Tidal effects up to next-to-next-to-leading post-Newtonian order in massless scalar-tensor theories

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In this paper, we study the tidal effects in the gravitationally bound two-body system at next-to-next-to leading post-Newtonian order for spinless sources in massless scalar-tensor theories. We compute the conservative dynamics, using both a Fokker Lagrangian approach and effective field theory with the post-Newtonian effective field theory formalism. We also compute the ten conserved quantities at the same next-to-next-to leading order. Finally, we extend our results from simple scalar-tensor theories to Einstein-scalar-Gauss-Bonnet gravity. Such results are important in preparation of the science case of the next generation of gravitational wave detectors.

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I. INTRODUCTION

Gravitational wave astronomy is becoming a mature field, nourished by the current and future gravitational wave experiments such as LIGO-Virgo-KAGRA, LISA, or the Einstein Telescope. To accompany the development of the field, it is crucial to provide the community with all the necessary tools to achieve a very high standard science interpretation. Among this, one will explore the strong-field and highly dynamical regime of gravity which will allow us to test fundamental physics.

Compact binary systems are the most common sources of gravitational waves, and their detection and parameter estimation heavily rely on our ability to model their waveform at a very high precision. To test our gravitational paradigm, such a program should be done not only in general relativity (GR) but also in a representative selection of alternative theories of gravity. In this work, we focus on the scalar-tensor (ST) class of theories in which a single massless scalar field is introduced in addition to the gravitational field [1]. Although such a class is wide, with many different models that can usually be classified in the Degenerate Higher-Order Scalar-Tensor (DHOST) theories [2], we will focus here on the simplest one, namely, the generalized Brans-Dicke theories [3]. However, we will see that our results can easily be extended to other theories, such as Einstein-scalar-Gauss-Bonnet gravity [4].

Gravitational wave modeling in scalar-tensor theories has been developed for the different phases of the coalescence for several years. The merger part is being tackled using different numerical relativity approaches [5–10]. On the analytical side, which is used to model the inspiral phase, results have now reached a high post-Newtonian (PN) accuracy. Hence, the dynamics is now known at 3PN order,[1] while the waveform and flux have been obtained at 1.5PN order[2] [11–13]. In particular, tidal effects have also been investigated and derived at leading order [14]. Note that all these results were first obtained in the generalized Brans-Dicke (BD) framework and then extended to other theories like Einstein-Maxwell-Dilaton or Einstein-scalar-Gauss-Bonnet (EsGB) for which only the leading-order correction was necessary to achieve the same accuracy [15–20].

The purpose of this work is to compute the tidal effects at the next-to-next-to-leading order (NNLO). Tidal effects are particularly interesting in ST theories as they start at 3PN order compared to 5PN order in GR. This is due to the presence of a time-varying dipole moment that generates a scalar-induced tidal deformation of compact objects. The motivation to go to the NNLO is to reach a level where the gravitationally induced tidal deformations start contributing [21].

After a short subsection on the notations, the rest of the paper is organized as follows. First, in Sec. II, we present the massless-scalar-tensor theories that will be studied through this work. Then, in Sec. III, we explain how the post-Newtonian formalism is adapted to the treatment of

[1] We call nPN order the $2n$th order in an expansion in $1/c$, namely, nPN $= \mathcal{O}(\frac{1}{c^{2n}})$.

[2] In scalar-tensor theories, the leading-order flux is at -1PN order compared to GR due to the presence of dipolar emission. In this paper, we choose to refer PN orders with respect to the leading GR contribution. For example, 1PN order in the flux corresponds to 2PN order beyond the leading-order ST contribution.

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tidal effects in ST theories, and we present in Sec. IV an alternative calculation based on the PN effective field theory (EFT) formalism that we used to check our results. In Secs. V and VI, the NNLO Lagrangian and the conserved quantities are respectively presented. Before concluding, the short Sec. VII explains why our results are also valid for EsGB gravity. The paper ends with some Appendixes presenting technical details. We have relegated most of the lengthy results to the supplemental material [22].

A. Notations

In this section, we present the notation that will be used throughout the paper. Some quantities are related to ST theories and their generalized PPN parameters, while others are linked to compact objects and binary systems:

(i) We adopt the convention that the leading-order contribution due to tidal effect, which is formally at 3PN order, is noted as leading order (LO). The higher-order corrections will be called next-to-leading order (NLO) and NNLO, and they respectively correspond to 4PN and 5PN orders.

(ii) The two masses are indicated by m_1 and m_2. We denote by $\boldsymbol{y}_a(t)$ the two ordinary coordinate trajectories in a harmonic coordinate system $\{t, \mathbf{x}\}$, by $\boldsymbol{v}_a(t) = \mathrm{d}\boldsymbol{y}_a/\mathrm{d}t$ the two ordinary velocities, and by $\boldsymbol{a}_a(t) = \mathrm{d}\boldsymbol{v}_a/\mathrm{d}t$ the two ordinary accelerations. The ordinary separation vector reads $\boldsymbol{n}_{12} = (\boldsymbol{y}_1 - \boldsymbol{y}_2)/r_{12}$, where $r_{12} = |\boldsymbol{y}_1 - \boldsymbol{y}_2|$; ordinary scalar products are denoted by parentheses, e.g., $(n_{12}v_1) = \boldsymbol{n}_{12}\cdot\boldsymbol{v}_1$, while the three-dimensional Dirac function is denoted $\delta^{(3)}(\mathbf{x})$, and its value at the position $\boldsymbol{y}_a$ is written $\delta_a \equiv \delta^{(3)}(\mathbf{x} - \boldsymbol{y}_a)$. We denote by $L = i_1 \cdots i_\ell$ a multi-index with ℓ spatial indices; $\nabla_L = \nabla_{i_1} \cdots \nabla_{i_\ell}$ and so on; similarly, $n_L = n_{i_1} \cdots n_{i_\ell}$.

(iii) To express quantities in the center-of-mass (CM) frame, we introduce the notations $\boldsymbol{n} = \boldsymbol{n}_{\mathbf{12}}$, $r = r_{12}$ and define the relative position $\boldsymbol{x} = \boldsymbol{y}_{\mathbf{1}} - \boldsymbol{y}_{\mathbf{2}}$, velocity $\boldsymbol{v} = \boldsymbol{v}_{\mathbf{1}} - \boldsymbol{v}_{\mathbf{2}}$, and acceleration $\boldsymbol{a} = \boldsymbol{a}_{\mathbf{1}} - \boldsymbol{a}_{\mathbf{2}}$; we pose $v^2 = (vv) = \boldsymbol{v}\cdot\boldsymbol{v}$ and $\dot{r} = (nv) = \boldsymbol{n}\cdot\boldsymbol{v}$. In the CM frame, we use the total mass $m = m_1 + m_2$, the reduced mass $\mu = m_1 m_2/m$, the symmetric mass ratio $\nu = \mu/m \in]0, 1/4]$, and the relative mass difference $\delta = (m_1 - m_2)/m \in [0, 1[$. Note that the symmetric mass ratio and the relative mass difference are linked by the relation $\delta^2 = 1$–4ν. To reduce our expressions in the CM frame, we define convenient combinations of the tidal deformabilities, namely,

TABLE I. Parameters for the general ST theory and our notation for PN parameters.

	ST parameters	
General	$\omega_0 = \omega(\phi_0)$, $\omega'_0 = \frac{\mathrm{d}\omega}{\mathrm{d}\phi}\Big\vert_{\phi=\phi_0}$, ${\omega_0}'' = \frac{\mathrm{d}^2\omega}{\mathrm{d}\phi^2}\Big\vert_{\phi=\phi_0}$, $\varphi = \frac{\phi}{\phi_0}$, $\tilde{g}_{\mu\nu} = \varphi g_{\mu\nu}$, $\tilde{G} = \frac{G(4+2\omega_0)}{\phi_0(3+2\omega_0)}$, $\zeta = \frac{1}{4+2\omega_0}$, $\lambda_1 = \frac{\zeta^2}{(1-\zeta)}\frac{\mathrm{d}\omega}{\mathrm{d}\varphi}\Big\vert_{\varphi=1}$, $\lambda_2 = \frac{\zeta^3}{(1-\zeta)}\frac{\mathrm{d}^2\omega}{\mathrm{d}\varphi^2}\Big\vert_{\varphi=1}$, $\lambda_3 = \frac{\zeta^4}{(1-\zeta)}\frac{\mathrm{d}^3\omega}{\mathrm{d}\varphi^3}\Big\vert_{\varphi=1}$. $[1 \leftrightarrow 2]$ switches the particle's labels (note the index on the λ_i's is not a particle label)	
Sensitivities	$s_a = \frac{\mathrm{d}\ln m_a(\phi)}{\mathrm{d}\ln\phi}\Big\vert_{\phi=\phi_0}$, $s_a^{(k)} = \frac{\mathrm{d}^{k+1}\ln m_a(\phi)}{\mathrm{d}(\ln\phi)^{k+1}}\Big\vert_{\phi=\phi_0}$, $(a = 1, 2)$ $s'_a = s_a^{(1)}$, $s''_a = s_a^{(2)}$, $s'''_a = s_a^{(3)}$, $\mathcal{S}_+ = \frac{1-s_1-s_2}{\sqrt{\alpha}}$, $\mathcal{S}_- = \frac{s_2-s_1}{\sqrt{\alpha}}$.	
Order	**PN parameters**	
N	$\alpha = 1 - \zeta + \zeta(1-2s_1)(1-2s_2)$	
1PN	$\bar{\gamma} = -\frac{2\zeta}{\alpha}(1-2s_1)(1-2s_2)$, $\bar{\beta}_1 = \frac{\zeta}{\alpha^2}(1-2s_2)^2(\lambda_1(1-2s_1) + 2\zeta s'_1)$, $\bar{\beta}_2 = \frac{\zeta}{\alpha^2}(1-2s_1)^2(\lambda_1(1-2s_2) + 2\zeta s'_2)$, $\bar{\beta}_+ = \frac{\bar{\beta}_1+\bar{\beta}_2}{2}$, $\bar{\beta}_- = \frac{\bar{\beta}_1-\bar{\beta}_2}{2}$.	Degeneracy $\alpha(2+\bar{\gamma}) = 2(1-\zeta)$
2PN	$\bar{\delta}_1 = \frac{\zeta(1-\zeta)}{\alpha^2}(1-2s_1)^2$, $\bar{\delta}_2 = \frac{\zeta(1-\zeta)}{\alpha^2}(1-2s_2)^2$, $\bar{\delta}_+ = \frac{\bar{\delta}_1+\bar{\delta}_2}{2}$, $\bar{\delta}_- = \frac{\bar{\delta}_1-\bar{\delta}_2}{2}$, $\bar{\chi}_1 = \frac{\zeta}{\alpha^3}(1-2s_2)^3[(\lambda_2 - 4\lambda_1^2 + \zeta\lambda_1)(1-2s_1) - 6\zeta\lambda_1 s'_1 + 2\zeta^2 s''_1]$, $\bar{\chi}_2 = \frac{\zeta}{\alpha^3}(1-2s_1)^3[(\lambda_2 - 4\lambda_1^2 + \zeta\lambda_1)(1-2s_2) - 6\zeta\lambda_1 s'_2 + 2\zeta^2 s''_2]$, $\bar{\chi}_+ = \frac{\bar{\chi}_1+\bar{\chi}_2}{2}$, $\bar{\chi}_- = \frac{\bar{\chi}_1-\bar{\chi}_2}{2}$.	Degeneracy $16\bar{\delta}_1\bar{\delta}_2 = \bar{\gamma}^2(2+\bar{\gamma})^2$

$$\begin{aligned}
\lambda_{\pm}^{(n)} &= \frac{m_2}{m_1}\bar{\delta}_2\lambda_1^{(n)} \pm \frac{m_1}{m_2}\bar{\delta}_1\lambda_2^{(n)}, \qquad \Lambda_{\pm}^{(n)} = \frac{m_2}{m_1}(1-2s_2)\bar{\delta}_2\lambda_1^{(n)} \pm \frac{m_1}{m_2}(1-2s_1)\bar{\delta}_1\lambda_2^{(n)}, \\
\mu_{\pm}^{(n)} &= \frac{m_2}{m_1}\bar{\delta}_2\mu_1^{(n)} \pm \frac{m_1}{m_2}\bar{\delta}_1\mu_2^{(n)}, \\
c_{\pm}^{(n)} &= \frac{m_2}{m_1}\frac{1-\zeta}{\zeta}(1-\zeta+\zeta(1-2s_2))^2 c_1^{(n)} \pm \frac{m_1}{m_2}\frac{1-\zeta}{\zeta}(1-\zeta+\zeta(1-2s_1))^2 c_2^{(n)}, \\
\nu_{\pm}^{(n)} &= \frac{m_2}{m_1}(1-\zeta)(1-2s_2)(1-\zeta+\zeta(1-2s_2))\nu_1^{(n)} \pm \frac{m_1}{m_2}(1-\zeta)(1-2s_1)(1-\zeta+\zeta(1-2s_1))\nu_2^{(n)}.
\end{aligned} \tag{1}$$

(iv) Finally, to later present our results, following Ref. [11], we introduce a number of ST and post-Newtonian parameters. The ST parameters are defined based on the value ϕ_0 of the scalar field ϕ at spatial infinity, on the Brans-Dicke-like scalar function $\omega(\phi)$ and on the mass -functions $m_a(\phi)$. We pose $\varphi \equiv \phi/\phi_0$. The post-Newtonian parameters naturally extend and generalize the usual PPN parameters to the case of a general ST theory [23,24]. All these parameters are given and summarized in Table I.

II. MASSLESS SCALAR-TENSOR THEORIES

A. Gravitational action

We consider a generic class of scalar-tensor theories in which a single massless scalar field ϕ minimally couples to the metric $g_{\mu\nu}$. It is described by the action

$$S_{\rm ST} = \frac{c^3}{16\pi G}\int {\rm d}^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right] + S_{\rm m}(\mathfrak{M}, g_{\alpha\beta}), \tag{2}$$

where R and g are, respectively, the Ricci scalar and the determinant of the metric; ω is a function of the scalar field; and $\mathfrak{M}$ stands generically for the matter fields. The action for the matter $S_{\rm m}$ is a function only of the matter fields and the metric.

Equivalently, one can consider the same theory in another frame, the Einstein frame, which is more practical to perform the computations. We consider the following conformal transformation for the metric and the redefinition of the scalar field as

$$\tilde{g}_{\mu\nu} = \varphi g_{\mu\nu}, \qquad \varphi = \frac{\phi}{\phi_0}. \tag{3}$$

After some integration by parts, the action becomes

$$S_{\rm ST}^{\rm GF} = \frac{c^3\phi_0}{16\pi G}\int {\rm d}^4x\sqrt{-\tilde{g}}\left[\tilde{R} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\Gamma}^\mu\tilde{\Gamma}^\nu - \frac{3+2\omega(\phi)}{2\varphi^2}\tilde{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\right] + S_{\rm m}(\mathfrak{m}, g_{\alpha\beta}), \tag{4}$$

where we have introduced a gauge fixing term, $\propto \tilde{g}_{\mu\nu}\tilde{\Gamma}^\mu\tilde{\Gamma}^\nu$ with $\tilde{\Gamma}^\nu \equiv \tilde{g}^{\rho\sigma}\tilde{\Gamma}^\nu_{\rho\sigma}$, to enforce working in the harmonic gauge.

B. Matter action

To study the tidal effects, we will consider finite-size objects and go beyond the point-particle approximation that is commonly used. Following the post-Newtonian effective field theory approach, we model the coupling to matter by a worldline action describing the coupling of the two objects to gravity. First, as is usual in the PN formalism, we start with the point-particle action

$$S_{\rm pp} = -c\sum_{a=1,2}\int {\rm d}\tau_a m_a(\phi), \tag{5}$$

where ${\rm d}\tau_a$ is the proper time of particle a along its worldline y_a^μ, defined as ${\rm d}\tau_a \equiv c{\rm d}t\sqrt{-(g_{\mu\nu})_a\frac{v_a^\mu v_a^\nu}{c^2}}$. We have also introduced a dependence of the masses on the scalar field, $m_a(\phi)$, in order to take into account the internal self-gravity of each object with respect to the scalar field [25].

Then, going beyond the point-particle action, we construct a tidal action, decomposed as

$$S_{\rm tidal} = S_{\rm fs}^{(s)} + S_{\rm fs}^{(g)} + S_{\rm fs}^{(g-s)}. \tag{6}$$

The first piece of Eq. (6) encodes scalar-induced tidal effects, i.e., the response of each object with respect to an external scalar field. Still using the EFT approach, we consider the action for the scalar tidal contribution

$$S_{\rm fs}^{(s)} = -c\sum_{a=1,2}\int {\rm d}\tau_a\sum_{l=1}^{\infty}\frac{1}{2l!}\lambda_a^l(\phi)(\nabla_L^\perp\varphi)_a(\nabla_\perp^L\varphi)_a, \tag{7}$$

where $L = \mu_1\cdots\mu_l$ is a multi-index. We have introduced the projection onto the hypersurface orthogonal to the four velocity, namely, $\nabla_\mu^\perp \equiv (\delta_\mu^\nu + u_\mu u^\nu)\nabla_\nu$ and $\nabla_L^\perp = \nabla_{\mu_1}^\perp\cdots\nabla_{\mu_l}^\perp$. The coefficients λ_a^l are the lth-order scalar tidal deformability parameters. As for the masses, we have added an explicit dependence in ϕ to describe their internal response to the scalar field. In the following, we set $\lambda_a(\phi) \equiv \lambda_a^1(\phi)$ and $\mu_a(\phi) \equiv \lambda_a^2(\phi)$.

