Nonstationary energy of perfect fluid sources in general relativity

Emel Altas[1,*] and Bayram Tekin[2,†]
[1]*Department of Physics, Karamanoglu Mehmetbey University, 70100 Karaman, Turkey*
[2]*Department of Physics, Middle East Technical University, 06800 Ankara, Turkey*

(Received 7 December 2023; accepted 16 January 2024; published 1 February 2024)

The Arnowitt-Deser-Misner (ADM) energy for asymptotically flat spacetimes or its generalizations to asymptotically nonflat spacetimes measure the energy content of a stationary spacetime, such as a single black hole. Such a stationary energy is given as a geometric invariant of the spatial hypersurface of the spacetime and is expressed as an integral on the boundary of the hypersurface. For nonstationary spacetimes, there is a refinement of the ADM energy, the so-called Dain's invariant that measures the nonstationary part, the gravitational radiation component, of the total energy. Dain's invariant uses the metric and the extrinsic curvature of the spatial hypersurface together with the so-called approximate Killing initial data and vanishes for stationary spacetimes. In our earlier work [E. Altas and B. Tekin, Nonstationary energy in general relativity, Phys. Rev. D **101**, 024035 (2020)], we gave a reformulation of the nonstationary energy for vacuum spacetimes in the Hamiltonian form of general relativity written succinctly in the Fischer-Marsden form. That formulation is relevant for merging black holes or other compact sources. Here we extend this formulation to nonvacuum spacetimes with a perfect fluid source. This is expected to be relevant for spacetimes that have a compact star, say a neutron star colliding with a black hole or another nonvacuum object.

DOI: 10.1103/PhysRevD.109.044001

I. INTRODUCTION AND A BRIEF RECAPITULATION OF DAIN'S INVARIANT IN TWO DIFFERENT FORMULATIONS

The main purpose of this work is to derive an expression of non-stationary energy contained in a codimension-one spacelike hypersurface in matter-coupled general Relativity, where the matter sector is taken to be a perfect fluid. (Our formulation will be valid for any type of source, but we shall give explicit results only in the perfect fluid case.) As this discussion is a natural extension of the vacuum case, we will first recap what has been done so far in that case. Let us first note that some of what we shall briefly discuss here can also be found in our work [1], which we closely follow, but as the nonstationary energy concept and Dain's invariant [2] are not widely known, it pays to summarize it here.

On a spacelike hypersurface Σ of the spacetime, which we assume has the topology $\mathscr{M} = \mathbb{R}\times\Sigma$, one takes initial data to be the Riemannian metric γ and the extrinsic curvature K on Σ for Einstein's gravity. We shall work in some local coordinates and so denote the components of the hypersurface metric as γ_{ij} and the symmetric extrinsic curvature as K_{ij} with the indices taking values as $i, j = 1, 2, \ldots, D-1$. Let $\mathcal{D}_i$ be the covariant derivative compatible with γ_{ij}; and consider the usual lapse-shift decomposition of the metric as (see Fig. 1),

$$ds^2 = (N_iN^i - N^2)dt^2 + 2N_i dt dx^i + \gamma_{ij}dx^i dx^j. \quad (1)$$

In our conventions, the extrinsic curvature components read explicitly as

$$K_{ij} = \frac{1}{2N}(\dot{\gamma}_{ij} - \mathcal{D}_iN_j - \mathcal{D}_jN_i), \qquad \dot{\gamma}_{ij} \coloneqq \partial_t\gamma_{ij}. \quad (2)$$

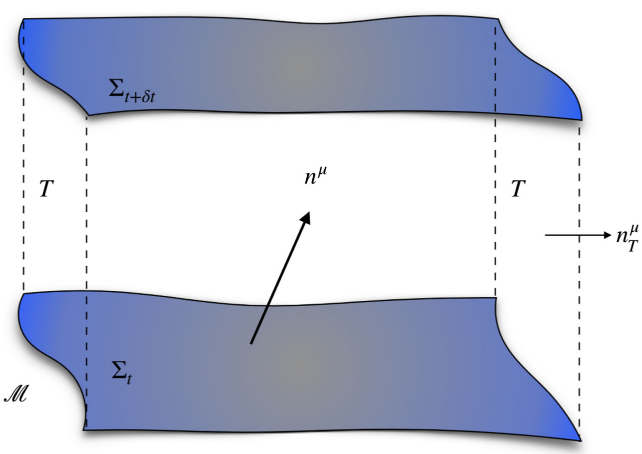

FIG. 1. Slicing of the spacetime in terms of codimension one spatial hypersurface.

*emelaltas@kmu.edu.tr
†btekin@metu.edu.tr

With this spacetime decomposition, one can now almost forget about the full covariant structure of spacetime and discuss everything in terms of the tensor fields living and evolving on the hypersurface. To this end, one can raise and lower the indices with the spatial metric and its inverse. In particular, the trace of the extrinsic curvature is defined as $K \coloneqq \gamma^{ij}K_{ij}$. For completeness, and not to disturb the flow of the paper, we give a rather comprehensive discussion of the Arnowitt-Deser-Misner (ADM) formulation [3] in the Appendixes.

Under the above decomposition of spacetime of which the details are given in the Appendixes, the D-dimensional Einstein equations with a cosmological constant and a source term,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3}$$

produce the following D constraints (the Hamiltonian and the momenta constraints) on the hypersurface:

$$\begin{aligned}\Phi_0(\gamma, K) &\coloneqq -{}^{\Sigma}R - K^2 + K^{ij}K_{ij} + 2\Lambda - 2\kappa T_{nn} = 0,\\ \Phi_i(\gamma, K) &\coloneqq -2\mathcal{D}_k K^k_i + 2\mathcal{D}_i K - 2\kappa T_{ni} = 0,\end{aligned} \tag{4}$$

where ${}^{\Sigma}R$ denotes the scalar curvature of the hypersurface; and the energy-momentum tensor of the matter sector has the following projections:

$$\begin{aligned}T_{nn} &= \frac{1}{N^2}(2N^i T_{0i} - T_{00} - N^i N^j T_{ij}),\\ T_{ni} &= \frac{1}{N}(N^j T_{ij} - T_{0i}).\end{aligned} \tag{5}$$

In addition to these constraints, we have the time-evolution equations which we shall give below.

A. Dain's invariant using the constraints

Let us briefly summarize Dain's original construction [2] of the nonstationary energy component of the total energy contained in the initial data surface for the vacuum case. Hence, one sets $T_{\mu\nu} = 0$, but we keep the cosmological constant slightly extending Dain's result, and we do the computation in generic D dimensions, extending the four-dimensional result of [2].

