Strong deflection gravitational lensing by the marginally unstable photon spheres of a wormhole

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A wormhole might have two new kinds of marginally unstable photon spheres. They could be formed either by the merger of the photon sphere and throat or by the coincidence of the photon sphere, antiphoton sphere, and throat. Both of them are unique for an object with throat. We investigate the strong deflection gravitational lensing near these marginally unstable photon spheres of a static and spherically symmetric wormhole. Its deflection angle in the strong deflection limit diverges in a manner of a power law with a specific exponent, rather than by the well-known logarithmic law for the photon sphere. We analytically obtain its observables, and apply them to a wormhole in the beyond Horndeski theory.

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I. INTRODUCTION

The detection of gravitational waves [1–6] and the direct images of the supermassive black holes in the centers of the galaxy M87 [7–12] and our Galaxy [13–18] suggest the widespread existence of black holes in the Universe. As the simplest object predicted by Einstein's general relativity, the black holes have been playing a significant role in the astrophysics, gravitational physics, and cosmology. However, the singularities and event horizons of the black holes might trigger the divergence of the curvature of the spacetime [19,20] and the information loss problem [21–24]. To address these problems, numerous ideas have been proposed by constructing a regular and/or horizonless spacetime [25–36]. One of them is a wormhole which could connect different regions of spacetime by its throat and mimic a black hole with similar observational signals [37]. Therefore, distinguishing these two kinds of objects will pave the way for the tests and understanding of the laws of gravitation.

There are two possible ways to distinguish the wormholes from the black holes by the gravitational waves and by the electromagnetic waves. However, the gravitational waves might only explore the existence of the photon sphere, which is an unstable photon orbit [38], rather than the event horizon, and it is difficult to rule out wormholes similar to black holes [39]. For the ground-based detectors, the event rate of the echoes, which are the prominent signature in the late-time waveforms of the gravitational waves emitted by an end product of a compact binary coalescence and might show the distinctive features between the wormholes and black holes [39–43], is small, and it requires more sophisticated searches [44]. Thus, we focus on the electromagnetic waves in this paper. The trajectory of a photon passing through a massive object might be deflected by its gravitational field, causing the gravitational lensing [45]. In the scenario of strong deflection gravitational lensing with its deflection angle much bigger than 1, a photon could be bent intensively and wind around the compact object by several times, producing the multiple relativistic images and a shadow [46–48]. The direct images of M87* [7–12] and Sgr A* [13–18] by the Event Horizon Telescope (EHT) indicate the possibility of observing these effects and its potential of examining the spacetime [49–73] and testing the gravitational theories [74–78].

To confront the theories of a wormhole with the observations, it is necessary to model its strong deflection gravitational lensing with detail. Its analytic description is built on special circular orbits of the photon, which include the photon sphere, the antiphoton sphere, and the throat in the wormhole spacetime [38,63]. The former two are the unstable and stable orbits of a photon, respectively, and the last one is the neck of the wormhole. The strong deflection gravitational lensing near the photon sphere in a static and spherically symmetric spacetime was analyzed in [48,79], and its deflection angle diverges logarithmically in the strong deflection limit $u \to u_{\mathrm{m}}$ as [48,79]

$$\hat{\alpha}(u) = -\bar{a}_{+} \log\left(\frac{u}{u_{\mathrm{m}}} - 1\right) + \bar{b}_{+} + \mathcal{O}\left[\left(\frac{u}{u_{\mathrm{m}}} - 1\right) \log\left(\frac{u}{u_{\mathrm{m}}} - 1\right)\right], \quad (1)$$

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where u and u_{m} are the impact parameter and its critical value at the photon sphere, respectively, and $\bar{a}_+$ and $\bar{b}_+$ are strong deflection coefficients. In a similar way, the strong deflection gravitational lensing near the antiphoton sphere [66] and the throat [38] were investigated. Their deflection angles also diverge in a manner of logarithm law similar to Eq. (1), but with different coefficients which are noted as $\bar{a}_-$, $\bar{b}_-$ [66] and $\bar{a}_{\mathrm{th}}$, $\bar{b}_{\mathrm{th}}$ [38], respectively.

The photon sphere and antiphoton sphere might merge into a marginally unstable photon sphere that could also trigger the strong deflection gravitational lensing [80]. Its deflection angle was found to be divergent as a power law as [80]

$$\hat{\alpha}(u) = \bar{a}_{\mathrm{T}}\left(\frac{u}{u_{\mathrm{m}}} - 1\right)^{-\frac{1}{6}} + \bar{b}_{\mathrm{T}} + \mathcal{O}(u - u_{\mathrm{m}})^{\frac{1}{6}}, \qquad (2)$$

where $\bar{a}_{\mathrm{T}}$ and $\bar{b}_{\mathrm{T}}$ are its strong deflection coefficients.

A wormhole might have two new kinds of marginally unstable photon spheres, both of them are unique for an object with throat. One is formed by the merger of the photon sphere and throat. Its deflection angle in the strong deflection limit of the Damour-Solodukhin wormhole [81] was calculated in Refs. [82,83], but its general formalism is still missing. The other one emerges when the photon sphere, antiphoton sphere, and throat coincide. The strong deflection gravitational lensing near that is still barely known in the literature. In this paper, we focus on these two marginally unstable photon spheres for a static and spherically symmetric wormhole, and obtain their deflection angles in the strong deflection limit as

$$\hat{\alpha}(u) = \bar{a}_k\left(\frac{u}{u_{\mathrm{m}}} - 1\right)^{-\frac{1}{k}} + \bar{b}_k + \mathcal{O}(u - u_{\mathrm{m}})^{\frac{1}{k}}, \qquad (3)$$

where $k = 4$ and $k = 3$ are for the former and latter cases, respectively, and $\bar{a}_k$ and $\bar{b}_k$ are the corresponding coefficients. It shows that these deflection angles also diverge as the power law rather than logarithm, which is similar to the case for the merger of the photon sphere and antiphoton sphere.

This paper is organized as follows. In Sec. II, we investigate the deflection angle of a photon for the strong deflection gravitational lensing near the marginally unstable photon spheres formed either by the merger of the photon sphere and throat or by the coincidence of the photon sphere, antiphoton sphere, and throat, respectively, for a static and spherically symmetric wormhole, and obtain their observables, such as the radius of the shadow, the angular separations and magnitude differences between the relativistic images. We apply our method to a wormhole in the beyond Horndeski theory in Sec. III and summarize our results in Sec. IV.