Through the second piece of Eq. (6), we also model the gravitational tidal effects, i.e., the usual gravitational response of each object with respect to the companion body. Following the approach [26] pioneered by Ref. [27], the corresponding action is given by

$$S_{\rm fs}^{(g)} = \sum_{a=1,2} \int \frac{d\tau_a}{c} \sum_{l=2}^{\infty} \frac{1}{2l!} \left[c_a^l(\phi) G_L^a G_a^L + \frac{l}{(l+1)c^2} d_a^l(\phi) H_L^a H_a^L \right]. \tag{8}$$

G_L and H_L are the tidal moments whose expressions read

$$G^a_{\mu_1\ldots\mu_l} = -c^2 [\nabla^\perp_{<\mu_1} \ldots \nabla^\perp_{\mu_l-2} C_{\mu_{l-1}\underline{\rho}\mu_l>\sigma}] u_a^\rho u_a^\sigma, \tag{9a}$$

$$H^a_{\mu_1\ldots\mu_l} = 2c^3 [\nabla^\perp_{<\mu_1} \ldots \nabla^\perp_{\mu_l-2} C^*_{\mu_{l-1}\underline{\rho}\mu_l>\sigma}] u_a^\rho u_a^\sigma, \tag{9b}$$

with $C_{\mu\nu\rho\sigma}$ the Weyl tensor and $C^*_{\mu\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\kappa} C^{\lambda\kappa}{}_{\rho\sigma}$, where $\varepsilon_{\mu\nu\lambda\kappa}$ denotes the completely antisymmetric Levi-Civita tensor. This action is formally the same as GR, with $\{c_a^{(l)}, d_a^{(l)}\}$ being, respectively, the lth-order mass-type and current-type tidal deformability parameters and where, once again, we have introduced an explicit dependence on the scalar field. In our case, as we are interested in the NNLO tidal effects with respect to the leading-order scalar contribution, it will be sufficient to consider the mass quadrupole tidal moment only, defined as

$$(G_{\mu\nu})_a = -c^2 (R_{\mu\rho\nu\sigma})_a u_a^\rho u_a^\sigma. \tag{10}$$

Note that in this definition the Weyl tensor in Eqs. (9) has been replaced by the Riemann tensor. This is due to the fact that the traces of the Riemann tensor do not impact the dynamics, as proved in Appendix B of Ref. [26]. In other words, using both definitions yields the same equations of motion for the system. As only the mass-type quadrupolar deformation will contribute at the NNLO, we set $c_a(\phi) \equiv c_a^2(\phi)$.

Finally, by adding the third piece of Eq. (6), we also introduce gravitoscalar tidal effects, i.e., the possibility to have a mixing between scalar and gravitational tidal effects. Such effects were already mentioned in Ref. [28]. The action is

$$S_{\rm fs}^{(g-s)} = -c \sum_{a=1,2} \int d\tau_a \sum_{l=2}^{\infty} \frac{1}{l!} \nu_a^l(\phi) G_a^L (\nabla_L^\perp \varphi)_a, \tag{11}$$

where $\nu_a^l(\phi)$ is the lth-order gravitoscalar tidal deformability parameter and possesses, as the other parameters, an explicit dependence on the scalar field. As for the purely gravitational case, only the $l=2$ mode contributes to the NNLO dynamics, so we will restrict to this case in the following and use the loose notation $\nu_a(\phi) \equiv \nu_a^2(\phi)$.

At this point, a clarification on the different frames is needed. In our formalism, we have coupled the matter fields directly to the (Jordan-frame) physical metric, the possible matter interaction being relegated to the dependence of the mass and tidal parameters on the scalar field. However, as will be clearer in Sec. IV, the definition of scalar and gravitational perturbations and the rest of the calculation are done in the conformal (Einstein) frame. As a consequence, it will introduce additional couplings between the matter and scalar fields as well as a mixing of the different type of tidal deformability parameters. In other work, such as Ref. [28], the coupling to matter is directly performed in the Einstein frame; hence, no such mixing is appearing. We will come back on this point in Sec. VII.

III. POST-NEWTONIAN FORMALISM

To derive the equations of motion for each object, we follow a Lagrangian approach, often referred to as the Fokker Lagrangian approach [29]. We remind the reader here the main steps for such a construction:

(i) Starting from the action (4), we derive the field equations for the metric and the scalar field perturbations, displayed in Eqs. (13).

(ii) We solve iteratively these equations up to a certain PN order, determined by the Fokker approach. Here, as we are interested only in the correction due to tidal effects, it is sufficient to know the point-particle contributions to the metric and scalar field perturbations; see Appendix A.

(iii) We inject these solutions in the total action up to the required order. It results in a generalized (Fokker) Lagrangian that depends not only on the positions and velocities of the particles but also on their higher-order derivatives. The result is presented in Sec. V.

(iv) Varying the generalized action with respect to the position of the particles, we obtain the equations of motion for each particle. In Sec. VI B, we display the resulting tidal correction to the equations of motion after reduction to the center-of-mass frame at the LO and NLO, and we relegate the NNLO result to Appendix C.

In the following subsections, we give more details on the implementation of the PN formalism in ST theories.

A. Field equations

First, after introducing the conformal gothic metric $\mathfrak{g}^{\mu\nu} \equiv \sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}$, we define the scalar perturbation ψ and the metric perturbation $h^{\mu\nu}$ as

$$\psi \equiv \varphi - 1, \quad \text{and} \quad h^{\mu\nu} \equiv \mathfrak{g}^{\mu\nu} - \eta^{\mu\nu}, \tag{12}$$

where $\eta^{\mu\nu}$ is the Minkowski metric. Then, from the harmonic gauge-fixed action (4), we get the field equations

$$\Box_\eta h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}, \tag{13a}$$

$$\Box_\eta \psi = -\frac{8\pi G}{c^4}\tau_s, \tag{13b}$$

where $\Box_\eta$ denotes the ordinary flat space-time d'Alembertian operator. The source terms read

$$\tau^{\mu\nu} = \frac{\varphi}{\phi_0}|g|(T^{\mu\nu} + \Delta T^{\mu\nu}) + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}, \tag{14a}$$

$$\tau_s = -\frac{\varphi}{\phi_0(3+2\omega)}\sqrt{-g}\left(T - 2\phi\left(\frac{\partial T}{\partial\phi} + \Delta S\right)\right) - \frac{c^4}{8\pi G}\Lambda_s. \tag{14b}$$

The nonlinearities in the source terms are encoded by $\Lambda^{\mu\nu}$ and Λ_s whose explicit expressions can be found in Ref. [13]. In Eq. (14), we have introduced the classical matter stress-energy tensor $T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S_{\rm pp}}{\delta g_{\mu\nu}}$, the tidal correction to this tensor $\Delta T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S_{\rm tidal}}{\delta g_{\mu\nu}}$, and their respective contractions with the metric $T \equiv g_{\mu\nu}T^{\mu\nu}$. For convenience, we have also defined $\frac{\partial T}{\partial\phi} \equiv \frac{\partial T(g_{\mu\nu},\phi)}{\partial\phi}\Big|_{g\,\rm fixed}$ and $\Delta S = \frac{1}{\sqrt{-g}}\frac{\delta S_{\rm tidal}}{\delta\phi}$.

The point-particle contribution to the stress energy tensor $T^{\mu\nu}$ reads

$$T^{\mu\nu}(t,\mathbf{x}) = \sum_{a=1,2}\frac{m_a(\phi)v_a^\mu v_a^\nu}{\sqrt{-g_{\mu\nu}v_a^\mu v_a^\nu/c^2}}\frac{\delta_a^{(3)}(\mathbf{x}-\mathbf{y_a}(\mathbf{t}))}{\sqrt{-g}}. \tag{15}$$

Then, we expand the mass $m_a(\phi)$ around the asymptotic value of the scalar field at infinity, ϕ_0. Using the definition of the sensitivities and higher-order sensitivities of Ref. [30],

$$s_a^{(n)} \equiv \frac{d^{n+1}\ln m_a(\phi)}{d\ln\phi^{n+1}}\Bigg|_{\phi=\phi_0}, \tag{16}$$

we obtain, at the minimal order required for this work,

$$m_a(\phi) = m_a\Bigg[1 + s_a\psi + \frac{1}{2}(s_a^2 + s_a' - s_a)\psi^2 + \frac{1}{6}(s_a'' + 3s_a's_a - 3s_a' + s_a^3 - 3s_a^2 + 2s_a)\psi^3 + \mathcal{O}(\psi^4)\Bigg]. \tag{17}$$

As explained at the beginning of Sec. III and in Appendix A, only the point-particle solution is required to compute the NNLO tidal correction to the dynamics. Hence, we do not display here the explicit expressions for $\Delta T^{\mu\nu}$ and ΔS. They can be found in the forthcoming companion article [31] in which we compute the NNLO gravitational and scalar fluxes and waveforms. Similarly to the mass, we expand the tidal deformation parameters as a function of the scalar perturbation ψ as

$$\lambda_a(\phi) = \sum_{n=0}^{\infty}\frac{\lambda_a^{(n)}}{n!}\phi_0{}^n\psi^n, \quad \mu_a(\phi) = \sum_{n=0}^{\infty}\frac{\mu_a^{(n)}}{n!}\phi_0{}^n\psi^n,$$
$$\nu_a(\phi) = \sum_{n=0}^{\infty}\frac{\nu_a^{(n)}}{n!}\phi_0{}^n\psi^n, \quad c_a(\phi) = \sum_{n=0}^{\infty}\frac{c_a^{(n)}}{n!}\phi_0{}^n\psi^n. \tag{18}$$

B. Metric and scalar-field decomposition

To solve the field equations, we further decompose the metric and the scalar field in terms of PN potentials that obey flat d'Alembertian equations [32]. We start by decomposing the metric perturbation in its component $h^{\mu\nu} = (h^{00ii}, h^{0i}, h^{ij})$, where $h^{00ii} \equiv h^{00} + h^{ii}$, which are in turn decomposed in terms of some PN potentials. The same applies for the scalar perturbation ψ; see Eqs. (20). The next step is to determine the minimal order that is required in order to get the dynamics at the NNLO in the tidal effects. Following Ref. [26], we start by noticing that it is enough to solve the point-particle field equations (i.e., neglecting the tidal corrections to the potentials) in order to get the corrections to the Lagrangian due to tidal effects. See Appendix A for the full reasoning. Hence, the potentials will be sourced by the point-particle matter action (5) only, and the tidal corrections will come from the injection of these field equation solutions into $S_{\rm tidal} = S_{\rm fs}^{(s)} + S_{\rm fs}^{(g)} + S_{\rm fs}^{(g-s)}$. Furthermore, to get the NNLO corrections for tidal effects, we formally need to know the metric and scalar field up to 2PN beyond the leading order, i.e., to order $(h^{00ii}, h^{0i}, h^{ij}; \psi) = \mathcal{O}(\frac{1}{c^6}, \frac{1}{c^5}, \frac{1}{c^6}; \frac{1}{c^6})$. However, as the leading term is sourced only by the scalar field, see Ref. [14], such a high order is only required for the scalar field, while one can go to one lower PN order for the metric components. At the end, we get that we should know the metric and the scalar field at the orders

$$(h^{00ii}, h^{0i}, h^{ij}; \psi) = \mathcal{O}\left(\frac{1}{c^4}, \frac{1}{c^3}, \frac{1}{c^4}; \frac{1}{c^6}\right). \tag{19}$$

Using the standard PN decomposition, we introduce the potentials $(V, V^i, \hat{W}^{ij})$ and $(\psi_{(0)}, \psi_{(1)})$ to parametrize the metric and scalar perturbations [11,32],

$$h^{00ii} = -\frac{4}{c^2}V - \frac{8}{c^4}V^2 + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{20a}$$

$$h^{0i} = -\frac{4}{c^3}V_i + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{20b}$$

$$h^{ij} = -\frac{4}{c^4}\left(\hat{W}_{ij} - \frac{1}{2}\delta_{ij}\hat{W}\right) + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{20c}$$

$$\psi = -\frac{2}{c^2}\psi_{(0)} + \frac{2}{c^4}\left(1 - \frac{\phi_0\omega_0'}{3+2\omega_0}\right)\psi_{(0)}^2 + \frac{1}{c^6}\left[-\frac{4}{3}\left(1 - \frac{4\phi_0\omega_0'}{3+2\omega_0} - \frac{\phi_0^2(-4(\omega_0')^2 + (3+2\omega_0)\omega_0'')}{(3+2\omega_0)^2}\right)\psi_{(0)}^3 + \psi_{(1)}\right] + \mathcal{O}\left(\frac{1}{c^8}\right), \tag{20d}$$

with $\hat{W} \equiv \hat{W}_{ii}$. Each PN potential satisfies a flat space-time wave equation

$$\Box V = -4\pi G\sigma, \tag{21a}$$

$$\Box V_i = -4\pi G\sigma_i, \tag{21b}$$

$$\Box \hat{W}_{ij} = -4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - \partial_i V \partial_i V - (3+2\omega_0)\partial_i\psi_{(0)}\partial_j\psi_{(0)}, \tag{21c}$$

$$\Box \psi_{(0)} = +4\pi G\sigma_s, \tag{21d}$$

$$\Box \psi_{(1)} = +16\pi G\sigma_s\hat{W} - 4\left(4V_i\partial_{ti}\psi_{(0)} + 2\hat{W}_{ij}\partial_{ij}\psi_{(0)} + 2V\partial_t^2\psi_{(0)} + 2\left(\partial_t V_i + \partial_j\hat{W}_{ij} - \frac{1}{2}\partial_i\hat{W}\right)\partial_i\psi_{(0)}\right), \tag{21e}$$

where we have introduced the compact-support matter source densities in the point-particle approximation

$$\sigma = \frac{1}{\phi_0\varphi^3}\frac{T^{00}+T^{ii}}{c^2}, \quad \sigma_i = \frac{1}{\phi_0\varphi^3}\frac{T^{0i}}{c}, \quad \sigma_{ij} = \frac{1}{\phi_0\varphi^3}T^{ij}, \tag{22a}$$

$$\sigma_s = -\frac{1}{c^2\phi_0}\frac{\sqrt{-g}}{\sqrt{(3+2\omega_0)(3+2\omega)}}\left(T - 2\varphi\frac{\partial T}{\partial\varphi}\right). \tag{22b}$$

Among these potentials, only $\psi_{(1)}$, which has a noncompact support source, is new to this paper. Note that the derivatives in Eqs. (21e) should be understood as Schwartz distributional derivatives [33]. Solving these equations does not introduce new difficulties, and it was achieved using the techniques already introduced in the literature [32]. In Table II, we summarize the orders at which each potential is needed in order to derive the Fokker Lagrangian up to the NNLO in the tidal effects.

TABLE II. Summary of the PN potentials and their required order for the computation of the NNLO tidal corrections in the Lagrangian.

	V	V_i	W_{ij}	$\psi_{(0)}$	$\psi_{(1)}$
N	×	×	×	×	×
1PN	×			×	
2PN				×	

We then inject the gravitational and scalar solutions in the total action in order to get a generalized Lagrangian that depends on the positions and their successive derivatives. As we are interested in the tidal corrections only, it is sufficient to incorporate the point-particle (p.p.) solutions in the finite-size actions (7), (8), (11). We present the result of the calculations in Sec. V.

IV. ALTERNATIVE DERIVATION: PN EFT FORMALISM

In parallel to the traditional post-Newtonian calculation presented in Sec. III, we have also computed the Lagrangian using an EFT method [34]. In addition to paving the way to perform other heavy calculations with this method, it provided us with an additional check of our PN results. We recall here the main steps of the EFT framework and explain how to adapt it to a ST theory. The core of the calculation, namely, the calculations of Feynman rules and diagrams, are, respectively, put in Appendix B and in the supplemental material [22].

Our goal is to work with a worldline theory and canonically normalized bulk fields so that we can exploit the PN EFT machinery [35–37]. Specifically, in addition to the conformal transformation (3), we perform a redefinition of the scalar field,

$$\frac{\psi}{\tilde{M}_{\rm pl}} = \sqrt{\frac{3+2\omega_0}{2}}\ln\frac{\phi}{\phi_0}, \tag{23}$$

where we have introduced the effective Planck mass $\tilde{M}_{\rm pl}^2 \equiv \frac{c^3\phi_0}{8\pi G}$ following closely the EFT vocabulary but replacing the gravitational constant by an effective one, $\frac{G}{\phi_0}$. As before, we expand all the field-dependent couplings around the asymptotic value ϕ_0 with respect to the canonically normalized field ψ, ending up with the following action,

$$
\begin{aligned}
S_{\rm EFT} = & \int \mathrm{d}^4 x \sqrt{-\tilde{g}} \left[\frac{\tilde{M}_{pl}^2}{2} \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \sum_{n=0}^{\infty} \frac{x_n}{n!} \left(\frac{\psi}{\tilde{M}_{pl}} \right)^n \right] - \sum_{a=1,2} \int \mathrm{d}t_a m_a \sqrt{-(\tilde{g}_{\mu\nu})_a v_a^\mu v_a^\nu} \sum_{n=0}^{\infty} \frac{\tilde{d}_n^{(a)}}{n!} \left(\frac{\psi}{\tilde{M}_{pl}} \right)^n \\
& - \frac{1}{2} \sum_{a=1,2} \int \mathrm{d}t_a \lambda_a^{(0)}(\phi_0) \sqrt{-(\tilde{g}_{\mu\nu})_a v_a^\mu v_a^\nu} \left(\tilde{g}^{\mu\nu} - \frac{v_a^\mu v_a^\nu}{\tilde{g}_{\mu\nu} v_a^\mu v_a^\nu} \right) \frac{\partial_\mu \psi \partial_\nu \psi}{\tilde{M}_{pl}^2} \sum_{n=0}^{\infty} \frac{\tilde{f}_n^{(a)}}{n!} \left(\frac{\psi}{\tilde{M}_{pl}} \right)^n \\
& - \frac{1}{4} \sum_{a=1,2} \int \mathrm{d}t_a \mu_a^{(0)} \tilde{f}_0^{(a)} \frac{\nabla_{\mu\nu}^{\perp} \psi \nabla_{\perp}^{\mu\nu} \psi}{\tilde{M}_{pl}^2} + \frac{1}{4} \sum_{a=1,2} \int \mathrm{d}t_a \nu_a^{(0)} \frac{\sqrt{2}}{\sqrt{3+2\omega_0}} \frac{\partial_\mu \partial_\nu \left(e^{-\sqrt{\frac{2}{3+2\omega_0}} \frac{\psi}{\tilde{M}_{pl}}} \tilde{g}_{00} \right) \nabla_{\perp}^{\mu\nu} \psi}{\tilde{M}_{pl} \left(e^{-\sqrt{\frac{2}{3+2\omega_0}} \frac{\psi}{\tilde{M}_{pl}}} \tilde{g}_{00} \right)} \\
& + \sum_{a=1,2} \int \mathrm{d}t_a \frac{c_a^{(0)}}{16} \frac{\partial_\mu \partial_\nu \left(e^{-\sqrt{\frac{2}{3+2\omega_0}} \frac{\psi}{\tilde{M}_{pl}}} \tilde{g}_{00} \right) \partial^\mu \partial^\nu \left(e^{-\sqrt{\frac{2}{3+2\omega_0}} \frac{\psi}{\tilde{M}_{pl}}} \tilde{g}_{00} \right)}{\left(e^{-\sqrt{\frac{2}{3+2\omega_0}} \frac{\psi}{\tilde{M}_{pl}}} \tilde{g}_{00} \right)^2},
\end{aligned}
\tag{24}
$$

where, only in this section, we have set $c = 1$. Note that, for the last three terms we are displaying only the leading-order contribution needed for this work. From the two last terms in Eq. (24), we directly observe the mixing between the different kinds of tidal deformability parameters. In particular, the purely gravitational contribution will also contain terms corresponding to purely scalar and gravitoscalar interactions. The coupling constants, x_n, $\tilde{d}_n$, and $\tilde{f}_n$, are given up to the order needed for our calculations by the expressions

$$
x_n = \left(\frac{2}{3+2\omega_0} \right)^{1+n/2} \sum_{m=1}^{n} (-1)^m \omega_0^{(m)} \phi_0^m \left(\sum_{l=1}^{m} \frac{l^{n-1}(-1)^l}{\Gamma(l)\Gamma(1+m-l)} \right) \quad \text{for } n \geq 1, \quad x_0 = 1, \tag{25}
$$

and

$$
\begin{aligned}
\tilde{d}_n^{(a)} = \frac{d_n^{(a)}}{2^n} \left(\frac{2}{3+2\omega_0} \right)^{n/2}, \qquad & d_0^{(a)} = 1, \\
& d_1^{(a)} = 2s_a - 1, \\
& d_2^{(a)} = 4s'_a + (2s_a - 1)^2, \\
& d_3^{(a)} = 8s''_a + 12s'_a(2s_a - 1) + (2s_a - 1)^3,
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\tilde{f}_n^{(a)} = f_n^{(a)} \left(\frac{2}{3+2\omega_0} \right)^{1+n/2}, \qquad & f_0^{(a)} = 1, \\
& f_1^{(a)} = \frac{\lambda_a^{(1)}}{\lambda_a^{(0)}} \phi_0 + 5/2, \\
& f_2^{(a)} = \frac{\lambda_a^{(2)}}{\lambda_a^{(0)}} \phi_0^2 + 6 \frac{\lambda_a^{(1)}}{\lambda_a^{(0)}} \phi_0 + 25/4.
\end{aligned}
\tag{27}
$$

