Symmetry analysis involving meson mixing for charmonium decay

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(Received 4 January 2024; accepted 6 February 2024; published 28 February 2024)

For charmonium's decaying to the final states involving merely light quarks, in light of $SU(3)$ flavor symmetry, the effective interaction Hamiltonian in tensor form is obtained by virtue of group representation theory. The strong and electromagnetic breaking effects are treated as a spurion octet so that the flavor singlet principle can be utilized as the criterion to determine the form of effective Hamiltonian for all charmonium two body decays. Moreover, a synthetic nonet is introduced to include both octet and singlet representations for meson description, and resorting to the mixing angle, the pure octet and singlet states are combined into the observable pseudoscalar and vector particles, so that the empirically effective Hamiltonian can be obtained in a concise way. As an application, by virtue of this scenario the relative phase between the strong and electromagnetic amplitudes is studied for vector-pseudoscalar meson final state. In data analysis of samples taken in an e^+e^- collider, the details of experimental effects, such as energy spread and initial state radiative correction, are taken into consideration in order to make full use of experimental information and acquire accurate and delicate results.

DOI: 10.1103/PhysRevD.109.036036

I. INTRODUCTION

Since the upgraded Beijing Electron-Positron Collider (BEPCII) and spectrometer detector (BESIII) started data taking in 2008 [1,2], colossal charm and charmonium data samples in the world were collected, especially the data at J/ψ and ψ' resonance peaks, which provide an unprecedented opportunity to acquire useful information for understanding the interaction dynamics by analyzing various decay final states.

Although the Standard Model (SM) has been accepted as a universally appreciated theory basis in the high energy community, it is still hard to calculate the wanted experimental observable from the first principle of the SM for a great many of processes, especially when the strong interaction is involved. Quantum chromodynamics (QCD) as a widely appreciated theory of strong interaction, has been proved to be very successful at high energy when the calculation can be executed perturbatively. Nevertheless, its validity at the nonperturbative regime, such as J/ψ and ψ' resonance regions, needs more experimental guidance. As an exploratory step, it is necessary to develop a reliable and extensively applicable phenomenological model (PM).

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The advantage of PM lies in that a well-defined PM contains few experimentally determined parameters that have a clear physical meaning; moreover, with only few parameters determined from experiment, PM could produce concrete results, which can be directly confirmed or falsified by experiment and may guide further experimental searches. Such a model has a good relation with the elementary principle of the theory and if applicable, can be used for further theoretical refinement. This point is noteworthy for the time being since the general QCD can hardly provide solutions for special problems; conversely, we have to establish certain effective empirical model to advance our understanding for a generic QCD principle.

As a matter of fact, many models are constructed for charmonium decay [3–12], and the parametrization of various decay modes are obtained, such as the pseudoscalar and pseudoscalar mesons (PP), vector and pseudoscalar mesons (VP), octet baryon-pair, and so on. Especially in Ref. [12], by virtue of $SU(3)$ flavor symmetry, the effective interaction Hamiltonian in the tensor form is obtained according to group representation theory. In the light of flavor singlet principle, systematical parametrization is realized for all charmonium two-body decays, including both baryonic and mesonic final states. The parametrizations of ψ' or J/ψ decaying to an octet baryon pair, decuplet baryon pair, decuplet-octet baryon final state, vector-pseudoscalar meson final state, and pseudoscalar-pseudoscalar meson final state are presented. However, in the previous study [12], mesons are merely treated as pure octet states while the actual particles are the mixing of both octet and singlet states. Therefore, this paper concentrates on the meson mixing issue. With the nonet concept and mixing angle, both octet and singlet states are mixed into mass eigenvalue states so that the pragmatically effective Hamiltonian can be constructed in a concise way. Such a treatment finalizes the parametrization scheme proposed in the previous symmetry analysis of charmonium decay.

In the next section, the parametrization scheme will be expounded first, a synthetic nonet is introduced to combine both octet and singlet representations for meson description, then the effective Hamiltonian is obtained consecutively. The section that follows discusses in detail the experimental character of the e^+e^- collider, then the successive section focuses on concrete data analysis for VP final states. A special section is used to discuss another fit method and some open questions on understanding fit results. After that is a summary section. Relegated to the Appendix are two kinds of materials that are mainly concerned on calculation details.

II. ANALYSIS FRAMEWORK

In the e^+e^- collider experiment, the initial state is obviously flavorless, then the final state must be flavor singlet. Moreover, only the Okubo-Zweig-Iizuka (OZI) rule suppressed processes are considered, and the final states merely involve light quarks, that is u, d, s quarks. Therefore, the $SU(3)$ group is employed for symmetry analysis. The key rule herein is the so-called "flavor singlet principle" that determines what kinds of terms are permitted in the effective interaction Hamiltonian. Resorting to the perturbation language, the Hamiltonian is written as

$$\mathcal{H}_{\rm eff} = H_0 + \Delta H, \tag{1}$$

where H_0 is the symmetry conserved term and ΔH the symmetry breaking term, which is generally small compare to H_0. Since we focus on two-body decay, merely two multiplets, say $\mathbf{n}$ and $\mathbf{m}$, need to be considered. In the light of group representation theory, the product of two multiplets can be decomposed into a series of irreducible representations, that is

$$\mathbf{n} \otimes \mathbf{m} = \mathbf{l}_1 \oplus \mathbf{l}_2 \oplus \cdots \oplus \mathbf{l}_k. \tag{2}$$

The singlet principle requires that among the $\mathbf{l}_j (j = 1, \ldots, k)$, only the singlet term, i.e., $\mathbf{l}_j = \mathbf{1}$ for certain j, is allowed in the Hamiltonian. Since this term is obviously $SU(3)$ invariant, it is called the symmetry conserved term, i.e., H_0.

Now turn to the $SU(3)$-breaking term. Following the recipe of the proceeding study [13–15], the $SU(3)$-breaking effect is treated as a "spurion" octet. With this notion, in order to pin down the breaking term in the Hamiltonian, the products of this spurion octet with the irreducible representations $\mathbf{l}_j (j = 1, \ldots, k)$ will be scrutinized; only the singlet term in the decomposition is valid in the Hamiltonian. Concretely,

$$\mathbf{l}_j \otimes \mathbf{8} = \mathbf{q}_1 \oplus \mathbf{q}_2 \oplus \cdots \oplus \mathbf{q}_k, \tag{3}$$

then if and only if $\mathbf{q}_i = 1$, the corresponding term is allowed in the Hamiltonian. Since such a kind of term violates $SU(3)$ invariance, it is called the symmetry breaking term. In a word, with the singlet principle, the effective interaction Hamiltonian can be determined unequivocally.

Generally speaking, the Hamiltonian term should be written as $\psi M_1 M_2$, where ψ indicates the charmonium state and M_1 and M_2 are two multiplet components. Since ψ is the same for the whole final state, and the relative strength of the multiplet final state really matters, ψ dependence is left implicit henceforth.

The tensor denotation is adopted in order to express all kinds of multiplets consistently. A tensor of rank (u, v) reads

$$T^{i_1 i_2 \cdots i_u}_{j_1 j_2 \cdots j_v},$$

or denoted as $T(u, v)$. The irreducible tensor of $SU(3)$ representation is traceless and totally symmetric in indices of the same type. The number of independent components in a multiplet is called its dimension, which is calculated by the formula,

$$d\{T(u, v)\} = \frac{1}{2}(u+1)(v+1)(u+v+2). \tag{4}$$

The baryon and meson of a multiplet can be expressed by the corresponding components of the irreducible tensor, the characteristics of which can be used to calculate the eigenvalues of particle charge (Q), hypercharge (Y), and the third component of isospin (I_3), that is

$$\begin{aligned} Q &= u_1 - v_1 - \frac{1}{3}(u - v), \\ Y &= -u_3 + v_3 + \frac{1}{3}(u - v), \\ I_3 &= \frac{1}{2}(u_1 - u_2) - \frac{1}{2}(v_1 - v_2), \end{aligned} \tag{5}$$

where u_i denotes the number of upper indices with value i, and v_j denotes the number of lower indices with value j. For $SU(3)$ group, i, $j = 1, 2, 3$ and $u = u_1 + u_2 + u_3$ and $v = v_1 + v_2 + v_3$. The values (Q, Y, I_3) of the tensor component indicate the corresponding physical particle; as examples, some commonly used octets of baryon and meson are displayed as follows [16–18]:

$$\mathbf{B}=\begin{pmatrix}\Sigma^0/\sqrt{2}+\Lambda/\sqrt{6} & \Sigma^+ & p\\ \Sigma^- & -\Sigma^0/\sqrt{2}+\Lambda/\sqrt{6} & n\\ \Xi^- & \Xi^0 & -2\Lambda/\sqrt{6}\end{pmatrix}, \tag{6}$$

$$\bar{\mathbf{B}}=\begin{pmatrix}\bar{\Sigma}^0/\sqrt{2}+\overline{\Lambda/}\sqrt{6} & \bar{\Sigma}^+ & \bar{\Xi}^+\\ \bar{\Sigma}^- & -\bar{\Sigma}^0/\sqrt{2}+\overline{\Lambda/}\sqrt{6} & \bar{\Xi}^0\\ \bar{p} & \bar{n} & -2\bar{\Lambda}/\sqrt{6}\end{pmatrix}, \tag{7}$$

$$\mathbf{V}=\begin{pmatrix}\rho^0/\sqrt{2}+\omega/\sqrt{6} & \rho^+ & K^{*+}\\ \rho^- & -\rho^0/\sqrt{2}+\omega/\sqrt{6} & K^{*0}\\ K^{*-} & \bar{K}^{*0} & -2\omega/\sqrt{6}\end{pmatrix}, \tag{8}$$

and

$$\mathbf{P}=\begin{pmatrix}\pi^0/\sqrt{2}+\eta/\sqrt{6} & \pi^+ & K^+\\ \pi^- & -\pi^0/\sqrt{2}+\eta/\sqrt{6} & K^0\\ K^- & \bar{K}^0 & -2\eta/\sqrt{6}\end{pmatrix}. \tag{9}$$

The corresponding tensor notations are, respectively, B^i_j, $\bar{B}^i_j$, V^i_j, and P^i_j, where the superscript denotes the row index of matrix and the subscript the column index.

The $SU(3)$-breaking effect is treated as a "spurion" octet; its additive quantum numbers, such as Q, Y, I_3, are set to be 0 while the multiplicative quantum numbers are set to be 1. Specially speaking, there are two kinds of spurion octet; one is denoted as $\mathbf{S}_m$ that indicates the strong breaking effect and is I-spin conservation breaking, and the other is denoted as $\mathbf{S}_e$ that indicates the electromagnetic breaking effect and is U-spin conservation breaking. The tensor expressions for these two special octets read

$$(\mathbf{S}_m)^i_j = g_m\delta^i_3\delta^3_j, \tag{10}$$

and

$$(\mathbf{S}_e)^i_j = g_e\delta^i_1\delta^1_j. \tag{11}$$

It is noted that if u and v are switched in Eq. (5), the values of Q, Y, I_3 just change their signs, which means a particle turns into its antiparticle. If the charge conjugate operator is denoted as $\hat{C}$, then

$$\hat{C}T(u,v)\to T(v,u).$$

This can be confirmed for $\mathbf{V}$ and $\mathbf{P}$. But for a baryon, since the particle and its antiparticle are in a different multiplet, the charge conjugate not only changes the signs of values (Q, Y, I_3) but also keeps the mass of particle unchanged. According to this requirement, it yields

$$\hat{C}T(u,v)\to \bar{T}(v,u),$$

which can be confirmed for $\mathbf{B}$ and $\bar{\mathbf{B}}$.