II. DEFLECTION ANGLE OF WORMHOLE IN STRONG DEFLECTION LIMIT

The metric of a general static and spherically symmetric Morris-Thorne wormhole spacetime can be written in the spherical polar coordinates as $(G = c = 1)$ [84]

$$\mathrm{d}s^2 = -e^{2\Phi(r)}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{\mathcal{B}(r)}{r}} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \qquad (4)$$

where $\Phi(r)$ and $\mathcal{B}(r)$ are the redshift function and wormhole shape function, respectively. The wormhole spacetime have the following properties:

(1) The throat of wormhole is determined by $(1 - \frac{\mathcal{B}(r)}{r})|_{r_{\mathrm{th}}} = 0$, where r_{th} is the radius of the wormhole throat.
(2) $\mathcal{B}(r)$ satisfies the flare-out condition, namely $\mathcal{B}'(r_{\mathrm{th}}) < 1$.
(3) $\Phi(r)$ must be finite from the throat to spatial infinity.

Without loss of the generality, the metric of the wormhole can also be written as

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + C(r)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \qquad (5)$$

where we demand the metric coefficients satisfy the conditions

$$\lim_{r\to\infty} A(r) = 1, \qquad \lim_{r\to\infty} B(r) = 1, \qquad \lim_{r\to\infty} C(r) = r^2. \qquad (6)$$

Comparing the metric Eq. (4) with Eq. (5), we have the relation for the throat

$$B(r_{\mathrm{th}})^{-1} = 0. \qquad (7)$$

In the static and spherically symmetric spacetime, a particle will move in the same plane when its motion is governed by the geodesic. Choosing the condition $\theta = \pi/2$, the Lagrangian for the motion of a photon can be written as

$$2\mathcal{L} = -A(r)\dot{t}^2 + B(r)\dot{r}^2 + C(r)\dot{\phi}^2, \qquad (8)$$

where an overdot represents a derivative with respect to the affine parameter. Since the Lagrangian is independent of t and ϕ, there are two constants of the motion

$$p_t = \frac{\partial\mathcal{L}}{\partial\dot{t}} = -A(r)\dot{t} = -E, \qquad p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = C(r)\dot{\phi} = L, \qquad (9)$$

where E and L are the energy and angular momentum of the photon, respectively.

Considering $B(r_{\mathrm{th}}) \to \infty$ at throat and with the condition $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$, we can rewrite the equation of motion of the photon near the wormhole as [38,63]

$$\dot{r}^2 + A^{-1}B^{-1}(V_{\rm eff}L^2 - E^2) = 0, \tag{10}$$

where the effective potential is defined as

$$V_{\rm eff}(r) = \frac{A(r)}{C(r)}. \tag{11}$$

In the gravitational lensing, a photon coming from a source may travel to an turning point r_0 near the central body and then escape again. The impact parameter of a light ray is defined as $u = L/E$, which remains constant throughout the trajectory of the photon. The turning point is indicated by $\dot{r}_0 = 0$, and, combining the motion equation, we have

$$u = \sqrt{\frac{C_0}{A_0}}, \tag{12}$$

where subscript "0" represents evaluation at the turning point r_0. The deflection angle for the photon $\hat{\alpha}(r_0)$ is given by [85]

$$\hat{\alpha}(r_0) = I(r_0) - \pi, \tag{13}$$

where

$$I(r_0) = 2\int_{r_0}^{\infty} \sqrt{\frac{B(r)}{\mathcal{R}(r,r_0)C(r)}}{\rm d}r, \tag{14}$$

with

$$\mathcal{R}(r,r_0) = \frac{A_0}{V_{\rm eff}(r)C_0} - 1. \tag{15}$$

When the light ray passes by the vicinity of a wormhole, the deflection angle $\hat{\alpha}$ will diverge as r_0 gradually decreases. It allows the photon to loop around the wormhole by several times before reaching the observer and produce the multiple relativistic images.

In order to calculate the deflection angle in the strong deflection limit, we use a variable z defined as [79]

$$z = 1 - \frac{r_0}{r}, \tag{16}$$

so that the integral function $I(r_0)$ in Eq. (14) can be rewritten as

$$I(r_0) = \int_0^1 f(z,r_0){\rm d}z, \tag{17}$$

where

$$f(z,r_0) = \frac{2r_0}{\sqrt{G(z,r_0)}}, \quad G(z,r_0) = \mathcal{R}\frac{C}{B}(1-z)^4. \tag{18}$$

In a wormhole spacetime, as $r_0 \to r_{\rm th}$, $B(r_0) \to \infty$. We redefine a $\bar{B}(r) = 1/B(r)$ so that $\bar{B}(r_{\rm th}) = 0$ at the throat [38]. Thus, $G(z,r_0)$ in Eq. (18) can be expressed as

$$G(z,r_0) = \bar{V}_{\rm eff}(z,r_0)C(1-z)^4, \tag{19}$$

where $\bar{V}_{\rm eff}$ is a modified effective potential of the photon as

$$\bar{V}_{\rm eff}(z,r_0) \equiv \mathcal{R}(z,r_0)\bar{B}(r). \tag{20}$$

Expanding $G(z,r_0)$ near $z=0$, we can obtain

$$G(z,r_0) = \sigma z + \eta z^2 + \gamma z^3 + \kappa z^4 + \mathcal{O}(z^5), \tag{21}$$

where

$$\sigma = \sigma_1 \bar{V}'_{\rm eff}(0,r_0), \tag{22}$$

$$\eta = \sum_{i=1}^{2} \eta_i \bar{V}^{(i)}_{\rm eff}(0,r_0), \tag{23}$$

$$\gamma = \sum_{i=1}^{3} \gamma_i \bar{V}^{(i)}_{\rm eff}(0,r_0), \tag{24}$$

$$\kappa = \sum_{i=1}^{4} \kappa_i \bar{V}^{(i)}_{\rm eff}(0,r_0), \tag{25}$$

and their expressions can be found in the Appendix.

The analytic description of the strong deflection gravitational lensing was built on the special circular orbits of the photon [38,48,66,80]. These circular orbits include the following:

(1) The photon sphere [48]. It is an unstable circular orbit for a photon at $r = r_{\rm m} > r_{\rm th}$ with a finite $B(r)$, which satisfies

$$V_{\rm eff}(r_{\rm m}) = \frac{E^2}{L^2}, \quad V'_{\rm eff}(r_{\rm m}) = 0, \quad V''_{\rm eff}(r_{\rm m}) < 0. \tag{26}$$

For $r_0 \to r_{\rm m}^+$, we have

$$\sigma_{\rm m} = \sigma|_{r_0=r_{\rm m}} = 0, \tag{27}$$

$$\eta_{\rm m} = \eta|_{r_0=r_{\rm m}} = -\frac{r_{\rm m}^2 C_{\rm m}^2}{2A_{\rm m}}\bar{B}_{\rm m}V''_{\rm eff}(r_{\rm m}), \tag{28}$$

where the subscript "m" indicates the quantities evaluated at $r_{\rm m}$, so that the leading term of $f(z,r_0)$ is z^{-1} and the integral $I(r_0)$ diverges logarithmically [48].

