}\right), \tag{150}$$

where $z = x/t_{\rm GP}$ and s_2 is the sign of $\sin(2\lambda K_\varphi)$; thus, we shall take $s_2 = -1$ for the collapsing region and $s_2 = +1$ for the time-reversed version. This equation has a general solution with one constant of integration, which, however, is an extremely long expression. Imposing the boundary condition $m(x = R(t)) = M$ determines the integration constant of the solution, which simplifies considerably and now reads

$$m = \frac{|t_{\rm GP}|}{8\sqrt{3}}\frac{1+\lambda^2}{\lambda^3}\left[-3-\lambda^2\left(3-4\left(\frac{x}{t_{\rm GP}}\right)^2\right) + 3\left(\frac{8\lambda^3}{\sqrt{27}}\left(\frac{x}{|t_{\rm GP}|}\right)^3+\sqrt{1+\lambda^2}\sqrt{1+\lambda^2\left(2-4\left(\frac{x}{t_{\rm GP}}\right)^2\right)+\lambda^4\left(1-4\left(\frac{x}{t_{\rm GP}}\right)^2+\frac{16}{3}\left(\frac{x}{t_{\rm GP}}\right)^4\right)}\right)^{2/3}\right] \times\left(\frac{8\lambda^3}{\sqrt{27}}\left(\frac{x}{|t_{\rm GP}|}\right)^3+\sqrt{1+\lambda^2}\sqrt{1+\lambda^2\left(2-4\left(\frac{x}{t_{\rm GP}}\right)^2\right)+\lambda^4\left(1-4\left(\frac{x}{t_{\rm GP}}\right)^2+\frac{16}{3}\left(\frac{x}{t_{\rm GP}}\right)^4\right)}\right)^{-1/3} \tag{151}$$

and also satisfies $m(x\to 0)\to 0$; more on this below. In the limit $\lambda\to 0$, the mass function recovers its classical form (118).

From Eq. (147), we obtain the dust momentum

$$P_T = \frac{m'}{\sqrt{1+\lambda^2}}. \tag{152}$$

The global conserved charge (23) with $\alpha = \sqrt{1+\lambda^2}$ is precisely the total mass of the star,

$$Q_0[\alpha] = \int {\rm d}x\sqrt{1+\lambda^2}P_T = m(x)|_0^R = M, \tag{153}$$

and the associated dust energy density is

$$\rho^{\rm dust} = \frac{|\cos(\lambda K_\varphi)|}{\sqrt{1+\lambda^2}}\frac{m'}{4\pi x^2}, \tag{154}$$

which shows that the dust is not uniformly distributed as in the classical solution.

Only the equation of motion of the dust variable X given by (104) is too complicated to be solved exactly. However, the solution to this variable is not relevant for most purposes, including ours. We will only need an approximate solution near the maximum-curvature surface discussed below.

Using (77) to compute the structure function, we obtain the star-interior spacetime metric,

$$ {\rm d}s^2 = -{\rm d}t_{\rm GP}^2 + \sec^2(\lambda K_\varphi)\left({\rm d}x - \sqrt{1+\lambda^2}\frac{\sin(2\lambda K_\varphi)}{2\lambda}{\rm d}t_{\rm GP}\right)^2 + x^2{\rm d}\Omega, \tag{155}$$

where the K_φ expressions are obtained from inverting

$$\sin^2(\lambda K_\varphi) = \frac{\lambda^2}{1+\lambda^2}\frac{2m}{x} \tag{156}$$

with the mass function (151). Taking the range $K_\varphi\in(0,-\pi/2\lambda)$—corresponding to $t_{\rm GP}<0$—the signs $s=s_2=-1$ of (145) and (150) are recovered, and thus, it describes the collapsing region of the star.