Since the effective Hamiltonian must be $\hat{C}$-parity conserved, it should be unchanged under $\hat{C}$-parity transformation. This will constrain the form of the effective Hamiltonian. Let us take the baryon octet as an example. According to group theory, the product of two octets can be reduced as follows:

$$\mathbf{8}\otimes\mathbf{8}=\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}\oplus\mathbf{10}^*\oplus\mathbf{27}. \tag{12}$$

The singlet (denoted as $[\bar{B}B]_0$) building from two octets reads

$$[\bar{B}B]_0 = \bar{B}^i_j B^j_i, \tag{13}$$

Here, Einstein summation convention is adopted, that is the repeated suffix, once as a subscript and once as a superscript, implies the summation. Under $\hat{C}$-parity transformation, $\hat{C}[\bar{B}B]_0 \to [B\bar{B}]_0$. In order to keep $\hat{C}$ parity, the effective Hamiltonian should take the form,

$$H_0 = g_0\cdot([\bar{B}B]_0 + [B\bar{B}]_0). \tag{14}$$

Next, by virtue of Eq. (12), there are two types of octet: an antisymmetric, or f type, and a symmetric, or d type, which are respectively constructed as

$$([\bar{B}B]_f)^i_j = \bar{B}^i_k B^k_j - \bar{B}^k_j B^i_k, \tag{15}$$

and

$$([\bar{B}B]_d)^i_j = \bar{B}^i_k B^k_j + \bar{B}^k_j B^i_k - \frac{2}{3}\delta^i_j\cdot\bar{B}^i_j B^j_i. \tag{16}$$

Under $\hat{C}$-parity transformation, $\hat{C}([\bar{B}B]_d)^i_j \to -([B\bar{B}]_d)^j_i$ and $\hat{C}([\bar{B}B]_f)^i_j \to ([B\bar{B}]_f)^j_i$. Here, the eigenvalues $\xi_{d,f}$ with values $\xi_d=+1$ and $\xi_f=-1$ (obviously $\xi_0=+1$) can be introduced to describe such effects. So in the effective Hamiltonian, the two terms involving $[\bar{B}B]_f$ and $[\bar{B}B]_d$ should have the forms,

$$[\bar{B}B]_{d,f} + \xi_{d,f}[B\bar{B}]_{d,f}.$$

Nevertheless, the C parity does not change the cross section of certain process; the terms with and without C-parity transformation furnish the same measurement results; as far as the relative strength is concerned, it proves expedient to keep only one kind of term. Therefore, the effective interaction Hamiltonian for octet-octet final state reads

$$\mathcal{H}_{\text{eff}} = g_0 \cdot [\bar{B}B]_0 + g_m \cdot ([\bar{B}B]_f)^3_3 + g'_m \cdot ([\bar{B}B]_d)^3_3 + g_e \cdot ([\bar{B}B]_f)^1_1 + g'_e \cdot ([\bar{B}B]_d)^1_1. \tag{17}$$

The other results for decuplet-decuplet and decuplet-octet final states can be obtained similarly; the detailed information can be referred to Refs. [11,12], together with the corresponding parametrization forms.

The effective Hamiltonian for meson final state can be obtained in the similar way, say, for **VP**, it reads

$$\mathcal{H}^{VP}_{\text{eff}} = g_0 \cdot [VP]_0 + g_m \cdot ([VP]_f)^3_3 + g'_m \cdot ([VP]_d)^3_3 + g_e \cdot ([VP]_f)^1_1 + g'_e \cdot ([VP]_d)^1_1. \tag{18}$$

However, unlike a baryon, the particle and antiparticle are contained in the same meson multiplet, so $\hat{C}\mathbf{V}/\mathbf{P} \to \mathbf{V}/\mathbf{P}$, which like the neutral particle such as π^0 with $\hat{C}\pi^0 \to \pi^0$. It is known that π^0 has the inherent C parity, that is $\hat{C}\pi^0 = \eta_{\pi^0}\pi^0$, with $\eta_{\pi^0} = +1$. Herein, the generalized inherent C parity for a multiplet is introduced, and its value is set to be equal to that of the neutral particle in the multiplet. That is to say,

$$\hat{C}\mathbf{P} = \eta_P \mathbf{P}, \quad \text{with} \quad \eta_P = \eta_{\pi^0} = +1;$$
$$\hat{C}\mathbf{V} = \eta_V \mathbf{V}, \quad \text{with} \quad \eta_V = \eta_{\rho^0} = -1. \tag{19}$$

With these conventions, it has

$$\hat{C}[VP]_f = \eta_V \eta_P \xi_f [VP]_f,$$
$$\hat{C}[VP]_d = \eta_V \eta_P \xi_d [VP]_d. \tag{20}$$

The invariant of the effective Hamiltonian under C-parity transformation requires that $\eta_V\eta_P\xi_f$ or $\eta_V\eta_P\xi_d$ must be equal to $\eta_{J/\psi,\psi'}$. Since $\eta_{J/\psi,\psi'} = -1$, only $\eta_{J/\psi,\psi'} = \eta_V\eta_P\xi_d$ is allowed, which means only the term $[VP]_d$ can exist in the effective Hamiltonian of **VP** final state, that is

$$\mathcal{H}^{VP}_{\text{eff}} = g_0 \cdot [VP]_0 + g_m \cdot ([VP]_d)^3_3 + g_e \cdot ([VP]_d)^1_1. \tag{21}$$

By virtue of the same criterion, the effective Hamiltonian of the **PP** final state reads

$$\mathcal{H}^{PP}_{\text{eff}} = g_m \cdot ([PP]_f)^3_3 + g_e \cdot ([PP]_f)^1_1. \tag{22}$$

With the components given in Eqs. (8) and (9), the corresponding parametrization can be obtained and summarized in Tables I and II, respectively. For the other octet final states, the parametrization scheme can be referred to the Appendix.

In light of Eq. (21), the first term $g_0 \cdot [VP]_0$ represents the $SU(3)$-conserved effect, while in Eq. (22) only the $SU(3)$-breaking terms exist. Therefore, for VP-like final states, the decay branching fractions are generally greater than those of PP-like final states.

TABLE I. Amplitude parametrization form for decays of the ψ' or J/ψ into VP final states. General expressions in terms of singlet A (by definition $A = g_0$), as well as charge-breaking ($D = g_e/3$) and mass-breaking terms ($D' = g_m/3$). The table can also be used for more similar decays by appropriate change in labeling, refer to the Appendix for more details.

Final state	Amplitude parametrization form
$\rho^\pm\pi^\mp$, $\rho^0\pi^0$	$A + D - 2D'$
$K^{*\pm}K^\mp$	$A + D + D'$
$K^{*0}\bar{K}^0$, $\bar{K}^{*0}K^0$	$A - 2D + D'$
$\omega\eta$	$A - D + 2D'$
$\omega\pi^0$	$\sqrt{3}D$
$\rho^0\eta$	$\sqrt{3}D$

TABLE II. Amplitude parametrization form for decays of the ψ' or J/ψ into PP final states. General expressions in terms of the charge-breaking one ($D = 2g_e$) and the mass-breaking one ($D' = -2g_m$). The table can also be used for more similar decays by appropriate change in labeling, refer to the Appendix for more details.

Final state	Amplitude parametrization form
$\pi^+\pi^-$	D
K^+K^-	$D + D'$
$K^0\bar{K}^0$	D'

In above analysis, it is prominent that in Eqs. (8) and (9), the mesons ω and η are treated as pure octet components, but the real or observable ω and η are actually the mixing of pure octet and singlet components with a certain mixing angle. According to PDG [19],

$$\phi = \omega^8 \cos\theta_V - \omega^1 \sin\theta_V,$$
$$\omega = \omega^8 \sin\theta_V + \omega^1 \cos\theta_V, \tag{23}$$

or its reverse,

$$\omega^8 = \cos\theta_V \phi + \sin\theta_V \omega,$$
$$\omega^1 = -\sin\theta_V \phi + \cos\theta_V \omega, \tag{24}$$

and

$$\eta = \eta^8 \cos\theta_P - \eta^1 \sin\theta_P,$$
$$\eta' = \eta^8 \sin\theta_P + \eta^1 \cos\theta_P, \tag{25}$$

or its reverse,

$$\eta^8 = \cos\theta_P\eta + \sin\theta_P\eta',$$
$$\eta^1 = -\sin\theta_P\eta + \cos\theta_P\eta'. \tag{26}$$

In above equations, the superscript 8 indicates the pure octet component and the superscript 1 the pure singlet component.

In order to obtain the actual effective Hamiltonian, both octet and singlet components are to be included. The concise and effective way is to introduce a nonet that merges a singlet with an octet [5]. Under such circumstances, the matrices in Eqs. (8) and (9) are recast as

$$\mathbf{V}_N = \begin{pmatrix} V_1^1 & \rho^+ & K^{*+} \\ \rho^- & V_2^2 & K^{*0} \\ K^{*-} & \bar{K}^{*0} & V_3^3 \end{pmatrix}, \tag{27}$$

with

$$V_1^1 = \rho^0/\sqrt{2} + \omega^8/\sqrt{6} + \omega^1/\sqrt{3},$$
$$V_2^2 = -\rho^0/\sqrt{2} + \omega^8/\sqrt{6} + \omega^1/\sqrt{3},$$
$$V_3^3 = -2\omega^8/\sqrt{6} + \omega^1/\sqrt{3}; \tag{28}$$

and

TABLE III. Amplitude parametrization form for decays of the ψ' or J/ψ into VP final states. General expressions in terms of singlet A (by definition $A = g_0$), as well as charge-breaking ($D = g_e/\sqrt{3}$) and mass-breaking terms ($D' = 2g_m/\sqrt{3}$). The shorthand symbols are defined as $s_\alpha = \sin\theta_\alpha$, $c_\alpha = \cos\theta_\alpha$, $s^\pm_{\alpha\beta} = \sin(\theta_\alpha \pm \theta_\beta)$, $c^\pm_{\alpha\beta} = \cos(\theta_\alpha \pm \theta_\beta)$, $s_{\gamma\alpha\beta} = \sin(\theta_\gamma + \theta_\alpha + \theta_\beta)$, $c_{\gamma\alpha\beta} = \cos(\theta_\gamma + \theta_\alpha + \theta_\beta)$, $s_\gamma = \sin\theta_\gamma \equiv \sqrt{1/3}$, and $c_\gamma = \cos\theta_\gamma \equiv \sqrt{2/3}$. It should be noted that the definitions of D and D' herein are different from those in Table I.