Let the constraint covector be $\mathbf{\Phi}(\gamma, K) \coloneqq (\Phi_0, \Phi_i)$, and let $\mathbf{D\Phi}(\gamma, K)$ be its linearization about a given solution (γ, K) to the constraints and $\mathbf{D\Phi}^*(\gamma, K)$ be the formal adjoint. Dain used an operator defined by Bartnik [4]. Bartnik's operator looks somewhat mysterious at first sight, but later in the computation one realizes that it naturally should appear, and in fact, it is necessary to use. Without further ado, let us define it as

$$\mathcal{P} \coloneqq \mathbf{D\Phi}(\gamma, K)\circ\begin{pmatrix}1 & 0\\ 0 & -\mathcal{D}^m,\end{pmatrix}, \qquad \text{Bartnik's operator.} \tag{6}$$

We also need the formal adjoint $\mathcal{P}^*$ of this operator to define the Dain's invariant on the codimension one hypersurface as

$$\mathscr{I}(\xi) \coloneqq \int_\Sigma dV\, \mathcal{P}^*(\xi)\cdot\mathcal{P}^*(\xi), \qquad \text{Dain's invariant.} \tag{7}$$

Here $\xi \coloneqq (N, N^i)$, thus $P^*(\xi) \coloneqq P^*\binom{N}{N^k}$ and the multiplication in (7) is defined componentwise as follows:

$$\begin{pmatrix}N\\ N^i\end{pmatrix}\cdot\begin{pmatrix}A\\ B_i\end{pmatrix} \coloneqq NA + N^i B_i. \tag{8}$$

The important point here is the following. The integral (7) is not to be computed for arbitrary lapse and shift (N, N^i) functions, but for specific vectors $\xi \coloneqq (N, N^i)$ that satisfy the following fourth-order partial differential equation (PDE):

$$\mathcal{P}\circ\mathcal{P}^*(\xi) = 0, \qquad \text{approximate KID equation.} \tag{9}$$

Dain dubbed this last equation as the "approximate Killing initial data" (KID) equation and in the case of time-symmetric initial data $(K_{ij} = 0)$, he showed that for any asymptotically flat three-manifold, the approximate KID equation has nontrivial solutions, that are solutions which only solve the full fourth-order equation. Of course, one must be careful here in stating what he proved; note that if ξ satisfies the second order equation $\mathcal{P}^*(\xi) = 0$, then it also automatically satisfies (9). But, after some reasonable decay assumptions at infinity, this second-order equation can be shown to be the same as the first-order KID equation, $\mathbf{D\Phi}^*(\gamma, K)(\xi) = 0$. An important result about this is due to Moncrief [5], who proved that ξ is a spacetime Killing vector satisfying $\mathcal{L}_\xi g = 0$, that is ξ generates infinitesimal isometries *if only if* it satisfies the KID equations:

$$\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0 \Longleftrightarrow \mathbf{D\Phi}^*(\gamma, K)(\xi) = 0, \qquad \text{KIDs are Killing vectors.} \tag{10}$$

From the physics vantage point Moncrief's theorem heuristically says that, as expected, the isometries of the full spacetime are certainly encoded in the initial data; and in this construction, the components of the Killing vector are given simply by the lapse and the shift functions.

From the above discussion, it is clear that Dain's invariant (7), by construction, vanishes identically when ξ is a Killing vector, i.e. $\mathcal{P}(\xi) = 0$ and the spacetime has exact symmetries. On the other hand, for approximate translational KIDs, Dain argued that for asymptotically flat spaces, and time-symmetric initial data, $\mathcal{I}(\xi)$ is a measure of the nonstationary energy contained in the hypersurface Σ. This nonstationary component is expected to evolve into gravitational radiation in spacetime. The extension to time nonsymmetric initial data was carried out by Kroon and Williams [6], where several important results on KIDs by

Moncrief [5] and Beig-Chruściel [7] were used. Another formulation of the nonstationary energy employing the time-evolution equations was given in [1] in generic D dimensions and for spacetimes that are not necessarily asymptotically flat. In the latter formulation, which makes use of the approximate KIDs and the Hamiltonian formulation of general relativity in the compact Fischer-Marsden form, the physical meaning of the invariant is more transparent. In both formulations, one can reduce the integral to a codimension-two spatial hypersurface after integration by parts. The final formula is somewhat cumbersome, the reader is referred to Eq. (53) of Ref. [1] for the final result and its various subcases. More recently, explicit details and extension of Dain's invariant to the initial data describing black holes were carried out by Sansom and Kroon [8].

As we shall need the basics of the formulation of Dain's invariant using the time-evolution equations, let us briefly summarize the relevant discussion given in [1] here.

B. Nonstationary energy via time-evolution equations

Let the canonical phase space fields be the spatial metric γ_{ij} and the canonical momenta π^{ij}. The Einstein-Hilbert Lagrangian in the ADM formulation up to a boundary term reads

$$\begin{aligned}\mathscr{L}_{\rm EH} &= \frac{1}{2\kappa}\sqrt{-g}(R-2\Lambda)\\ &= \frac{1}{2\kappa}\sqrt{\gamma}N({}^{\Sigma}R + K_{ij}K^{ij} - K^2 - 2\Lambda)\\ &\quad + \text{boundary terms}.\end{aligned} \tag{11}$$

Then, by definition, one has

$$\pi^{ij} := \frac{\delta \mathscr{L}_{\rm EH}}{\delta \dot{\gamma}_{ij}} = \frac{1}{2\kappa}\sqrt{\gamma}(K^{ij} - \gamma^{ij}K), \qquad \pi = \frac{1}{2\kappa}\sqrt{\gamma}(2-D)K, \tag{12}$$

with the reverse relations, for $D \neq 2$, given as

$$K^{ij} = \frac{2\kappa}{\sqrt{\gamma}}\left(\pi^{ij} - \frac{1}{D-2}\gamma^{ij}\pi\right), \qquad K = \frac{2\kappa}{\sqrt{\gamma}(2-D)}\pi. \tag{13}$$

The densitized version of the Hamiltonian and the momenta constraints (4) for the case of pure gravity (no matter fields) in terms of the canonical fields become,

$$\begin{aligned}\Phi_0(\gamma,\pi) &:= \frac{\sqrt{\gamma}}{2\kappa}(-{}^{\Sigma}R + 2\Lambda) + \frac{2\kappa}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \frac{\pi^2}{D-2}\right) = 0,\\ \Phi_i(\gamma,\pi) &:= -2\gamma_{ik}\mathcal{D}_j\pi^{kj} = 0.\end{aligned} \tag{14}$$