Two regions are of interest: near the $x=0$ axis and the $t_{\rm GP}=0$ axis. To analyze them, we perform a Taylor expansion around $x/t_{\rm GP}=0$ for the former,

$$\frac{2m}{x} = \frac{4}{9}\left(\frac{x}{t_{\rm GP}}\right)^2 + \frac{16}{27}\frac{\lambda^2}{1+\lambda^2}\left(\frac{x}{t_{\rm GP}}\right)^4 + O\left(\frac{x}{t_{\rm GP}}\right)^6, \tag{157}$$

and an expansion around $t_{\rm GP}/x=0$ for the latter,

$$\frac{2m}{x} = \frac{1+\lambda^2}{\lambda^2}\left(1-\frac{1}{4}\frac{1+\lambda^2}{\lambda^2}\left(\frac{t_{\rm GP}}{x}\right)^2 - \frac{1}{48}\frac{(1+\lambda^2)^2}{\lambda^4}\left(\frac{t_{\rm GP}}{x}\right)^4\right) + O\left(\frac{t_{\rm GP}}{x}\right)^6. \tag{158}$$

This shows that in the limit $x\to 0$, with nonzero $t_{\rm GP}$, the mass function vanishes and one obtains a flat spacetime. On the other hand, the spacetime metric diverges in the limit $t_{\rm GP}/x\to 0$; thus, this coordinate chart is strictly valid only for the region $[t_{\rm GP}<0, 0<x<R(t_{\rm GP})]$. Using (156), we find that this divergent surface corresponds precisely to a maximum-curvature surface. As explained in Sec. V, this is also a reflection-symmetry surface. Thus, the range $K_\varphi\in(-\pi/2\lambda,-\pi/\lambda)$—corresponding to $t_{\rm GP}>0$ and the sign $s_2=-1$, which is a solution too—describes the time-reversed process of an exploding star.

3. Near maximum-curvature surface and causal structure

We now analyze the causal structure. Null rays follow ${\rm d}s^2=0$, or

$$\left.\frac{{\rm d}x}{{\rm d}t_{\rm GP}}\right|_{\rm null} = s_{(i)}|\cos(\lambda K_\varphi)| + \sqrt{1+\lambda^2}\frac{\sin(2\lambda K_\varphi)}{2\lambda}, \tag{159}$$

where $s_{(\mathrm{in})} = -1$ for ingoing rays and $s_{(\mathrm{out})} = +1$ for outgoing ones. The causal structure then changes at $x = 2m$ [that is, $\sin(\lambda K_\varphi) = -\lambda/\sqrt{1+\lambda^2}$], just as in the classical case, defining a causal horizon where the outgoing null rays become ingoing. While exactly solving the line $t_{\mathrm{GP}}(x)$ for the causal horizon is too complicated, near the maximum-curvature surface we can use the expansion (158), such that the causal horizon is approximately given by the line

$$t_{\mathrm{GP}} = -2\frac{\lambda}{1+\lambda^2}x. \tag{160}$$

There are two important characteristics of the maximum-curvature surface given by $K_\varphi \to -\pi/(2\lambda)$—the would-be classical singularity. First, using the expansion (158), we find that the velocities of both ingoing and outgoing null rays do not diverge at this surface, but vanish; second, the acceleration of null rays remains finite,

$$\left.\frac{\mathrm{d}^2x}{\mathrm{d}t_{\mathrm{GP}}^2}\right|_{\mathrm{null}} \to \frac{1}{2x}\frac{\sqrt{1+\lambda^2}}{\lambda}\left(\pm 1 + \frac{\sqrt{1+\lambda^2}}{\lambda}\right), \tag{161}$$

and positive, since $\lambda \ll 1$.

While the previous analysis of null rays suggests that this surface can be crossed, in this expansion the Ricci scalar is given by

$$\mathcal{R} = \frac{4}{t_{\mathrm{GP}}^2}\left(1 + \frac{1+13\lambda^2}{24\lambda^2}\left(\frac{t_{\mathrm{GP}}}{x}\right)^2 - \frac{(8-\lambda^2)(1+\lambda^2)}{72\lambda^4}\left(\frac{t_{\mathrm{GP}}}{x}\right)^4 + O\left(\frac{t_{\mathrm{GP}}}{x}\right)^6\right). \tag{162}$$

Therefore, the singularity at the maximum-curvature surface is a true geometric singularity, and no coordinate transformation can remove it.