States	A	D	D'
$\rho^0\pi^0$	1	$1/\sqrt{3}$	$-1/\sqrt{3}$
$\rho^+\pi^-$	1	$1/\sqrt{3}$	$-1/\sqrt{3}$
$\rho^+\pi^-$	1	$1/\sqrt{3}$	$-1/\sqrt{3}$
$K^{*+}K^-$	1	$1/\sqrt{3}$	$1/2\sqrt{3}$
$K^{*-}K^+$	1	$1/\sqrt{3}$	$1/2\sqrt{3}$
$K^{*0}\bar{K}^0$	1	$-2/\sqrt{3}$	$1/2\sqrt{3}$
$\bar{K}^{*0}K^0$	1	$-2/\sqrt{3}$	$1/2\sqrt{3}$
$\phi\eta$	c^-_{VP}	$-s_{\gamma VP} - s_\gamma s_V s_P$	$s_{\gamma VP} + s_\gamma s_V s_P$
$\phi\eta'$	$-s^-_{VP}$	$c_{\gamma VP} + s_\gamma s_V c_P$	$-c_{\gamma VP} - s_\gamma s_V c_P$
$\omega\eta$	s^-_{VP}	$c_{\gamma VP} + s_\gamma c_V s_P$	$-c_{\gamma VP} - s_\gamma c_V c_P$
$\omega\eta'$	c^-_{VP}	$s_{\gamma VP} - s_\gamma c_V s_P$	$-s_{\gamma VP} + s_\gamma c_V c_P$
$\rho^0\eta$	0	$\sqrt{3}\cdot s^-_{\gamma P}$	0
$\rho^0\eta'$	0	$\sqrt{3}\cdot c^-_{\gamma P}$	0
$\phi\pi^0$	0	$\sqrt{3}\cdot s^-_{\gamma V}$	0
$\omega\pi^0$	0	$\sqrt{3}\cdot c^-_{\gamma V}$	0

$$\mathbf{P}_N = \begin{pmatrix} P_1^1 & \pi^+ & K^+ \\ \pi^- & P_2^2 & K^0 \\ K^- & \bar{K}^0 & P_3^3 \end{pmatrix}, \tag{29}$$

with

$$P_1^1 = \pi^0/\sqrt{2} + \eta^8/\sqrt{6} + \eta^1/\sqrt{3},$$
$$P_2^2 = -\pi^0/\sqrt{2} + \eta^8/\sqrt{6} + \eta^1/\sqrt{3},$$
$$P_3^3 = -2\eta^8/\sqrt{6} + \eta^1/\sqrt{3}. \tag{30}$$

With these new nonets $\mathbf{V}_N$ and $\mathbf{P}_N$, the effective Hamiltonian in Eq. (21) can be used formally to acquire the corresponding parametrization for **VP** final states, with both octet components, say ω^8 and η^8, and singlet components, say ω^1 and η^1. Then with Eqs. (24) and (26), the pure octet and singlet components ω^8, ω^1, η^8, and η^1 will be replaced by the actual particles ω, ϕ, η, and η'. The final parametrization results are summarized in Table III.

Besides the nonet approach, the singlet component can be treated separately, and the final results are essentially the same. The details are degenerated into the Appendix.

III. EXPERIMENTAL SECTION

An electron-positron collider experiment has its special character. When analyzing the data taken in e^+e^- collider, the important experimental effects, such as the initial state radiative (ISR) correction and the effect due to energy spread of accelerator, must be dealt with carefully.

A. Born section

For e^+e^- colliding experiments, there is the inevitable continuum amplitude [20],

$$e^+e^- \to \gamma^* \to \text{hadrons}$$

which may produce the same final state as the resonance does. In $e^+e^- \to \mathbf{VP}$ at J/ψ or ψ' resonance, the Born order cross section for the final state f is [21–25]

$$\sigma_{\text{Born}} = \frac{4\pi\alpha^2}{s^{3/2}}|A_f(s)|^2\mathcal{P}_f(s), \tag{31}$$

where $\mathcal{P}_f(s) = q_f^3/3$, with q_f being the momentum of either the **V** or the **P** particle, viz.

$$q_f = \frac{[(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)]^{1/2}}{2\sqrt{s}}, \tag{32}$$

where $\sqrt{s}$ is the center of mass energy, m_1 and m_2 are the masses of two **VP** final state mesons. The total amplitude reads

$$A_f(s) = a_{3g}(s) + a_\gamma(s) + a_c(s), \tag{33}$$

which consists of three kinds of amplitudes corresponding to (a) the strong interaction $[a_{3g}(s)]$ presumably through three-gluon annihilation, (b) the electromagnetic interaction $[a_\gamma(s)]$ through the annihilation of $c\bar{c}$ pair into a virtual photon, and (c) the electromagnetic interaction $[a_c(s)]$ due to one-photon continuum process. For the **VP** final state, the amplitudes have the forms,

$$a_c(s) = Y_f \cdot \mathcal{F}(s), \tag{34}$$

$$a_\gamma(s) = Y_f \cdot B(s) \cdot \mathcal{F}(s), \tag{35}$$

$$a_{3g}(s) = X_f \cdot B(s) \cdot \mathcal{F}(s), \tag{36}$$

with the definition,

$$B(s) \equiv \frac{3\sqrt{s}\Gamma_{ee}/\alpha}{s - M^2 + iM\Gamma_t}, \tag{37}$$

where α is the QED fine structure constant; M and Γ_t are the mass and the total width of the ψ' or J/ψ; Γ_{ee} is the partial width to e^+e^-. $\mathcal{F}(s)$ is the form factor and takes the form $1/s$ for **VP** final state. X_f and Y_f are the functions of the amplitude parameters of final state f, that is A, D, D', s_P, and s_V, à la Table III, viz.

$$Y = Y(D, s_P, s_V), \tag{38}$$

$$X = X(A, D', s_P, s_V)e^{i\phi}. \tag{39}$$

By virtue of Eqs. (23) and (25), $s_P = \sin\theta_P$ and $s_V = \sin\theta_V$, where θ_P and θ_V are respectively the mixing angle of pseudoscalar meson between η and η', and that of vector meson between ϕ and ω. The concrete form of X or Y depends on the decay mode, as examples, for $\rho\pi$ final state, $X_{\rho\pi} = A - D'/\sqrt{3}$ and $Y_{\rho\pi} = D/\sqrt{3}$, while for the $\omega\pi^0$ final state, $X_{\omega\pi^0} = 0$ and $Y_{\omega\pi^0} = D \cdot \sqrt{3} \cdot c_{\gamma V}^{-}$, according to the parametrization form in Table III. In principle, the parameters A, D, and D' could be complex arguments, each with a magnitude together with a phase. Conventionally, it is assumed that there is not relative phases among the strong-originated amplitudes A, D', and the sole phase [denoted by ϕ in Eq. (39)] is between the strong and electromagnetic interactions, that is between X and Y, as indicated in Eqs. (38) and (39), where A, D, and D' are actually treated as real numbers.

B. Observed section

In an e^+e^- collision, the Born order cross section is modified by the initial state radiation in the way [26],

$$\sigma_{r.c.}(s) = \int_0^{x_m} dx F(x,s) \frac{\sigma_{\rm Born}(s(1-x))}{|1-\Pi(s(1-x))|^2}, \tag{40}$$

where $x_m = 1 - s'/s$. $F(x,s)$ is the radiative function been calculated to an accuracy of 0.1% [26–28], and $\Pi(s)$ is the vacuum polarization factor. In the upper limit of the integration, $\sqrt{s'}$ is the experimentally required minimum invariant mass of the final particles. If $x_m = 1$, it corresponds to no requirement for invariant mass; if $x_m = 0.2$, it corresponds to invariant mass cut of 3.3 GeV for ψ' resonance. The concrete value of x_m should be determined by the cut of invariant mass, which is adopted in actual event selection.

By convention, Γ_{ee} has the QED vacuum polarization in its definition [29,30]. Here, it is natural to extend this convention to the partial widths of other pure electromagnetic decays, that is

$$\Gamma_f = \frac{\tilde{\Gamma}_{ee} q_f^3}{M} |\mathcal{F}(M^2)|^2, \tag{41}$$

where

$$\tilde{\Gamma}_{ee} \equiv \frac{\Gamma_{ee}}{|1-\Pi(M^2)|^2}$$

with the vacuum polarization effect included.

The e^+e^- collider has a finite energy resolution, which is much wider than the intrinsic width of narrow resonances such as ψ' and J/ψ [31,32]. Such an energy resolution is usually a Gaussian distribution [33],

$$G(W, W') = \frac{1}{\sqrt{2\pi}\Delta} e^{-\frac{(W-W')^2}{2\Delta^2}},$$

where $W = \sqrt{s}$ and Δ, a function of energy, is the standard deviation of Gaussian distribution. The experimentally observed cross section is the radiative corrected cross section folded with the energy resolution function,

$$\sigma_{\rm obs}(W) = \int_0^\infty dW' \sigma_{r.c.}(W') G(W', W). \tag{42}$$

In fact, as pointed out in Ref. [22], the radiative correction and the energy spread of the collider are two important factors, both of which reduce the height of the resonance and shift the position of the maximum cross section. Although the ISR are the same for all e^+e^- experiments, the energy spread is quite different for different accelerators, even different for the same accelerator at different running periods. As an example, for the CLEO data used in this paper, the energy spread varies due to different accelerator lattices [34]: one (for CLEO III

TABLE IV. Breakdown of experiment conditions correspond to different detectors and accelerators. The energy spread is the effective one, according to which the calculated maximum cross section satisfies the relation $N_{\rm tot} = \sigma_{\rm max} \cdot \mathcal{L}$. The number with star ($*$) is the equivalent luminosity calculated by the relation $\mathcal{L} = N_{\rm tot}/\sigma_{\rm max}$.