As explained in detail in [1], the Hamiltonian form of the Einstein-Hilbert action, when extremized, leads to the Fischer-Marsden form [9] of the field equations,

$$\frac{d}{dt}\begin{pmatrix}\gamma\\ \pi\end{pmatrix} = J\circ \mathbf{D\Phi}^*(\gamma,\pi)(\mathcal{N}), \qquad J := \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{15}$$

Here $\mathcal{N}$ is the lapse-shift vector with components (N, N^i). A crucial point here is that the formal adjoint of the linearized constraint map $\mathbf{D\Phi}^*(\gamma,\pi)$ appears in the time evolution instead of the operator itself, and the constraints not only determine the initial data, they also determine the time evolution. The symplectic structure of the Hamiltonian equations is also evident from the J matrix. The constraints (14) augmented with tensor equations (15) constitute constrained dynamical systems for a given lapse-shift vector (N, N^i). This form of the equations is the most suitable one for our purpose since, as discussed above, if $\mathbf{D\Phi}^*(\gamma,\pi)(\mathcal{N}) = 0$, that is $\mathcal{N} = \xi$ is a Killing vector, then the time evolution is trivial. On the other hand, if the lapse-shift vector is not a Killing vector, then one can ask how much it fails to be a Killing vector by the following equation:

$$\mathbf{D\Phi}^*(\gamma,\pi)(\mathcal{N}) = J^{-1}\circ\frac{d}{dt}\begin{pmatrix}\gamma\\ \pi\end{pmatrix}. \tag{16}$$

In particular, one can try to understand that the approximate KID equation as defined by Dain in terms of the data on the hypersurface can now have a different representation. For this purpose, we still need to work a little more. For example, to match the dimensions, and to get a number out of the above matrix, it was argued in [1] that one necessarily introduces the adjoint of the Bartnik's operator [4]:

$$\begin{aligned}\mathcal{P}^*(\mathcal{N}) &:= \begin{pmatrix}1 & 0\\ 0 & \mathcal{D}_m\end{pmatrix}\circ\mathbf{D\Phi}^*(\gamma,\pi)(\mathcal{N})\\ &= \begin{pmatrix}1 & 0\\ 0 & \mathcal{D}_m\end{pmatrix}\circ J^{-1}\circ\frac{d}{dt}\begin{pmatrix}\gamma\\ \pi\end{pmatrix},\end{aligned} \tag{17}$$

which at the end boils down to a very simple form $\mathcal{P}^*(\mathcal{N}) = (-\dot{\pi}, \mathcal{D}_m\dot{\gamma})$. This is still not sufficient yet, as π is a tensor density, we define,

$$\tilde{\mathcal{P}}^*(\mathcal{N}) := \begin{pmatrix}\gamma^{-1/2} & 0\\ 0 & 1\end{pmatrix}\circ\mathcal{P}^*(\mathcal{N}). \tag{18}$$

Finally, we have another representation of Dain's invariant (actually its generalization) which makes use of the time derivatives of the canonical phase space variables:

$$\begin{aligned}\mathscr{I}(\mathcal{N}) &= \int_\Sigma dV\,\tilde{\mathcal{P}}^*(\mathcal{N})\cdot\tilde{\mathcal{P}}^*(\mathcal{N})\\ &= \int_\Sigma dV\left(|\mathcal{D}_m\dot{\gamma}_{ij}|^2 + \frac{1}{\gamma}|\dot{\pi}^{ij}|^2\right),\end{aligned} \tag{19}$$

where we have used the shorthand notations for squares as $|\mathcal{D}_m\dot{\gamma}_{ij}|^2 \coloneqq \gamma^{mn}\gamma^{ij}\gamma^{kl}\mathcal{D}_m\dot{\gamma}_{ik}\mathcal{D}_n\dot{\gamma}_{jl}$ and $|\dot{\pi}^{ij}|^2 \coloneqq \gamma_{ij}\gamma_{kl}\dot{\pi}^{ik}\dot{\pi}^{jl}$. Several remarks are apt here: the above integral is valid for any lapse-shift vector and in the presence of a cosmological constant. Observe that the time derivative of the canonical momentum appears in the integral as well as the time derivative of the spatial covariant derivative of the spatial metric, both of which vanish for the stationary case. The integrand is explicitly positive definite. Moreover, when $(\mathcal{N})$ is an approximate KID, then something special happens and one can turn this volume integral into a surface integral reproducing the case of Dain. For details on this see Ref. [1].

II. TIME-EVOLUTION EQUATIONS, INCLUSION OF MATTER

To be able to extend the discussion of nonstationary energy to the nonvacuum case, here, we first find the time-evolution equations directly, without using the linearized constraint map. Starting from the definition of the extrinsic curvature, time evolution of the dynamical variable γ_{ij}, the spatial metric reads,

$$\frac{d\gamma_{ij}}{dt} = 2NK_{ij} + 2\mathcal{D}_{(i}N_{j)}, \tag{20}$$

where we use the symmetrization notation with a $1/2$ factor. Equivalently, in terms of the conjugate momenta, one has

$$\frac{d\gamma_{ij}}{dt} = 4\kappa N\mathcal{G}_{ijkl}\pi^{kl} + 2\mathcal{D}_{(i}N_{j)}, \tag{21}$$

where the *DeWitt metric* [10] $\mathcal{G}_{ijkl}$ in D dimensions reads,

$$\mathcal{G}_{ijkl} = \frac{1}{2\sqrt{\gamma}}\left(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \frac{2}{D-2}\gamma_{ij}\gamma_{kl}\right). \tag{22}$$

To find the evolution of the conjugate momentum, we consider the purely spatial components of the cosmological Einstein equations

$$R_{ij} - \frac{1}{2}R\gamma_{ij} + \Lambda\gamma_{ij} = \kappa T_{ij}. \tag{23}$$