Finally, using the expansion (158), we can solve (104) to the lowest order to obtain the remaining dust variable,

$$X \approx f_X(x), \tag{163}$$

with arbitrary f_X, which can be fixed by demanding compatibility (replacing boundary conditions) with the local spatial coordinate, $f_X = x$.

The spacetime diagram of the resulting collapse with holonomy effects in the GP coordinate system developed above is shown in Fig. 2.

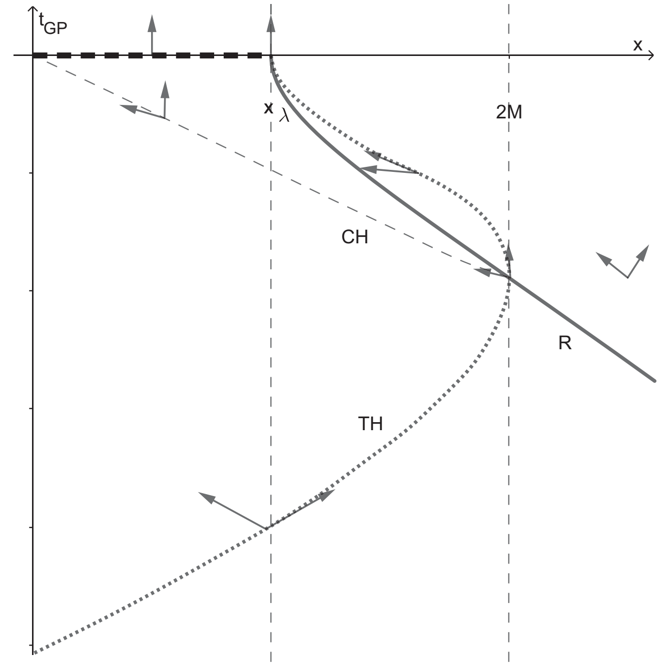


FIG. 2. Holonomy collapse of a dust star with infinite initial radius $R_0 \to \infty$ in GP coordinates. The star radius R crosses its own horizon at the time $t_{\mathrm{GP}} = -\frac{4M}{3}(1+2\lambda^2/(1+\lambda^2))/\sqrt{1+\lambda^2}$. Unlike the classical case, the star's radius does not collapse all the way to $x = 0$, but rather to x_λ with vanishing velocity and finite positive acceleration. This is similarly the case for the trapping horizon TH. The spacelike singularity of the spacetime lies on the surface ($t_{\mathrm{GP}} = 0, x \le x_\lambda$). Some light cones have been represented with arrows for reference on the causal structure.

4. Nonsingular canonical dynamics

While at the maximum-curvature surface the Ricci scalar diverges, implying a true geometric singularity, the canonical formulation of EMG does not assume the spacetime to be a fundamental entity; it is rather an emergent one, and furthermore, it is not the spacetime, but the structure function, the lapse, and shift, that are canonically defined. Indeed, all of the canonical variables are finite on the maximum-curvature surface:

$$m \to \frac{1+\lambda^2}{\lambda^2}\frac{x}{2}, \qquad N^x \to 0,$$
$$P_T \to \frac{\sqrt{1+\lambda^2}}{\lambda^2}\frac{1}{2}, \qquad q^{xx} \to 0. \tag{164}$$

This differs from the classical canonical values at $K_\varphi \to -\infty$ (and necessarily $x \to R \to 0$), where $m \to M$, $N^x \to \infty$, $P_T \to \infty$, and $K_x \to -\infty$. The nonsingular canonical dynamics of the phase-space variables, which does not rely on the spatial metric, allows us to extend the solution past this surface to complete the full range of the curvature $K_\varphi \in (0, -\pi/\lambda)$, corresponding to the whole $t_{\mathrm{GP}} \in (-\infty, \infty)$. The first term in the null rays' velocity expression (159) shows that ingoing (outgoing) rays remain ingoing (outgoing) after crossing the maximum-curvature surface as determined by the sign $s_{(i)}$; however, since the second term differs by a sign when crossing the surface, the causal structure past the maximum-curvature surface is instead that of a white hole. The resulting spacetime,