Detector	Accelerator	Center of mass energy spread (MeV)	Data taking position[a] (GeV)	Maximum section (nb)	Total event ($\times 10^6$)	Integral luminosity (pb^{-1})	References
BES II	BEPC	1.23	3.68623	712.9	14.0 ± 0.6	19.72	[35]
		1.23	3.65	⋯	⋯	6.42	[36]
BESIII	BECPII	1.343	3.68624	662.16	107.0 ± 0.8	161.63	[37]
		1.343	3.65	⋯	⋯	43.88	[37]
		1.318	3.68624	672.74	341.1 ± 2.1	506.92	[37]
		1.324	3.68624	670.17	448.1 ± 2.9	668.55	[37]
		1.324	3.65	⋯	⋯	48.8	[38]
CLEO-c	CESR	1.68	3.68627	557.23	3.08	5.63	[39]
		1.68	3.67	⋯	⋯	20.46	
BESIII	BECPII	1.131	3.097014	2808.63	223.7 ± 1.4	79.63	[40]
		1.131	3.08	⋯	⋯	0.282	[41]
		0.898	3.096990	3447.87	1086.9 ± 6.0	315.02	[42]
		0.937	3.096993	3320.35	1310.6 ± 7.0	394.65	[42]
		0.937	3.08	⋯	⋯	153.8	[38]
BES II	BEPC	0.85	3.09700	3631.8	57.7 ± 2.72	15.89$*$	[43]
		0.85	3.07	⋯	⋯	2.3473	[44]
DM II	DCI	1.98	3.097114	1702.0	8.6 ± 1.3	5.053$*$	[45]

[a]The data taking position is the energy which yield the maximum inclusive hadronic cross section.

detector) with a single wiggler magnet and a center-of-mass energy spread $\Delta = 1.5$ MeV, the other (for CLEOc detector) with the first half of its full complement (12) of wiggler magnets and $\Delta = 2.3$ MeV. The two Δ's lead to two maximum total cross sections 602 nb and 416 nb, respectively. All these subtle effects must be taken into account in data analysis. In the following analysis, all data are assumed to be taken at the energy point, which yields the maximum inclusive hadron cross sections instead of the nominal resonance mass [20,22]. Besides the factors considered above, the resonance parameters can also affect the evaluation results. Since the present central values of resonance parameters can be obviously distinct from those of some time before, the calculated maximum inclusive hadron cross sections will be consequently different. In order to ensure the relation $N_{\rm tot} = \sigma_{\rm max} \cdot \mathcal{L}$, some adjustments are mandatory. The principle is as follows: if the luminosity is available, the energy spread will be tuned to give a consistent maximum cross section; otherwise, the effective luminosity is evaluated by the relation $\mathcal{L} = N_{\rm tot}/\sigma_{\rm max}$ according to the corresponding accelerator parameters. All relevant experimental details are summarized in Table IV, which are crucial for the following data analysis. At last, the resonance parameters adopted in this paper for J/ψ and ψ' are respectively [19],

$$M_R = 3096.900 \pm 0.006\ \mathrm{MeV},\quad \Gamma_t = 92.9 \pm 2.8\ \mathrm{keV},\quad \Gamma_{ee} = 5.53 \pm 0.10\ \mathrm{keV}; \tag{43}$$

and

$$M_R = 3686.10 \pm 0.06\ \mathrm{MeV},\quad \Gamma_t = 294 \pm 8\ \mathrm{keV},\quad \Gamma_{ee} = 2.33 \pm 0.04\ \mathrm{keV}. \tag{44}$$

IV. DATA ANALYSIS

The Standard Model mainly consists of two parts. One is the Salam-Weinberg model that depicts the electroweak interaction, which can usually accommodate an accurate enough evaluation for certain processes. Another part is QCD, the validity of which at nonperturbative regime needs more experimental guidance. The production and decay of charmonium states benefit such a study.

As one of the important and interesting steps, it is a good start point to study the relative phase between the strong and electromagnetic (EM) interaction amplitudes, which provides us a new viewpoint to explore the quarkonium decay dynamics and profound our understanding on QCD. Studies have been carried out for many J/ψ and ψ' two-body mesonic decay modes with various spin parities: 1^-0^- [24,45–47], 0^-0^- [48–52], and 1^-1^- [52]. These analyses reveal that there exists a relative orthogonal phase between the EM and strong decay amplitudes. There is also a conjecture to claim that such an orthogonal phase is universal for all quarkonia decays [53,54].

The systematical parametrization scheme of charmonium decay modes facilitates the study of the relative phase. In Ref. [12], the phase is measured for various charmonium decay modes, including ψ' and/or J/ψ decay to octet baryon pair, decuplet baryon pair, decuplet-octet baryon final state, and pseudoscalar-pseudoscalar meson final state. In this section, the study is devoted to the vector-pseudoscalar (**VP**) meson final state according to the parametrization of Table III.

Since our analysis involves the experimental details as indicated by the description in preceding section, some measurements are not suitable in the following study due to the lack of necessary information of detectors and/or accelerators. In addition, at a different energy point, the status parameters of accelerators are also distinctive, so the studies of phase angle for ψ' and J/ψ decay are performed separately for the sake of clarity.

A. $\psi' \to VP$ decay

There are lots of measurements concerned with ψ' decaying to **VP** final states. However, the results of Ref. [55] were obtained forty years ago, and moreover, only the upper limits of $K^{*\pm}K^{\mp}$ and $\rho\pi$ final states were given based on 1 million ψ' events. The results from Ref. [56] are merely the upper limits for $\rho\pi$ and $\gamma\eta'$ final states based on 0.2 million ψ' events. Therefore, these kinds of measurements are not adopted in our study. The results of experimental measurements obtained in this century are collected in Table V, which are mainly due to CLEO and BES Collaborations.

First, let us focus on $\rho\pi$ final state. By virtue of Table V, the branching fraction of $\psi' \to \pi^+\pi^-\pi^0$ agrees very well for both CLEO and BES experiments, while that of $\psi' \to \rho\pi$

TABLE V. Experimental data of ψ' decaying to VP final states. For branching ratios, the first uncertainties are statistical, and the second systematic. The peak position is assumed at $\sqrt{s} = 3.686$ GeV, while the continuum position is assumed at $\sqrt{s} = 3.65$ GeV for BES and $\sqrt{s} = 3.67$ GeV for CLEO. The efficiency indicates the selection efficiency for resonance events. The number in parenthesis is not quoted in the original literature but is evaluated according to the information given therein.

Mode	$N^{\rm obs}$ (peak)	$N^{\rm obs}$ (continuum)	Efficiency (%)	Branching ratio ($\times 10^{-5}$)	Detector
$\pi^+\pi^-\pi^0$	7771 ± 88	220.6 ± 14.8	30.5	$2.14 \pm 0.03^{+0.08}_{-0.07}$	BESIII [41]
	216.7	85	33.5	$18.8^{+1.6}_{-1.5} \pm 2.8$	CLEO [39]
	260 ± 19	10.0 ± 4.2	9.02	$18.1 \pm 1.8 \pm 1.9$	BESII [57]
$\rho\pi$	34.4	47	28.8	$2.4^{+0.8}_{-0.7} \pm 0.2$	CLEO [39]
	64.12 ± 6.44	$\cdots$	9.02	$5.1 \pm 0.7 \pm 1.1$	BESII [57]
$K^{*+}K^- +$ c.c.	7.7	4	16.7	$1.3^{+1.0}_{-0.7} \pm 0.3$	CLEO [39]
	9.6 ± 4.2	$\cdots$	7.3	$2.9^{+1.3}_{-1.7} \pm 0.4$	BESII [58]
	224 ± 21	(54.7 ± 7.6)	20.25	$3.18 \pm 0.30^{+0.26}_{-0.31}$	BESIII [59]
$K^{*0}\bar{K}^0 +$ c.c.	34.5	36	8.7	$9.2^{+2.7}_{-2.2} \pm 0.9$	CLEO [39]
	65.6 ± 9.0	$2.5^{+2.6}_{-1.8}$	9.7	$12.3^{+2.4}_{-2.6} \pm 1.7$	BESII [58]
$\phi\eta$	6.6	3	9.4	$2.0^{+1.5}_{-1.1} \pm 0.4$	CLEO [39]
	16.7 ± 5.6	$\cdots$	18.9	$3.3 \pm 1.1 \pm 0.5$	BESII [60]
	216 ± 16	(7.0 ± 2.5)	33.53	$3.14 \pm 0.23 \pm 0.23$	BESIII [59]
$\phi\eta'$	8.4 ± 3.7	$\cdots$	8.4	$3.1 \pm 1.4 \pm 0.7$	BESII [60]
	201 ± 15	221 ± 15[a]	26.8	$1.51 \pm 0.16 \pm 0.12$	BESIII [61]
$\omega\eta$	< 0	3	10.2	< 1.1	CLEO [39]
	< 9.7	$\cdots$	6.3	< 3.1	BESII [60]
$\omega\eta'$	$4.2^{+3.2}_{-2.7}$	$\cdots$	2.3	$3.2^{+2.4}_{-2.0} \pm 0.7$	BESII [60]
$\rho^0\eta$	28.1	38	19.3	$3.0^{+1.1}_{-0.9} \pm 0.2$	CLEO [39]
	$29.2^{+7.5}_{-6.8}$	$2.3^{+2.1}_{-1.4}$	(12.06)	$1.87^{+0.68}_{-0.62} \pm 0.18$	BESII [62]
$\rho^0\eta'$	$5.4^{+3.3}_{-2.2}$	< 4.4	(4.92)	$1.87^{+1.64}_{-1.11} \pm 0.33$	BESII [62]
	(211 ± 16)[b]	5.06 ± 2.01	18.7	$1.02 \pm 0.11 \pm 0.24$	BESIII [38]
	(148 ± 18)[c]	5.06 ± 2.01	18.7	$0.569 \pm 0.128 \pm 0.236$	BESIII [38]
$\phi\pi^0$	< 0	3	15.8	< 0.7	CLEO [39]
	< 4.4	$\cdots$	16.1	< 0.4	BESII [60]
	< 6	$\cdots$	35.63	< 0.04	BESIII [59]
$\omega\pi^0$	29.1	55	19.1	$2.5^{+1.2}_{-1.0} \pm 0.2$	CLEO [39]
	$31.2^{+7.7}_{-6.9}$	$7.3^{+3.3}_{-2.7}$	(5.45)	$1.78^{+0.67}_{-0.62} \pm 0.28$	BESII [62]

[a]The number of event is obtained at $\sqrt{s} = 3.773$ GeV.
[b]The solution of destructive interference between ρ and nonresonant components.
[c]The solution of constructive interference between ρ and nonresonant components.

TABLE VI. Data of $\psi' \to VP$ decays from CLEO and BES experiments. The error is merely the statistical and the efficiency is the effective one as depicted in the text.