The action (24) is the starting point to develop the machinery of the PN EFT formalism. To do so, we also perform a Kaluza-Klein decomposition of the metric as

$$
\tilde{g}_{\mu\nu} = e^{4\frac{\phi_g}{\tilde{M}_{pl}}} \begin{pmatrix} -1 & 2A_j/\tilde{M}_{pl} \\ 2A_i/\tilde{M}_{pl} & e^{-8\frac{\phi_g}{\tilde{M}_{pl}}} \gamma_{ij} - 4A_i A_j/\tilde{M}_{pl}^2 \end{pmatrix}, \tag{28}
$$

where $\gamma_{ij} = \delta_{ij} + \sigma_{ij}/\Lambda$. In the following, we will work with the Kaluza-Klein gravitational fields, $(\phi_g, A_i, \sigma_{ij})$, instead of the perturbation $h^{\mu\nu}$. We derive all the Feynman rules with respect to the scalar, vector, and tensor modes of the Kaluza-Klein decomposition and the canonically normalized massless scalar field ψ, that we introduced earlier, which can be found in Appendix B. For completeness, we have also put all the Feynman diagrams and their values in the supplemental material [22].

Summing together all the diagrams, we get the NNLO tidal Lagrangian computed from EFT techniques. First, to absorb terms nonlinear in the accelerations, we add a double-zero term of the form [38]

$$-\frac{G^2 m_1^2 \lambda_2^{(0)}}{r^2}\tilde{f}_0^{(2)}(\tilde{d}_1^{(1)})^2(2na_1^{\mathrm{LO}}(2na_2^{\mathrm{LO}}-na_1^{\mathrm{LO}}) - a_1^{\mathrm{LO}}\cdot a_2^{\mathrm{LO}}) + [1\leftrightarrow 2], \quad (29)$$

where a_i^{LO} denotes the replacement of the LO equations of motion (EOM).

Then, to compare directly our results at the level of the Lagrangian, we need to make another operation. Indeed, in a separate computation within PN EFT, we have computed the 2PN-order Lagrangian in the point-particle approximation in ST theories [39]. We found explicitly that in order to bring the 2PN ST point-particle Lagrangian to the form displayed in Ref. [12], we have to add a double-zero term of the form

$$\delta L^{2\mathrm{PN}} = \frac{\alpha\tilde{G}m_1m_2r}{8}na_1^{\mathrm{LO}}\cdot na_2^{\mathrm{LO}} - (15+8\bar{\gamma})\frac{\alpha\tilde{G}m_1m_2r}{8}a_1^{\mathrm{LO}}\cdot a_2^{\mathrm{LO}}, \quad (30)$$

where the constants are defined according to Table I. Such a double-zero term, when we include the tidal effects to the LO EOM, contributes to the NNLO tidal Lagrangian as

$$\delta L_{\mathrm{tidal}}^{2\mathrm{PN}} = -2(7-2\tilde{d}_1^{(1)}\tilde{d}_1^{(2)})\frac{G^3}{r^4}[m_2^2\lambda_1^{(0)}\tilde{f}_0^{(1)}(\tilde{d}_1^{(2)})^2 + m_1^2\lambda_2^{(0)}\tilde{f}_0^{(2)}(\tilde{d}_1^{(1)})^2]\left[m_1na_1 - m_2na_2 + 2\frac{Gm_1m_2}{r^2}(1+2\tilde{d}_1^{(1)}\tilde{d}_1^{(2)})\right]. \quad (31)$$

Finally, after taking into account the above contribution, we find complete agreement for the Lagrangian computed with the traditional PN formalism.

V. NNLO CONSERVATIVE LAGRANGIAN

The Lagrangian describing the conservative dynamics of compact binary systems in ST theories at 3PN order in harmonic coordinates and the leading-order scalar tidal correction have been previously derived in Refs. [11,14]. In the present work, we have pushed the computation of the tidal correction up to the NNLO associated to the 2PN dynamics. The resulting Lagrangian is displayed below. As explained in the previous section, it has been obtained both with the traditional PN formalism and the most recent PN EFT one. The resulting Fokker Lagrangian is a generalized one, meaning that it is a function of the positions $y_a^i(t)$, the velocities $v_a^i(t)$, and also of the accelerations $a_a^i(t)$ and their successive derivatives. To get a Lagrangian that is at most linear in the acceleration, we have applied a reduction procedure which consists in iteratively adding total time derivatives, to remove the dependence in higher-order derivative of the acceleration, and double-zero terms, to remove terms nonlinear in acceleration [38]. After this procedure, the Fokker Lagrangian in its reduced form is completely equivalent to the original one, meaning that they both yield to the same equations of motion in a harmonic gauge. We write the total Lagrangian as

$$L = L_{\mathrm{pp}}^{2\mathrm{PN}} + L_{\mathrm{tidal}}, \quad L_{\mathrm{tidal}} = L_{\mathrm{LO}} + L_{\mathrm{NLO}} + L_{\mathrm{NNLO}}, \quad (32)$$

where $L_{\mathrm{pp}}^{2\mathrm{PN}}$ is the 2PN-order Lagrangian in the point-particle approximation in ST theories displayed in Ref. [12]. The NNLO result is further split in increasing powers of $\tilde{G}$ as

$$L_{\mathrm{NNLO}} = \tilde{G}^2 L_{\mathrm{NNLO}}^{(2)} + \tilde{G}^3 L_{\mathrm{NNLO}}^{(3)} + \tilde{G}^4 L_{\mathrm{NNLO}}^{(4)}. \quad (33)$$

The complete result for each component is then

$$L_{\mathrm{LO}} = \frac{\alpha^2\tilde{G}^2}{c^2 r_{12}^4}m_1m_2\frac{-2\zeta}{1-\zeta}\frac{m_2}{m_1}\bar{\delta}_2\lambda_1^{(0)} + [1\leftrightarrow 2], \quad (34a)$$

$$L_{\mathrm{NLO}} = \frac{\alpha^3\tilde{G}^3}{c^4 {r_{12}}^5}\frac{\zeta}{1-\zeta}(2+\bar{\gamma})\left[\bar{\delta}_2\lambda_1^{(0)}\left(\frac{4m_1{m_2}^2(\bar{\gamma}-4\bar{\beta}_2)}{\bar{\gamma}(2+\bar{\gamma})} + {m_2}^3\left(3+\frac{(5\zeta-4\lambda_1)(1-2s_2)}{-1+\zeta}\right)\right) + \frac{2\zeta {m_2}^3\bar{\delta}_2\phi_0\lambda_1^{(1)}(1-2s_2)}{-1+\zeta}\right] + \frac{\alpha^2\tilde{G}^2}{c^4{r_{12}}^4}\frac{\zeta}{1-\zeta}{m_2}^2\bar{\delta}_2\lambda_1^{(0)}\left(-2(n_{12}v_1)^2 + v_1^2 + 4(n_{12}v_1)(n_{12}v_2) + 2(n_{12}v_2)^2\right) + [1\leftrightarrow 2], \quad (34b)$$

$$\begin{aligned} L_{\mathrm{NNLO}}^{(2)} = \alpha^2{m_2}^2\Bigg[&\frac{1}{{r_{12}}^6}\left(c_1^{(0)} - 4\nu_1^{(0)}\frac{\zeta(1-2s_2)}{1-2\zeta s_2} - 4\frac{\mu_1^{(0)}}{c^2}\left(\frac{\zeta(1-2s_2)}{1-2\zeta s_2}\right)^2\right)\frac{3}{8}(2+\bar{\gamma})^2\left(1+\frac{\zeta}{1-\zeta}(1-2s_2)\right)^2 \\ &+ \frac{1}{c^6{r_{12}}^3}\frac{-\zeta}{1-\zeta}\bar{\delta}_2\lambda_1^{(0)}\Bigg(2\Big(-2(a_2v_1)(n_{12}v_1) + 7(a_2n_{12})(n_{12}v_1)^2 - 2(a_2n_{12})v_1^2 + 2(n_{12}v_1)(a_2v_2) + 2(a_2v_1)(n_{12}v_2) \\ &- 12(a_2n_{12})(n_{12}v_1)(n_{12}v_2) - 2(a_2v_2)(n_{12}v_2) + 7(a_2n_{12})(n_{12}v_2)^2 + 4(a_2n_{12})(v_1v_2) - 2(a_2n_{12})v_2^2\Big) \\ &+ \frac{1}{4r_{12}}\Big\{4(n_{12}v_1)^2v_1^2 - v_1^4 - 8(n_{12}v_1)(n_{12}v_2)v_1^2 - 24(n_{12}v_1)^2(n_{12}v_2)^2 + 12(n_{12}v_2)^2v_1^2 + 48(n_{12}v_1)(n_{12}v_2)^3 \\ &+ 16(n_{12}v_1)(n_{12}v_2)(v_1v_2) - 16(n_{12}v_2)^2(v_1v_2) - 16(n_{12}v_1)(n_{12}v_2)v_2^2\Big\}\Bigg)\Bigg] + [1\leftrightarrow 2], \end{aligned} \quad (34c)$$

$$L^{(3)}_{\rm NNLO}=\frac{\alpha^3}{c^6 r_{12}{}^5}\frac{\zeta}{1-\zeta}(2+\bar\gamma)\bar\delta_2\Bigg[\lambda_1^{(0)}\Bigg(m_1 m_2{}^2\Bigg(-\frac{(43\bar\gamma+44\bar\gamma^2+80\bar\beta_2)}{\bar\gamma(2+\bar\gamma)}(n_{12}v_1)^2+\frac{(7\bar\gamma+8\bar\gamma^2+16\bar\beta_2)}{\bar\gamma(2+\bar\gamma)}v_1^2$$
$$+\frac{3(7\bar\gamma+16\bar\gamma^2+40\bar\beta_2)}{\bar\gamma(2+\bar\gamma)}(n_{12}v_1)(n_{12}v_2)-\frac{4(-3+\bar\gamma)}{2+\bar\gamma}(n_{12}v_2)^2-\frac{(5\bar\gamma+8\bar\gamma^2+8\bar\beta_2)}{\bar\gamma(2+\bar\gamma)}(v_1v_2)-\frac{4v_2^2}{2+\bar\gamma}\Bigg)$$
$$+m_2{}^3\Bigg(-(n_{12}v_1)^2+\frac{1}{2}v_1^2+\frac{(88+69\bar\gamma)}{2(2+\bar\gamma)}(n_{12}v_1)(n_{12}v_2)-\frac{(57+41\bar\gamma)}{2+\bar\gamma}(n_{12}v_2)^2-\frac{(32+21\bar\gamma)}{2(2+\bar\gamma)}(v_1v_2)$$
$$+\frac{(24+17\bar\gamma)}{2(2+\bar\gamma)}v_2^2-(1-2s_2)\frac{(5\zeta-4\lambda_1)}{1-\zeta}\left\{(n_{12}v_1)^2-\frac{v_1^2}{2}-\frac{9(n_{12}v_1)(n_{12}v_2)}{2}+(n_{12}v_2)^2+\frac{(v_1v_2)}{2}-\frac{v_2^2}{2}\right\}\Bigg)\Bigg)$$
$$+\frac{\zeta}{1-\zeta}m_2{}^3\phi_0\lambda_1^{(1)}(1-2s_2)\Big(-2(n_{12}v_1)^2+v_1^2+9(n_{12}v_1)(n_{12}v_2)-2(n_{12}v_2)^2-(v_1v_2)+v_2^2\Big)\Bigg]+[1\leftrightarrow 2],\quad(34\text{d})$$

$$L^{(4)}_{\rm NNLO}=\frac{\alpha^4}{c^6 r_{12}{}^6}\frac{\zeta}{1-\zeta}\Bigg[\lambda_1^{(0)}\Bigg(m_1{}^2m_2{}^2\Bigg(\bar\gamma(2+\bar\gamma)^2\left(\frac{4}{20}-\frac{3}{20}\bar\gamma\right)-\frac{\bar\delta_2(22\bar\gamma^2+12\bar\gamma^3+3\bar\gamma^4-160\bar\gamma\bar\beta_2+20\bar\gamma^2\bar\beta_2+160(\bar\beta_2)^2+80\bar\gamma\bar\chi_2)}{5\bar\gamma^2}\Bigg)$$
$$+m_2{}^4\Bigg(-(\bar\delta_2)^2\frac{1}{(1-\zeta)\zeta}(4\zeta+21\zeta^2-58\zeta\lambda_1+40(\lambda_1)^2-8\lambda_2)+\bar\delta_2(2+\bar\gamma)^2\left(-\frac{13}{4}+\frac{3}{2(1-\zeta)}(5\zeta-4\lambda_1)(1-2s_2)\right)\Bigg)$$
$$+m_1m_2{}^3\Bigg((\bar\delta_2)^2\frac{1}{(1-\zeta)\bar\gamma^2}6(\bar\gamma^2+8\bar\beta_2)(5\zeta-4\lambda_1)(1-2s_1)+\bar\delta_2\Bigg(-\frac{79\bar\gamma^2+41\bar\gamma^3+8\bar\gamma^2\bar\beta_1-64\bar\gamma\bar\beta_2-24\bar\gamma^2\bar\beta_2+64\bar\beta_1\bar\beta_2}{\bar\gamma^2}$$
$$+\frac{3}{2(1-\zeta)}(2+\bar\gamma)^2(5\zeta-4\lambda_1)(1-2s_2)\Bigg)\Bigg)\Bigg)$$
$$+\frac{\zeta}{1-\zeta}\bar\delta_2\phi_0\lambda_1^{(1)}\left(-\frac{4m_2{}^4\bar\delta_2(6\zeta-5\lambda_1)}{\zeta}+3(2+\bar\gamma)^2m_2{}^4(1-2s_2)+\frac{6(2+\bar\gamma)m_1m_2{}^3(\bar\gamma-4\bar\beta_2)(1-2s_2)}{\bar\gamma}\right)$$
$$+\frac{-4\zeta}{1-\zeta}m_2{}^4(\bar\delta_2)^2\phi_0{}^2\lambda_1^{(2)}\Bigg]+[1\leftrightarrow 2].\quad(34\text{e})$$

Note that the tidal corrections in the Lagrangian do not show directly the PN order to which they contribute. This is a well-known feature already in GR that comes from the fact that the tidal parameters $(\lambda_a,\mu_a,\nu_a,c_a)$ are dimensionful parameters. It can be better understood by a simple scaling argument. The couplings $(\lambda_a,\mu_a,\nu_a,c_a)$, respectively, scale as $(ML^2, ML^4, ML^2T^2, ML^2T^2)$. It leads us to define the dimensionless parameters,

$$k^{(n)}_{a,\lambda}\equiv\lambda_a^{(n)}\cdot\frac{\tilde G\alpha}{c^2R_a^3}\cdot\left(\frac{\tilde G\alpha M_a}{c^2R_a}\right)^n,\qquad k^{(0)}_{a,\mu}\equiv\mu_a^{(0)}\cdot\frac{\tilde G\alpha}{c^2R_a^5},\qquad k^{(0)}_{a,\nu}\equiv\nu_a^{(0)}\cdot\frac{\tilde G\alpha}{R_a^5},\qquad k^{(0)}_{a,c}\equiv c_a^{(0)}\cdot\frac{\tilde G\alpha}{R_a^5}.\quad(35)$$

Substituting these parameters in the Lagrangian and using the compacity argument, namely, that $\frac{\tilde G\alpha M_a}{c^2R_a}\sim 1$, we see that the leading-order contribution of each family of tidal parameters is indeed $\mathcal{O}(\frac{1}{c^6},\frac{1}{c^{10}},\frac{1}{c^{10}},\frac{1}{c^{10}})$ as expected. Hence, we have computed the correction to the p. p. Lagrangian up to the NNLO, i.e., 5PN order.