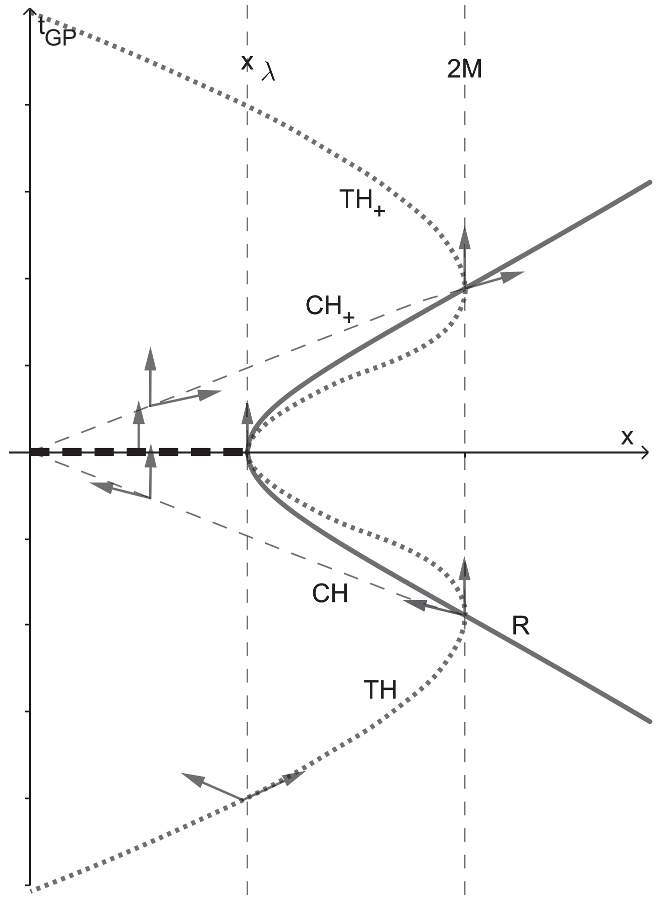


FIG. 3. Extension of the star-interior holonomy collapse in GP coordinates. The dashed line ($t_{\rm GP} = 0, x < x_\lambda$) is the maximum-curvature surface containing the spacetime singularity. The arrows represent light cones for reference on the causal structure. The surface TH_+ is an antitrapping horizon. The surface CH_+ is an anticausal horizon, beyond which the ingoing null ray's velocity becomes negative once again, and the constant x curves become timelike again.

up to the divergent surface, can then be pictured by the region $0 < x < R$ in Fig. 3, which describes a collapsing star of dust followed by its explosion after it crosses the maximum-curvature surface. The star-exterior region $x > R$ of Fig. 3, however, is not uniquely determined by the canonical dynamics, since the star-interior solution can be glued to a vacuum solution in different ways, leading to different physical phenomena. We will explore two such possibilities, one corresponding to the formation of a wormhole, and the other to a black-to-white-hole transition. To this end, we will make use of the Kruskal maximal extension adapted to the modified theory.

5. Vacuum maximal extension

Starting with either the modified Schwarzschild metric (127) or the modified Gullstrand-Painlevé metric (133)—giving both the exact same result—we perform the frame transformation to null coordinates (97) to put it in the Kruskal-Szekeres form:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{x}\right)\mathrm{d}u\mathrm{d}v, \tag{165}$$

where the Schwarzschild coordinates are related to the null ones, using (99), by

$$u = t - x_*, \qquad v = t + x_*, \tag{166}$$

with

$$\begin{aligned} x_* &= 2M\sqrt{1+\lambda^2}\ln\left(c\left(\frac{1+\sqrt{1-x_\lambda/x}}{1-\sqrt{1-x_\lambda/x}}\right)^{\frac{1+x_\lambda/(4M)}{\sqrt{1+\lambda^2}}}\right. \\ &\quad\left.\times\left|\frac{\sqrt{1+\lambda^2}\sqrt{1-x_\lambda/x}-1}{\sqrt{1+\lambda^2}\sqrt{1-x_\lambda/x}+1}\right| e^{\frac{x}{2M}\frac{\sqrt{1-x_\lambda/x}}{\sqrt{1+\lambda^2}}}\right) \\ &=: 2M\sqrt{1+\lambda^2}\ln\Gamma(x), \end{aligned} \tag{167}$$