Mode	Energy (GeV)	$N^{\rm obs}$	Efficiency (%)	Detector
$\rho\pi$	3.686	54.24 ± 11.53	33.5	CLEO [39]
	3.686	64.12 ± 6.44	9.02	BESII [57]
$K^{*+}K^{-}$ +c.c.	3.686	7.7 ± 5.9	16.7	CLEO [39]
	3.67	4 ± 3.1	16.7	
	3.686	9.6 ± 4.2	2.34	BESII [58]
	3.686	224 ± 21	6.65	BESIII [59]
$K^{*0}\bar{K}^{0}$ +c.c.	3.686	34.5 ± 10.1	8.7	CLEO [39]
	3.67	36 ± 10.6	8.7	
	3.686	65.6 ± 9.0	3.11	BESII [58]
	3.65	2.5 ± 2.6	3.11	
$\phi\eta$	3.686	6.6 ± 5.0	9.4	CLEO [39]
	3.67	3 ± 2.3	9.4	
	3.686	16.7 ± 5.6	3.67	BESII [60]
	3.686	216 ± 16	6.50	BESIII [59]
$\phi\eta'$	3.686	8.4 ± 3.7	1.92	BESII [60]
	3.686	201 ± 15	2.21	BESIII [61]
$\omega\eta$	3.686	4.2 ± 3.2	0.95	BESII [60]
$\rho^0\eta$	3.686	28.1 ± 10.3	19.3	CLEO [39]
	3.67	38 ± 13.9	19.3	
	3.686	29.2 ± 7.5	4.75	BESII [62]
	3.65	2.3 ± 2.1	4.29	
$\rho^0\eta'$	3.686	5.4 ± 3.3	0.86	BESII [62]
	3.686	(211 ± 16)[a]	3.116	BESIII [38]
	3.686	(148 ± 18)[b]	3.116	BESIII [38]
	3.65	5.06 ± 2.01	3.116	BESIII [38]
$\omega\pi^0$	3.686	29.1 ± 14.0	19.1	CLEO [39]
	3.67	55 ± 26.4	19.1	
	3.686	31.2 ± 7.7	4.87	BESII [62]
	3.65	7.3 ± 3.3	4.55	

[a]The destructive solution.
[b]The constructive solution.

are rather different. As pointed out in Ref. [57], the partial wave analysis indicates that in all $\pi^+\pi^-\pi^0$ events, only 28% are due to $\rho\pi$ final state. It seems that the result from BES takes more information into account. If we adopt the same proportion of $\rho\pi$ for CLEO data, the branching fraction $\psi' \to \rho\pi$ is around 5.3×10^{-5}, which is fairly consist with BES result. Therefore, the modified data from CLEO are adopted in the following analysis; the details are presented in Table VI.

Second, in Ref. [38] based on 448 million ψ' events, $\pi^+\pi^-\eta'$ final state is studied resorting to the partial wave analysis technique. The interference between ρ and nonresonant components is observed. The constructive and destructive interferences lead to two possible solutions of branching fraction, that is $(5.69\pm1.28\pm2.36)\times10^{-6}$ and $(1.02\pm0.11\pm0.24)\times10^{-5}$, respectively. In the following fit, two results will be dealt with separately.

Third, for many **VP** decay modes, there are the intermediate states. Take $\phi\eta$ mode as an example, $\phi\to K^+K^-$ and $\eta\to\gamma\gamma$, in calculation of branching ratio, the intermediate decay branching ratios must be taken into consideration. Such kinds of effects could also be included in the Monte Carlo simulation. Therefore, it must be careful to figure out how such kinds of effects are taken into account. Moreover, there are some efficiency correction factors due to the detector or Monte Carlo simulation, which should also be considered. In a word, the efficiency in the following analysis is the one that includes all kinds of necessary effects and is termed as the effective efficiency.

Fourth, as far as the aforementioned principle is concerned, the energy spread will be tuned to give the maximum cross section that can satisfy the relation $N_{\rm tot}=\sigma_{\rm max}\cdot\mathcal{L}$. CLEO data [39] are composed of two sets, one with luminosity $2.74\ {\rm pb}^{-1}$ and the other $2.89\ {\rm pb}^{-1}$, which are taken with energy spreads 1.5 and 2.3 MeV, respectively. In the following analysis, the data are treated as one set with total luminosity $5.63\ {\rm pb}^{-1}$ corresponding to the effective energy spread 1.68 MeV as displayed in Table IV. It it worthy of noticing that unlike branching fraction evaluation, the contribution due to QED continuum should not be subtracted from the observed number of events, since the QED contribution is included in the calculation of observed cross section. At last, since the error of number of events are needed in analysis, for CLEO data, the maximum relative statistical error of branching fraction is used to evaluate the corresponding error of the number of events.

The chi-square method is adopted to fit the experimental data. The estimator is constructed as

$$\chi^2=\sum_i\frac{[N_i-n_i(\vec{\eta})]^2}{(\delta N_i)^2}, \tag{45}$$

where N with the corresponding error (δN) denotes the experimentally measured number of events while n the theoretically calculated number of events,

$$n=\mathcal{L}\cdot\sigma_{\rm obs}\cdot\epsilon, \tag{46}$$

where $\mathcal{L}$ is the integrated luminosity, ϵ is the effective efficiency, and $\sigma_{\rm obs}$ the observed cross section calculated according to formula (42), which contains the parameters to be fit, such as A, D, D', s_P, s_V, and the phase angle ϕ. All these parameters are denoted by the parameter vector $\vec{\eta}$ in Eq. (45). The concrete form is determined by the parametrization form in Table III. All observed numbers of events together with the corresponding efficiencies displayed in Table VI are employed as input information. The data can be grouped into four sets: two from BESIII, one with total luminosity $668.55\ {\rm pb}^{-1}$, the other with luminosity $161.63\ {\rm pb}^{-1}$; one from CLEO, with total luminosity $5.63\ {\rm pb}^{-1}$, and one from BESII with luminosity

19.72 pb^{-1}. There might be some systematic difference among those datasets, so normalization factors are introduced to take into account these systematic effects. However, only three relative (relative to the greatest dataset of BESIII) factors of luminosity are introduced with the belief that the relative relations of measurements of each experiment group is more reliable than the corresponding absolute values. The fit values of three factors f_{cleo}, f_{bes2}, and f_{bes3a} indicate that there indeed exists certain obvious differences, the inconsistencies of these experiments from the highest precision one range from 10% to 70%.

The fitting yields a χ^2 of 18.91 with the number of degrees of freedom being 19 for the destructive inference case,

$$\begin{aligned} \phi &= -131.55° \pm 13.05°, \\ A &= 0.577 \pm 0.053, \\ D &= 0.334 \pm 0.026, \\ D' &= -0.025 \pm 0.078, \\ s_P &= -0.277 \pm 0.055, \\ \theta_P &= -16.10°, \\ s_V &= 0.279 \pm 0.195, \\ \theta_V &= 16.19°, \\ f_{\mathrm{cleo}} &= 0.937 \pm 0.167, \\ f_{\mathrm{bes2}} &= 1.277 \pm 0.208, \\ f_{\mathrm{bes3a}} &= 1.361 \pm 0.252; \end{aligned} \tag{47}$$

and a χ^2 of 15.82 with the number of degrees of freedom being 19 for the constructive inference case,

$$\begin{aligned} \phi &= -144.31° \pm 20.93°, \\ A &= 0.545 \pm 0.047, \\ D &= 0.300 \pm 0.027, \\ D' &= -0.030 \pm 0.068, \\ s_P &= -0.307 \pm 0.058, \\ \theta_P &= -17.88°, \\ s_V &= 0.386 \pm 0.221, \\ \theta_V &= 22.73°, \\ f_{\mathrm{cleo}} &= 1.141 \pm 0.220, \\ f_{\mathrm{bes2}} &= 1.516 \pm 0.264, \\ f_{\mathrm{bes3a}} &= 1.703 \pm 0.364, \end{aligned} \tag{48}$$

where $\theta_\alpha = \arcsin s_\alpha$ $(\alpha = P, V)$.

The scan for the parameter ϕ displays merely one minimum, which is a rather uncommon case for multiple solution theory. When fitting cross sections with several resonances or interfering background and resonances, one usually obtains multiple solutions of parameters with equal fitting quality. Such a phenomenon was firstly noticed experimentally [63,64], then some studies are performed from a mathematical point of view [65–68]. Especially in Ref. [68], the source of multiple solutions for a combination of several resonances or interfering background and resonances is found by analyzing the mathematical structure of the Breit-Wigner function. It is proved that there are exactly 2^{n-1} fitting solutions with equal quality for n amplitudes, and the multiplicity of the interfering background function and resonance amplitudes depends on zeros of the amplitudes in the complex plane. Our study involves the interference between strong and electromagnetic amplitudes, corresponding to $n = 2$ case; therefore, it is expected that there are two solutions for the phase angle ϕ. For the present fit result, we are not sure if this is the special feature of $\psi' \to \mathbf{VP}$ decay, or the current data are not accurate enough to differentiate two solutions that are close to each other.

Besides the fit for all data in Table VI, we also perform the fit for part of it. In the light of parametrization of Table III, it is obvious that $\rho^0\eta$, $\rho^0\eta'$, $\phi\pi^0$, and $\omega\pi^0$ are only concerned with D, s_P, and s_V; therefore, the fit of the data relevant to these final states can be used to determine the three parameters. Based on the information of Table VI related to $\rho^0\eta$, $\rho^0\eta'$, and $\omega\pi^0$ final states, the fitting yields a χ^2 of 7.08 with the number of degrees of freedom being 8 for the destructive inference case,

$$\begin{aligned} D &= 0.369 \pm 0.023, \\ s_P &= -0.335 \pm 0.044, \\ \theta_P &= -19.57°, \\ s_V &= 0.853 \pm 0.136, \\ \theta_V &= 58.54°; \end{aligned} \tag{49}$$

and a χ^2 of 4.68 with the number of degrees of freedom being 8 for the constructive inference case,

$$\begin{aligned} D &= 0.351 \pm 0.025, \\ s_P &= -0.405 \pm 0.044, \\ \theta_P &= -23.90°, \\ s_V &= 0.342 \pm 0.400, \\ \theta_V &= 20.02°. \end{aligned} \tag{50}$$

It can be seen that the fit value of θ_P is marginally consistent with the overall fit, which means that the value of θ_P is dominantly determined by $\rho^0\eta$ and $\rho^0\eta'$ final states. On the contrary, since only the data of $\omega\pi^0$ final state are available, the fluctuation of fit value of θ_V is fairly prominent. Anyway, if the precise measurements of $\rho^0\eta$,

$\rho^0\eta'$, $\phi\pi^0$, and $\omega\pi^0$ final states can be obtained, the mixing angle of θ_P and θ_V is expected to be determined definitely.

B. $J/\psi \to VP$ decay

For J/ψ decaying to the **VP** final state, there are lots of experimental results, some of which are summarized in Table VII. However, a few of measurements were obtained forty years ago with low statistic samples. In Refs. [69,70], the branching fraction of $\rho\pi$ is measured to be $(1.3 \pm 0.3) \times 10^{-2}$ and $(1.6 \pm 0.4) \times 10^{-2}$ based on 50 and 84 thousand J/ψ events, respectively. In Refs. [55,71], the branching fraction of $\rho\pi$ is measured to be $(1.3 \pm 0.3) \times 10^{-2}$ and $(1.0 \pm 0.2) \times 10^{-2}$ based on 0.4 and 0.87 million J/ψ events, respectively. In Ref. [72], the study is mainly focused on J/ψ radiative decay into $\pi\pi\gamma$ and $KK\gamma$. With 1.71 million J/ψ events, the branching fraction of $\rho\pi$ is merely obtained as a consistency check. These kinds of measurements involving $\rho\pi$ final state are excluded from the following analysis. The measurements of $J/\psi \to K^{*+}K^- + \text{c.c.}$ from Ref. [55], $J/\psi \to \phi\pi^0$ from Ref. [73], $J/\psi \to \rho^\pm\pi^\mp$, $K^{*\pm}K^\mp$ from Ref. [74], and $J/\psi \to K^{*+}K^- + \text{c.c.}$, $K^{*0}\bar{K}^0 + \text{c.c.}$ from Ref. [75] are not adopted either due to low statistic.