(2) The antiphoton sphere [66]. It is a stable circular orbit at $r = r_a > r_{th}$, which satisfies [66]

$$V_{eff}(r_a) = \frac{E^2}{L^2}, \quad V'_{eff}(r_a) = 0, \quad V''_{eff}(r_a) > 0. \quad (29)$$

For $r_0 \to r_c^-$, we have [66]

$$\sigma_m = \sigma|_{r_0=r_c} = 0, \quad (30)$$

$$\eta_m = \eta|_{r_0=r_c} = -\frac{r_m^2 C_m^2}{2A_m}\bar{B}_m V''_{eff}(r_m), \quad (31)$$

where r_c is defined to meet $A(r_c)/C(r_c) = A_m/C_m$, and the variable z defined in Eq. (16) should be modified as $z = 1 - r_m/r$ [66]. Then, the leading term of $f(z, r_0)$ is as z^{-1}, and its integral diverges logarithmically [66].

(3) The one merged by the photon and antiphoton spheres [80]. It is a marginally unstable photon sphere at $r_m = r_a \equiv r_{ma}$ [80]. For a wormhole, the radius r_{ma} satisfies the following relations:

$$\frac{\bar{B}(r_{ma})}{V_{eff}(r_{ma})} > 0, \quad (32)$$

$$\bar{V}'_{eff}(0, r_{ma}) = V'_{eff}(r_{ma}) = 0, \quad (33)$$

$$\bar{V}''_{eff}(0, r_{ma}) = V''_{eff}(r_{ma}) = 0, \quad (34)$$

$$\bar{V}'''_{eff}(0, r_{ma}) = -\frac{\bar{B}(r_{ma})}{V_{eff}(r_{ma})} V'''_{eff}(r_{ma}) > 0. \quad (35)$$

The strong deflection gravitational lensing of the light rays near the marginally unstable photon sphere has been studied [80]. It was found that [80]

$$\sigma_{ma} = \sigma|_{r_0=r_{ma}} = 0,$$
$$\eta_{ma} = \eta|_{r_0=r_{ma}} = 0,$$
$$\gamma_{ma} = \gamma|_{r_0=r_{ma}} = -\frac{r_{ma}^3 C_{ma}^2}{6A_{ma}}\bar{B}_{ma} V'''_{eff}(r_{ma}), \quad (36)$$

where the subscript "ma" denotes the evaluation at r_{ma}, so that the leading term of $f(z, r_0)$ is $z^{-\frac{3}{2}}$, and its integral $I(r_0)$ diverges as $z^{-\frac{1}{2}}$ [80].

(4) The throat of a wormhole [38]. Due to $\bar{B}(r_{th}) = 0$ at the throat, we can have [38]

$$\sigma_{th} = \sigma|_{r_0=r_{th}} = 0,$$
$$\eta_{th} = \eta|_{r_0=r_{th}} = -\frac{r_{th}^2 C_{th}^2}{A_{th}}\bar{B}'_{th} V'_{eff}(r_{th}), \quad (37)$$

where the subscript "th" represents the evaluation at r_{th}. The leading term of $f(z, r_0)$ is z^{-1}, and its integral $I(r_0)$ also diverges logarithmically [38].

In addition, a wormhole might have two new kinds of marginally unstable photon spheres which could be formed either by the merger of the photon sphere and throat or by the coincidence of the photon sphere, antiphoton sphere, and throat. The strong deflection gravitational lensing near these orbits is still barely known, and we will investigate in detail these special cases in the following parts.

A. Merger of the photon sphere and throat

When the photon sphere and throat merger into one at $r_m = r_{th} \equiv r_{mth}$, we can have

$$\bar{B}(r_{mth}) = 0, \quad V'_{eff}(r_{mth}) = 0, \quad V''_{eff}(r_{mth}) < 0, \quad (38)$$

which are equivalent to

$$\bar{V}'_{eff}(0, r_{mth}) = 0, \quad \bar{V}''_{eff}(0, r_{mth}) = 0, \quad \bar{V}'''_{eff}(0, r_{mth}) > 0. \quad (39)$$

It indicates that such a merger forms a marginally unstable photon sphere.

For the strong deflection gravitational lensing near this marginally unstable orbit, i.e., $r_0 \to r_{mth}^+$, we can obtain the coefficients in the expansion of $G(z, r_0)$ as

$$\sigma_{mth} = \sigma|_{r_0=r_{mth}} = 0, \quad (40)$$

$$\eta_{mth} = \sigma|_{r_0=r_{mth}} = 0, \quad (41)$$

$$\gamma_{mth} = \gamma|_{r_0=r_{mth}} = \frac{r_{mth}^3}{2}\bar{B}'_{mth} C_{mth} D''_{mth}, \quad (42)$$

where the subscript "mth" denotes the evaluation at r_{mth},

$$D''_{mth} = \frac{C''_{mth}}{C_{mth}} - \frac{A''_{mth}}{A_{mth}}, \quad (43)$$

and the following relation has been used

$$V''_{eff}(r_{mth}) V_{eff}^{-1}(r_{mth}) = -D''_{mth}. \quad (44)$$

Therefore, we can have

$$G(z, r_{mth}) = \gamma_{mth} z^3 + \mathcal{O}(z^4), \quad (45)$$

so that the leading term of $f(z, r_0)$ is $z^{-\frac{3}{2}}$, and its integral $I(r_0)$ would diverge as $z^{-\frac{1}{2}}$.