We have performed a number of consistency checks on our results. First, the leading-order Lagrangian (34) is in total agreement with the leading-order result from Ref. [14]. Then, by taking the GR limit, i.e., when $\omega_0\to\infty$ and $\phi_0\to 1$, we have verified that we retrieve the LO tidal correction in GR computed in Ref. [26]. Finally, we have checked that the equations of motion for each particle, derived from the Lagrangian (32), are indeed Lorentz invariant. To keep this paper short and readable, we have relegated the result for the equations of motion in harmonic coordinates up to the NNLO in the supplementary material [22].

VI. NOETHERIAN QUANTITIES AND CENTER-OF-MASS FRAME

In this section, we aim at computing the ten conserved Noetherian quantities, namely, the energy E, the linear and angular momenta P^i and J^i, and the boost K^i. We then use them to define the center-of-mass frame by solving $G^i=0$ where G^i is the center-of-mass position. We finally reduce the expressions of the relative acceleration and the conserved quantities to the center-of-mass frame. We start by briefly recalling the reasoning to derive such quantities.

In Sec. V, we have obtained the Lagrangian up to the NNLO as a function of only the positions $y_a^i(t)$, velocities

$v_a^i(t)$, and accelerations $a_a^i(t)$. If we consider an infinitesimal transformation for the body A at some time t, namely, $\delta y_a(t) = y'_a(t) - y_a(t)$, the Lagrangian should transform at linear order as

$$\delta L = \frac{dQ}{dt} + \sum \frac{\delta L}{\delta y_a}\delta y_a + \mathcal{O}(\delta y_a^2), \tag{36}$$

where the function derivative $\delta L/\delta y_a$ is zero "on shell." The quantity Q is defined as

$$Q = \sum[(p_a \delta y_a) + (q_a \delta v_a)], \tag{37}$$

where p_a^i and q_a^i are, respectively, the conjugate momenta to the positions and velocities, namely,

$$p_a^i \equiv \frac{\delta L}{\delta v_a^i} = \frac{\partial L}{\partial v_a^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial a_a^i}\right), \tag{38}$$

$$q_a^i \equiv \frac{\delta L}{\delta a_a^i} = \frac{\partial L}{\partial a_a^i}. \tag{39}$$

The Lagrangian is invariant under the Poincaré group. Hence, under arbitrary infinitesimal time translation $\delta t = \tau$, spatial translation $\delta y_a^i = \epsilon^i$, and spatial rotation $\delta y_a^i = w^i{}_j y_a^j$, we have $\delta L = 0$. This yields to the conservation on shell of the energy and the linear and angular momenta, defined as [40]

$$E = \sum_a [(p_a v_a) + (q_a a_a)] - L, \tag{40}$$

$$P^i = \sum_a p_a^i, \tag{41}$$

$$J^i = \varepsilon_{ijk} \sum_a [(p_a y_a) + (q_a v_a)]. \tag{42}$$

In addition, the Lagrangian is also invariant under an infinitesimal Lorentz boost. At the linear order in the boost velocity W^i, the transformation of the body trajectories reads

$$\delta y_a^i = -W^i t - \frac{1}{c^2}(W r_a) v_a^i + \mathcal{O}(W^2), \tag{43}$$

where r_a is the distance between the field point and the body a. Following Ref. [40], under such a linear transformation, there should exist a functional Z^i such that $\delta L = (W dZ/dt) + \mathcal{O}(W^2)$ plus some "double-zero" terms which give zero on shell by the Noether theorem. The transformation (43) is associated with the conservation of the Noetherian integral $K^i = G^i - P^i t$, where P^i is the linear momentum (41) and G^i stands for the center-of-mass position,

$$G^i = -Z^i + \sum_a \Bigg(-q_a^i + \frac{1}{c^2}[(p_a v_a) y_a^i + (q_a a_a) y_a^i + (q_a v_a) v_a^i]\Bigg). \tag{44}$$

The existence of such a boost symmetry of the Lagrangian is confirmed since our equations of motion at the NNLO in the tidal effects are Lorentz invariant, as mentioned in Sec. V.

In the following, we present the results for the NNLO tidal correction to all the conserved quantities. To stay concise, we are only displaying the center-of-mass position G^i in generic harmonic coordinates while the relative acceleration a^i, and the conserved quantities (E, J^i), will be given only in the center-of-mass frame. The results in the generic coordinates are displayed in the supplemental material [22].

A. Center-of-mass frame

First, using Eq. (44), we have obtained the position of the center of mass at NNLO and in a generic harmonic frame,

$$G^i_{\rm LO} = 0, \tag{45a}$$

$$G^i_{\rm NLO} = \frac{\alpha^2 \tilde{G}^2}{c^4 r_{12}{}^4} \frac{2\zeta}{1-\zeta} m_2{}^2 \bar{\delta}_2 \lambda_1^{(0)} y_1{}^i + [1 \leftrightarrow 2], \tag{45b}$$

$$\begin{aligned} G^i_{\rm NNLO} = {}& \frac{\alpha^2 \tilde{G}^2}{c^6 r_{12}{}^3} \frac{\zeta}{1-\zeta} m_2{}^2 \bar{\delta}_2 \lambda_1^{(0)} \Big[2 n_{12}{}^i (7 (n_{12} v_1)^2 - 2 v_1^2 - 14 (n_{12} v_1)(n_{12} v_2) + 5 (n_{12} v_2)^2 + 4 (v_1 v_2) - 2 v_2) \\ & + \frac{y_1{}^i}{r_{12}} (-2 (n_{12} v_1)^2 + v_1^2 + 4 (n_{12} v_1)(n_{12} v_2) + 2 (n_{12} v_2)^2) - 4 (v_1{}^i - v_2{}^i)((n_{12} v_1) - (n_{12} v_2)) \Big] \\ & + \frac{\alpha^3 \tilde{G}^3}{c^6 r_{12}{}^4} \frac{\zeta}{1-\zeta} \Bigg[n_{12}{}^i \Bigg(\bar{\delta}_2 \lambda_1^{(0)} \Bigg(\frac{m_1 m_2{}^2 (13\bar{\gamma} + 8\bar{\gamma}^2 + 8\bar{\beta}_2)}{\bar{\gamma}} + m_2{}^3 \Bigg(\frac{1}{2}(-16 - 13\bar{\gamma}) + (5\zeta - 4\lambda_1) \frac{(2+\bar{\gamma})(1-2s_2)}{2(-1+\zeta)} \Bigg) \Bigg) \\ & + \bar{\delta}_2 \phi_0 \lambda_1^{(1)} m_2{}^3 \frac{\zeta (2+\bar{\gamma})(1-2s_2)}{-1+\zeta} \Bigg) + \frac{y_1{}^i}{r_{12}} \Bigg(\bar{\delta}_2 \lambda_1^{(0)} \Bigg(-\frac{4 m_1 m_2{}^2 (\bar{\gamma} - 4\bar{\beta}_2)}{\bar{\gamma}} \\ & + m_2{}^3 \Bigg(-3(2+\bar{\gamma}) - (5\zeta - 4\lambda_1) \frac{(2+\bar{\gamma})(1-2s_2)}{-1+\zeta} \Bigg) \Bigg) - \bar{\delta}_2 \phi_0 \lambda_1^{(1)} m_2{}^3 \frac{2\zeta (2+\bar{\gamma})(1-2s_2)}{-1+\zeta} \Bigg) \Bigg] + [1 \leftrightarrow 2]. \end{aligned} \tag{45c}$$

We note that at leading order the center of mass is zero and the first correction enters only at NLO. From this, we determine the center-of-mass frame by solving the equation $G^i = 0$ iteratively at each PN order. We obtain the positions $y^i_{a,CM}$ in the center-of-mass frame as

$$y^i_{1,CM} = \left(\frac{m_2}{m} + \nu\mathcal{P}\right) n^i r + \nu\mathcal{Q} v^i, \tag{46}$$

$$y^i_{2,CM} = \left(-\frac{m_1}{m} + \nu\mathcal{P}\right) n^i r + \nu\mathcal{Q} v^i. \tag{47}$$

Dividing the quantities $\mathcal{P}$ and $\mathcal{Q}$ into a 2PN point-particle contribution and a tidal one, $\mathcal{P} = \mathcal{P}_{2PN} + \mathcal{P}_{\rm tidal}$ and $\mathcal{Q} = \mathcal{Q}_{2PN} + \mathcal{Q}_{\rm tidal}$. The point-particle results up to the 2PN order $\mathcal{P}_{2PN}$ and $\mathcal{Q}_{2PN}$ can be found in Ref. [12], and we display here the tidal corrections up the NNLO, further dividing $\mathcal{P}_{\rm tidal}$ and $\mathcal{Q}_{\rm tidal}$ as

$$\mathcal{P}_{\rm tidal} = \mathcal{P}_{\rm LO} + \mathcal{P}_{\rm NLO} + \mathcal{P}_{\rm NNLO}, \tag{48}$$

$$\mathcal{Q}_{\rm tidal} = \mathcal{Q}_{\rm LO} + \mathcal{Q}_{\rm NLO} + \mathcal{Q}_{\rm NNLO}. \tag{49}$$

We then get

$$\mathcal{P}_{\rm LO} = 0, \tag{50a}$$

$$\mathcal{P}_{\rm NLO} = \frac{\alpha^2\tilde{G}^2}{c^4 r^4}\frac{-\zeta}{1-\zeta} m(\lambda_-^{(0)} - \delta\lambda_+^{(0)}), \tag{50b}$$

$$\begin{aligned}\mathcal{P}_{\rm NNLO} = {}& \frac{\alpha^2\tilde{G}^2}{c^6 r^4}\frac{\zeta}{2(-1+\zeta)} m[v^2((-7-6\nu)\lambda_-^{(0)} - \delta(1-6\nu)\lambda_+^{(0)}) + (nv)^2(2(11+10\nu)\lambda_-^{(0)} + 2\delta(-1+6\nu)\lambda_+^{(0)})] \\ &+ \frac{\alpha^3\tilde{G}^3}{c^6 r^5}\frac{\zeta}{4(-1+\zeta)} m^2\Bigg\{\frac{1}{\bar{\gamma}}[(16\bar{\beta}^+ - 2\bar{\gamma} - 3\bar{\gamma}^2 - 16\bar{\beta}^-\delta + (64\bar{\beta}^+ + 12\bar{\gamma} + 12\bar{\gamma}^2)\nu)\lambda_-^{(0)} \\ &+ (-16\bar{\beta}^- + (16\bar{\beta}^+ + 54\bar{\gamma} + 35\bar{\gamma}^2)\delta + (-64\bar{\beta}^- - 8\bar{\gamma}\delta)\nu)\lambda_+^{(0)}] \\ &+ \frac{(2+\bar{\gamma})(4\lambda_1(\Lambda_-^{(0)} - 4\nu\Lambda_-^{(0)} - \delta\Lambda_+^{(0)}) + \zeta(5(-1+4\nu)\Lambda_-^{(0)} + 5\delta\Lambda_+^{(0)} + 2\phi_0((-1+4\nu)\Lambda_-^{(1)} + \delta\Lambda_+^{(1)})))}{-1+\zeta}\Bigg\}\end{aligned} \tag{50c}$$

and

$$\mathcal{Q}_{\rm LO} = 0, \tag{51a}$$

$$\mathcal{Q}_{\rm NLO} = 0, \tag{51b}$$

$$\mathcal{Q}_{\rm NNLO} = \frac{\alpha^2\tilde{G}^2}{c^6 r^3}\frac{4\zeta}{1-\zeta} m(nv)\lambda_-^{(0)}. \tag{51c}$$

We have introduced some *plus* and *minus* quantities for the ST and tidal parameters as defined for the notation in Sec. I A.

B. Acceleration in the CM

Once we have determined the coordinates in the center-of-mass frame, we can inject it into the relative acceleration $a^i \equiv a^i_1 - a^i_2$. In the CM frame, we get up to the NLO

$$a^i_{\rm CM,LO} = \frac{\alpha^2\tilde{G}^2 m}{c^2 r^5}\frac{8\zeta}{1-\zeta}\lambda_+^{(0)} n^i, \tag{52a}$$

$$\begin{aligned}a^i_{\rm CM,NLO} = {}& \frac{\alpha^2\tilde{G}^2}{c^4 r^5}\frac{\zeta}{1-\zeta} m[n^i(12(nv)^2(\delta\lambda_-^{(0)} - 2\nu\lambda_+^{(0)}) - 2v^2(\delta\lambda_-^{(0)} + 3\lambda_+^{(0)} - 12\nu\lambda_+^{(0)})) - 4(nv)v^i(\delta\lambda_-^{(0)} + \lambda_+^{(0)} - 4\nu\lambda_+^{(0)})] \\ &+ \frac{\alpha^3\tilde{G}^3}{c^4 r^6}\frac{\zeta}{2(1-\zeta)} m^2 n^i\Bigg[-\frac{80\bar{\beta}^-\lambda_-^{(0)}}{\bar{\gamma}} + \frac{(80\bar{\beta}^+ - 96\bar{\gamma} - 47\bar{\gamma}^2 - 52\bar{\gamma}\nu)\lambda_+^{(0)}}{\bar{\gamma}} + \delta\left(\frac{(80\bar{\beta}^+ + 16\bar{\gamma} + 15\bar{\gamma}^2)\lambda_-^{(0)}}{\bar{\gamma}} - \frac{80\bar{\beta}^-\lambda_+^{(0)}}{\bar{\gamma}}\right) \\ &+ \frac{5(2+\bar{\gamma})(4\lambda_1(-\delta\Lambda_-^{(0)} + \Lambda_+^{(0)}) + \zeta(5\delta\Lambda_-^{(0)} + 2\phi_0\delta\Lambda_-^{(1)} - 5\Lambda_+^{(0)} - 2\phi_0\Lambda_+^{(1)}))}{-1+\zeta}\Bigg].\end{aligned} \tag{52b}$$

The expression for the NNLO tidal correction to the relative acceleration is displayed in Appendix C 1.

C. Conserved quantities in the CM

1. Energy

In the center-of-mass frame, the conserved energy at NLO is given by

$$E_{\rm LO} = \frac{\alpha^2\tilde{G}^2 m^2\nu}{c^2 r^4}\frac{2\zeta}{1-\zeta}\lambda_+^{(0)}, \tag{53a}$$

$$\begin{aligned} E_{\rm NLO} = {} & \frac{\alpha^2\tilde{G}^2}{c^4 r^4}\frac{-\zeta}{2(1-\zeta)}m^2\nu[(nv)^2(-4\delta\lambda_-^{(0)} + 8\nu\lambda_+^{(0)}) + v^2(\delta\lambda_-^{(0)} + (-1+2\nu)\lambda_+^{(0)})] \\ & + \frac{\alpha^3\tilde{G}^3}{c^4 r^5}\frac{\zeta}{1-\zeta}m^3\nu\Bigg[\left\{-\frac{8\bar{\beta}^-}{\bar{\gamma}} + \frac{(16\bar{\beta}^+ + \bar{\gamma}(2+3\bar{\gamma}))\delta}{2\bar{\gamma}}\right\}\lambda_-^{(0)} + \left\{\frac{16\bar{\beta}^+ - \bar{\gamma}(10+3\bar{\gamma})}{2\bar{\gamma}} - \frac{8\bar{\beta}^-\delta}{\bar{\gamma}}\right\}\lambda_+^{(0)} \\ & + \frac{(2+\bar{\gamma})(4\lambda_1(-\delta\Lambda_-^{(0)} + \Lambda_+^{(0)}) + \zeta(5\delta\Lambda_-^{(0)} + 2\phi_0\delta\Lambda_-^{(1)} - 5\Lambda_+^{(0)} - 2\phi_0\Lambda_+^{(1)}))}{2(-1+\zeta)}\Bigg]. \end{aligned} \tag{53b}$$

Again, to lighten the main text, we have relegated the NNLO tidal contribution to the CM energy in Appendix C 2

2. Angular momentum

Finally, the tidal correction to conserved angular momentum in the center-of-mass frame up to the NNLO can be written as

$$J^i_{\rm tidal} = (\mathcal{J}_{\rm LO} + \mathcal{J}_{\rm NLO} + \mathcal{J}_{\rm NNLO})(n\times v)^i, \tag{54}$$

where $(n\times v)^i$ denotes a vector product and with

$$\mathcal{J}_{\rm LO} = 0, \tag{55a}$$

$$\mathcal{J}_{\rm NLO} = \frac{\alpha^2\tilde{G}^2}{c^4 r^3}\frac{\zeta}{1-\zeta}m^2\nu(-\delta\lambda_-^{(0)} + (1-2\nu)\lambda_+^{(0)}), \tag{55b}$$

$$\begin{aligned} \mathcal{J}^i_{\rm NNLO} = {} & \frac{\alpha^2\tilde{G}^2}{c^6 r^3}\frac{-\zeta}{2(1-\zeta)}m^2\nu(2(nv)^2(\delta(21+8\nu)\lambda_-^{(0)} + (23+6\nu-20\nu^2)\lambda_+^{(0)}) + v^2(-\delta(11+6\nu)\lambda_-^{(0)} \\ & - (13-8\nu+18\nu^2)\lambda_+^{(0)})) + \frac{\alpha^3\tilde{G}^3}{c^6 r^4}\frac{-\zeta}{2(1-\zeta)}m^3\nu\Bigg[\left\{(6+\bar{\gamma})\delta + \left(\frac{48\bar{\beta}^-}{\bar{\gamma}} + \frac{(16\bar{\beta}^+ + 4\bar{\gamma} + 3\bar{\gamma}^2)\delta}{\bar{\gamma}}\right)\nu\right\}\lambda_-^{(0)} \\ & + \left\{2-\bar{\gamma} + \left(-\frac{3(16\bar{\beta}^+ + 52\bar{\gamma} + 33\bar{\gamma}^2)}{\bar{\gamma}} - \frac{16\bar{\beta}^-\delta}{\bar{\gamma}}\right)\nu + 8\nu^2\right\}\lambda_+^{(0)} \\ & + \frac{(2+\bar{\gamma})(-1+\nu)(4\lambda_1(-\delta\Lambda_-^{(0)} + \Lambda_+^{(0)}) + \zeta(5\delta\Lambda_-^{(0)} + 2\delta\phi_0\Lambda_-^{(1)} - 5\Lambda_+^{(0)} - 2\phi_0\Lambda_+^{(1)}))}{-1+\zeta}\Bigg]. \end{aligned} \tag{55c}$$

VII. TIDAL EFFECTS IN EINSTEIN-SCALAR-GAUSS-BONNET THEORY

While our results have been computed within the simple framework of generalized Brans-Dicke theories, see Sec. II, it can easily be generalized to more involved theories. An example that has been extensively studied in the past years is Einstein-scalar-Gauss-Bonnet theory, which arises as a low-energy limit of several quantum completions of gravity, see Refs. [41–43]. An interesting feature of these theories is that black hole solutions are different from that of GR, having nontrivial scalar solutions [44].