where we have absorbed the coefficients $v_t^{(i)}$ into the null coordinates, as we have chosen them to be the constant Killing quantities of null geodesics, and c is a constant. In the limit $x \to 2M$, the null coordinates take the values $u \to \infty$ and $v \to -\infty$, as the defined function takes the value $\Gamma \to 0$. We overcome this by the usual coordinate transformation

$$U = -e^{-u/(4M\sqrt{1+\lambda^2})}, \qquad V = e^{v/(4M\sqrt{1+\lambda^2})}, \tag{168}$$

so that the region $r > 2M$ corresponds to $U \in (-\infty, 0)$, $V \in (0, \infty)$. The metric becomes

$$\mathrm{d}s^2 = -F_{\rm E}(x)\mathrm{d}U\mathrm{d}V, \tag{169}$$

with

$$F_{\rm E}(x) = 16M^2(1+\lambda^2)\left(1 - \frac{2M}{x}\right)/\Gamma(x), \tag{170}$$

and $x = x(U, V)$ given by the solution to

$$UV = -\Gamma(x). \tag{171}$$

In the limit $x \to 2M$, we obtain $U, V \to 0$ and finite $F_{\rm E}$:

$$F_{\rm E} \underset{x\to 2M}{\longrightarrow} \frac{64M^2}{c}\frac{1+\lambda^2}{\lambda^2}e^{-1/(1+\lambda^2)}\left(\frac{\sqrt{1+\lambda^2}-1}{\sqrt{1+\lambda^2}+1}\right)^{(1+\frac{1}{2}\frac{\lambda^2}{1+\lambda^2})/\sqrt{1+\lambda^2}}. \tag{172}$$

Therefore, we can extend the null coordinates to $x < 2M$ by taking negative values for U, V, achieving the modified

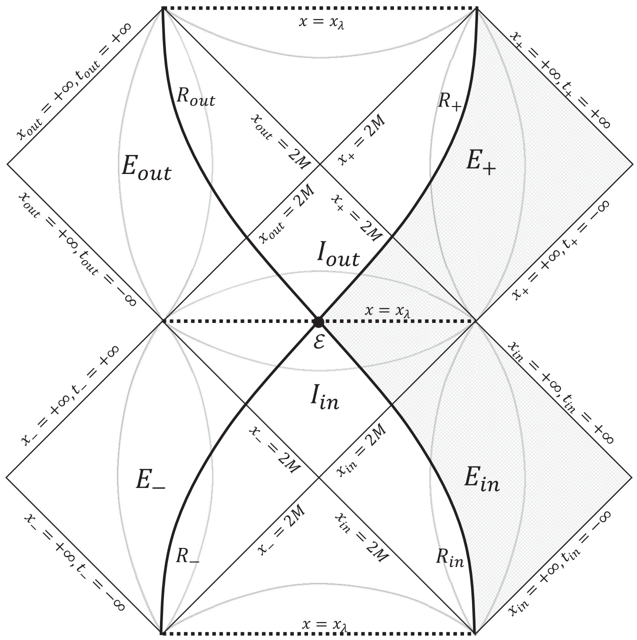


FIG. 4. Maximal extension of the wormhole solution in vacuum. The faint gray lines denote constant-x curves. The solid line $R_{\rm in}$ represents a timelike geodesic at rest at spatial infinity in the remote past of the exterior $E_{\rm in}$. The corresponding geodesics for the other regions are also shown, all intersecting at the point $\mathcal{E}$ on the minimum-radius surface.