Both Refs. [46] and [47] made a systematical measurement for J/ψ decaying into **VP** final state. The latter data, consisting of 5.8×10^6 produced J/ψ's, represent a twofold increase over the former data (2.7×10^6); therefore, the better accuracy is realized in the latter analysis. Although all **VP** channels, viz., $\rho\pi$, $\rho^0\eta$, $\rho^0\eta'$, $\omega\pi^0$, $\omega\eta$, $\omega\eta'$, $\phi\pi^0$, $\phi\eta$, $\phi\eta'$, $K^{*+}K^- + \text{c.c.}$, and $K^{*0}\bar{K}^0 + \text{c.c.}$, are measured, the number of events and the corresponding efficiencies are not provided, which leads to impossibility to include these results in this analysis.

TABLE VII. Experimental data of J/ψ decaying to VP final states. For branching ratios, the first uncertainties are statistical, and the second systematic. The peak position is assumed at $\sqrt{s} = 3.097$ GeV, while the continuum position is assumed at $\sqrt{s} = 3.07$ or $\sqrt{s} = 3.08$ GeV for BES measurement. The effective efficiency is presented and the one with star ($*$) is evaluated by virtue of the observed number of events N^{obs}, the total number of resonance events, and the corresponding of branching fraction. The symbol "$n.g.b.$" indicates that the continuum background is negligible.

Mode	N^{obs} (peak)	N^{obs} (continuum)	Efficiency (%)	Branching ratio ($\times 10^{-3}$)	Detector
$\pi^+\pi^-\pi^0$	1849852 ± 1360	31.0 ± 5.6	38.13	$21.37 \pm 0.04^{+0.58}_{-0.56}$	BESIII [41]
	219691.0 ± 503.0		17.83	$21.84 \pm 0.05 \pm 2.01$	BESII [76]
	166		2.68	15.0 ± 2.0	MARKII [55]
$\rho\pi$	149.7		2.68	13.0 ± 3.0	MARKII [55]
	543.0 ± 105.6		6.08	10 ± 2	DESY-H. [71]
$K^{*+}K^- + \text{c.c.}$	2285.0 ± 43.1		5.814^*	$4.57 \pm 0.17 \pm 0.70$	DM2 [45]
	24		2.13	2.6 ± 0.8	MARKII [55]
$K^{*0}\bar{K}^0 + \text{c.c.}$	1192.0 ± 39.1		3.500^*	$3.96 \pm 0.15 \pm 0.60$	DM2 [45]
$\phi\eta$	346.0 ± 21.2		6.286^*	$0.64 \pm 0.04 \pm 0.11$	DM2 [45]
	2418.0 ± 65.2		4.667^*	$0.898 \pm 0.024 \pm 0.089$	BESII [73]
$\phi\eta'$	167.0 ± 13.5		4.736^*	$0.41 \pm 0.03 \pm 0.08$	DM2 [45]
	728.0 ± 40.5		2.311^*	$0.546 \pm 0.031 \pm 0.056$	BESII [73]
	31321 ± 201		4.690	$0.510 \pm 0.003 \pm 0.032$	BESIII [77]
$\omega\eta$	378.0 ± 26.9		3.074^*	$1.43 \pm 0.10 \pm 0.21$	DM2 [45]
	4927.0 ± 91.0	$n.g.b.$	3.631^*	2.352 ± 0.273	BESII [44]
$\omega\eta'$	6.0 ± 2.5		0.388^*	$0.18^{+0.10}_{-0.08} \pm 0.03$	DM2 [45]
	218.0 ± 32.8	$n.g.b.$	1.672^*	0.226 ± 0.043	BESII [44]
	137 ± 20	$n.g.b.$	0.050	$0.208 \pm 0.030 \pm 0.014$	BESIII [38]
$\rho^0\eta$	299.0 ± 34.0		17.921^*	$0.194 \pm 0.017 \pm 0.029$	DM2 [45]
$\rho^0\eta'$	19.2 ± 7.5		2.690^*	$0.083 \pm 0.030 \pm 0.012$	DM2 [45]
	3621 ± 83	57.13 ± 11.03	3.381	$0.0790 \pm 0.0019 \pm 0.0049$	BESIII [38]
$\phi\pi^0$	24		8.08	$< 6.4 \times 10^{-3}$ (90% C.L.)	BESII [73]
	(838.5 ± 45.8)[a]	$n.g.b.$	21.79	$(2.94 \pm 0.16 \pm 0.16) \times 10^{-3}$	BESIII [78]
	(35.3 ± 9.3)[b]	$n.g.b.$	21.79	$(1.24 \pm 0.33 \pm 0.30) \times 10^{-4}$	BESIII [78]
$\omega\pi^0$	222.0 ± 19.0		7.171^*	$0.360 \pm 0.028 \pm 0.054$	DM2 [45]
	2090.0 ± 67.3	6.88 ± 2.85	6.57	$0.538 \pm 0.012 \pm 0.065$	BESII [44]

[a]The solution of constructive interference between $J/\psi \to \phi\pi^0$ and $J/\psi \to K^+K^-\pi^0$ decays.
[b]The solution of destructive interference between $J/\psi \to \phi\pi^0$ and $J/\psi \to K^+K^-\pi^0$ decays.

In addition, in Ref. [79], cross sections of $\rho\pi$ final state are measured at 29 different energy points covering a 40 MeV interval spanning the J/ψ resonance. Based on this data sample, which corresponds to a total integrated luminosity of 238 nb^{-1}, the branching fraction is determined to be $(1.21 \pm 0.20)\%$. Such information is too distinctive to be combined with those in Table VII. In Ref. [80], resorting to partial wave analysis technique measured is the branching fraction of $J/\psi \to K^{*+}K^- + \mathrm{c.c.}$, whose feature is too different to be merged with other information. In both Refs. [81] and [82], the initial state radiation technique is used to obtain the branching fractions of $J/\psi \to \pi^+\pi^-\pi^0$ and $J/\psi \to \omega\eta$. Such kinds of results can not be utilized in the present scheme of analysis.

Based on a sample of 1.31 billion J/ψ events [78], the K^+K^- mass spectrum is scrutinized, and the observation is a clear structure due to the interference between $J/\psi \to \phi\pi^0$ and $J/\psi \to K^+K^-\pi^0$ decays. Such a interference yields two possible solutions of branching fraction, that is $(2.94 \pm 0.16 \pm 0.16) \times 10^{-6}$ and $(1.24 \pm 0.33 \pm 0.30) \times 10^{-7}$, which correspond to the constructive and destructive interferences, respectively. In the following fit, two results will be dealt with separately.

The minimization estimator for J/ψ is similar to that of ψ' as defined in Eq. (45) and the fit yields

$$\begin{aligned}
\phi &= -66.52° \pm 1.82°, \quad \text{or} \quad +71.42° \pm 1.82°,\\
A &= 2.622 \pm 0.011, \quad \text{or} \quad 2.697 \pm 0.011,\\
D &= 0.523 \pm 0.006, \quad \text{or} \quad 0.538 \pm 0.006,\\
D' &= -0.475 \pm 0.017, \quad \text{or} \quad -0.489 \pm 0.017,\\
s_P &= -0.341 \pm 0.004, \quad \text{or} \quad -0.341 \pm 0.004,\\
\theta_P &= -20.17°, \quad \text{or} \quad -19.96°,\\
s_V &= 0.557 \pm 0.002, \quad \text{or} \quad 0.557 \pm 0.002,\\
\theta_V &= 33.86°, \quad \text{or} \quad 33.82°,\\
f_{dm2} &= 0.624 \pm 0.011, \quad \text{or} \quad 0.590 \pm 0.010,\\
f_{\mathrm{bes2}} &= 1.246 \pm 0.010, \quad \text{or} \quad 1.177 \pm 0.010,\\
f_{\mathrm{bes3}a} &= 0.613 \pm 0.010, \quad \text{or} \quad 0.579 \pm 0.009,\\
f_{\mathrm{bes3}b} &= 1.305 \pm 0.010, \quad \text{or} \quad 1.233 \pm 0.010;
\end{aligned} \tag{51}$$

with $\chi^2 = 4302$ for the destructive interference solution, and

$$\begin{aligned}
\phi &= -66.92° \pm 1.77°, \quad \text{or} \quad +71.82° \pm 1.77°,\\
A &= 2.633 \pm 0.011, \quad \text{or} \quad 2.713 \pm 0.011,\\
D &= 0.536 \pm 0.006, \quad \text{or} \quad 0.552 \pm 0.006,\\
D' &= -0.501 \pm 0.017, \quad \text{or} \quad -0.516 \pm 0.018,\\
s_P &= -0.397 \pm 0.004, \quad \text{or} \quad -0.397 \pm 0.004,\\
\theta_P &= -23.38°, \quad \text{or} \quad -23.38°,\\
s_V &= 0.492 \pm 0.002, \quad \text{or} \quad 0.492 \pm 0.002,\\
\theta_V &= 29.47°, \quad \text{or} \quad 29.47°,\\
f_{dm2} &= 0.620 \pm 0.011, \quad \text{or} \quad 0.584 \pm 0.010,\\
f_{\mathrm{bes2}} &= 1.224 \pm 0.010, \quad \text{or} \quad 1.152 \pm 0.010,\\
f_{\mathrm{bes3}a} &= 0.664 \pm 0.011, \quad \text{or} \quad 0.625 \pm 0.010,\\
f_{\mathrm{bes3}b} &= 1.282 \pm 0.010, \quad \text{or} \quad 1.207 \pm 0.010;
\end{aligned} \tag{52}$$

with $\chi^2 = 4200$ for the constructive interference solution, where $\theta_\alpha = \arcsin s_\alpha (\alpha = P, V)$.

From a pure viewpoint of a hypothesis test [83,84], the ratio of the chi square value to the number of degrees of freedom should approximate one for a good fit, but the values of χ^2 for fitting results in Eqs. (51) and (52) is horrendously large. Since the accuracy of J/ψ data is generally higher than that of ψ', the discrepancies between the different experiments become much more prominent. The data summarized in Table VI can be grouped into four sets: two from BESIII, one with total luminosity 394.65 pb^{-1}, the other with luminosity 79.63 pb^{-1}; one from BESII with luminosity 15.89 pb^{-1} and one from MD2, with total luminosity 5.053 pb^{-1}. Four normalization factors of luminosity are introduced to alleviate the possible inconsistency among the data from different experiment groups. Nevertheless, such a kind of treatment is obviously not sufficient enough so that certain great deviations still exist which lead to huge chi square value.