Firstly, inserting the ADM decomposition of the corresponding tensor fields

$$R_{ij} = {}^{\Sigma}R_{ij} + KK_{ij} - 2K_{ik}K^k_j + \frac{1}{N}(\dot{K}_{ij} - N^k\mathcal{D}_kK_{ij} - \mathcal{D}_i\mathcal{D}_jN - 2K_{k(i}\mathcal{D}_{j)}N^k), \tag{24}$$

and

$$R = {}^{\Sigma}R + K^2 + K_{ij}K^{ij} + \frac{2}{N}(\dot{K} - \mathcal{D}_k\mathcal{D}^kN - N^k\mathcal{D}_kK) \tag{25}$$

in (23), and then using the Hamiltonian constraint, one arrives at

$$\begin{aligned}\dot{K}_{ij} - \gamma_{ij}\dot{K} = {}& N(-{}^{\Sigma}R_{ij} - KK_{ij} + 2K_{ik}K^k_j + \gamma_{ij}K^2_{kl}) + N^k\mathcal{D}_kK_{ij} + \mathcal{D}_iD_jN \\ &+ 2K_{k(i}\mathcal{D}_{j)}N^k - \gamma_{ij}(\mathcal{D}_k\mathcal{D}^kN + N^k\mathcal{D}_kK) + \kappa NT_{ij} - \frac{\kappa}{N}\gamma_{ij}(2N^kT_{ok} - T_{00} - N^lN^kT_{lk}).\end{aligned} \tag{26}$$

Adding $-\dot{\gamma}_{ij}K$ to both sides of the last equation, one has

$$\begin{aligned}\frac{d}{dt}(K_{ij} - \gamma_{ij}K) = {}& N(-{}^{\Sigma}R_{ij} - 3KK_{ij} + 2K_{ik}K^k_j + \gamma_{ij}K^2_{kl}) + N^k\mathcal{D}_kK_{ij} \\ &+ \mathcal{D}_i\mathcal{D}_jN - 2K\mathcal{D}_{(i}N_{j)} + 2K_{k(i}\mathcal{D}_{j)}N^k - \gamma_{ij}(\mathcal{D}_k\mathcal{D}^kN + N^k\mathcal{D}_kK) \\ &+ \kappa NT_{ij} - \frac{\kappa}{N}\gamma_{ij}(2N^kT_{ok} - T_{00} - N^lN^kT_{lk}).\end{aligned} \tag{27}$$

To obtain $\dot{\pi}_{ij}$, we multiply the result by $\sqrt{\gamma}$ and use

$$\sqrt{\gamma}\frac{d}{dt}(K_{ij} - \gamma_{ij}K) = 2\kappa\dot{\pi}_{ij} - \sqrt{\gamma}(NKK_{ij} + K_{ij}\mathcal{D}_kN^k - \gamma_{ij}NK^2 - \gamma_{ij}K\mathcal{D}_kN^k). \tag{28}$$

Then, Eq. (27) can be rewritten in terms of conjugate momentum as

$$\frac{d\pi_{ij}}{dt} = \frac{\sqrt{\gamma}}{2\kappa}(-N^{\Sigma}R_{ij} + \mathcal{D}_i\mathcal{D}_jN - \gamma_{ij}\mathcal{D}_k\mathcal{D}^kN) + \mathcal{D}_k(\pi_{ij}N^k) + 2\pi_{k(i}\mathcal{D}_{j)}N^k + N\frac{2\kappa}{\sqrt{\gamma}}\left(2\left(\pi_{ik}\pi_j^k - \frac{\pi\pi_{ij}}{D-2}\right) + \gamma_{ij}\left(\pi_{kl}^2 - \frac{\pi^2}{D-2}\right)\right) + \frac{\sqrt{\gamma}}{2N}(N^2T_{ij} - \gamma_{ij}(2N^kT_{ok} - T_{00} - N^lN^kT_{lk})). \quad (29)$$

We will also need the up-up indices version of this. In terms of the DeWitt metric, it reads

$$\frac{d\pi^{ij}}{dt} = \frac{\sqrt{\gamma}}{2\kappa}(-N^{\Sigma}R^{ij} + \mathcal{D}^i\mathcal{D}^jN - \gamma^{ij}\mathcal{D}_k\mathcal{D}^kN) + \mathcal{D}_k(\pi^{ij}N^k) - 2\pi_k{}^{(i}\mathcal{D}_kN^{j)} + N\frac{2\kappa}{\sqrt{\gamma}}(-2\mathcal{G}_{klmn}\gamma^{ik}\pi^{jl}\pi^{mn} + \gamma^{ij}\mathcal{G}_{klmn}\pi^{kl}\pi^{mn}) + \frac{\sqrt{\gamma}}{2N}(N^2T_{ij} - \gamma_{ij}(2N^kT_{ok} - T_{00} - N^lN^kT_{lk})). \quad (30)$$

Using the Hamiltonian constraint one more time, one can express the last equation as

$$\frac{d\pi^{ij}}{dt} = \frac{\sqrt{\gamma}}{2\kappa}(-N^{\Sigma}\mathcal{G}^{ij} + \mathcal{D}^i\mathcal{D}^jN - \gamma^{ij}\mathcal{D}_k\mathcal{D}^kN) + \mathcal{L}_N\pi^{ij} + \pi^{ij}\mathcal{D}_kN^k + N\frac{2\kappa}{\sqrt{\gamma}}\left(-2\left(\pi^{ik}\pi^j{}_k - \frac{\pi\pi^{ij}}{D-2}\right) + \frac{1}{2}\gamma^{ij}\left(\pi_{kl}^2 - \frac{\pi^2}{D-2}\right)\right) + \sqrt{\gamma}\frac{N}{2}T^{ij}, \quad (31)$$

where we have used the hypersurface Einstein tensor given as ${}^{\Sigma}\mathcal{G}^{ij} = {}^{\Sigma}R^{ij} - \frac{1}{2}\gamma^{ij}{}^{\Sigma}R + \Lambda\gamma^{ij}$, and we also used the Lie-derivative along the shift-vector, $\mathcal{L}_N\pi^{ij}$, that reads explicitly as

$$\mathcal{L}_N\pi^{ij} \coloneqq N^k\mathcal{D}_k\pi^{ij} - \pi^{ki}\mathcal{D}_kN^j - \pi^{kj}\mathcal{D}_kN^i. \quad (32)$$