It suggests that the integral $I(r_0)$ can be separated into two parts of a divergent part $I_D(r_0)$ and a regular part $I_R(r_0)$, namely $I(r_0) = I_D(r_0) + I_R(r_0)$. The divergent integral $I_D(r_0)$ is defined as

$$I_D(r_0) = \int_0^1 f_D(z, r_0) \mathrm{d}z, \tag{46}$$

with

$$f_D(z, r_0) = \frac{2r_0}{\sqrt{\sigma z + \eta z^2 + \gamma z^3}}, \tag{47}$$

and the regular part $I_R(r_0)$ is defined by

$$I_R(r_0) = \int_0^1 f_R(z, r_0) \mathrm{d}z, \tag{48}$$

with

$$f_R(z, r_0) = f(z, r_0) - f_D(z, r_0). \tag{49}$$

For the strong deflection limit, as $r_0 \to r_{\mathrm{mth}}^+$, we can obtain

$$f_D(z, r_{\mathrm{mth}}) = \frac{2r_{\mathrm{mth}}}{\sqrt{\gamma_{\mathrm{mth}} z^3}}, \tag{50}$$

and its integral as

$$\begin{aligned} I_D(r_0) &= \int_0^1 f_D(z, r_0) \mathrm{d}z, \\ &= \frac{4r_{\mathrm{mth}}}{\sqrt{\gamma_{\mathrm{mth}}}} \frac{1}{\sqrt{\frac{r_0}{r_{\mathrm{mth}}} - 1}} - \frac{4r_{\mathrm{mth}}}{\sqrt{\gamma_{\mathrm{mth}}}} + \mathcal{O}\left(\sqrt{\frac{r_0}{r_{\mathrm{mth}}} - 1}\right). \end{aligned} \tag{51}$$

Meanwhile, the impact parameter can be expanded as

$$\begin{aligned} u &= \sqrt{\frac{C_0}{A_0}} \\ &= u_{\mathrm{mth}}\left[1 + \frac{1}{4} D''_{\mathrm{mth}} (r_0 - r_{\mathrm{mth}})^2\right] + \mathcal{O}(r_0 - r_{\mathrm{mth}})^3. \end{aligned} \tag{52}$$

The divergent part $I_D(r_0)$ can be written in terms of the impact parameter u as

$$\begin{aligned} I_D(r_0) &= \frac{(2r_{\mathrm{mth}})^{\frac{3}{2}}}{\sqrt{\gamma_{\mathrm{mth}}}} (D''_{\mathrm{mth}})^{\frac{1}{4}} \left(\frac{u}{u_{\mathrm{mth}}} - 1\right)^{-\frac{1}{4}} - \frac{4r_{\mathrm{mth}}}{\sqrt{\gamma_{\mathrm{mth}}}} \\ &\quad + \mathcal{O}\left(\frac{u}{u_{\mathrm{mth}}} - 1\right)^{\frac{1}{4}}, \end{aligned} \tag{53}$$

while the regular integral $I_R(r_0)$ can be expanded as

$$I_R(r_0) = \int_0^1 f_R(z, r_{\mathrm{mth}}) \mathrm{d}z + \mathcal{O}(r_0 - r_{\mathrm{mth}})^{\frac{1}{2}}. \tag{54}$$

In the strong deflection limit $r_0 \to r_{\mathrm{mth}}^+$ or $u \to u_{\mathrm{mth}}^+$, its deflection angle is

$$\hat{\alpha}(u) = \bar{a}_{\mathrm{mth}} \left(\frac{u}{u_{\mathrm{mth}}} - 1\right)^{-\frac{1}{4}} + \bar{b}_{\mathrm{mth}} + \mathcal{O}(u - u_{\mathrm{mth}})^{\frac{1}{4}}, \tag{55}$$

where

$$\bar{a}_{\mathrm{mth}} = 4(\bar{B}'^2_{\mathrm{mth}} C^2_{\mathrm{mth}} D''_{\mathrm{mth}})^{-\frac{1}{4}}, \tag{56}$$

$$\bar{b}_{\mathrm{mth}} = -\bar{a}_{\mathrm{mth}} \sqrt{\frac{2}{r_{\mathrm{mth}}}} (D''_{\mathrm{mth}})^{-\frac{1}{4}} + I_R(r_{\mathrm{mth}}) - \pi, \tag{57}$$

and

$$\begin{aligned} I_R(r_{\mathrm{mth}}) = \int_0^1 &\left[\frac{2r_{\mathrm{mth}}}{(1-z)^2} \sqrt{\frac{AC_{\mathrm{mth}}}{\bar{B}C(A_{\mathrm{mth}}C - AC_{\mathrm{mth}})}} \right. \\ &\left. - \frac{\bar{a}_{\mathrm{mth}}}{\sqrt{2r_{\mathrm{mth}}}} (D''_{\mathrm{mth}})^{-\frac{1}{4}} z^{-\frac{3}{2}}\right] \mathrm{d}z. \end{aligned} \tag{58}$$

As a check of our results, we apply this method to the Damour-Solodukhin wormhole, whose metric reads [81]

$$A(r) = 1 - \frac{2M}{r}, \tag{59}$$

$$B(r) = \left[1 - \frac{2M(1+\chi^2)}{r}\right]^{-1}, \tag{60}$$

$$C(r) = r^2, \tag{61}$$

with M and χ being positive parameters. The radii of the photon sphere and throat are $r_{\mathrm{m}} = 3M$ and $r_{\mathrm{th}} = 2M(1+\chi^2)$. When the parameter $\chi = \chi_{\mathrm{mth}} = \sqrt{2}/2$, the photon sphere and throat merge into a marginally unstable photon sphere at $r_{\mathrm{mth}} = 3M$ [82,83]. Substituting the metric and r_{mth} into Eqs. (56)–(58), we have

$$\bar{a}^{\mathrm{DS}}_{\mathrm{mth}} = \frac{2^{\frac{5}{4}}[r_{\mathrm{mth}}(r_{\mathrm{mth}} - 2)]^{\frac{1}{4}}}{(1+\chi^2_{\mathrm{mth}})^{\frac{1}{2}}} = 2^{\frac{7}{4}} 3^{-\frac{1}{4}} = 2.5558, \tag{62}$$

$$\bar{b}^{\mathrm{DS}}_{\mathrm{mth}} = -2.1083, \tag{63}$$

which are consistent with the results of Refs. [82,83], although our expression of $\bar{b}_{\mathrm{mth}}$ is slightly different from the one in Refs. [82,83] that is specified for the Damour-Solodukhin wormhole.