In order to figure out the effect due to the discrepancy of different experiments on the value of χ^2, the fits for different datasets are performed respectively, viz., DM2 data, DM2 and BESII data, DM2 and BESIII data, and all dataset; the results are tabulated in Table VIII. The value of χ^2 is minimum for a sole experiment dataset (DM2 data), and the values increase when more experiment datasets are fit together (DM2 and BESII datasets or DM2 and BESIII datasets). It can be seen when all data are fit together, the value of χ^2/n_d enhances almost one order of magnitude. Moreover, after some filtration trial it is found that the inconsistency of the measurement of $\psi' \to \omega\eta$ decay from BESII [44] from the other measurements is rather obviously. The fit is performed for all data except for this one; the results are presented in the second column of Table VIII. The comparison of chi squares of the first two columns indicates that the sole measurement of $\omega\eta$ channel from BESII contributes more than half of chi square value, which seriously deteriorates the χ^2 probability of the fit. If we compare the two branching fractions from DM2 [45] and BESII [44], that is $(1.43 \pm 0.10 \pm 0.21) \times 10^{-3}$ and $(2.352 \pm 0.273) \times 10^{-3}$, the former is only about

TABLE VIII. Comparison of various fitting results for the destructive interference solution data and positive angle case. The data with star (⋆) exclude the measurement of $\psi' \to \omega\eta$ decay from BESII [44]. n_d indicates the degree of freedom.

$\vec{\eta}$ & χ^2	All data	All data⋆	DM2 data	DM2 and BESII data⋆	DM2 and BESIII data
χ^2	4302.444	1962.792	91.254	345.825	239.325
χ^2/n_d	330.6	163.6	45.63	49.40	29.92
ϕ	$(71.42 \pm 1.82)°$	$(64.47 \pm 1.91)°$	$(70.92 \pm 3.43)°$	$(61.60 \pm 3.46)°$	$(69.52 \pm 3.18)°$
A	2.697 ± 0.011	2.631 ± 0.011	2.013 ± 0.209	2.460 ± 0.033	2.914 ± 0.018
D	0.538 ± 0.006	0.562 ± 0.006	0.395 ± 0.044	0.497 ± 0.010	0.690 ± 0.017
D'	-0.489 ± 0.017	-0.638 ± 0.018	-0.223 ± 0.049	-0.679 ± 0.032	0.171 ± 0.069
s_P	-0.341 ± 0.004	-0.472 ± 0.005	-0.530 ± 0.046	-0.755 ± 0.010	-0.208 ± 0.028
θ_P	$-19.96°$	$-28.17°$	$-32.76°$	$-48.98°$	$-11.98°$
s_V	0.557 ± 0.002	0.557 ± 0.001	0.308 ± 0.045	0.221 ± 0.010	0.657 ± 0.025
θ_V	$33.82°$	$33.85°$	$17.94°$	$12.76°$	$41.10°$
f_{dm2}	0.590 ± 0.010	0.603 ± 0.011	1.020 ± 0.212	0.698 ± 0.022	0.422 ± 0.014
f_{bes2}	1.177 ± 0.010	1.166 ± 0.010		1.295 ± 0.034	
f_{bes3a}	0.579 ± 0.009	0.686 ± 0.015			0.281 ± 0.012
f_{bes3b}	1.233 ± 0.010	$1.864 \pm 0.01-$			1.326 ± 0.011

the half of the latter. The great discrepancy renders huge chi square, which can only be settled down by further more accurate experimental measurement.

Of course, there is a factor that has been neglected for data analysis. In the fitting solely considered are the statistic uncertainties, if the systematic uncertainties are included as well, it is expected that the chi square could be decreased to one half or one third of the present value. As a matter of fact, we also performed a fit with increased error of $\omega\eta$ channel. If 4927.0 ± 571.0 instead of 4927.0 ± 91.0 is used, $\chi^2 = 2055$ instead of $\chi^2 = 4302$. Anyway, even so the value of χ^2 is still too large to be satisfied from a point of statistical view.

It also exists the possibility that the present parametrization form is not exquisite enough to describe all data perfectly, but only more precise and consistent experimental data can furnish quantitative evidence for or against the present phenomenology model and pin down the problem we come across here. The absence of a set of ideally experimental data is keenly felt.

Last but not least, we also perform the fit for data related to $\rho^0\eta$, $\rho^0\eta'$, $\omega\pi^0$, and $\phi\pi^0$ final states, the fitting yields a χ^2 of 3.91 with the number of degrees of freedom being 3 for the destructive interference case,

$$
\begin{aligned}
D &= 0.404 \pm 0.037,\\
s_P &= -0.288 \pm 0.068,\\
\theta_P &= -16.73°,\\
s_V &= 0.595 \pm 0.003,\\
\theta_V &= 36.50°,\\
f_{dm2} &= 1.096 \pm 0.248,\\
f_{\mathrm{bes2}} &= 1.773 \pm 0.329; \qquad (53)
\end{aligned}
$$

and a χ^2 of 3.97 with the number of degrees of freedom being 3 for the constructive interference case,

$$
\begin{aligned}
D &= 0.403 \pm 0.036,\\
s_P &= -0.285 \pm 0.068,\\
\theta_P &= -16.57°,\\
s_V &= 0.660 \pm 0.010,\\
\theta_V &= 41.30°,\\
f_{dm2} &= 1.112 \pm 0.252,\\
f_{\mathrm{bes2}} &= 1.805 \pm 0.337. \qquad (54)
\end{aligned}
$$

For the destructive interference case, the fit values of θ_P and θ_V are marginally consist with those of overall fit, which means that these four channels are necessary, sufficient, and efficient for measurement of the mixing angles of pseudoscalar and vector mesons.

V. DISCUSSION

Besides the cross section approach (CSA) used to analyze the data, there is another way to deal with the information involving $\psi', J/\psi \to$ **VP** decay, which is call branching ratio method (BRM). The idea is fairly simple, the so-called "reduced branching ratio" is related to the square of amplitude directly [5,52], that is

$$\tilde{B}(\psi \to f) = |X_f + Y_f|^2, \qquad (55)$$

where X_f and Y_f are defined in Eqs. (38) and (39) for a certain **VP** final state f, and the reduced branching ratio is defined as follows:

TABLE IX. Branching ratios of $\psi', J/\psi \to VP$ extracted from PDG2022 [85]. Q_h is defined in Eq. (57) and calculated by the ratio of $B(\psi' \to VP)$ to $B(J/\psi \to VP)$. $(K^{*+}K^-)_{\rm c.c.}$ and $(K^{*0}\bar{K}^0)_{\rm c.c.}$ indicate $K^{*+}K^- +$ c.c. and $K^{*0}\bar{K}^0 +$ c.c. respectively.

Mode	$B(\psi' \to VP)$ $(\times 10^{-5})$	$B(J/\psi \to VP)$ $(\times 10^{-3})$	Q_h (%)
$\rho\pi$	3.2 ± 1.2	16.9 ± 1.5	0.2 ± 0.1
$(K^{*+}K^-)_{\rm c.c.}$	2.9 ± 0.4	6.0 ± 1.0	1.1 ± 0.2
$(K^{*0}\bar{K}^0)_{\rm c.c.}$	10.9 ± 2.0	4.2 ± 0.4	2.6 ± 0.5
$\phi\eta$	3.10 ± 0.31	0.74 ± 0.08	4.2 ± 0.6
$\phi\eta'$	1.54 ± 0.20	0.46 ± 0.05	3.3 ± 0.6
$\omega\eta$	< 1.1	1.74 ± 0.20	< 0.632
$\omega\eta'$	3.2 ± 2.5	0.189 ± 0.018	16.9 ± 13.3
$\rho^0\eta$	2.2 ± 0.6	0.193 ± 0.023	11.4 ± 3.4
$\rho^0\eta'$	1.87 ± 1.67	0.081 ± 0.008	23.1 ± 20.7
$\omega\pi^0$	< 0.004	0.45 ± 0.05	< 0.0089
$\phi\pi^0$	2.1 ± 0.6	$(2.94 \pm 0.23) \times 10^{-3}$[a]	$(7.1 \pm 2.1) \times 10^2$
		$(1.24 \pm 0.45) \times 10^{-4}$[b]	$(1.7 \pm 0.8) \times 10^4$

[a]The constructive solution.
[b]The destructive solution.

$$\tilde{B}(\psi \to f) = \frac{B(\psi \to f)}{q_f^3}, \tag{56}$$

where $B(\psi \to f)$ is the branching ratio of $\psi(\psi = J/\psi, \psi')$ decays to the final state f, and q_f defined in Eq. (32), is the momentum of either particle in the center of mass system for two-body decay. According to the formula (55) together with the information in Table IX, the fits are performed and the fitting results are displayed in Table X.

Comparing results of two kinds of fits, for ψ' case, the values of χ^2 are both reasonable, merely one solution is found. Some results are similar such as A, D, s_P, while others are different such as D', s_V. For J/ψ case, the similarity is much worse, especially for the destruction solution; only one minimum instead of two is found for the phase angle. As a matter of fact, since the branching ratios herein are averaged results that combine the various measurements due to many experiments, such an admixture blurs the discrepancy between distinctive experimental analysis. Therefore, the following discussion is based on the results due to CSA instead of BRM.

Table XI summarizes various fitting results of phase angle for ψ' and J/ψ two-body decays. The most exceptional solution is due to $\psi' \to \mathbf{VP}$ decay. Such a situation can not help reminding of the famous "$\rho\pi$ puzzle" in charmonium decays.

Theoretically, the OZI (Okubo-Zweig-Iizuka) [86] suppressed decays of J/ψ and ψ' to hadrons are via three gluons or a photon, in either case, the perturbative QCD (pQCD) provides a relation [87],

$$Q_h = \frac{\mathcal{B}_{\psi' \to h}}{\mathcal{B}_{J/\psi \to h}} = \frac{\mathcal{B}_{\psi' \to e^+e^-}}{\mathcal{B}_{J/\psi \to e^+e^-}} \approx (13.28 \pm 0.29)\%, \tag{57}$$

where $\mathcal{B}_{\psi' \to e^+e^-}$ and $\mathcal{B}_{J/\psi \to e^+e^-}$ are taken from PDG2022 [85]. This relation is expected to be held to a reasonable

TABLE X. Fit results of branching ratio method for $\psi', J/\psi \to VP$ decay. $n_{\rm d.o.f.}$ indicates degree of freedom.

$\vec{\eta}$ & χ^2	$\psi' \to VP$	$J/\psi \to VP$		
		Constructive solution		Destructive solution
χ^2	1.995	184.075	184.075	169.651
$n_{\rm d.o.f.}$	3	5	5	5
ϕ	$(-133.56 \pm 13.54)^\circ$	$(61.65 \pm 31.87)^\circ$	$(-61.65 \pm 20.89)^\circ$	$(-152.18 \pm 44.56)^\circ$
A	0.887 ± 0.092	0.796 ± 0.027	0.796 ± 0.031	0.714 ± 0.040
D	0.618 ± 0.079	-0.192 ± 0.008	-0.192 ± 0.008	0.202 ± 0.007
D'	0.247 ± 0.166	-0.277 ± 0.088	-0.277 ± 0.060	-0.233 ± 0.071
s_P	-0.227 ± 0.074	-0.585 ± 0.018	-0.585 ± 0.022	-0.439 ± 0.051
θ_P	-13.10°	-35.84°	-35.84°	-26.04°
s_V	0.029 ± 0.141	0.493 ± 0.002	0.493 ± 0.005	0.594 ± 0.003
θ_V	1.64°	29.52°	29.52°	36.44°

TABLE XI. The fit results of phase angle for ψ' and J/ψ two-body decays, some of which are from Ref. [12]. The subscripts d and c indicate the results for destructive and constructive cases, respectively.