III. NONSTATIONARY ENERGY OF PERFECT FLUIDS

As a concrete and a useful application, let us study the case of a perfect fluid source with the energy-momentum tensor given as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \quad (33)$$

where p is the pressure, ρ is the energy density and u^μ is the D-velocity of the perfect fluid. Following [11], we take

$$(\rho + p) = nh, \quad (34)$$

where n denotes the baryon number and h denotes the enthalpy. The fluid velocity can be taken as

$$u^\mu = hJ^\mu, \quad (35)$$

and can be decomposed according to the hypersurface as

$$J^\mu = J^\perp n^\mu + J^\mu_\parallel. \quad (36)$$

The hypersurface orthogonal part reads $J^\perp \coloneqq -J^\sigma n_\sigma = J_\perp$. Contracting the last equation with n_μ and using $n_\mu n^\mu = -1$, we obtain $n_\mu J^\mu_\parallel = 0$. Since

$$n_\mu = (-N, 0), \qquad n^\mu = (1/N, -N^i/N), \quad (37)$$

we can write $J^0_\parallel = 0$ and $J_{\parallel 0} = N^iJ_{\parallel i}$. If we evaluate the zeroth component, we find $J^0 = J^\perp/N$, and the lower index case reads

$$J_0 = -NJ^\perp + N^iJ_{\parallel i}, \quad (38)$$

which yields

$$J_i = J_{\parallel i}. \quad (39)$$

These results reduce the spatial component of the energy momentum tensor to

$$T_{ij} = (\rho + p)u_iu_j + pg_{ij} = nh^3J_{\parallel i}J_{\parallel j} + p\gamma_{ij}. \quad (40)$$

Then, the time-evolution equation for the conjugate momentum reads,

$$\frac{d\pi^{ij}}{dt} = \frac{\sqrt{\gamma}}{2\kappa}(-N^{\Sigma}\mathcal{G}^{ij} + \mathcal{D}^i\mathcal{D}^j N - \gamma^{ij}\mathcal{D}_k\mathcal{D}^k N) + \mathcal{L}_N\pi^{ij} + \pi^{ij}\mathcal{D}_k N^k \\ + N\frac{2\kappa}{\sqrt{\gamma}}\left(-2\left(\pi^{ik}\pi^j{}_k - \frac{\pi\pi^{ij}}{D-2}\right) + \frac{1}{2}\gamma^{ij}\left(\pi^2_{kl} - \frac{\pi^2}{D-2}\right)\right) + \sqrt{\gamma}\frac{N}{2}(nh^3 J^i_{\parallel}J^j_{\parallel} + p\gamma^{ij}). \tag{41}$$

Inserting these results in our formulation of the nonstationary energy formula (19)

$$\mathscr{I}(\mathcal{N}) = \int_{\Sigma} dV\left(|\mathcal{D}_m\dot{\gamma}_{ij}|^2 + \frac{1}{\gamma}|\dot{\pi}^{ij}|^2\right), \tag{42}$$

we have

$$|\mathcal{D}_m\dot{\gamma}_{ij}|^2 = \frac{16N^2\kappa^2}{\gamma}\left(\mathcal{D}_m\pi_{ij}\mathcal{D}^m\pi^{ij} + \frac{D-3}{D-2}\partial_m\pi\partial^m\pi\right) + 4\mathcal{D}_m\mathcal{D}_{(i}N_{j)}\mathcal{D}^m\mathcal{D}^{(i}N^{j)} + \frac{16N\kappa}{\sqrt{\gamma}}\left(\mathcal{D}_m\pi_{ij} - \frac{\gamma_{ij}}{D-2}\partial_m\pi\right)\mathcal{D}^m\mathcal{D}^{(i}N^{j)}. \tag{43}$$

To recast the equation in a more compact form, let us define

$$X^{ij} := \frac{\sqrt{\gamma}}{2\kappa}(-N^{\Sigma}\mathcal{G}^{ij} + \mathcal{D}^i\mathcal{D}^j N - \gamma^{ij}\mathcal{D}_k\mathcal{D}^k N), \\ Y^{ij} := N\frac{2\kappa}{\sqrt{\gamma}}\left(-2\left(\pi^{ik}\pi^j{}_k - \frac{\pi\pi^{ij}}{D-2}\right) + \frac{1}{2}\gamma^{ij}\left(\pi^2_{kl} - \frac{\pi^2}{D-2}\right)\right), \tag{44}$$

so that we have

$$\frac{d\pi^{ij}}{dt} = X^{ij} + \mathcal{L}_N\pi^{ij} + \pi^{ij}\mathcal{D}_k N^k + Y^{ij} + \sqrt{\gamma}\frac{N}{2}T^{ij}, \tag{45}$$

which yields

$$|\dot{\pi}^{ij}|^2 = X^2_{ij} + \gamma_{ik}\gamma_{jl}\mathcal{L}_N\pi^{kl}\mathcal{L}_N\pi^{ij} + \pi^2_{ij}\mathcal{D}_k N^k\mathcal{D}_l N^l + Y^2_{ij} + \gamma\frac{N^2}{4}T^{ij}T_{ij} \\ + 2X_{ij}\left(\mathcal{L}_N\pi^{ij} + \pi^{ij}\mathcal{D}_k N^k + Y^{ij} + \sqrt{\gamma}\frac{N}{2}T^{ij}\right) + 2\mathcal{L}_N\pi^{ij}\left(\pi_{ij}\mathcal{D}_k N^k + Y_{ij} + \sqrt{\gamma}\frac{N}{2}T_{ij}\right) \\ + 2\pi^{ij}\mathcal{D}_k N^k\left(Y_{ij} + \sqrt{\gamma}\frac{N}{2}T_{ij}\right) + \sqrt{\gamma}NY_{ij}T^{ij}, \tag{46}$$

where $X^2_{ij} = X_{ij}X^{ij}$. Collecting all the pieces, one arrives at the following:

$$\mathscr{I}(\xi) = \int_{\Sigma} dV\left(\frac{16N^2\kappa^2}{\gamma}\left(\mathcal{D}_m\pi_{ij}\mathcal{D}^m\pi^{ij} + \frac{D-3}{D-2}\partial_m\pi\partial^m\pi\right) + 4\mathcal{D}_m\mathcal{D}_{(i}N_{j)}\mathcal{D}^m\mathcal{D}^{(i}N^{j)} + \frac{16N\kappa}{\sqrt{\gamma}}\left(\mathcal{D}_m\pi_{ij} - \frac{\gamma_{ij}}{D-2}\partial_m\pi\right)\mathcal{D}^m\mathcal{D}^{(i}N^{j)} \right. \\ + \frac{1}{\gamma}\left(X^2_{ij} + \gamma_{ik}\gamma_{jl}\mathcal{L}_N\pi^{kl}\mathcal{L}_N\pi^{ij} + \pi^2_{ij}\mathcal{D}_k N^k\mathcal{D}_l N^l + Y^2_{ij} + \gamma\frac{N^2}{4}T^2_{ij}\right) + \frac{2X_{ij}}{\gamma}\left(\mathcal{L}_N\pi^{ij} + \pi^{ij}\mathcal{D}_k N^k + Y^{ij} + \sqrt{\gamma}\frac{N}{2}T^{ij}\right) \\ \left. + \frac{2}{\gamma}\mathcal{L}_N\pi^{ij}\left(\pi_{ij}\mathcal{D}_k N^k + Y_{ij} + \sqrt{\gamma}\frac{N}{2}T_{ij}\right) + \frac{2}{\gamma}\pi^{ij}\mathcal{D}_k N^k\left(Y_{ij} + \sqrt{\gamma}\frac{N}{2}T_{ij}\right) + \frac{N}{\sqrt{\gamma}}Y_{ij}T^{ij}\right), \tag{47}$$

where we took ξ to be an approximate KID satisfying (9).