B. Coincidence of the photon sphere, antiphoton sphere, and throat

There is a another type of marginally unstable photon sphere where the photon sphere, antiphoton sphere, and throat coincide, i.e., $r_{\mathrm{m}} = r_{\mathrm{a}} = r_{\mathrm{th}} \equiv r_{\mathrm{mat}}$. We can have

$$\bar{B}(r_{\mathrm{mat}}) = 0, \qquad V'_{\mathrm{eff}}(r_{\mathrm{mat}}) = 0, \tag{64}$$

$$V''_{\mathrm{eff}}(r_{\mathrm{mat}}) = 0, \qquad V'''_{\mathrm{eff}}(r_{\mathrm{mat}}) < 0, \tag{65}$$

which are equivalent to

$$\bar{V}'_{\mathrm{eff}}(0, r_{\mathrm{mat}}) = 0, \qquad \bar{V}''_{\mathrm{eff}}(0, r_{\mathrm{mat}}) = 0, \tag{66}$$

$$\bar{V}'''_{\mathrm{eff}}(0, r_{\mathrm{mat}}) = 0, \qquad \bar{V}^{(4)}_{\mathrm{eff}}(0, r_{\mathrm{mat}}) > 0. \tag{67}$$

As $r_0 \to r^+_{\mathrm{mat}}$, combining Eqs. (64) and (22), we can obtain

$$\begin{aligned}
&\sigma_{\mathrm{mat}} = \sigma|_{r_0=r_{\mathrm{mat}}} = 0, \qquad \eta_{\mathrm{mat}} = \eta|_{r_0=r_{\mathrm{mat}}} = 0,\\
&\gamma_{\mathrm{mat}} = \gamma|_{r_0=r_{\mathrm{mat}}} = 0,\\
&\kappa_{\mathrm{mat}} = \kappa|_{r_0=r_{\mathrm{mat}}} = \frac{r^4_{\mathrm{mat}}}{6} C_{\mathrm{mat}} \bar{B}'_{\mathrm{mat}} D'''_{\mathrm{mat}},
\end{aligned} \tag{68}$$

with

$$D'''_{\mathrm{mat}} = \frac{C'''_{\mathrm{mat}}}{C_{\mathrm{mat}}} - \frac{A'''_{\mathrm{mat}}}{A_{\mathrm{mat}}}, \tag{69}$$

where "mat" indicates evaluation at $r = r_{\mathrm{mat}}$ and the following relation has been used

$$V'''_{\mathrm{eff}}(r_{\mathrm{mat}}) V^{-1}_{\mathrm{eff}}(r_{\mathrm{mat}}) = -D'''_{\mathrm{mat}}. \tag{70}$$

Therefore, $G(z, r_0)$ becomes

$$G(z, r_{\mathrm{mth}}) = \kappa_{\mathrm{mat}} z^4 + \mathcal{O}(z^5), \tag{71}$$

so that the leading term of $f(z, r_0)$ is z^{-2} and its integral $I(r_0)$ would diverge as z^{-1}.

Similarly, the integral $I(r_0)$ can be separated into two parts of a divergent part $I_D(r_0)$ and a regular part $I_R(r_0)$. For the strong deflection limit, as $r_0 \to r^+_{\mathrm{mat}}$, we know that

$$f_D(z, r_{\mathrm{mat}}) = \frac{2r_{\mathrm{mat}}}{\sqrt{\kappa_{\mathrm{mat}} z^4}}, \tag{72}$$

so the divergent integral $I_D(r_0)$ can be written as

$$\begin{aligned}
I_D(r_0) &= \int_0^1 f_D(z, r_0) \mathrm{d}z,\\
&= \frac{2r_{\mathrm{mat}}}{\sqrt{\kappa_{\mathrm{mat}}}} \left(\frac{r_0}{r_{\mathrm{mat}}} - 1\right)^{-1} + \mathcal{O}\left(\frac{r_0}{r_{\mathrm{mth}}} - 1\right).
\end{aligned} \tag{73}$$

With the expansion of the impact parameter

$$u = u_{\mathrm{mat}} \left[1 + \frac{1}{12} D'''_{\mathrm{mat}} (r_0 - r_{\mathrm{mat}})^3\right] + \mathcal{O}(r_0 - r_{\mathrm{mat}})^4, \tag{74}$$

the divergent part $I_D(r_0)$ takes the form as

$$\begin{aligned}
I_D(r_0) = &\frac{2^{\frac{1}{3}} 3^{-\frac{1}{3}} r^2_{\mathrm{mat}}}{\sqrt{\kappa_{\mathrm{mat}}}} (D'''_{\mathrm{mat}})^{\frac{1}{3}} \left(\frac{u}{u_{\mathrm{mat}}} - 1\right)^{-\frac{1}{3}}\\
&+ \mathcal{O}\left(\frac{u}{u_{\mathrm{mat}}} - 1\right)^{\frac{4}{3}},
\end{aligned} \tag{75}$$

while the regular part $I_R(r_0)$ can be expanded as

$$I_R(r_0) = \int_0^1 f_R(z, r_{\mathrm{mat}}) \mathrm{d}z + \mathcal{O}(r_0 - r_{\mathrm{mat}}). \tag{76}$$

Thus, in the strong deflection limit $r_0 \to r^+_{\mathrm{mat}}$ or $u \to u^+_{\mathrm{mat}}$, its deflection angle is

$$\hat{\alpha}(u) = \bar{a}_{\mathrm{mat}} \left(\frac{u}{u_{\mathrm{mat}}} - 1\right)^{-\frac{1}{3}} + \bar{b}_{\mathrm{mat}} + \mathcal{O}(u - u_{\mathrm{mat}})^{\frac{1}{3}}, \tag{77}$$

where

$$\bar{a}_{\mathrm{mat}} = \frac{2^{\frac{5}{6}} 3^{\frac{1}{6}}}{\sqrt{\bar{B}'_{\mathrm{mat}} C_{\mathrm{mat}}}} (D'''_{\mathrm{mat}})^{-\frac{1}{6}}, \tag{78}$$

$$\bar{b}_{\mathrm{mat}} = I_R(r_{\mathrm{mat}}) - \pi, \tag{79}$$

and

$$\begin{aligned}
I_R(r_{\mathrm{mat}}) = \int_0^1 &\left[\frac{2r_{\mathrm{mat}}}{(1-z)^2} \sqrt{\frac{AC_{\mathrm{mat}}}{\bar{B}C(A_{\mathrm{mat}} C - AC_{\mathrm{mat}})}}\right.\\
&\left. - \frac{2}{r_{\mathrm{mat}}} \sqrt{\frac{6}{\bar{B}'_{\mathrm{mat}} C_{\mathrm{mat}}}} (D'''_{\mathrm{mat}})^{-\frac{1}{2}} \frac{1}{z^2}\right] \mathrm{d}z.
\end{aligned} \tag{80}$$

C. Observables

Having the deflection angle for these two marginally unstable photon spheres, we are able to obtain their observables. We consider a light ray which comes from a source S with the impact parameter u, takes a turn near the gravitational lens L, and escapes to a distant observer O, and the source and observer are faraway from the gravitational object. In the strong deflection gravitational lensing, the photons might round the object several times before arriving the observer and form the relativistic images [48]. Under the thin lens approximation, the lens equation is given by [47]

$$\beta = \theta - \frac{D_{\mathrm{LS}}}{D_{\mathrm{OS}}} \Delta\hat{\alpha}_n, \tag{81}$$

where D_{LS} and D_{OS} are the distances between the lens and source and between the source and observer, respectively, β is the angular separation between the source and lens, θ is the angular separation between the lens and images, $\Delta\hat{\alpha} = \hat{\alpha}(\theta) - 2\pi n$ is the offset of the deflection angle after subtracting all the n loops by the photon.