Kruskal extension. Finally, performing the conformal transformation

$$\bar{U} = \arctan(U), \qquad \bar{V} = \arctan(V), \tag{173}$$

we obtain the metric

$$\mathrm{d}s^2 = -\bar{F}_{\rm E}(x)\mathrm{d}\bar{U}\mathrm{d}\bar{V}, \tag{174}$$

and the usual Penrose diagram follows (see Fig. 4), which was first obtained in [7]. Unlike the classical solution, this one is singularity free in the vacuum. The maximum-curvature surface at $x = x_\lambda$ is regular and can be crossed. This surface is one of reflection symmetry, as explained in Sec. V; thus, on the other side of x_λ, we simply have the reflected solution of the black hole with the only difference being the flow of time, which gives this region the properties of a white hole. The global structure is then that of an interuniversal wormhole connecting two different asymptotic regions belonging to the exteriors $E_{\rm in}$ and $E_{\rm out}$ via an interior $I = I_{\rm in} \cup I_{\rm out}$. Furthermore, the maximal extension allows for the periodic behavior seen in Fig. 4 such that not only two, but four asymptotic regions can be joined by the same interior I of the wormhole. The curve $R_{\rm in}$ (or R_-) is precisely the trajectory that the star's radius follows in its collapse stage, while $R_{\rm out}$ (or R_+) is the trajectory it follows in its exploding stage.

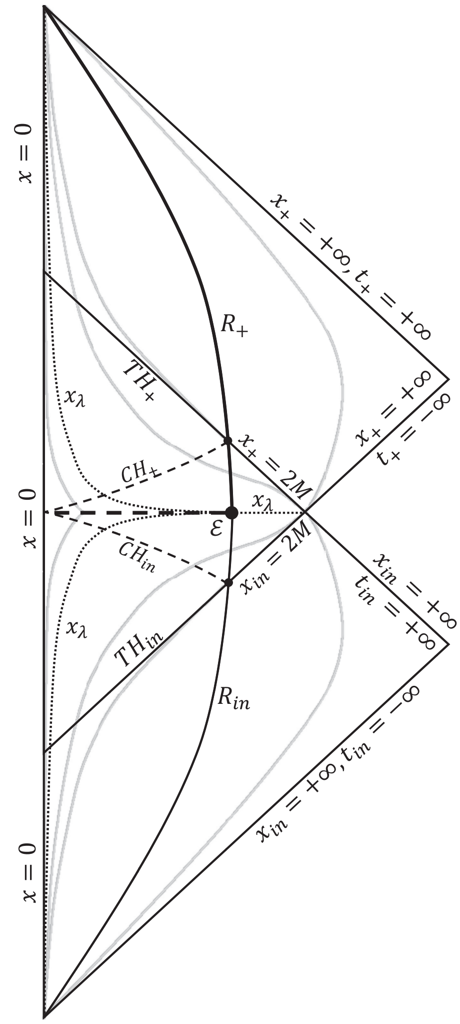


FIG. 5. Formation of a wormhole from the collapse. For reference, the faint gray lines represent constant-x curves, except for $x = x_\lambda$, which is the dotted line. The dashed line at the center is the $t_{\rm GP} = 0$ maximum-curvature surface. The curve $R = R_{\rm in} \cup R_+$ is the full trajectory of the star's radius.

VII. ON THE POSSIBLE PHYSICAL OUTCOMES OF THE COLLAPSE

A. Formation of an interuniversal wormhole

Consider the vacuum solution shaded in gray in Fig. 4 corresponding to the region $E_{\rm in} \cup I \cup E_{\rm out}$, restricted to $x_{\rm in} > R_{\rm in}$ and $x_+ > R_+$, with boundary $R = R_{\rm in} \cup R_+$. One can then continuously glue this solution to that of the dust collapse, given by the star interior of Fig. 3, by their mutual boundary R. The resulting spacetime can then be represented in a conformal diagram (see Fig. 5). The global structure is well defined everywhere, except at the $t_{\rm GP} = 0$ surface joining $x = 0$ to $\mathcal{E}$ where the spacetime curvature diverges. The gluing of the boundary R is everywhere continuous. This is the spacetime of an interuniversal wormhole joining only two asymptotic regions, rather than the four of the vacuum solution. The time flow and the causal structure allows an observer in $E_{\rm in}$ to communicate with another observer in E_+ by sending signals that traverse the wormhole, but an observer in E_+ cannot send a signal back to $E_{\rm in}$.