In these theories, the scalar field is nonminimally coupled to gravity through the Gauss-Bonnet topological invariant, $\mathcal{R}^2_{\rm GB} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. The action is

$$\begin{aligned} S_{\rm EsGB} = {} & \frac{c^3}{16\pi G}\int {\rm d}^4x\sqrt{-g}[R - 2g^{\alpha\beta}\partial_\alpha\hat{\psi}\partial_\beta\hat{\psi} + \alpha f(\hat{\psi})\mathcal{R}^2_{\rm GB}] \\ & + S_{\rm m}^{\rm EsGB}[\mathfrak{m}, \mathcal{A}^2(\hat{\psi})g_{\alpha\beta}], \end{aligned} \tag{56}$$

where α is the coupling constant and has a dimension of $[\text{length}]^2$. The action is directly in the Einstein frame with a canonical kinetic term for the scalar field and a coupling to

TABLE III. Translation map for tidal coefficients between simple ST and EsGB theories.

	Simple ST–Refs. [11–13]	EsGB–Refs. [17,20,44]
LO	$\lambda_a^{(0)}$	$\frac{\mathcal{A}_0}{4\alpha_0^2}\lambda_a^{(0)}{}_{\vert\mathrm{EsGB}}$
NLO	$\lambda_a^{(1)}$	$\frac{\mathcal{A}_0^3}{8\alpha_0^3}(\lambda_a^{(0)}{}_{\vert\mathrm{EsGB}}(2\frac{\beta_0}{\alpha_0}-5\alpha_0)-\lambda_a^{(1)}{}_{\vert\mathrm{EsGB}})$
NNLO	$\lambda_a^{(2)}$	$\frac{\mathcal{A}_0^5}{16\alpha_0^4}(\lambda_a^{(0)}{}_{\vert\mathrm{EsGB}}(35\alpha_0^2-24\beta_0+8\frac{\beta_0^2}{\alpha_0^2}-2\frac{\beta_0'}{\alpha_0})+\lambda_a^{(1)}{}_{\vert\mathrm{EsGB}}(12\alpha_0-5\frac{\beta_0}{\alpha_0})+\lambda_a^{(2)}{}_{\vert\mathrm{EsGB}})$
	$\mu_a^{(0)}$	$\frac{\mathcal{A}_0^3}{4\alpha_0^2}(\mu_a^{(0)}{}_{\vert\mathrm{EsGB}}+2\alpha_0c^2\nu_a^{(0)}{}_{\vert\mathrm{EsGB}}-\alpha_0^2c^2c_a^{(0)}{}_{\vert\mathrm{EsGB}})$
	$\nu_a^{(0)}$	$-\frac{\mathcal{A}_0^3}{2\alpha_0}(\nu_a^{(0)}{}_{\vert\mathrm{EsGB}}-\alpha_0c_a^{(0)}{}_{\vert\mathrm{EsGB}})$
	$c_a^{(0)}$	$\mathcal{A}_0^3c_a^{(0)}{}_{\vert\mathrm{EsGB}}$

matter through the conformal metric $\mathcal{A}^2(\hat{\psi})g_{\alpha\beta}$. As displayed in Appendix A of Ref. [45], the action (56) has been obtained from Eq. (2) by using the redefinitions

$$\hat{g}_{\mu\nu}=\mathcal{A}^2(\hat{\psi})g_{\mu\nu}, \tag{57}$$

$$3+2\omega(\phi)=\left(\frac{\mathrm{d}\ln\mathcal{A}}{\mathrm{d}\hat{\psi}}\right)^{-2}, \tag{58}$$

where $\mathcal{A}=1/\sqrt{\phi}$. Note that in this section we call $\hat{g}_{\mu\nu}$ the metric associated to the ST action (2) in the Jordan frame and $g_{\mu\nu}$ the metric associated to the EsGB action (56) in the Einstein frame. As for the scalar fields, ϕ is the same as the one displayed in Eq. (2), and we call $\hat{\psi}$ the scalar field in the EsGB theory.

In recent years, waveform modeling in EsGB theories has been developed for all the stages of the coalescence. Advances in the well-posed formulation of initial conditions have allowed performing numerical simulations and studying the merger of these objects [8,46–48]. Results for the inspiral evolution of compact binaries have also been obtained, relying on the fact that the leading-order correction due to the new GB term is at 3PN order [15–20,45].[3] Hence, using previous results in "simple" ST theories and adding only the leading-order correction allows having the full dynamics at 3PN order and the waveform up to 2.5PN order, including the leading tidal effect.

Following the convention in Ref. [20], the matter action can be further decomposed in a point-particle part and a tidal one, $S_m^{\mathrm{EsGB}}=S_{\mathrm{pp}}^{\mathrm{EsGB}}+S_{\mathrm{tid}}^{\mathrm{EsGB}}$, with

$$S_{\mathrm{pp}}^{\mathrm{EsGB}}=-c\sum_{a=1,2}\int\mathrm{d}s_a m_a^{\mathrm{EsGB}}(\hat{\psi}), \tag{59}$$

$$S_{\mathrm{fs}}^{\mathrm{EsGB}}=-\frac{c}{2}\sum_{a=1,2}\int\mathrm{d}s_a\bigg\{\lambda_a^{\mathrm{EsGB}}(\hat{\psi})(\nabla_\alpha^\perp\hat{\psi})_a(\nabla_\perp^\alpha\hat{\psi})_a+\frac{1}{2}\mu_a^{\mathrm{EsGB}}(\hat{\psi})(\nabla_{\alpha\beta}^\perp\hat{\psi})_a(\nabla_\perp^{\alpha\beta}\hat{\psi})_a$$
$$+\nu_a^{\mathrm{EsGB}}(\hat{\psi})(\nabla_{\alpha\beta}^\perp\hat{\psi})_aG_a^{\alpha\beta}-\frac{1}{2c^2}c_a^{\mathrm{EsGB}}(\hat{\psi})G_{\alpha\beta}^aG_a^{\alpha\beta}\bigg\}, \tag{60}$$

where $\mathrm{d}\tau_a\equiv\mathcal{A}(\hat{\psi})\mathrm{d}s_a$ and we have added the superscript (EsGB) to the masses and tidal deformability parameters to indicate that they are not the same as in the rest of the paper. Notably, we use the definition $m_a^{\mathrm{EsGB}}(\hat{\psi})\equiv\mathcal{A}(\hat{\psi})m_a^{\mathrm{ST}}(\phi)$ [45], which highlights the fact that in this section the matter is coupled to the Einstein frame metric.

Using the results obtained in the present paper, one can directly get the NNLO order for tidal effects in EsGB gravity. To do so, one needs to translate the notation in the present work to the one used in Refs. [17,45]. Such a map has already been computed and can be found in Appendix A of Ref. [45], in which the point-particle Lagrangian at 2PN order is presented for EsGB in Appendix B. Also, the leading-order tidal correction was derived in Ref. [20] for EsGB and was found to be in perfect agreement with Ref. [14]. To derive such a result to the NNLO, we need to extend the map for the tidal coefficients. The result is presented in Table III. At this point, we stress that the parametrization used in Ref. [20] is not exactly the same as in Ref. [45]. In this section, our results are presented using the parametrization and the ST

[3] Formally, corrections from the Gauss-Bonnet invariant scale as $\frac{\alpha}{c^2}$, but as α has a dimension of $[\mathrm{length}]^2$, it reduces to a 3PN correction when introducing a dimensionless coupling constant.

coefficients introduced in Ref. [45]. However, as the latter work did not focus on tidal effect, the tidal deformation parameters are the ones introduced in Ref. [20] or in Eq. (3.12) of Ref. [28] for the higher-order ones. Note that there is some mixing between the different types of Love number. This is due to the fact that we are coupling the matter to the metric in different frames.

To illustrate the use of this map, we present the LO and NLO tidal corrections to the acceleration in the CM frame in EsGB:

$$a^i_{\mathrm{CM,LO}|\mathrm{EsGB}} = \frac{G_{12}^2 M}{c^2 \nu r^5}[-(-1+m_- + 2\nu)\delta_2 \lambda^{(0)}_1{}_{|\mathrm{EsGB}} + (1+m_- - 2\nu)\delta_1 \lambda^{(0)}_2{}_{|\mathrm{EsGB}}] n^i, \tag{61a}$$

$$\begin{aligned} a^i_{\mathrm{CM,NLO}|\mathrm{EsGB}} = {}& \frac{G_{12}^3 M^2}{c^4 r^6}\Bigg[\Bigg(-\frac{80\bar\beta^- - 80\bar\beta^+ - 102\bar\gamma_{12} - 77\bar\gamma_{12}^2}{8\bar\gamma_{12}} + \frac{5(16\bar\beta^- - 16\bar\beta^+ + 2\bar\gamma_{12} - 3\bar\gamma_{12}^2)m_-}{8\bar\gamma_{12}} \\ &+ \frac{\frac{1}{8}(-56-31\bar\gamma_{12}) + \frac{1}{8}(56+31\bar\gamma_{12})m_-}{\nu} + \frac{13}{2}\nu\Bigg)\delta_2\lambda^{(0)}_1{}_{|\mathrm{EsGB}} \\ &+ \Bigg(\frac{15}{8}(2+\bar\gamma_{12})\alpha^0_2 - \frac{5}{8}(2+\bar\gamma_{12})m_-\alpha^0_2 - \frac{\frac{5}{8}(2+\bar\gamma_{12})\alpha^0_2 - \frac{5}{8}(2+\bar\gamma_{12})m_-\alpha^0_2}{\nu}\Bigg)\delta_2\lambda^{(1)}_1{}_{|\mathrm{EsGB}} \\ &+ \Bigg(\frac{80\bar\beta^- + 80\bar\beta^+ + 102\bar\gamma_{12} + 77\bar\gamma_{12}^2}{8\bar\gamma_{12}} + \frac{5(16\bar\beta^- + 16\bar\beta^+ - 2\bar\gamma_{12} + 3\bar\gamma_{12}^2)m_-}{8\bar\gamma_{12}} \\ &+ \frac{\frac{1}{8}(-56-31\bar\gamma_{12}) + \frac{1}{8}(-56-31\bar\gamma_{12})m_-}{\nu} + \frac{13}{2}\nu\Bigg)\delta_1\lambda^{(0)}_2{}_{|\mathrm{EsGB}} \\ &+ \Bigg(\frac{15}{8}(2+\bar\gamma_{12})\alpha^0_1 + \frac{5}{8}(2+\bar\gamma_{12})m_-\alpha^0_1 - \frac{\frac{5}{8}(2+\bar\gamma_{12})\alpha^0_1 + \frac{5}{8}(2+\bar\gamma_{12})m_-\alpha^0_1}{\nu}\Bigg)\delta_1\lambda^{(1)}_2{}_{|\mathrm{EsGB}}\Bigg] n^i \\ &+ \frac{G_{12}^2 M}{c^4 r^5}\Bigg[n^i\Bigg(\Bigg(\Bigg(3 - \frac{\frac{3}{2}-\frac{3}{2}m_-}{\nu} + 6\nu\Bigg)\delta_2\lambda^{(0)}_1{}_{|\mathrm{EsGB}} + \Bigg(3 - \frac{\frac{3}{2}+\frac{3}{2}m_-}{\nu} + 6\nu\Bigg)\delta_1\lambda^{(0)}_2{}_{|\mathrm{EsGB}}\Bigg)(nv)^2 \\ &+ \Bigg(\Bigg(\frac{7}{2} - \frac{5}{2}m_- - \frac{\frac{1}{2}-\frac{1}{2}m_-}{\nu} - 6\nu\Bigg)\delta_2\lambda^{(0)}_1{}_{|\mathrm{EsGB}} + \Bigg(\frac{7}{2} + \frac{5}{2}m_- - \frac{\frac{1}{2}+\frac{1}{2}m_-}{\nu} - 6\nu\Bigg)\delta_1\lambda^{(0)}_2{}_{|\mathrm{EsGB}}\Bigg)v^2\Bigg) \\ &+ ((1-m_- - 4\nu)\delta_2\lambda^{(0)}_1{}_{|\mathrm{EsGB}} + (1+m_- - 4\nu)\delta_1\lambda^{(0)}_2{}_{|\mathrm{EsGB}})(nv)v^i\Bigg]. \end{aligned} \tag{61b}$$

The full result for tidal effects in EsGB gravity at NNLO can be found in the supplementary material [22]. As expected, the leading-order correction (61) agrees with Ref. [20], while the higher-order results are new to this paper.

An important point is that there are no specific corrections coming from the Gauss-Bonnet term as, mimicking the reasoning performed for tidal corrections in Appendix A, such a correction would enter at the order $\mathcal{O}(\epsilon_{\mathrm{tid}}\alpha^2)$ which is at least a second-order correction.[4]

VIII. CONCLUSIONS

Tidal effects are one of the most promising tool to perform tests of gravity with the next generation of gravitational wave detectors. Indeed, when an additional scalar field is present in a theory, it will induce a varying dipole moment that will in turn induce a tidal deformation of the other body. Such an effect starts at 3PN order, a much lower effect than in GR, which makes it very important to include these new effects with a sufficient accuracy. Notably, by taking into account current constraints on the ST parameters, coming from the nonobservation of dipolar emission, it was shown that such an effect could contribute up to $\mathcal{O}(1)$ cycle in the waveform for suitable binaries in the LISA band [14]. In the case of EsGB, it is even more important to take those effects into account as one expects black holes to have a scalar hair and hence to have nonvanishing tidal Love numbers.

In the present work, we have tackled this program up to the NNLO in the dynamics both in generalized BD theory and in EsGB gravity. Such an accuracy allowed us to reach the order at which the usual gravitational tidal deformability enters. In a subsequent work, we will extend our result to the computation of the fluxes and waveform modes at the same NNLO in the tidal corrections [31]. It will then permit us to perform a more quantitative analysis

[4]Note, however, that, strictly speaking, it is not really the case, as in EsGB the scalar charge is sourced by the Gauss-Bonnet term.

of the impact of tidal effects based on the comparison between waveforms.

An important other direction to be taken is to compute the value of the scalar and gravitational Love numbers for realistic models of compact objects in these theories. Such a program has been initiated in Ref. [28]. Notably, the authors found that the $l = 1$ scalar Love number can be comparable to the gravitational one, showing the importance of introducing such terms in the waveforms.

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APPENDIX A: THE FOKKER APPROACH

In this Appendix, we present the Fokker reasoning in the case of a tidal perturbation with respect to the point-particle contribution. Our goal is to show that it is sufficient to solve the point-particle field equations [i.e., Eqs. (13), in which we neglect the tidal contributions] in order to derive the NNLO tidal corrections to the Fokker Lagrangian. We start with the action

$$S[y_a, v_a, h, \psi] = S_{\mathrm{ST}}[y_a, v_a, h, \psi] + S_{\mathrm{pp}}[y_a, v_a, h, \psi] + S_{\mathrm{tidal}}[y_a, v_a, h, \psi], \tag{A1}$$

where S_{tidal} is the tidal action defined by Eq. (6). By varying the action (A1) with respect to the metric and scalar perturbations, and solving iteratively the field equations

$$\frac{\delta S}{\delta h}[y_a, v_a, h, \psi] = 0, \tag{A2a}$$

$$\frac{\delta S}{\delta \psi}[y_a, v_a, h, \psi] = 0, \tag{A2b}$$

we obtain the solutions

$$h = h_{\mathrm{pp}} + h_{\mathrm{tidal}}, \tag{A3a}$$

$$\psi = \psi_{\mathrm{pp}} + \psi_{\mathrm{tidal}}. \tag{A3b}$$

Their tidal correction should be at least of order

$$(h^{00ii}_{\mathrm{tidal}}, h^{0i}_{\mathrm{tidal}}, h^{ij}_{\mathrm{tidal}}; \psi_{\mathrm{tidal}}) = \mathcal{O}\left(\frac{\epsilon_{\mathrm{tidal}}}{c^2}, \frac{\epsilon_{\mathrm{tidal}}}{c^3}, \frac{\epsilon_{\mathrm{tidal}}}{c^4}; \frac{\epsilon_{\mathrm{tidal}}}{c^2}\right), \tag{A4}$$

with $\mathcal{O}(\epsilon_{\mathrm{tidal}}) = 1/c^6$, since the tidal effects in ST theories enter formally at 3PN order [14]. Because Eqs. (A3) are exact solutions to the field equations (A2), this implies that the functional derivative of the Fokker action with respect to the scalar and metric perturbations, evaluated with the point-particle solutions, will have the orders

$$\frac{\delta S}{\delta h}[y_a, v_a, h_{\mathrm{pp}}, \psi_{\mathrm{pp}}] = \mathcal{O}(c^2\epsilon_{\mathrm{tidal}}, c\epsilon_{\mathrm{tidal}}, \epsilon_{\mathrm{tidal}}; c^2\epsilon_{\mathrm{tidal}}), \tag{A5a}$$

$$\frac{\delta S}{\delta \psi}[y_a, v_a, h_{\mathrm{pp}}, \psi_{\mathrm{pp}}] = \mathcal{O}(c^2\epsilon_{\mathrm{tidal}}, c\epsilon_{\mathrm{tidal}}, \epsilon_{\mathrm{tidal}}; c^2\epsilon_{\mathrm{tidal}}). \tag{A5b}$$

By Taylor expanding the Fokker action around the point-particle solution, we get

$$\begin{aligned} S_F[y_a, v_a, h, \psi] &= S_F[y_a, v_a, h_{\mathrm{pp}}, \psi_{\mathrm{pp}}] \\ &\quad + \int \mathrm{d}^4x \left(\frac{\delta S}{\delta h}[y_a, v_a, h_{\mathrm{pp}}, \psi_{\mathrm{pp}}]h_{\mathrm{tidal}} \right. \\ &\quad \left. + \frac{\delta S}{\delta \psi}[y_a, v_a, h_{\mathrm{pp}}, \psi_{\mathrm{pp}}]\psi_{\mathrm{tidal}}\right) \\ &\quad + \mathcal{O}(h^2_{\mathrm{tidal}}, \psi^2_{\mathrm{tidal}}) \\ &= S_F[y_a, v_a, h_{\mathrm{pp}}, \psi_{\mathrm{pp}}] + \mathcal{O}(\epsilon^2_{\mathrm{tidal}}). \end{aligned} \tag{A6}$$

The contributions of the order of $\mathcal{O}(\epsilon^2_{\mathrm{tidal}}) = 1/c^{12}$ at least equivalent to a next-to-next-to-next-to-leading (NNNLO) or 6PN tidal effect. This shows that it is sufficient to inject in the action the point-particle solutions to the field equations to know the Fokker action up to the NNLO in the tidal effects.