For $n \geq 1$, we define

$$\hat{\alpha}(\theta_n^0) = 2\pi n. \tag{82}$$

Expanding the deflection angle $\hat{\alpha}(\theta)$ near $\theta = \theta_n^0$, we have

$$\hat{\alpha}(\theta) = \hat{\alpha}(\theta_n^0) + \frac{d\hat{\alpha}}{d\theta}\bigg|_{\theta=\theta_n^0}(\theta - \theta_n^0) + \mathcal{O}(\theta - \theta_n^0)^2. \tag{83}$$

Using the relation between impact parameter and the angular position of image $u = \theta D_{\rm OL}$, where $D_{\rm OL}$ is the distance between the observer and lens, the deflection angles (55) and (77) can be written in an unified form as

$$\hat{\alpha}(\theta) = \bar{a}_k\left(\frac{\theta}{\theta_\infty} - 1\right)^{-\frac{1}{k}} + \bar{b}_k + \mathcal{O}(u - u_{\rm m})^{\frac{1}{k}}, \tag{84}$$

where

$$\theta_\infty = \frac{u_{\rm m}}{D_{\rm OL}} \tag{85}$$

is the angular position of the relativistic image formed at the photon sphere. For the marginally unstable photon sphere merged by the photon sphere and throat, we have $k = 4$ so that $\bar{a}_4 = \bar{a}_{\rm mth}$ and $\bar{b}_4 = \bar{b}_{\rm mth}$; and for the coincidence of the photon sphere, antiphoton sphere, and throat, we have $k = 3$ so that $\bar{a}_3 = \bar{a}_{\rm mat}$ and $\bar{b}_3 = \bar{b}_{\rm mat}$. Based on Eq. (84), we can have

$$\frac{d\hat{\alpha}}{d\theta}\bigg|_{\theta=\theta_n^0} = -\frac{\bar{a}_k}{k\theta_\infty}\left(\frac{\theta_n^0}{\theta_\infty} - 1\right)^{-\frac{1+k}{k}}, \tag{86}$$

and after combining Eqs. (82) and (84), we can obtain

$$\theta_n^0 = \left[1 + \left(\frac{\bar{a}_k}{2\pi n - \bar{b}_k}\right)^k\right]\theta_\infty. \tag{87}$$

Substituting Eqs. (82), (86), and (87) into Eq. (83), the offset of the deflection angle can be found as

$$\Delta\hat{\alpha}(\theta_n) = \frac{(2\pi n - \bar{b}_k)^{1+k}}{k\theta_\infty \bar{a}_k^k}(\theta_n^0 - \theta_n). \tag{88}$$

Using this expression and the lens equation (81), we can obtain the angular position of the relativistic images as

$$\theta_n(\beta) = \theta_n^0 + \frac{k\bar{a}_k^k D_{\rm OS}\theta_\infty(\beta - \theta_n^0)}{D_{\rm LS}(2\pi n - \bar{b}_k)^{1+k}}, \tag{89}$$

and the corresponding magnification is [86]

$$\mu_n \equiv \left|\frac{\theta_n}{\beta}\frac{d\theta_n}{d\beta}\right| = \frac{k\bar{a}_k^k D_{\rm OS}\theta_\infty^2 J_k(n)}{D_{\rm LS}\beta}, \tag{90}$$

where

$$J_k(n) = \frac{\bar{a}_k^k + (2\pi n - \bar{b}_k)^k}{(2\pi n - \bar{b}_k)^{2k+1}}. \tag{91}$$

Assuming only the outmost relativistic image can be resolved from the packed others on the photon sphere, we can define two additional observables, i.e., the angular separation s and the magnitude difference Δm between the outmost relativistic image and the rest as [48]

$$s = \theta_1 - \theta_\infty = \left(\frac{\bar{a}_k}{2\pi - \bar{b}_k}\right)^k \theta_\infty \tag{92}$$

and

$$\Delta m = 2.5\log_{10}\left(\frac{\mu_1}{\sum_{n=2}^\infty \mu_n}\right), \\ = 2.5\log_{10}\left[\frac{J_k(1)}{\sum_{n=2}^\infty J_k(n)}\right]. \tag{93}$$

III. APPLICATION

We will apply our method introduced in the Sec. II to the wormhole in the beyond Horndeski theory [87].

A. Metric

The metric of a traversable wormhole in the beyond Horndeski theories is given by [87]

$$A(r) = h, \tag{94}$$

$$B(r) = \left[1 - \frac{r_{\rm th}}{\lambda r}(1 - \sqrt{h})\right]^{-1} h^{-1}, \tag{95}$$

$$C(r) = r^2, \tag{96}$$

where

$$h = 1 + \frac{r^2}{2\eta_\bullet^2}\left(1 - \sqrt{1 + \frac{8\eta_\bullet^2 m_\bullet}{r^3}}\right) \tag{97}$$

$\eta_\bullet$ is the coupling parameter and λ is a positive dimensionless constant. At the throat of the wormhole, we have [87]

$$h(r_{\rm th}) = (1 - \lambda)^2. \tag{98}$$

The existence of the throat requires the dimensionless constant $\lambda \in (0, 1)$ [87]. By solving Eq. (98), we obtain the radius of the throat as [87]

$$r_{\rm th} = \frac{m_\bullet + \sqrt{m_\bullet^2 - \eta_\bullet^2 \xi^3}}{\xi}, \tag{99}$$

with

$$\xi = \lambda(2-\lambda) \in (0,1). \tag{100}$$

A positive $r_{\rm th}$ demands [87]

$$\eta_\bullet \le \xi^{-\frac{3}{2}} m_\bullet \in (m_\bullet, +\infty). \tag{101}$$