B. Black-to-white-hole transition: Progress in an effective description

Following [36,43], one can obtain a spacetime describing a black-to-white-hole transition by slicing a wormhole spacetime of the form of Fig. 5 such that a single asymptotic region remains. This slicing can be done as shown in Fig. 6. The shaded area is a valid solution to the vacuum equations of motion with two boundaries: one given by the star's radius R, and the other given by S. The two surfaces TO_+ and $TO_{\rm in}$ can then be smoothly joined, because the spacetime surrounding them is locally identical. The resulting conformal diagram is shown in Fig. 7.

The details on the surface TS and the precise positions of the points ∇ and Δ are not provided by the canonical equations of EMG, and they must be provided by considerations of a particular quantum gravity approach (or whatever underlying theory is being modeled), so we leave them as undetermined in our effective approach—for instance, the construction of [36] estimates the coordinates of the point Δ to be $(t = 0, x = 7M/6)$ in classical Schwarzschild coordinates.

The effective theory thus provides a picture compatible with the black-to-white-hole transition proposal as a quantum gravity phenomenon in this particular gluing of

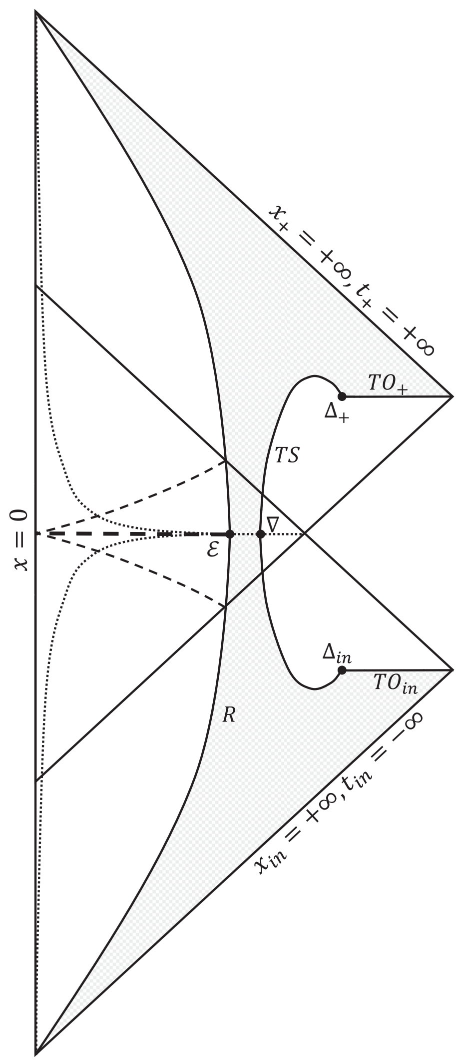


FIG. 6. Star exterior of a black-to-white-hole transition spacetime diagram. The shaded region is a solution to the star-exterior modified equations with the two boundaries R and $S = TO_{\rm in} \cup TS \cup TO_+$. The point where TS intersects the minimum-radius surface is denoted by ∇.