IV. CONCLUSIONS

The Fischer-Marsden form of Einstein equations can be seen as the failure of initial data to possess an exact time translation symmetry. This simple observation led us earlier [1] to give another representation of Dain's invariant [2] which was originally given in terms of the constraints and the approximate Killing initial data. In this work, we extended our discussion to the nonvacuum case and specifically discussed the nonstationary energy that can be assigned to a spacetime with a perfect fluid source. Of course, as expected the final formula (47) is rather cumbersome, and further progress requires evaluating this expression in a given (numerical) solution.

Finally, let us note that while we pursued and generalized Dain's approach to nonstationary case based on the notion of approximate Killing initial data and the Fischer-Marsden form of the Einstein equations, there are other approaches to gravitational radiation; the two most prominent ones being due to Newman-Penrose [12] and Penrose [13] that are based on the conformal compactification of null infinity $\mathcal{I}^+$; and the Bondi-Metzner-Sachs (BMS) [14,15] approach based on the asymptotic structure of future null infinity germane to outgoing radiation. There must be an intimate connection between these approaches and the one we presented here. Especially, the BMS approach, which also gave rise to the recent work [16] regarding asymptotic symmetries, gravitational memory, and soft charges, seems so close in spirit to the formalism outlined here, but these connections are subtle at this stage and more work is needed.[1]

APPENDIX A: ADM SPLIT OF EINSTEIN'S EQUATIONS IN D DIMENSIONS

As our computations depend on the space + time splitting of Einstein's equations and all the relevant tensors, we here give the relevant details. Using the $[(D-1)+1]$-dimensional decomposition of the metric (1) we have

$$g_{00} = -N^2 + N_i N^i, \qquad g_{0i} = N_i, \qquad g_{ij} = \gamma_{ij}, \tag{A1}$$

and the inverse metric as

$$g^{00} = -\frac{1}{N^2}, \qquad g^{0i} = \frac{1}{N^2} N^i, \qquad g^{ij} = \gamma^{ij} - \frac{1}{N^2} N^i N^j. \tag{A2}$$

And the determinant of the metric reads,

$$\sqrt{-g} = N\sqrt{\gamma}, \tag{A3}$$

where we have used $g = \det g_{\mu\nu}$ and also $\gamma = \det \gamma_{ij}$.

Let $\Gamma^{\mu}_{\nu\rho}$ denote the Christoffel symbol of the D dimensional spacetime,

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}), \tag{A4}$$

and let ${}^{\Sigma}\Gamma^{k}_{ij}$ the Christoffel symbol of the $(D-1)$-dimensional hypersurface, that is compatible with the spatial metric γ_{ij}:

$${}^{\Sigma}\Gamma^{k}_{ij} = \frac{1}{2}\gamma^{kp}(\partial_i \gamma_{jp} + \partial_j \gamma_{ip} - \partial_p \gamma_{ij}). \tag{A5}$$

Then one can show the following relations:

$$\Gamma^0_{00} = \frac{1}{N}(\dot{N} + N^k(\partial_k N + N^i K_{ik})); \tag{A6}$$

$$\Gamma^0_{0i} = \frac{1}{N}(\partial_i N + N^k K_{ik}); \quad \Gamma^0_{ij} = \frac{1}{N} K_{ij}; \quad \Gamma^k_{ij} = {}^{\Sigma}\Gamma^k_{ij} - \frac{N^k}{N} K_{ij}; \tag{A7}$$

$$\Gamma^i_{0j} = -\frac{1}{N} N^i(\partial_j N + K_{kj} N^k) + N K_j{}^i + \mathcal{D}_j N^i; \tag{A8}$$

$$\Gamma^i_{00} = -\frac{N^i}{N}(\dot{N} + N^k(\partial_k N + N^l K_{kl})) + N(\partial^i N + 2N^k K_k{}^i) + \dot{N}^i + N^k \mathcal{D}_k N^i. \tag{A9}$$

To compute the decomposition of the field equations, we need to express additional tensor quantities such as the Ricci tensor components, the scalar curvature.

APPENDIX B: ADM SPLIT OF THE RICCI TENSOR AND THE SCALAR CURVATURE

Starting with the definition of the D-dimensional Ricci tensor,

$$R_{\rho\sigma} = \partial_\mu \Gamma^\mu_{\rho\sigma} - \partial_\rho \Gamma^\mu_{\mu\sigma} + \Gamma^\mu_{\mu\nu}\Gamma^\nu_{\rho\sigma} - \Gamma^\mu_{\sigma\nu}\Gamma^\nu_{\mu\rho}, \tag{B1}$$

one has

$$R_{ij} = \partial_0 \Gamma^0_{ij} + \partial_k \Gamma^k_{ij} - \partial_i(\Gamma^0_{0j} + \Gamma^k_{kj}) + \Gamma^0_{ij}(\Gamma^0_{00} + \Gamma^k_{k0}) + \Gamma^k_{ij}\Gamma^0_{0k} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^0_{0j}\Gamma^0_{0i} - \Gamma^0_{kj}\Gamma^k_{0i} - \Gamma^0_{ki}\Gamma^k_{0j} - \Gamma^k_{jl}\Gamma^l_{ki},$$

which yields

$$R_{ij} = {}^{\Sigma}R_{ij} + K K_{ij} - 2K_{ik}K^k_j + \frac{1}{N}(\dot{K}_{ij} - N^k \mathcal{D}_k K_{ij} - \mathcal{D}_i \mathcal{D}_j N - K_{ki}\mathcal{D}_j N^k - K_{kj}\mathcal{D}_i N^k), \tag{B2}$$

where ${}^{\Sigma}R_{ij}$ denotes the ij component of the Ricci tensor on the hypersurface given as

$${}^{\Sigma}R_{ij} = \partial_k {}^{\Sigma}\Gamma^k_{ij} - \partial_i {}^{\Sigma}\Gamma^k_{kj} + {}^{\Sigma}\Gamma^k_{kl}{}^{\Sigma}\Gamma^l_{ij} - {}^{\Sigma}\Gamma^k_{kl}{}^{\Sigma}\Gamma^l_{ki}. \tag{B3}$$