Using the condition Eqs. (26) and (29), we obtain the radii of its photon and antiphoton spheres as

$$r_{\rm m} = 2\sqrt{3}\sin\left[\frac{\pi}{6} + \frac{1}{3}\arccos\left(\frac{4\sqrt{3}}{9}\eta^2\right)\right]m_\bullet, \tag{102}$$

$$r_{\rm a} = 2\sqrt{3}\sin\left[\frac{\pi}{6} - \frac{1}{3}\arccos\left(\frac{4\sqrt{3}}{9}\eta^2\right)\right]m_\bullet, \tag{103}$$

where the dimensionless coupling constant is

$$\eta = \frac{\eta_\bullet}{m_\bullet}. \tag{104}$$

The existence of photon sphere and antiphoton sphere demands the coupling constant satisfies

$$0 \le \eta \le \frac{3^{\frac{3}{4}}}{2} \equiv \eta_{\rm c}. \tag{105}$$

The corresponding ranges of the radius of the photon sphere and antiphoton sphere are

$$r_{\rm m} \in [\sqrt{3}m_\bullet, 3m_\bullet], \tag{106}$$

$$r_{\rm a} \in [0, \sqrt{3}m_\bullet]. \tag{107}$$

When the photon sphere and throat merge into a marginally unstable photon sphere at $r_{\rm m} = r_{\rm th} = r_{\rm mth}$, with the help of Eqs. (100)–(102) and (105), we can find

$$\lambda_{\rm mth} \in \left[1 - \frac{\sqrt{3}}{3}, 1 - \frac{1}{3}\sqrt{6\sqrt{3}-9}\right) \tag{108}$$

and

$$\eta_{\rm mth} = \sqrt{\frac{9\lambda^4 - 36\lambda^3 + 54\lambda^2 - 36\lambda + 8}{\lambda^3(\lambda-2)^3}}. \tag{109}$$

For some specific $\lambda_{\rm mth}$ and $\eta_{\rm mth}$, we can obtain the radius of the marginally unstable photon sphere $r_{\rm mth}$. In addition, there is another marginally unstable photon sphere of the wormhole in the beyond Horndeski theory where the photon sphere, antiphoton sphere, and throat coincide at $r_{\rm mat} = \sqrt{3}m_\bullet$ for $\eta_{\rm mat} = \eta_{\rm c}$ and $\lambda_{\rm mat} = 1 - \sqrt{6\sqrt{3}-9}/3$.

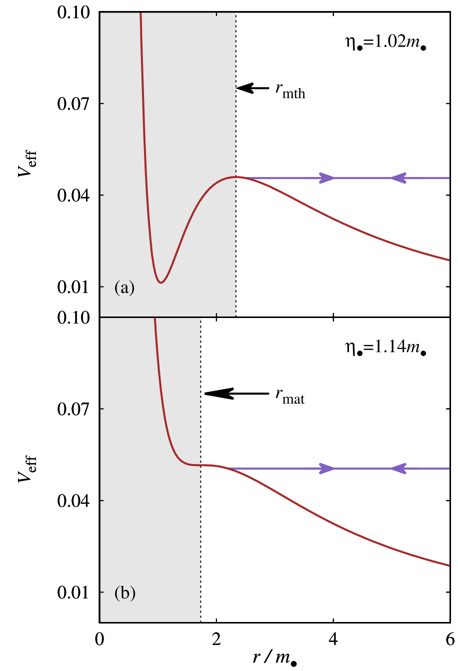

FIG. 1. The effective potential of a photon near the marginally unstable photon sphere formed either by the merger of the photon sphere and throat (top) or by the coincidence of the photon sphere, antiphoton sphere, and throat (bottom).

The effective potential of a photon near these two marginally unstable photon spheres are shown in Fig. 1.

B. Observables for the marginally unstable photon sphere at $r_{\rm mth}$

Assuming the supermassive black hole Sgr A* in the Galactic Center is a wormhole in the beyond Horndeski theory and using the method introduced in Sec. II A, we can calculate the deflection angle and its coefficients $\bar{a}_{\rm mth}$ and $\bar{b}_{\rm mth}$ for the strong deflection gravitational lensing near the marginally unstable photon sphere when the photon sphere and throat merge, where the mass of the wormhole and the distance between the lens and observer are $m_\bullet = 4.26 \times 10^6 M_\odot$, $D_{\rm OL} = 8.25$ kpc [88]. Its observables in the strong deflection limit can be obtained by using Eqs. (85), (92), and (93) in Sec. II C. Figure 2 shows the angular radius of shadow θ_∞, the angular separation s and magnitude difference Δm between the outmost relativistic image and the rest. The angular radius of the shadow decreases with dimensionless constant λ down to 22.5 μas. It deviates from the one of the Schwarzschild black hole more than 4 μas, which might be detectable by EHT's observation on the shadow of Sgr A* [13]. With the increasing of the dimensionless parameter λ, the angular separation s could grow to 263 nas and drop after that. It is too small to detect for current technology. The magnitude difference Δm decreases with λ, whose minimum could reach −2.1 mag and maximum deviation with the

Schwarzschild black hole approaches −9 mag. However, due to the unresloved relativistic images, it is impossible to detect this observable now.

C. Observables for the marginally unstable photon sphere at r_{mat}

For the marginally unstable photon sphere at r_{mat}, by using the method introduced in Sec. II B, we can calculate the coefficients as

$$\bar{a}_{\mathrm{mat}} = (2^{\frac{3}{4}}3^{\frac{17}{24}} + 2^{\frac{1}{4}}3^{\frac{11}{24}} - 2^{\frac{1}{4}}3^{\frac{23}{24}})\Delta \approx 2.766, \tag{110}$$

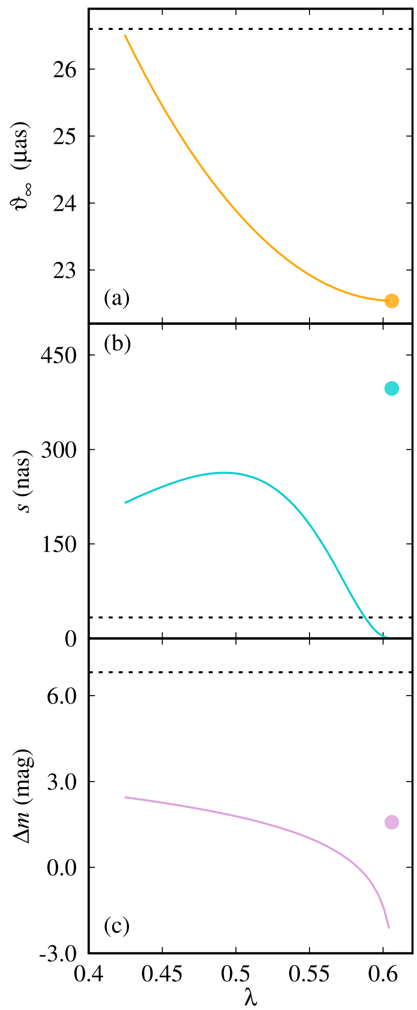

FIG. 2. Assuming Sgr A* is a wormhole in the beyond Horndeski theory, it shows the angular radius of shadow θ_∞, the angular separation s, and the magnitude difference Δm between the outmost relativistic image and the rest for the strong deflection gravitational lensing near the marginally unstable photon sphere that is merged by the photon sphere and throat (line) or coincided by the photon sphere, antiphoton sphere, and throat (circle). The black dash lines represent the corresponding observables of the Schwarzschild black hole.