APPENDIX B: FEYNMAN RULES FOR THE PN EFT CALCULATIONS

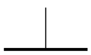
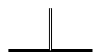
In this Appendix, we give the Feynman rules that are relevant for the computation of tidal effects at NNLO. The starting point for the PN EFT machinery is the action (24) accompanied with the Kaluza-Klein decomposition of the metric (28). Since we know that at this sector we do not encounter divergences, we can readily set $d = 3$ for simplicity. Below, we present the Feynman rules with respect to the Kaluza-Klein modes and the canonically normalized scalar field ψ relevant for the computation at this order. The propagators for the potential modes are

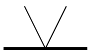
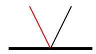
 $= \frac{-i}{8}\frac{1}{k^2-\partial_{t_a}\partial_{t_b}}$,

$= \frac{i}{2}\frac{\delta_{ij}}{k^2-\partial_{t_a}\partial_{t_b}}$,

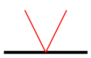
$= \frac{i}{2}\frac{P_{ijkl}}{k^2-\partial_{t_a}\partial_{t_b}}$, $= -i\frac{1}{k^2-\partial_{t_a}\partial_{t_b}}$

Then, the worldline vertices are

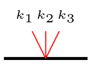
 $= -2\frac{im_a}{\tilde{M}_{pl}}(1+\frac{3}{2}v_a^2)$,  $= 2\frac{im_a}{\tilde{M}_{pl}}v_a^i(1+\frac{1}{2}v_a^2)$,  $= \frac{im_a}{\tilde{M}_{pl}}v_a^iv_a^j(1+\frac{1}{2}v_a^2)$,  $= -\tilde{d}_1^{(a)}\frac{im_a}{\tilde{M}_{pl}}(1-\frac{1}{2}v_a^2-\frac{1}{8}v_a^4)$,

 $= -2\frac{im_a}{\tilde{M}_{pl}^2}\Big(1-\frac{c_a^{(0)}}{m_a}\frac{(k_1\cdot k_2)^2}{2}\Big)$,  $= -2\tilde{d}_1^{(a)}\frac{im_a}{\tilde{M}_{pl}^2}\Big(1+\frac{3}{2}v_a^2+\frac{(c_a^{(0)}-2\nu_a^{(0)})}{m_a d_1^{(a)}}\frac{(k_1\cdot k_2)^2}{2}\Big)$,

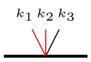
 $= 2\tilde{d}_1^{(a)}\frac{im_a}{\tilde{M}_{pl}^2}v_a^i(1+\frac{1}{2}v_a^2)$,

 $= -\frac{1}{2}\tilde{d}_2^{(a)}\frac{im_a}{\tilde{M}_{pl}^2}(1-\frac{1}{2}v_a^2)+\frac{1}{2}\frac{i\tilde{f}_0^{(a)}\lambda_a^{(0)}}{\tilde{M}_{pl}^2}\Big(k_1\cdot k_2(1-\frac{v_a^2}{2}-\frac{v_a^4}{8})+k_1\cdot v_a\ k_2\cdot v_a(1+\frac{v_a^2}{2})-\partial_{t_1}\partial_{t_2}v_a^2-v_a^i\big(k_1(i\partial_{t_2})+k_2(i\partial_{t_1})\big)_i(1+\frac{v_a^2}{2})\Big)$

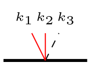
$$-\frac{i\tilde{f}_0^{(a)}}{\tilde{M}_{pl}^2}(4\mu_a^{(0)}+4\nu_a^{(0)}-c_a^{(0)})\frac{(k_1\cdot k_2)^2}{16},$$

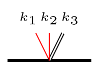


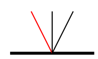
$= -\frac{\tilde{d}_3^{(a)}}{3!}\frac{im_a}{\tilde{M}_{pl}^3}+\frac{1}{3!}\frac{i\tilde{f}_1^{(a)}\lambda_a^{(0)}}{\tilde{M}_{pl}^3}\Big((k_1k_2+k_1k_3+k_2k_3)_{ij}(\delta^{ij}+v_a^iv_a^j)-iv_a^i\big(k_1(\partial_{t_2}+\partial_{t_3})+k_2(\partial_{t_1}+\partial_{t_3})+k_3(\partial_{t_1}+\partial_{t_2})\big)_i\Big)$,

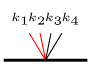


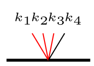
$= -\tilde{d}_2^{(a)}\frac{im_a}{\tilde{M}_{pl}^3}+\frac{i\tilde{f}_0^{(a)}\lambda_a^{(0)}}{\tilde{M}_{pl}^3}\Big(k_1\cdot k_2(1+v_a^2/2)-k_1\cdot v_1\ k_2\cdot v_2+2v_a^i\big(k_1(i\partial_{t_2})+k_2(i\partial_{t_1})\big)_i\Big)$,



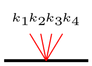
$= -\frac{i\tilde{f}_0^{(a)}\lambda_a^{(0)}}{\tilde{M}_{pl}^3}\Big(k_1^i(i\partial_{t_2})+k_2^i(i\partial_{t_1})+v_a^ik_1\cdot k_2\Big)$, 

$= -\frac{i\tilde{f}_0^{(a)}\lambda_a^{(0)}}{\tilde{M}_{pl}^3}k_1^ik_2^j$,  $= -2\tilde{d}_1^{(a)}\frac{im_a}{\tilde{M}_{pl}^3}$



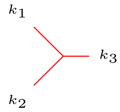
$= 9\frac{i\tilde{f}_0^{(a)}\lambda_a^{(0)}}{\tilde{M}_{pl}^4}(k_1\cdot k_2)$, 

$= \frac{i\tilde{f}_1^{(a)}\lambda_a^{(0)}}{\tilde{M}_{pl}^4}\frac{(k_1+k_2+k_3)^2-k_1^2-k_2^2-k_3^2}{2}$,

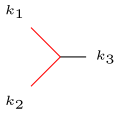


$= \frac{i\tilde{f}_2^{(a)}\lambda_a^{(0)}}{\tilde{M}_{pl}^4}\frac{(k_1+k_2+k_3+k_4)^2-k_1^2-k_2^2-k_3^2-k_4^2}{4!2}$,

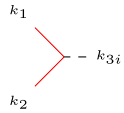
where all the momenta are outgoing. Finally, the bulk interaction vertices are



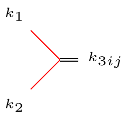
$$= -\frac{i}{\tilde{M}_{pl}}\frac{x_1}{2}\Big((k_1^2+k_2^2+k_3^2)-2(\partial_{t_1}\partial_{t_2}+\partial_{t_1}\partial_{t_3}+\partial_{t_2}\partial_{t_3})\Big),$$



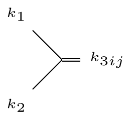
$$= -8\frac{i}{\tilde{M}_{pl}}\partial_{t_1}\partial_{t_2},$$



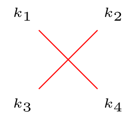
$$= -2\frac{i}{\tilde{M}_{pl}}\Big(k_1(i\partial_{t_2})+k_2(i\partial_{t_1})\Big)_i,$$



$$= \frac{1}{2c_d}\times$$



$$= -\frac{i}{\tilde{M}_{pl}}\Big(k_1^k k_2^l Q_{ijkl}+\partial_{t_1}\partial_{t_2}\delta_{ij}\Big),$$



$$= -\frac{i}{\tilde{M}_{pl}^2}\frac{x_2}{2}(k_1^2+k_2^2+k_3^2+k_4^2)$$

where $P_{ijkl} = -(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}+(2-c_d)\delta_{ij}\delta_{kl})$, $Q_{ijkl} = I_{ijkl}-\delta_{ij}\delta_{kl}$, and $I_{ijkl} = \delta_{il}\delta_{kj}+\delta_{ik}\delta_{jl}$.

APPENDIX C: NNLO RESULTS IN THE CM FRAME

1. Acceleration in the CM frame

The NNLO tidal correction to the relative acceleration in the center-of-mass frame is given, after separating it in powers of $\tilde{G}$, by

$$\begin{aligned}
a^{i,(2)}_{\text{CM,NNLO}} = \alpha^2\tilde{G}^2\frac{\zeta}{1-\zeta}m\Bigg[&-\frac{9(2+\bar{\gamma})^2}{4(1-\zeta)^2r^7}(c_+^{(0)}-4\nu_+^{(0)})n^i+\frac{36}{c^2r^7}\mu_+^{(0)}n^i\\
&+\frac{1}{c^6r^5}\Bigg(v^i((nv)v^2(4\delta(18+\nu)\lambda_-^{(0)}-4(-2+\nu)(9+4\nu)\lambda_+^{(0)})\\
&+(nv)^3(-6\delta(23+2\nu)\lambda_-^{(0)}+6(-23-2\nu+8\nu^2)\lambda_+^{(0)})\\
&+n^i((nv)^2v^2(-3\delta(87+10\nu)\lambda_-^{(0)}+3(3+2\nu)(-29+16\nu)\lambda_+^{(0)})\\
&+v^4\left(\frac{1}{2}\delta(45+4\nu)\lambda_-^{(0)}+\frac{1}{2}(45+20\nu-64\nu^2)\lambda_+^{(0)}\right)\\
&+(nv)^4(48\delta(7+\nu)\lambda_-^{(0)}-48(-7-\nu+3\nu^2)\lambda_+^{(0)})\Bigg)\Bigg],
\end{aligned} \tag{C1a}$$

$$\begin{aligned}
a^{i,(3)}_{\text{CM,NNLO}} = \frac{\alpha^3\tilde{G}^3}{c^6r^6}\frac{\zeta}{1-\zeta}m^2\Bigg[&(nv)v^i\Bigg(\Bigg(\Bigg\{\frac{40\bar{\beta}^-}{\bar{\gamma}}-\frac{2(20\bar{\beta}^++19\bar{\gamma})\delta}{\bar{\gamma}}+\left(-\frac{240\bar{\beta}^-}{\bar{\gamma}}+\frac{(80\bar{\beta}^++6\bar{\gamma}+15\bar{\gamma}^2)\delta}{\bar{\gamma}}\right)\nu\Bigg\}\lambda_-^{(0)}\\
&+\left\{-\frac{2(20\bar{\beta}^++19\bar{\gamma})}{\bar{\gamma}}+\frac{40\bar{\beta}^-\delta}{\bar{\gamma}}+\left(\frac{240\bar{\beta}^++314\bar{\gamma}+229\bar{\gamma}^2}{\bar{\gamma}}-\frac{80\bar{\beta}^-\delta}{\bar{\gamma}}\right)\nu+52\nu^2\right\}\lambda_+^{(0)}\\
&\times\frac{5(2+\bar{\gamma})\nu(4\lambda_1(-\delta\Lambda_-^{(0)}+\Lambda_+^{(0)})+\zeta(5\delta\Lambda_-^{(0)}+2\delta\phi_0\Lambda_-^{(1)}-5\Lambda_+^{(0)}-2\phi_0\Lambda_+^{(1)}))}{-1+\zeta}\Bigg)\\
&+n^i\Bigg(v^2\Bigg(\Bigg\{\frac{20\bar{\beta}^-}{\bar{\gamma}}+\frac{(-40\bar{\beta}^+-36\bar{\gamma}+15\bar{\gamma}^2)\delta}{2\bar{\gamma}}+\left(-\frac{200\bar{\beta}^-}{\bar{\gamma}}+\frac{(240\bar{\beta}^++46\bar{\gamma}+45\bar{\gamma}^2)\delta}{2\bar{\gamma}}\right)\nu\Bigg\}\lambda_-^{(0)}
\end{aligned}$$

$$
\begin{aligned}
&+\left\{\frac{5(-8\bar{\beta}^{+}-4\bar{\gamma}+5\bar{\gamma}^{2})}{2\bar{\gamma}}+\frac{20\bar{\beta}^{-}\delta}{\bar{\gamma}}+\left(\frac{400\bar{\beta}^{+}-54\bar{\gamma}+31\bar{\gamma}^{2}}{2\bar{\gamma}}-\frac{120\bar{\beta}^{-}\delta}{\bar{\gamma}}\right)\nu+14\nu^{2}\right\}\lambda_{+}^{(0)}\\
&-\frac{3(2+\bar{\gamma})(1-5\nu)(4\lambda_{1}(-\delta\Lambda_{-}^{(0)}+\Lambda_{+}^{(0)})+\zeta(5\delta\Lambda_{-}^{(0)}+2\phi_{0}\delta\Lambda_{-}^{(1)}-5\Lambda_{+}^{(0)}-2\phi_{0}\Lambda_{+}^{(1)}))}{2(-1+\zeta)}\Bigg)\\
&+(nv)^{2}\Bigg(\left\{-\frac{5}{2}(-32+17\bar{\gamma})\delta+\left(\frac{700\bar{\beta}^{-}}{\bar{\gamma}}-\frac{(560\bar{\beta}^{+}+38\bar{\gamma}+105\bar{\gamma}^{2})\delta}{4\bar{\gamma}}\right)\nu\right\}\lambda_{-}^{(0)}\\
&+\left\{-\frac{7}{2}(-32+5\bar{\gamma})+\left(-\frac{2800\bar{\beta}^{+}+2906\bar{\gamma}+2295\bar{\gamma}^{2}}{4\bar{\gamma}}+\frac{140\bar{\beta}^{-}\delta}{\bar{\gamma}}\right)\nu-83\nu^{2}\right\}\lambda_{+}^{(0)}\\
&-\frac{7(2+\bar{\gamma})(2+5\nu)(4\lambda_{1}(-\delta\Lambda_{-}^{(0)}+\Lambda_{+}^{(0)})+\zeta(5\delta\Lambda_{-}^{(0)}+2\delta\phi_{0}\Lambda_{-}^{(1)}-5\Lambda_{+}^{(0)}-2\phi_{0}\Lambda_{+}^{(1)}))}{4(-1+\zeta)}\Bigg)\Bigg)\Bigg], \qquad \text{(C1b)}
\end{aligned}
$$