The $0i$ component can be written as

[1]We would like to thank an astute referee who brought our attention to these issues.

$$R_{0i} = \partial_0 \Gamma^0_{0i} + \partial_k \Gamma^k_{0i} - \partial_i(\Gamma^0_{00} + \Gamma^k_{k0}) + \Gamma^0_{0i}\Gamma^k_{k0} + \Gamma^k_{kl}\Gamma^l_{i0} - \Gamma^k_{00}\Gamma^0_{ki} - \Gamma^k_{0l}\Gamma^l_{ki}, \quad (B4)$$

and this expression gives us the following simple result:

$$R_{0i} = N^j R_{ij} + N(\mathcal{D}_m K^m_i - \mathcal{D}_i K). \quad (B5)$$

Similarly, the 00 component,

$$R_{0i} = \partial_k \Gamma^k_{00} - \partial_0 \Gamma^k_{0k} + \Gamma^0_{00}\Gamma^k_{k0} + \Gamma^k_{kl}\Gamma^l_{00} - \Gamma^k_{00}\Gamma^0_{k0} - \Gamma^k_{0l}\Gamma^l_{k0}, \quad (B6)$$

can be written in a compact form as

$$R_{00} = N^i N^j R_{ij} - N^2 K_{ij}K^{ij} + N(\mathcal{D}_k \mathcal{D}^k N - \dot{K} - N^k \mathcal{D}_k K + 2N^k \mathcal{D}_m K^m_k). \quad (B7)$$

Then, the scalar curvature of the spacetime, $R = g^{\mu\nu}R_{\mu\nu}$, can be expressed in terms of the scalar curvature of the spatial hypersurface, ${}^\Sigma R = \gamma^{ij\Sigma}R_{ij}$, as

$$R = {}^\Sigma R + K^2 + K_{ij}K^{ij} + \frac{2}{N}(\dot{K} - \mathcal{D}_k \mathcal{D}^k N - N^k \mathcal{D}_k K). \quad (B8)$$

APPENDIX C: ADM LAGRANGIAN DENSITY

The Einstein-Hilbert Lagrangian density reads

$$\mathscr{L}_{\mathrm{EH}} = \frac{1}{2\kappa}\sqrt{-g}(R - 2\Lambda). \quad (C1)$$

Inserting (B8), using the relation $\sqrt{-g} = N\sqrt{\gamma}$ together with

$$2\sqrt{\gamma}\dot{K} = \partial_0(2K\sqrt{\gamma}) - \sqrt{\gamma}(2NK^2 + 2K\mathcal{D}_k N^k), \quad (C2)$$

one obtains the Lagrangian density as

$$\mathscr{L}_{\mathrm{EH}} = \frac{1}{\kappa}(\partial_0(K\sqrt{\gamma}) - \mathcal{D}_k(\sqrt{\gamma}N^k K + \sqrt{\gamma}\partial^k N)) + \frac{1}{2\kappa}\sqrt{\gamma}N({}^\Sigma R + K^2_{ij} - K^2 - 2\Lambda). \quad (C3)$$

Ignoring the boundary expression, we get

$$\mathscr{L}_{\mathrm{EH}} = \frac{1}{2\kappa}\sqrt{\gamma}N({}^\Sigma R + K^2_{ij} - K^2 - 2\Lambda). \quad (C4)$$

The canonical momenta is, as usual, defined as follows:

$$\pi^{ij} \coloneqq \frac{\delta \mathscr{L}_{\mathrm{EH}}}{\delta \dot{\gamma}_{ij}}, \quad (C5)$$

and equivalently can be written as

$$\pi^{ij} \coloneqq \frac{\delta \mathscr{L}_{\mathrm{EH}}}{\delta \dot{\gamma}_{ij}} = \frac{\delta \mathscr{L}_{\mathrm{EH}}}{\delta K_{kl}}\frac{\delta K_{kl}}{\delta \dot{\gamma}_{ij}}, \quad (C6)$$

where the variation of the Lagrangian density yields

$$\frac{\delta \mathscr{L}_{\mathrm{EH}}}{\delta K_{kl}} = \frac{N\sqrt{\gamma}}{\kappa}(K^{kl} - \gamma^{kl}K). \quad (C7)$$

Also, due to definition of the extrinsic curvature one obtains

$$\frac{\delta K_{kl}}{\delta \dot{\gamma}_{ij}} = \frac{1}{2N}\delta^i_k \delta^j_l. \quad (C8)$$

Collecting the pieces, one ends up with

$$\pi^{ij} = \frac{1}{2\kappa}\sqrt{\gamma}(K^{ij} - \gamma^{ij}K). \quad (C9)$$

Taking the trace one has

$$\pi = \frac{1}{2\kappa}\sqrt{\gamma}(2 - D)K. \quad (C10)$$

APPENDIX D: ADM HAMILTONIAN DENSITY

The Einstein-Hilbert Hamiltonian density reads,

$$\mathcal{H}_{\mathrm{EH}} = \pi^{ij}\dot{\gamma}_{ij} - \mathscr{L}_{\mathrm{EH}}. \quad (D1)$$

Using the previous results, it is straightforward to find it explicitly

$$\mathcal{H}_{\mathrm{EH}} = \frac{1}{\kappa}(\mathcal{D}_k(\sqrt{\gamma}N^i K^k_i + \sqrt{\gamma}\partial^k N) - \partial_0(K\sqrt{\gamma})) + \frac{\sqrt{\gamma}N}{2\kappa}(-{}^\Sigma R + K^2_{ij} - K^2 + 2\Lambda) + \frac{\sqrt{\gamma}N^i}{2\kappa}(\mathcal{D}_i K - \mathcal{D}_k K^k_i). \quad (D2)$$

Here the first three terms on the right-hand side of the equality are boundary terms and they do not contribute to the constraint equations.