$$\bar{b}_{\mathrm{mat}} \approx -4.345, \tag{111}$$

where

$$\Delta = \frac{1}{2}\left(\frac{7}{6}\sqrt[4]{27}\sqrt{2} + 2\sqrt{2}\sqrt[4]{3} + \frac{5}{6}\sqrt{3} + \frac{3}{2}\right)^{\frac{1}{2}}(\sqrt{3}-1)^{\frac{5}{6}}. \tag{112}$$

Taking Sgr A* as an example, the angular radius of shadow θ_∞, the angular separation s and magnitude difference Δm between the outmost relativistic image and the packed ones are shown in Fig. 2. We find that the radius of the shadow formed by this marginally unstable photon sphere as $\theta_\infty = 22.5$ μas, which also deviates from the one of the Schwarzschild black hole more than 4 μas. The angular separation $s = 397.3$ nas and magnitude difference $\Delta m = 1.58$ mag, and their deviation from those of the Schwarzschild black hole $\delta s = 364$ nas and $\delta\Delta m = -5.24$ mag, respectively. However, because of the limited resolution, it is impossible to detect these two observables currently.

IV. CONCLUSIONS

A static, spherically symmetric wormhole might have two new kinds of marginally unstable photon spheres, which could be formed either by the merger of the photon sphere and throat or by the coincidence of the photon sphere, antiphoton sphere, and throat. We investigate the strong deflection gravitational lensing near these marginally unstable photon spheres and find that their deflection angles diverge in a manner of a power law with a specific exponent, rather than by the well-known logarithmic law for the photon sphere. We obtain the analytic expressions of the observables.

Applying our method to the wormhole in the beyond Horndeski theory and taking Sgr A* as the lens, we obtain the observables in strong deflection limit near these two marginally unstable photon spheres. Our results indicate that the angular radius of the shadow of the wormhole in the beyond Horndeski theory can be measured now, and its deviation from the one of the Schwarzschild black hole might be detected by EHT.

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APPENDIX: COEFFICIENTS IN THE EXPANSION OF $G(z, r_0)$

In the expansion of the function $G(z, r_0)$ near $z = 0$, $\bar{V}_{\rm eff}{}^{(i)}$ and other parameters are

$$\bar{V}'_{\rm eff}(0, r_0) = -V'_{\rm eff}(r_0)\bar{B}(r_0)V^{-1}_{\rm eff}(r_0), \tag{A1}$$

$$\bar{V}''_{\rm eff}(0, r_0) = -[V''_{\rm eff}(r_0)\bar{B}(r_0) + 2V'_{\rm eff}(r_0)\bar{B}'(r_0)]V^{-1}_{\rm eff}(r_0) + 2V'_{\rm eff}{}^2(r_0)\bar{B}(r_0)V^{-2}_{\rm eff}(r_0), \tag{A2}$$

$$\begin{aligned}\bar{V}'''_{\rm eff}(0, r_0) = &-[V'''_{\rm eff}(r_0)\bar{B}(r_0) + 3V''_{\rm eff}(r_0)\bar{B}'(r_0) + 3V'_{\rm eff}(r_0)\bar{B}''(r_0)]V^{-1}_{\rm eff}(r_0)\\ &+ 6V'_{\rm eff}(r_0)[V''_{\rm eff}(r_0)\bar{B}(r_0) + V'_{\rm eff}(r_0)\bar{B}'(r_0)]V^{-2}_{\rm eff}(r_0) - 6V'_{\rm eff}{}^3(r_0)\bar{B}(r_0)V^{-3}_{\rm eff}(r_0),\end{aligned} \tag{A3}$$

$$\begin{aligned}\bar{V}^{(4)}_{\rm eff}(0, r_0) = &-[V^{(4)}_{\rm eff}(r_0)\bar{B}(r_0) + 4V'''_{\rm eff}(r_0)\bar{B}'(r_0) + 6V''_{\rm eff}(r_0)\bar{B}''(r_0) + 4V'_{\rm eff}(r_0)\bar{B}'''(r_0)]V^{-1}_{\rm eff}(r_0)\\ &+ [8V'''_{\rm eff}(r_0)V'_{\rm eff}(r_0)\bar{B}(r_0) + 24V''_{\rm eff}(r_0)V'_{\rm eff}(r_0)\bar{B}'(r_0) + 12V'_{\rm eff}{}^2(r_0)\bar{B}''(r_0)\\ &+ 6V''_{\rm eff}{}^2\bar{B}(r_0)]V^{-2}_{\rm eff}(r_0) - 12V'_{\rm eff}{}^2(r_0)[3V''_{\rm eff}(r_0)\bar{B}(r_0) + 2V'_{\rm eff}(r_0)\bar{B}'(r_0)]V^{-3}_{\rm eff}(r_0)\\ &+ 24V'_{\rm eff}{}^4(r_0)\bar{B}(r_0)V^{-4}_{\rm eff}(r_0),\end{aligned} \tag{A4}$$

$$\sigma_1 = r_0C_0, \tag{A5}$$

$$\eta_1 = r_0^2C'_0 - 3r_0C_0, \tag{A6}$$

$$\eta_2 = \frac{1}{2}r_0^2C_0, \tag{A7}$$

$$\gamma_1 = 3r_0C_0 - 2r_0^2C'_0 + \frac{1}{2}r_0^3C''_0, \tag{A8}$$

$$\gamma_2 = -r_0^2C_0 + \frac{1}{2}r_0^3C'_0, \tag{A9}$$

$$\gamma_3 = \frac{1}{6}r_0^3C_0, \tag{A10}$$

$$\kappa_1 = -r_0C_0 + r_0^2C'_0 - \frac{1}{2}r_0^3C''_0 + \frac{1}{6}r_0^4C'''_0, \tag{A11}$$

$$\kappa_2 = \frac{1}{2}r_0^2C_0 - \frac{1}{2}r_0^3C'_0 + \frac{1}{4}r_0^4C''_0, \tag{A12}$$

$$\kappa_3 = -\frac{1}{6}r_0^3C_0 + \frac{1}{6}r_0^4C'_0, \tag{A13}$$

$$\kappa_4 = \frac{1}{24}r_0^4C_0. \tag{A14}$$

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