$$
\begin{aligned}
a^{i,(4)}_{\mathrm{CM,NNLO}}={}&\frac{\alpha^{4}\tilde{G}^{4}}{c^{6}r^{7}}\frac{\zeta}{(1-\zeta)^{2}}m^{3}n^{i}\Bigg[(2+\bar{\gamma})\Bigg(\left\{\zeta(34+19\bar{\gamma})\delta+\left(-\frac{144\bar{\beta}^{-}\zeta}{\bar{\gamma}}+16\zeta\delta\right)\nu\right\}\phi_{0}\Lambda_{-}^{(1)}\\
&+(5\zeta-4\lambda_{1})\left(\left\{\frac{1}{2}(34+19\bar{\gamma})\delta+\left(-\frac{72\bar{\beta}^{-}}{\bar{\gamma}}+8\delta\right)\nu\right\}\Lambda_{-}^{(0)}+\left\{\frac{1}{2}(-34-19\bar{\gamma})+\frac{(72\bar{\beta}^{+}-8\bar{\gamma}+9\bar{\gamma}^{2})\nu}{\bar{\gamma}}\right\}\Lambda_{+}^{(0)}\right)\\
&+\left\{-\zeta(34+19\bar{\gamma})+\frac{2\zeta(72\bar{\beta}^{+}-8\bar{\gamma}+9\bar{\gamma}^{2})\nu}{\bar{\gamma}}\right\}\phi_{0}\Lambda_{+}^{(1)}\Bigg)\\
&+\frac{1}{\bar{\gamma}^{2}}\Bigg(\Bigg\{\frac{1}{5}(-960\bar{\beta}^{-}\bar{\beta}^{+}+960\bar{\beta}^{-}\bar{\beta}^{+}\zeta+1080\bar{\beta}^{-}\bar{\gamma}-1080\bar{\beta}^{-}\zeta\bar{\gamma}+370\bar{\beta}^{-}\bar{\gamma}^{2}-370\bar{\beta}^{-}\zeta\bar{\gamma}^{2}-240\bar{\gamma}\bar{\chi}_{-}+240\zeta\bar{\gamma}\bar{\chi}_{-}\\
&-96\zeta\bar{\gamma}\mathcal{S}_{-}\mathcal{S}_{+}+96\zeta^{2}\bar{\gamma}\mathcal{S}_{-}\mathcal{S}_{+}-96\zeta\bar{\gamma}^{2}\mathcal{S}_{-}\mathcal{S}_{+}-654\zeta^{2}\bar{\gamma}^{2}\mathcal{S}_{-}\mathcal{S}_{+}-24\zeta\bar{\gamma}^{3}\mathcal{S}_{-}\mathcal{S}_{+}-351\zeta^{2}\bar{\gamma}^{3}\mathcal{S}_{-}\mathcal{S}_{+}+1740\zeta\bar{\gamma}^{2}\mathcal{S}_{-}\mathcal{S}_{+}\lambda_{1}\\
&+870\zeta\bar{\gamma}^{3}\mathcal{S}_{-}\mathcal{S}_{+}\lambda_{1}-1200\bar{\gamma}^{2}\mathcal{S}_{-}\mathcal{S}_{+}(\lambda_{1})^{2}-600\bar{\gamma}^{3}\mathcal{S}_{-}\mathcal{S}_{+}(\lambda_{1})^{2}+240\bar{\gamma}^{2}\mathcal{S}_{-}\mathcal{S}_{+}\lambda_{2}+120\bar{\gamma}^{3}\mathcal{S}_{-}\mathcal{S}_{+}\lambda_{2})\\
&+\frac{1}{20}\delta(1920\bar{\beta}^{-2}+1920\bar{\beta}^{+2}-1920\bar{\beta}^{-2}\zeta-1920\bar{\beta}^{+2}\zeta-4320\bar{\beta}^{+}\bar{\gamma}+4320\bar{\beta}^{+}\zeta\bar{\gamma}-1056\bar{\gamma}^{2}\\
&-1480\bar{\beta}^{+}\bar{\gamma}^{2}+1056\zeta\bar{\gamma}^{2}+1480\bar{\beta}^{+}\zeta\bar{\gamma}^{2}-1596\bar{\gamma}^{3}+1596\zeta\bar{\gamma}^{3}-459\bar{\gamma}^{4}+459\zeta\bar{\gamma}^{4}+960\bar{\gamma}\bar{\chi}_{+}-960\zeta\bar{\gamma}\bar{\chi}_{+}-192\zeta\bar{\gamma}\mathcal{S}_{-}{}^{2}\\
&+192\zeta^{2}\bar{\gamma}\mathcal{S}_{-}{}^{2}-192\zeta\bar{\gamma}^{2}\mathcal{S}_{-}{}^{2}-1308\zeta^{2}\bar{\gamma}^{2}\mathcal{S}_{-}{}^{2}-48\zeta\bar{\gamma}^{3}\mathcal{S}_{-}{}^{2}-702\zeta^{2}\bar{\gamma}^{3}\mathcal{S}_{-}{}^{2}-192\zeta\bar{\gamma}\mathcal{S}_{+}{}^{2}+192\zeta^{2}\bar{\gamma}\mathcal{S}_{+}{}^{2}-192\zeta\bar{\gamma}^{2}\mathcal{S}_{+}{}^{2}\\
&-1308\zeta^{2}\bar{\gamma}^{2}\mathcal{S}_{+}{}^{2}-48\zeta\bar{\gamma}^{3}\mathcal{S}_{+}{}^{2}-702\zeta^{2}\bar{\gamma}^{3}\mathcal{S}_{+}{}^{2}+3480\zeta\bar{\gamma}^{2}\mathcal{S}_{-}{}^{2}\lambda_{1}+1740\zeta\bar{\gamma}^{3}\mathcal{S}_{-}{}^{2}\lambda_{1}+3480\zeta\bar{\gamma}^{2}\mathcal{S}_{+}{}^{2}\lambda_{1}+1740\zeta\bar{\gamma}^{3}\mathcal{S}_{+}{}^{2}\lambda_{1}\\
&-2400\bar{\gamma}^{2}\mathcal{S}_{-}{}^{2}(\lambda_{1})^{2}-1200\bar{\gamma}^{3}\mathcal{S}_{-}{}^{2}(\lambda_{1})^{2}-2400\bar{\gamma}^{2}\mathcal{S}_{+}{}^{2}(\lambda_{1})^{2}-1200\bar{\gamma}^{3}\mathcal{S}_{+}{}^{2}(\lambda_{1})^{2}\\
&+480\bar{\gamma}^{2}\mathcal{S}_{-}{}^{2}\lambda_{2}+240\bar{\gamma}^{3}\mathcal{S}_{-}{}^{2}\lambda_{2}+480\bar{\gamma}^{2}\mathcal{S}_{+}{}^{2}\lambda_{2}+240\bar{\gamma}^{3}\mathcal{S}_{+}{}^{2}\lambda_{2})\\
&+\nu(2(-1+\zeta)\bar{\gamma}(64\bar{\beta}^{+}+11\bar{\gamma}+12\bar{\gamma}^{2})\delta-\frac{2}{5}(-960\bar{\beta}^{-}\bar{\beta}^{+}+960\bar{\beta}^{-}\bar{\beta}^{+}\zeta-1120\bar{\beta}^{-}\bar{\gamma}+1120\bar{\beta}^{-}\zeta\bar{\gamma}-480\bar{\beta}^{-}\bar{\gamma}^{2}\\
&+480\bar{\beta}^{-}\zeta\bar{\gamma}^{2}-240\bar{\gamma}\bar{\chi}_{-}+240\zeta\bar{\gamma}\bar{\chi}_{-}-96\zeta\bar{\gamma}\mathcal{S}_{-}\mathcal{S}_{+}+96\zeta^{2}\bar{\gamma}\mathcal{S}_{-}\mathcal{S}_{+}-96\zeta\bar{\gamma}^{2}\mathcal{S}_{-}\mathcal{S}_{+}-654\zeta^{2}\bar{\gamma}^{2}\mathcal{S}_{-}\mathcal{S}_{+}-24\zeta\bar{\gamma}^{3}\mathcal{S}_{-}\mathcal{S}_{+}\\
&-351\zeta^{2}\bar{\gamma}^{3}\mathcal{S}_{-}\mathcal{S}_{+}+1740\zeta\bar{\gamma}^{2}\mathcal{S}_{-}\mathcal{S}_{+}\lambda_{1}+870\zeta\bar{\gamma}^{3}\mathcal{S}_{-}\mathcal{S}_{+}\lambda_{1}-1200\bar{\gamma}^{2}\mathcal{S}_{-}\mathcal{S}_{+}(\lambda_{1})^{2}-600\bar{\gamma}^{3}\mathcal{S}_{-}\mathcal{S}_{+}(\lambda_{1})^{2}+240\bar{\gamma}^{2}\mathcal{S}_{-}\mathcal{S}_{+}\lambda_{2}\\
&+120\bar{\gamma}^{3}\mathcal{S}_{-}\mathcal{S}_{+}\lambda_{2}))\Bigg\}\lambda_{-}^{(0)}\\
&+\{-12\zeta\bar{\gamma}^{2}(2+\bar{\gamma})\mathcal{S}_{-}\mathcal{S}_{+}(6\zeta-5\lambda_{1})-6\zeta\bar{\gamma}^{2}(2+\bar{\gamma})(\mathcal{S}_{-}{}^{2}+\mathcal{S}_{+}{}^{2})\delta(6\zeta-5\lambda_{1})+24\zeta\bar{\gamma}^{2}(2+\bar{\gamma})\mathcal{S}_{-}\mathcal{S}_{+}\nu(6\zeta-5\lambda_{1})\}\phi_{0}\lambda_{-}^{(1)}\\
&+\{-12\zeta^{2}\bar{\gamma}^{2}(2+\bar{\gamma})\mathcal{S}_{-}\mathcal{S}_{+}-6\zeta^{2}\bar{\gamma}^{2}(2+\bar{\gamma})(\mathcal{S}_{-}{}^{2}+\mathcal{S}_{+}{}^{2})\delta+24\zeta^{2}\bar{\gamma}^{2}(2+\bar{\gamma})\mathcal{S}_{-}\mathcal{S}_{+}\nu\}\phi_{0}{}^{2}\lambda_{-}^{(2)}\\
&+\Bigg\{\frac{3}{5}\delta(-320\bar{\beta}^{-}\bar{\beta}^{+}+320\bar{\beta}^{-}\bar{\beta}^{+}\zeta+360\bar{\beta}^{-}\bar{\gamma}-360\bar{\beta}^{-}\zeta\bar{\gamma}+130\bar{\beta}^{-}\bar{\gamma}^{2}-130\bar{\beta}^{-}\zeta\bar{\gamma}^{2}-80\bar{\gamma}\bar{\chi}_{-}+80\zeta\bar{\gamma}\bar{\chi}_{-}-32\zeta\bar{\gamma}\mathcal{S}_{-}\mathcal{S}_{+}\\
&+32\zeta^{2}\bar{\gamma}\mathcal{S}_{-}\mathcal{S}_{+}+48\zeta\bar{\gamma}^{2}\mathcal{S}_{-}\mathcal{S}_{+}+202\zeta^{2}\bar{\gamma}^{2}\mathcal{S}_{-}\mathcal{S}_{+}+32\zeta\bar{\gamma}^{3}\mathcal{S}_{-}\mathcal{S}_{+}+93\zeta^{2}\bar{\gamma}^{3}\mathcal{S}_{-}\mathcal{S}_{+}-580\zeta\bar{\gamma}^{2}\mathcal{S}_{-}\mathcal{S}_{+}\lambda_{1}-290\zeta\bar{\gamma}^{3}\mathcal{S}_{-}\mathcal{S}_{+}\lambda_{1}\\
&+400\bar{\gamma}^{2}\mathcal{S}_{-}\mathcal{S}_{+}(\lambda_{1})^{2}+200\bar{\gamma}^{3}\mathcal{S}_{-}\mathcal{S}_{+}(\lambda_{1})^{2}-80\bar{\gamma}^{2}\mathcal{S}_{-}\mathcal{S}_{+}\lambda_{2}-40\bar{\gamma}^{3}\mathcal{S}_{-}\mathcal{S}_{+}\lambda_{2})\\
&+\frac{1}{20}(1920\bar{\beta}^{-2}+1920\bar{\beta}^{+2}-1920\bar{\beta}^{-2}\zeta-1920\bar{\beta}^{+2}\zeta-4320\bar{\beta}^{+}\bar{\gamma}+4320\bar{\beta}^{+}\zeta\bar{\gamma}+3264\bar{\gamma}^{2}-1560\bar{\beta}^{+}\bar{\gamma}^{2}-3264\zeta\bar{\gamma}^{2}\\
&+1560\bar{\beta}^{+}\zeta\bar{\gamma}^{2}+3164\bar{\gamma}^{3}-3164\zeta\bar{\gamma}^{3}+851\bar{\gamma}^{4}-851\zeta\bar{\gamma}^{4}+960\bar{\gamma}\bar{\chi}_{+}-960\zeta\bar{\gamma}\bar{\chi}_{+}-192\zeta\bar{\gamma}\mathcal{S}_{-}{}^{2}+192\zeta^{2}\bar{\gamma}\mathcal{S}_{-}{}^{2}+288\zeta\bar{\gamma}^{2}\mathcal{S}_{-}{}^{2}\\
&+1212\zeta^{2}\bar{\gamma}^{2}\mathcal{S}_{-}{}^{2}+192\zeta\bar{\gamma}^{3}\mathcal{S}_{-}{}^{2}+558\zeta^{2}\bar{\gamma}^{3}\mathcal{S}_{-}{}^{2}-192\zeta\bar{\gamma}\mathcal{S}_{+}{}^{2}+192\zeta^{2}\bar{\gamma}\mathcal{S}_{+}{}^{2}+288\zeta\bar{\gamma}^{2}\mathcal{S}_{+}{}^{2}+1212\zeta^{2}\bar{\gamma}^{2}\mathcal{S}_{+}{}^{2}+192\zeta\bar{\gamma}^{3}\mathcal{S}_{+}{}^{2}\\
&+558\zeta^{2}\bar{\gamma}^{3}\mathcal{S}_{+}{}^{2}-3480\zeta\bar{\gamma}^{2}\mathcal{S}_{-}{}^{2}\lambda_{1}-1740\zeta\bar{\gamma}^{3}\mathcal{S}_{-}{}^{2}\lambda_{1}-3480\zeta\bar{\gamma}^{2}\mathcal{S}_{+}{}^{2}\lambda_{1}-1740\zeta\bar{\gamma}^{3}\mathcal{S}_{+}{}^{2}\lambda_{1}+2400\bar{\gamma}^{2}\mathcal{S}_{-}{}^{2}(\lambda_{1})^{2}
\end{aligned}
$$

$$
\begin{aligned}
&+1200\bar{\gamma}^3\mathcal{S}_-{}^2(\lambda_1)^2+2400\bar{\gamma}^2\mathcal{S}_+{}^2(\lambda_1)^2+1200\bar{\gamma}^3\mathcal{S}_+{}^2(\lambda_1)^2-480\bar{\gamma}^2\mathcal{S}_-{}^2\lambda_2-240\bar{\gamma}^3\mathcal{S}_-{}^2\lambda_2-480\bar{\gamma}^2\mathcal{S}_+{}^2\lambda_2-240\bar{\gamma}^3\mathcal{S}_+{}^2\lambda_2)\\
&+\nu(4\bar{\beta}^-(-1+\zeta)\bar{\gamma}(-32+11\bar{\gamma})\delta+\frac{1}{10}(-5760\bar{\beta}^{-2}+1920\bar{\beta}^{+2}+5760\bar{\beta}^{-2}\zeta-1920\bar{\beta}^{+2}\zeta-4480\bar{\beta}^+\bar{\gamma}+4480\bar{\beta}^+\zeta\bar{\gamma}\\
&+4176\bar{\gamma}^2-600\bar{\beta}^+\bar{\gamma}^2-4176\zeta\bar{\gamma}^2+600\bar{\beta}^+\zeta\bar{\gamma}^2+1216\bar{\gamma}^3-1216\zeta\bar{\gamma}^3-231\bar{\gamma}^4+231\zeta\bar{\gamma}^4-960\bar{\gamma}\bar{\chi}_++960\zeta\bar{\gamma}\bar{\chi}_+\\
&+192\zeta\bar{\gamma}\mathcal{S}_-{}^2-192\zeta^2\bar{\gamma}\mathcal{S}_-{}^2-288\zeta\bar{\gamma}^2\mathcal{S}_-{}^2-1212\zeta^2\bar{\gamma}^2\mathcal{S}_-{}^2-192\zeta\bar{\gamma}^3\mathcal{S}_-{}^2-558\zeta^2\bar{\gamma}^3\mathcal{S}_-{}^2+192\zeta\bar{\gamma}\mathcal{S}_+{}^2-192\zeta^2\bar{\gamma}\mathcal{S}_+{}^2\\
&-288\zeta\bar{\gamma}^2\mathcal{S}_+{}^2-1212\zeta^2\bar{\gamma}^2\mathcal{S}_+{}^2-192\zeta\bar{\gamma}^3\mathcal{S}_+{}^2-558\zeta^2\bar{\gamma}^3\mathcal{S}_+{}^2+3480\zeta\bar{\gamma}^2\mathcal{S}_-{}^2\lambda_1\\
&+1740\zeta\bar{\gamma}^3\mathcal{S}_-{}^2\lambda_1+3480\zeta\bar{\gamma}^2\mathcal{S}_+{}^2\lambda_1+1740\zeta\bar{\gamma}^3\mathcal{S}_+{}^2\lambda_1-2400\bar{\gamma}^2\mathcal{S}_-{}^2(\lambda_1)^2-1200\bar{\gamma}^3\mathcal{S}_-{}^2(\lambda_1)^2-2400\bar{\gamma}^2\mathcal{S}_+{}^2(\lambda_1)^2\\
&-1200\bar{\gamma}^3\mathcal{S}_+{}^2(\lambda_1)^2+480\bar{\gamma}^2\mathcal{S}_-{}^2\lambda_2+240\bar{\gamma}^3\mathcal{S}_-{}^2\lambda_2+480\bar{\gamma}^2\mathcal{S}_+{}^2\lambda_2+240\bar{\gamma}^3\mathcal{S}_+{}^2\lambda_2))\Big\}\lambda_+^{(0)}\\
&+\{6\zeta\bar{\gamma}^2(2+\bar{\gamma})(\mathcal{S}_-{}^2+\mathcal{S}_+{}^2)(6\zeta-5\lambda_1)+\delta(12\zeta\bar{\gamma}^2(2+\bar{\gamma})\mathcal{S}_-\mathcal{S}_+(6\zeta-5\lambda_1))\\
&-\nu(12\zeta\bar{\gamma}^2(2+\bar{\gamma})\phi_0(\mathcal{S}_-{}^2+\mathcal{S}_+{}^2)(6\zeta-5\lambda_1))\}\phi_0\lambda_+^{(1)}\\
&+\{6\zeta^2\bar{\gamma}^2(2+\bar{\gamma})(\mathcal{S}_-{}^2+\mathcal{S}_+{}^2)+\delta(12\zeta^2\bar{\gamma}^2(2+\bar{\gamma})\mathcal{S}_-\mathcal{S}_+)-\nu(12\zeta^2\bar{\gamma}^2(2+\bar{\gamma})(\mathcal{S}_-{}^2+\mathcal{S}_+{}^2))\}\phi_0{}^2\lambda_+^{(2)}\Big)\Big]. \qquad \text{(C1c)}
\end{aligned}
$$