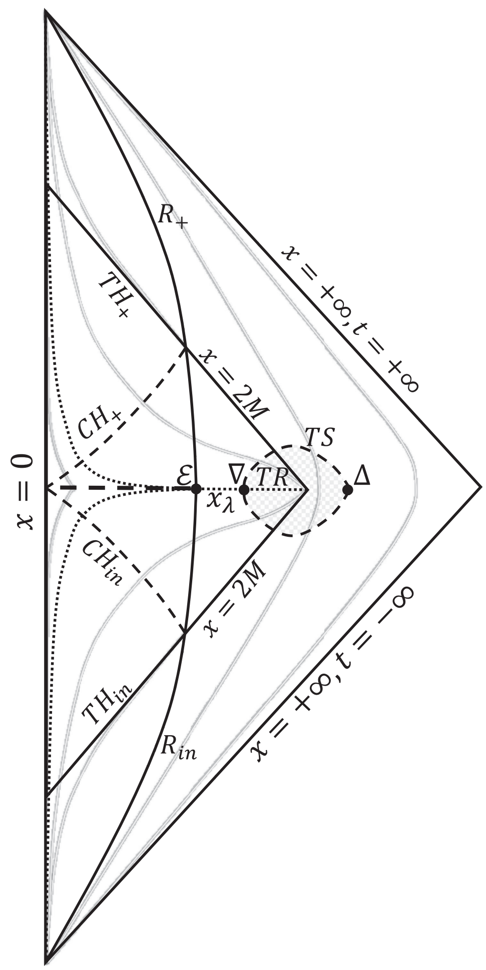


FIG. 7. Black-to-white-hole transition conformal diagram. The dashed line $(t_{\rm GP} = 0, x < x_\lambda)$ at the center is the maximum-curvature surface containing the spacetime singularity. We refer to the region TR as the transition region, bounded by the surface TS. The point Δ is chosen close to the horizon at the coordinates $(t_{\rm GP} = 0, x = 2M + \delta)$, with small, positive δ, but otherwise undetermined. While the transition region has an undetermined geometry, one can schematically extrapolate the trajectories of constant-x curves.

the dust and vacuum regions. However, the region TR bounded by the surface TS is still unsolved, and more work is needed to determine whether it can be described by EMG as the effective theory, or if more details of quantum gravity are needed.

If we take the divergence on the maximum-curvature surface as a sign of the breakdown of EMG as an effective theory, then we can insert a second transition region surrounding the surface joining $x = 0$ to $\mathcal{E}$, or simply extend the single-transition region to surround the maximum-curvature surface by placing ∇ at $x = 0$.

VIII. CONCLUSIONS

We have outlined a procedure to couple matter to EMG on purely canonical terms, ensuring general covariance of both the spacetime and matter fields. We have explicitly done so for the perfect fluid, obtaining the general form of the perfect fluid contribution to the constraint, and found that no modifications are allowed except for the use of an emergent metric and a nontrivial gravitational dependence of the pressure function. Finally, we have also outlined a procedure to study gravitational collapse in spherically symmetric EMG using a comoving coordinate system corresponding to the canonical formulation of the Oppenheimer-Snyder model, and we have explicitly computed the collapse of dust.

The dynamical solution to the gravitational collapse of dust in EMG, modeling certain effects of quantum gravity, predicts that a spacetime singularity is formed at the maximum-curvature surface. However, despite such a singularity, and unlike the classical case, the (ingoing and outgoing) null geodesics acquire a vanishing velocity and positive, finite acceleration at the maximum-curvature surface. Furthermore, all the canonical variables remain finite too, suggesting that the canonical solution can be extended past this surface and "bounce." Consequently, we have obtained a spacetime metric that describes the entire region below the star's radius. However, since a consistent gluing of the star-interior solution to the star-exterior one describing the vacuum does not follow uniquely from solving the equations of motion, the physical result of the collapse will depend on the specific choice of the gluing process.

The most straightforward gluing leads to an interuniversal wormhole spacetime with two distinct asymptotic regions that can be crossed in a single direction. This wormhole connects a nascent black hole as the result of the collapse in the first universe to an evaporating white hole as a result of the matter emerging out of it in the second universe. The second gluing, which results from slicing out part of the wormhole spacetime, consists of a single asymptotic region that fits with the black-to-white-hole transition proposal suggested by some quantum gravity approaches. As a compromise of such slicing, a finite transition region TR becomes undefined, which, together with the divergent geometry of the maximum-curvature surface, presents an ongoing challenge to this proposal in an effective description. The resolution of this transition region might be available only in a full quantum gravity treatment or, at least, with more input to the effective approach. Instabilities of the black-to-white-hole transition when perturbations are present studied with a classical geometry can be tamed by a time-asymmetric scenario [53]; thus, future work will be devoted to ensure the EMG model cures these instabilities too.

Though more work is needed to resolve the above issues, and to determine whether they appear in the collapse of different forms of matter, it is encouraging to learn that new properties of gravitational collapse and the ultimate fate of black holes can be found in covariant modified gravity theories such as emergent modified gravity.

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