APPENDIX E: ADM HAMILTONIAN AND CONSTRAINT EQUATIONS

Up to a boundary expression ADM Hamiltonian yields the constraints,

$$H_{\mathrm{EH}} = \int_V dV\,\mathcal{H}_{\mathrm{EH}} = \int_V dV\,(N\Phi_0 + N^i\Phi_i), \quad (E1)$$

where Φ_0 denotes the Hamiltonian constraint and Φ_i denotes the momentum constraint. One explicitly gets

$$H_{\rm EH} = \int_V dV\left(\frac{\sqrt{\gamma}N}{2\kappa}(-{}^{\Sigma}R + K_{ij}^2 - K^2 + 2\Lambda) + \frac{\sqrt{\gamma}N^i}{2\kappa}(\mathcal{D}_i K - \mathcal{D}_k K_i^k)\right), \tag{E2}$$

yielding the Hamiltonian constraint as

$$\Phi_0(\gamma, K) = \frac{\sqrt{\gamma}}{2\kappa}(-{}^{\Sigma}R + K_{ij}^2 - K^2 + 2\Lambda), \tag{E3}$$

and also the momentum constraint as

$$\Phi_i(\gamma, K) = \frac{\sqrt{\gamma}}{2\kappa}(\mathcal{D}_i K - \mathcal{D}_k K_i^k). \tag{E4}$$

In terms of the conjugate momenta, using the reverse relations

$$K^{ij} = \frac{2\kappa}{\sqrt{\gamma}}(\pi^{ij} - \gamma^{ij}\pi), \qquad K^{ij} = -\frac{2\kappa}{\sqrt{\gamma}(D-2)}\pi, \tag{E5}$$

we can equivalently write the following equations:

$$\Phi_0(\gamma, \pi) = \frac{\sqrt{\gamma}}{2\kappa}(-{}^{\Sigma}R + 2\Lambda) + \frac{2\kappa}{\sqrt{\gamma}}\left(\pi_{ij}^2 - \frac{\pi^2}{D-2}\right), \tag{E6}$$

$$\Phi_i(\gamma, \pi) = -2\mathcal{D}_k \pi_i^k. \tag{E7}$$

APPENDIX F: CONSTRAINT EQUATIONS VIA FIELD EQUATIONS

We can also get the constraints directly from the field Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{F1}$$

which split into the constraints and the evolution equations. Obviously we have

$$R_{ij} - \frac{1}{2}\gamma_{ij}R + \Lambda\gamma_{ij} = \kappa T_{ij}, \tag{F2}$$

which can be used to simplify the constraint equations. Starting from the $0i$ component, we have

$$R_{0i} - \frac{1}{2}g_{0i}R + \Lambda g_{0i} = \kappa T_{0i}. \tag{F3}$$

Using $g_{0i} = N_i$ and plugging the ADM decomposition of the $0i$ component of the Ricci tensor one obtains,

$$N^j\left(R_{ij} - \frac{1}{2}\gamma_{ij}R + \Lambda\gamma_{ij}\right) + N(\mathcal{D}_k K_i^k - \mathcal{D}_i K) = \kappa T_{0i}. \tag{F4}$$

Inserting (F2) we arrive at the momentum constraint,

$$\Phi_i(\gamma, K) = \frac{\sqrt{\gamma}}{2\kappa}(\mathcal{D}_i K - \mathcal{D}_k K_i^k) - \frac{\sqrt{\gamma}}{N}(N^j T_{ij} - T_{0i}) = 0. \tag{F5}$$

Similarly, the Hamiltonian constraint can be obtained via the 00 component of the field equations. We write

$$R_{00} - \frac{1}{2}g_{00}R + \Lambda g_{00} = \kappa T_{00}. \tag{F6}$$

We insert (B7), (F2) and use $g_{00} = N_i N^i - N^2$ to arrive at

$$\frac{N^2}{2}(R - 2K_{ij}^2 - 2\Lambda) + N(\mathcal{D}_k\mathcal{D}^k N - \dot{K} - N^k\mathcal{D}_k K + 2N^K\mathcal{D}_i K_i^k) - \kappa(T_{00} + N^i N^j T_{ij}) = 0. \tag{F7}$$

Moreover using (B8) together with the momentum constraint one gets the Hamiltonian constraint

$$\Phi_0(\gamma, K) = \frac{\sqrt{\gamma}}{2\kappa}(-{}^{\Sigma}R + K_{ij}^2 - K^2 + 2\Lambda) - \frac{\sqrt{\gamma}}{N^2}(2N^i T_{0i} - T_{00} - N^i N^j T_{ij}) = 0. \tag{F8}$$

[1] E. Altas and B. Tekin, Nonstationary energy in general relativity, Phys. Rev. D **101**, 024035 (2020).

[2] S. Dain, A new geometric invariant on initial data for the Einstein equations, Phys. Rev. Lett. **93**, 231101 (2004).

[3] R. Arnowitt, S. Deser, and C. Misner, The dynamics of general relativity, Phys. Rev. **116**, 1322 (1959); 117, 1595 (1960); in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).

[4] R. Bartnik, Phase space for the Einstein equations, Commun. Anal. Geom. **13**, 845 (2005).

[5] V. Moncrief, Spacetime symmetries and linearization stability of the Einstein equations. I, J. Math. Phys. (N.Y.) **16**, 493 (1975).

[6] J. A. V. Kroon and J. L. Williams, Dain's invariant on non-time symmetric initial data sets, Classical Quantum Gravity **34**, 125013 (2017).

[7] R. Beig and P. T. Chruściel, Killing initial data, Classical Quantum Gravity **14**, A83 (1997).

[8] R. Sansom and J. A. V. Kroon, Dain's invariant for black hole initial data, Classical Quantum Gravity **40**, 115002 (2023).

[9] A. E. Fischer and J. E. Marsden, Linearization stability of the Einstein equations, Bull. Am. Math. Soc. **79**, 997 (1973).

[10] B. S. DeWitt, Quantum theory of gravity. 1. The canonical theory, Phys. Rev. **160**, 1113 (1967).

[11] D. Bao, J. Marsden, and R. Walton, The Hamiltonian structure of general relativistic perfect fluids, Commun. Math. Phys. **99**, 319 (1985).

[12] E. Newman and R. Penrose, An approach to gravitational radiation by a method of spin coefficients, J. Math. Phys. (N.Y.) **3**, 566 (1962).

[13] R. Penrose, Asymptotic properties of fields and space-times, Phys. Rev. Lett. **10**, 66 (1963).

[14] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems, Proc. R. Soc. A **269**, 21 (1962).

[15] R. K. Sachs, Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times, Proc. R. Soc. A **270**, 103 (1962).

[16] A. Strominger and A. Zhiboedov, Gravitational memory, BMS supertranslations and soft theorems, J. High Energy Phys. 01 (2016) 086.