2. Energy in the CM frame

The NNLO tidal correction to the conserved energy in the center-of-mass frame is given, after separating it in power of $\tilde{G}$, by

$$
\begin{aligned}
E^{(2)}_{\rm NNLO}=&\,\alpha^2\tilde{G}^2\frac{\zeta}{1-\zeta}m^2\nu\Bigg[-\frac{3(2+\bar{\gamma})^2(c_+^{(0)}-4\nu_+^{(0)})}{8(-1+\zeta)^2r^6}+\frac{6\mu_+^{(0)}}{c^2r^6}\\
&+\frac{1}{c^6r^4}\Bigg((nv)^4((42\delta+6\delta\nu)\lambda_-^{(0)}+(42+6\nu-18\nu^2)\lambda_+^{(0)})\\
&+v^4\Bigg(\left(\frac{29}{8}\delta+\frac{9}{4}\delta\nu\right)\lambda_-^{(0)}+\left(\frac{35}{8}-3\nu+\frac{27}{4}\nu^2\right)\lambda_+^{(0)}\Bigg)\\
&+(nv)^2v^2\Bigg(\left(-\frac{71}{2}\delta-12\delta\nu\right)\lambda_-^{(0)}+\left(-\frac{77}{2}-9\nu+30\nu^2\right)\lambda_+^{(0)}\Bigg)\Bigg)\Bigg], \qquad \text{(C2a)}
\end{aligned}
$$

$$
\begin{aligned}
E^{(3)}_{\rm NNLO}=&\,\frac{\alpha^3\tilde{G}^3}{c^6r^5}\frac{\zeta}{1-\zeta}m^3\nu\Bigg[v^2\Bigg(\left(\left(\frac{1}{4}(-2-\bar{\gamma})\delta+\left(-\frac{12\bar{\beta}^-}{\bar{\gamma}}-\frac{(16\bar{\beta}^++4\bar{\gamma}+3\bar{\gamma}^2)\delta}{4\bar{\gamma}}\right)\nu\right)\lambda_-^{(0)}\right.\\
&+\left(\frac{1}{4}(2+\bar{\gamma})+\left(\frac{48\bar{\beta}^++212\bar{\gamma}+131\bar{\gamma}^2}{4\bar{\gamma}}+\frac{4\bar{\beta}^-\delta}{\bar{\gamma}}\right)\nu-2\nu^2\right)\lambda_+^{(0)}\\
&-\frac{(2+\bar{\gamma})(-1+\nu)(4\lambda_1(-\delta\Lambda_-^{(0)}+\Lambda_+^{(0)})+\zeta(5\delta\Lambda_-^{(0)}+2\phi_0\delta\Lambda_-^{(1)}-5\Lambda_+^{(0)}-2\phi_0\Lambda_+^{(1)}))}{4(-1+\zeta)}\Bigg)\\
&+(nv)^2\Bigg(\left(\left(\frac{1}{2}(10-3\bar{\gamma})\delta+\left(\frac{100\bar{\beta}^-}{\bar{\gamma}}-\frac{(80\bar{\beta}^+-4\bar{\gamma}+15\bar{\gamma}^2)\delta}{4\bar{\gamma}}\right)\nu\right)\lambda_-^{(0)}\right.\\
&+\left(\frac{1}{2}(6-5\bar{\gamma})+\left(-\frac{400\bar{\beta}^++484\bar{\gamma}+369\bar{\gamma}^2}{4\bar{\gamma}}+\frac{20\bar{\beta}^-\delta}{\bar{\gamma}}\right)\nu-18\nu^2\right)\lambda_+^{(0)}\\
&-\frac{(2+\bar{\gamma})(2+5\nu)(4\lambda_1(-\delta\Lambda_-^{(0)}+\Lambda_+^{(0)})+\zeta(5\delta\Lambda_-^{(0)}+2\phi_0\delta\Lambda_-^{(1)}-5\Lambda_+^{(0)}-2\phi_0\Lambda_+^{(1)}))}{4(-1+\zeta)}\Bigg)\Bigg], \qquad \text{(C2b)}
\end{aligned}
$$

$$
\begin{aligned}
E^{(4)}_{\rm NNLO} = {} & \frac{\alpha^4\tilde{G}^4}{c^6 r^6}\frac{\zeta}{(1-\zeta)^2} m^4\nu\Bigg[(2+\bar\gamma)^2\Bigg(\frac{3}{2}\zeta\phi_0\delta\Lambda_-^{(1)} + (5\zeta-4\lambda_1)\Bigg(\frac{3}{4}\delta\Lambda_-^{(0)} - \frac{3}{4}\Lambda_+^{(0)}\Bigg) - \frac{3}{2}\zeta\phi_0\Lambda_+^{(1)}\Bigg) \\
& + (2+\bar\gamma)\Bigg(-\frac{12\bar\beta^-\nu(5\zeta-4\lambda_1)\Lambda_-^{(0)}}{\bar\gamma} - \frac{24\bar\beta^-\zeta\phi_0\nu\Lambda_-^{(1)}}{\bar\gamma} + \frac{3(8\bar\beta^+ + \bar\gamma^2)\nu(5\zeta-4\lambda_1)\Lambda_+^{(0)}}{2\bar\gamma} + \frac{3\zeta(8\bar\beta^+ + \bar\gamma^2)\phi_0\nu\Lambda_+^{(1)}}{\bar\gamma}\Bigg) \\
& \times\frac{1}{\bar\gamma^2}\Bigg(\Bigg\{\frac{1}{5}\nu(320\bar\beta^-\bar\beta^+ - 320\bar\beta^-\bar\beta^+\zeta + 160\bar\beta^-\bar\gamma - 160\bar\beta^-\zeta\bar\gamma + 180\bar\beta^-\bar\gamma^2 - 180\bar\beta^-\zeta\bar\gamma^2 + 80\bar\gamma\bar\chi_- \\
& - 80\zeta\bar\gamma\bar\chi_- + 32\zeta\bar\gamma\mathcal{S}_-\mathcal{S}_+ - 32\zeta^2\bar\gamma\mathcal{S}_-\mathcal{S}_+ + 32\zeta\bar\gamma^2\mathcal{S}_-\mathcal{S}_+ + 218\zeta^2\bar\gamma^2\mathcal{S}_-\mathcal{S}_+ + 8\zeta\bar\gamma^3\mathcal{S}_-\mathcal{S}_+ + 117\zeta^2\bar\gamma^3\mathcal{S}_-\mathcal{S}_+ - 580\zeta\bar\gamma^2\mathcal{S}_-\mathcal{S}_+\lambda_1 \\
& - 290\zeta\bar\gamma^3\mathcal{S}_-\mathcal{S}_+\lambda_1 + 400\bar\gamma^2\mathcal{S}_-\mathcal{S}_+(\lambda_1)^2 + 200\bar\gamma^3\mathcal{S}_-\mathcal{S}_+(\lambda_1)^2 - 80\bar\gamma^2\mathcal{S}_-\mathcal{S}_+\lambda_2 - 40\bar\gamma^3\mathcal{S}_-\mathcal{S}_+\lambda_2) \\
& + \frac{1}{10}(-320\bar\beta^-\bar\beta^+ + 320\bar\beta^-\bar\beta^+\zeta + 160\bar\beta^-\bar\gamma - 160\bar\beta^-\zeta\bar\gamma - 20\bar\beta^-\bar\gamma^2 + 20\bar\beta^-\zeta\bar\gamma^2 - 80\bar\gamma\bar\chi_- + 80\zeta\bar\gamma\bar\chi_- - 32\zeta\bar\gamma\mathcal{S}_-\mathcal{S}_+ \\
& + 32\zeta^2\bar\gamma\mathcal{S}_-\mathcal{S}_+ - 32\zeta\bar\gamma^2\mathcal{S}_-\mathcal{S}_+ - 218\zeta^2\bar\gamma^2\mathcal{S}_-\mathcal{S}_+ - 8\zeta\bar\gamma^3\mathcal{S}_-\mathcal{S}_+ - 117\zeta^2\bar\gamma^3\mathcal{S}_-\mathcal{S}_+ + 580\zeta\bar\gamma^2\mathcal{S}_-\mathcal{S}_+\lambda_1 + 290\zeta\bar\gamma^3\mathcal{S}_-\mathcal{S}_+\lambda_1 \\
& - 400\bar\gamma^2\mathcal{S}_-\mathcal{S}_+(\lambda_1)^2 - 200\bar\gamma^3\mathcal{S}_-\mathcal{S}_+(\lambda_1)^2 + 80\bar\gamma^2\mathcal{S}_-\mathcal{S}_+\lambda_2 + 40\bar\gamma^3\mathcal{S}_-\mathcal{S}_+\lambda_2) \\
& + \frac{1}{40}\delta(640\bar\beta^{-2} + 640\bar\beta^{+2} - 640\bar\beta^{-2}\zeta - 640\bar\beta^{+2}\zeta - 640\bar\beta^+\bar\gamma + 640\bar\beta^+\zeta\bar\gamma - 172\bar\gamma^2 + 80\bar\beta^+\bar\gamma^2 + 172\zeta\bar\gamma^2 - 80\bar\beta^+\zeta\bar\gamma^2 \\
& - 212\bar\gamma^3 + 212\zeta\bar\gamma^3 - 53\bar\gamma^4 + 53\zeta\bar\gamma^4 + 320\bar\gamma\bar\chi_+ - 320\zeta\bar\gamma\bar\chi_+ - 64\zeta\bar\gamma{\mathcal{S}_-}^2 + 64\zeta^2\bar\gamma{\mathcal{S}_-}^2 - 64\zeta\bar\gamma^2{\mathcal{S}_-}^2 - 436\zeta^2\bar\gamma^2{\mathcal{S}_-}^2 \\
& - 16\zeta\bar\gamma^3{\mathcal{S}_-}^2 - 234\zeta^2\bar\gamma^3{\mathcal{S}_-}^2 - 64\zeta\bar\gamma{\mathcal{S}_+}^2 + 64\zeta^2\bar\gamma{\mathcal{S}_+}^2 - 64\zeta\bar\gamma^2{\mathcal{S}_+}^2 - 436\zeta^2\bar\gamma^2{\mathcal{S}_+}^2 - 16\zeta\bar\gamma^3{\mathcal{S}_+}^2 - 234\zeta^2\bar\gamma^3{\mathcal{S}_+}^2 \\
& + 1160\zeta\bar\gamma^2{\mathcal{S}_-}^2\lambda_1 + 580\zeta\bar\gamma^3{\mathcal{S}_-}^2\lambda_1 + 1160\zeta\bar\gamma^2{\mathcal{S}_+}^2\lambda_1 + 580\zeta\bar\gamma^3{\mathcal{S}_+}^2\lambda_1 - 800\bar\gamma^2{\mathcal{S}_-}^2(\lambda_1)^2 - 400\bar\gamma^3{\mathcal{S}_-}^2(\lambda_1)^2 \\
& - 800\bar\gamma^2{\mathcal{S}_+}^2(\lambda_1)^2 - 400\bar\gamma^3{\mathcal{S}_+}^2(\lambda_1)^2 + 160\bar\gamma^2{\mathcal{S}_-}^2\lambda_2 + 80\bar\gamma^3{\mathcal{S}_-}^2\lambda_2 + 160\bar\gamma^2{\mathcal{S}_+}^2\lambda_2 + 80\bar\gamma^3{\mathcal{S}_+}^2\lambda_2)\Bigg\}\lambda_-^{(0)} \\
& + \{-2\zeta\bar\gamma^2(2+\bar\gamma)\mathcal{S}_-\mathcal{S}_+(6\zeta-5\lambda_1) - \zeta\bar\gamma^2(2+\bar\gamma)({\mathcal{S}_-}^2 + {\mathcal{S}_+}^2)\delta(6\zeta-5\lambda_1) + 4\zeta\bar\gamma^2(2+\bar\gamma)\mathcal{S}_-\mathcal{S}_+\nu(6\zeta-5\lambda_1)\}\phi_0\lambda_-^{(1)} \\
& + \{-2\zeta^2\bar\gamma^2(2+\bar\gamma)\mathcal{S}_-\mathcal{S}_+ - \zeta^2\bar\gamma^2(2+\bar\gamma)({\mathcal{S}_-}^2 + {\mathcal{S}_+}^2)\delta + 4\zeta^2\bar\gamma^2(2+\bar\gamma)\mathcal{S}_-\mathcal{S}_+\nu\}{\phi_0}^2\lambda_-^{(2)} \\
& + \Bigg\{\frac{1}{10}\delta(-320\bar\beta^-\bar\beta^+ + 320\bar\beta^-\bar\beta^+\zeta + 160\bar\beta^-\bar\gamma - 160\bar\beta^-\zeta\bar\gamma - 20\bar\beta^-\bar\gamma^2 + 20\bar\beta^-\zeta\bar\gamma^2 - 80\bar\gamma\bar\chi_- + 80\zeta\bar\gamma\bar\chi_- - 32\zeta\bar\gamma\mathcal{S}_-\mathcal{S}_+ \\
& + 32\zeta^2\bar\gamma\mathcal{S}_-\mathcal{S}_+ + 48\zeta\bar\gamma^2\mathcal{S}_-\mathcal{S}_+ + 202\zeta^2\bar\gamma^2\mathcal{S}_-\mathcal{S}_+ + 32\zeta\bar\gamma^3\mathcal{S}_-\mathcal{S}_+ + 93\zeta^2\bar\gamma^3\mathcal{S}_-\mathcal{S}_+ - 580\zeta\bar\gamma^2\mathcal{S}_-\mathcal{S}_+\lambda_1 - 290\zeta\bar\gamma^3\mathcal{S}_-\mathcal{S}_+\lambda_1 \\
& + 400\bar\gamma^2\mathcal{S}_-\mathcal{S}_+(\lambda_1)^2 + 200\bar\gamma^3\mathcal{S}_-\mathcal{S}_+(\lambda_1)^2 - 80\bar\gamma^2\mathcal{S}_-\mathcal{S}_+\lambda_2 - 40\bar\gamma^3\mathcal{S}_-\mathcal{S}_+\lambda_2) \\
& + \frac{1}{40}(640\bar\beta^{-2} + 640\bar\beta^{+2} - 640\bar\beta^{-2}\zeta - 640\bar\beta^{+2}\zeta - 640\bar\beta^+\bar\gamma + 640\bar\beta^+\zeta\bar\gamma + 348\bar\gamma^2 + 80\bar\beta^+\bar\gamma^2 - 348\zeta\bar\gamma^2 - 80\bar\beta^+\zeta\bar\gamma^2 + 308\bar\gamma^3 \\
& - 308\zeta\bar\gamma^3 + 77\bar\gamma^4 - 77\zeta\bar\gamma^4 + 320\bar\gamma\bar\chi_+ - 320\zeta\bar\gamma\bar\chi_+ - 64\zeta\bar\gamma{\mathcal{S}_-}^2 + 64\zeta^2\bar\gamma{\mathcal{S}_-}^2 + 96\zeta\bar\gamma^2{\mathcal{S}_-}^2 + 404\zeta^2\bar\gamma^2{\mathcal{S}_-}^2 + 64\zeta\bar\gamma^3{\mathcal{S}_-}^2 \\
& + 186\zeta^2\bar\gamma^3{\mathcal{S}_-}^2 - 64\zeta\bar\gamma{\mathcal{S}_+}^2 + 64\zeta^2\bar\gamma{\mathcal{S}_+}^2 + 96\zeta\bar\gamma^2{\mathcal{S}_+}^2 + 404\zeta^2\bar\gamma^2{\mathcal{S}_+}^2 + 64\zeta\bar\gamma^3{\mathcal{S}_+}^2 + 186\zeta^2\bar\gamma^3{\mathcal{S}_+}^2 - 1160\zeta\bar\gamma^2{\mathcal{S}_-}^2\lambda_1 \\
& - 580\zeta\bar\gamma^3{\mathcal{S}_-}^2\lambda_1 - 1160\zeta\bar\gamma^2{\mathcal{S}_+}^2\lambda_1 - 580\zeta\bar\gamma^3{\mathcal{S}_+}^2\lambda_1 + 800\bar\gamma^2{\mathcal{S}_-}^2(\lambda_1)^2 \\
& + 400\bar\gamma^3{\mathcal{S}_-}^2(\lambda_1)^2 + 800\bar\gamma^2{\mathcal{S}_+}^2(\lambda_1)^2 + 400\bar\gamma^3{\mathcal{S}_+}^2(\lambda_1)^2 - 160\bar\gamma^2{\mathcal{S}_-}^2\lambda_2 - 80\bar\gamma^3{\mathcal{S}_-}^2\lambda_2 - 160\bar\gamma^2{\mathcal{S}_+}^2\lambda_2 - 80\bar\gamma^3{\mathcal{S}_+}^2\lambda_2) \\
& + \frac{1}{20}\nu(-1920\bar\beta^{-2} + 640\bar\beta^{+2} + 1920\bar\beta^{-2}\zeta - 640\bar\beta^{+2}\zeta - 640\bar\beta^+\bar\gamma + 640\bar\beta^+\zeta\bar\gamma + 1232\bar\gamma^2 - 400\bar\beta^+\bar\gamma^2 - 1232\zeta\bar\gamma^2 \\
& + 400\bar\beta^+\zeta\bar\gamma^2 + 512\bar\gamma^3 - 512\zeta\bar\gamma^3 - 77\bar\gamma^4 + 77\zeta\bar\gamma^4 - 320\bar\gamma\bar\chi_+ + 320\zeta\bar\gamma\bar\chi_+ + 64\zeta\bar\gamma{\mathcal{S}_-}^2 - 64\zeta^2\bar\gamma{\mathcal{S}_-}^2 - 96\zeta\bar\gamma^2{\mathcal{S}_-}^2 \\
& - 404\zeta^2\bar\gamma^2{\mathcal{S}_-}^2 - 64\zeta\bar\gamma^3{\mathcal{S}_-}^2 - 186\zeta^2\bar\gamma^3{\mathcal{S}_-}^2 + 64\zeta\bar\gamma{\mathcal{S}_+}^2 - 64\zeta^2\bar\gamma{\mathcal{S}_+}^2 - 96\zeta\bar\gamma^2{\mathcal{S}_+}^2 - 404\zeta^2\bar\gamma^2{\mathcal{S}_+}^2 - 64\zeta\bar\gamma^3{\mathcal{S}_+}^2 \\
& - 186\zeta^2\bar\gamma^3{\mathcal{S}_+}^2 + 1160\zeta\bar\gamma^2{\mathcal{S}_-}^2\lambda_1 + 580\zeta\bar\gamma^3{\mathcal{S}_-}^2\lambda_1 + 1160\zeta\bar\gamma^2{\mathcal{S}_+}^2\lambda_1 + 580\zeta\bar\gamma^3{\mathcal{S}_+}^2\lambda_1 - 800\bar\gamma^2{\mathcal{S}_-}^2(\lambda_1)^2 - 400\bar\gamma^3{\mathcal{S}_-}^2(\lambda_1)^2 \\
& - 800\bar\gamma^2{\mathcal{S}_+}^2(\lambda_1)^2 - 400\bar\gamma^3{\mathcal{S}_+}^2(\lambda_1)^2 + 160\bar\gamma^2{\mathcal{S}_-}^2\lambda_2 + 80\bar\gamma^3{\mathcal{S}_-}^2\lambda_2 + 160\bar\gamma^2{\mathcal{S}_+}^2\lambda_2 + 80\bar\gamma^3{\mathcal{S}_+}^2\lambda_2)\Bigg\}\lambda_+^{(0)} \\
& + \{\zeta\bar\gamma^2(2+\bar\gamma)({\mathcal{S}_-}^2 + {\mathcal{S}_+}^2)(6\zeta-5\lambda_1) + 2\zeta\bar\gamma^2(2+\bar\gamma)\mathcal{S}_-\mathcal{S}_+\delta(6\zeta-5\lambda_1) - 2\zeta\bar\gamma^2(2+\bar\gamma)({\mathcal{S}_-}^2 + {\mathcal{S}_+}^2)\nu(6\zeta-5\lambda_1)\}\phi_0\lambda_+^{(1)} \\
& + \{\zeta^2\bar\gamma^2(2+\bar\gamma)({\mathcal{S}_-}^2 + {\mathcal{S}_+}^2) + 2\zeta^2\bar\gamma^2(2+\bar\gamma)\mathcal{S}_-\mathcal{S}_+\delta - 2\zeta^2\bar\gamma^2(2+\bar\gamma)({\mathcal{S}_-}^2 + {\mathcal{S}_+}^2)\nu\}{\phi_0}^2\lambda_+^{(2)}\Bigg)\Bigg].
\end{aligned}
\tag{C2c}
$$

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