Decay mode	Phase angle ϕ (in degree)	
$\psi' \to B_8\bar{B}_8$	-94.59 ± 1.31	$+85.42 \pm 2.25$
$J/\psi \to B_8\bar{B}_8$	-84.81 ± 0.70	$+95.19 \pm 0.70$
$\psi' \to B_{10}\bar{B}_{10}$	-75.51 ± 4.87	$+104.49 \pm 4.91$
$J/\psi \to B_{10}\bar{B}_{10}$	-96.28 ± 17.23	$+83.27 \pm 11.38$
$J/\psi \to B_{10}\bar{B}_8$	-89.97 ± 37.17	$+101.20 \pm 71.87$
$\psi' \to PP$	-58.19 ± 5.47	$+92.82 \pm 5.62$
$J/\psi \to PP$	-87.25 ± 8.60	$+92.14 \pm 8.61$
$\psi' \to VP$	$(-131.55 \pm 13.05)_d$	
	$(-144.31 \pm 20.93)_c$	
$J/\psi \to VP$	$(-66.52 \pm 1.82)_d$	$(+71.42 \pm 1.82)_d$
	$(-66.92 \pm 1.77)_c$	$(+71.42 \pm 1.77)_c$

good degree for both inclusive and exclusive decays. The so-called "$\rho\pi$ puzzle" is that the prediction by Eq. (57) is severely violated in the $\rho\pi$ and several other decay channels. The first evidence for this effect was found by Mark-II Collaboration in 1983 [55]. From then on, many theoretical explanations have been put forth to decipher this puzzle; refer to the treatise [88] for a detailed review.

From the theoretical point of view, since the Q value is smaller than the expectation for $\rho\pi$, it may be caused either by enhanced J/ψ or suppressed ψ' decay rate. Another possibility is by both. Then the relevant theoretical speculations can be classified into three categories:

(1) J/ψ-enhancement hypothesis, which attributes the small Q value to the enhanced branching fraction of J/ψ decays [89–98].
(2) ψ'-suppress hypothesis, which attributes the small Q value to the suppressed branching fraction of ψ' decays [99–105].
(3) Other hypotheses, which are not included in the above two categories [106–113].

With more and more experimental data are obtained, many theoretical explanations have been ruled out, and some still need to be tested. However, "$\rho\pi$ puzzle" seems still a puzzle since no propose can explain all existing experimental data satisfactorily and naturally. If we scrutinize the Q_h values in Table IX, it can seen that many values are suppressed relative to the expected value 13.28% while some are enhanced, *a fortiori* for $\phi\pi^0$ channel, the Q_h value is several orders of magnitude greater than the expectation. Furthermore, the deviation seems rather arbitrary, no regularity can be found. At the same time, if we investigate Table XI, it can be seen that the phase angle for $\psi' \to$ **VP** decay is rather abnormal from the other decays. So the more we think about various information, more problems spring up. It seems to trigger a Pandora's Box of questions:

(1) ψ' and J/ψ are both S-state of charmonium; the decay pattern is similar for some channels but rather different for the others; the prominent example is about $\rho\pi$ final state, and some studies of which have been performed in Ref. [57]. What is the reason for such grotesquery situations?
(2) Is the abnormal feature only for **VP** mode, or for all VP-like modes, such as SV, SA^-, PA^-, VT, VA^+, TA^-, and A^+A^-? (S, P, V, A, and T denote scalar, pseudoscalar, vector, axial vector, and tensor, respectively; refer to Tables XII and XIII for more details.)
(3) Does the exceptional phase angle have a connection with the abnormal Q_h value? Is there a profound rationale for such a connection?
(4) From the viewpoint of phase angle and by virtue of fit result, the baryonic mode seems more normal than the mesonic mode. What is the reason for such a difference?
(5) From the viewpoint of phase angle and by virtue of fit result, J/ψ decay seems more normal than ψ'. What is the reason for such a difference?
(6) Does such a bizarre situation also exist for a bottomonium decay?
(7) There is a suggestion that ψ' is not a pure state but a mixing of $\psi(2^3S_1)$ state and $\psi(1^3D_1)$. Such a opinion is adopted to explain the decrease Q_h value for the $\rho\pi$ channel [105] and the increase Q_h value for the $K^0_S K^0_L$ channel [114]. Anyway, could this suggestion explain decay behaviors for all kinds of modes?
(8) Glueball was once put forth to explain "$\rho\pi$ puzzle" [90–96], but the experimental research is unfavorable to the detailed analyses and deductions due to this kind of scenario. However, the eccentric decay pattern of the **VP** mode shown by their various Q_h values indeed implies the peculiar feature of this mode. The doubt is aggravated by the fitted mixing angles of θ_P and θ_V, none of which is consist with theoretical expectation. According to mass matrix analysis, $\theta_P = -11.3°$ or $-24.5°$ and $\theta_V = 39.2°$ or $36.5°$, corresponding to the quadratic or linear mass assumptions, respectively. The deviation from expectation maybe indicate the admixture of pseudoscalar meson with some glueball-like component. This is an issue need to be studied further.

It is true that a lot of measurements are available now, but even more precise and systematical measurements are needed to clarify the dynamics of charmonium decay.

VI. SUMMARY

The flavor-singlet principle, with assuming the flavor symmetry breaking effects (both strong and electromagnetic breaking effects) as a special $SU(3)$ octet furnishes a criterion to figure out the effective interaction Hamiltonian in tensor form for all kinds of two-body final states decaying from a charmonium resonance. The generalized inherent C-parity for a multiplet is introduced, which plays a crucial role for determining the form of effective Hamiltonian, especially for mesonic final states. Resorting to the nonet notion, both octet and singlet representations for meson description are synthesized together to acquire the effective Hamiltonian in a concise way, then solving the meson mixing problem in charmonium decay. As an application, by virtue of this scenario, the relative phase between the strong and electromagnetic amplitudes is measured for vector-pseudoscalar meson final state; more information involving the interaction coupling coefficients is obtained, all of which deepen our understanding of the dynamics of charmonium decay. Furthermore, the fit results indicate that the measurements of four final states, that is $\rho^0\eta$, $\rho^0\eta'$, $\omega\pi^0$, and $\phi\pi^0$, are necessary, sufficient, and efficient for the study of mixing angles of pseudoscalar and vector multiplets.

In the analysis of data samples taken in an e^+e^- collider, the details of experimental effects, such as energy spread and initial state radiative correction are taken into consideration in order to make full advantage of experimental information and acquire the comprehensive results. However, the discrepancy between different experimental measurement leads to large chi square, which is fairly unfavorable from the statistic viewpoint. The data analysis of this paper makes it urgent that further more precise and systematic experimental measurements should be performed based on BESIII colossal data sample of charmonium decay, in order to figure out the unclear issues we come across here.

By virtue of present analysis, the uniform parametrization scheme provides a general description for various kinds of charmonium two-body decays and lays an extensive foundation for more profound dynamics exploration in the future.

ACKNOWLEDGMENTS

This work is supported in part by National Key Research and Development Program of China under Contracts No. 2023YFA1606003, No. 2023YFA1606000, No. 2020YFA0406302, and No. 2020YFA0406403.

APPENDIX

This appendix is devoted to two issues. The first one is about the C-parity transformation of meson octet.

For two meson octets, denoted respectively by O_1 and O_2, defined are the following terms, which may be allowed or forbidden in the effective Hamiltonian:

$$[O_1O_2]_0 = (O_1)^i_j(O_2)^j_i, \tag{A1}$$

$$([O_1O_2]_f)^i_j = (O_1)^i_k(O_2)^k_j - (O_1)^k_j(O_2)^i_k, \tag{A2}$$

and

$$([O_1O_2]_d)^i_j = (O_1)^i_k(O_2)^k_j + (O_1)^k_j(O_2)^i_k - \frac{2}{3}\delta^i_j \cdot (O_1)^i_j(O_2)^j_i. \tag{A3}$$

The aforementioned generalized inherent C parity for meson octet is introduced as a criterion for Hamiltonian terms, and its value (denoted as η_O) is set to be equal to that of the neutral particle of the corresponding octet. In Table XII, listed are some observed light mesons that are classified into distinctive octets. The value of generalized C parity is equal to the C parity presented in the table.

A remark is in order here. The physical isoscalars are mixtures of the $SU(3)$ wave function ψ_8 and ψ_1,

$$f' = \psi_8 \cos\theta - \psi_1 \sin\theta,$$
$$f = \psi_8 \sin\theta + \psi_1 \cos\theta, \tag{A4}$$

where θ is the nonet mixing angle and

$$\psi_8 = (u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6},$$
$$\psi_1 = (u\bar{u} + d\bar{d} + s\bar{s})/\sqrt{3}. \tag{A5}$$

These mixing relations are often rewritten to exhibit the $u\bar{u} + d\bar{d}$ and $s\bar{s}$ components, which decouple for the "ideal" mixing angle, such that $\tan\theta_i = 1/\sqrt{2}$ (or $\theta_i = 35.3°$). Defining $\alpha = \theta + 54.7°$, one obtains the physical isoscalar in the flavor basis,

TABLE XII. Some meson octet particles. For $I = 0$ meson, the mixing between singlet and octet always exists. The f' and f are mixing states as defined in Eq. (A4) or in Eqs. (A6) and (A7). In addition, K_{1A} and K_{1B} are nearly equal (45°) mixtures of the $K_1(1270)$ and $K_1(1400)$.

		$I = 1$	$I = 1/2$	$I = 0$	$I = 0$
Octet	J^{PC}	$u\bar{d}, \bar{u}d, (d\bar{d} - u\bar{u})/\sqrt{2}$	$u\bar{s}, d\bar{s}; \bar{d}s, -\bar{u}s$	f'	f
P	0^{-+}	π	K	η	$\eta'(958)$
V	1^{--}	$\rho(770)$	$K^*(892)$	$\phi(1020)$	$\omega(782)$
A^-	1^{+-}	$b_1(1235)$	K_{1B}	$h_1(1380)$	$h_1(1170)$
S	0^{++}	$a_0(1450)$	$K_0^*(1430)$	$f_0(1710)$	$f_0(1370)$
A^+	1^{++}	$a_1(1260)$	K_{1A}	$f_1(1420)$	$f_1(1285)$
T	2^{++}	$a_2(1320)$	$K_2^*(1430)$	$f_2'(1525)$	$f_2(1270)$

$$f' = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})\cos\alpha - s\bar{s}\sin\alpha, \tag{A6}$$

and

$$f = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})\sin\alpha + s\bar{s}\cos\alpha, \tag{A7}$$

which is the orthogonal partner of f' (replace α by $\alpha - 90°$). Thus, for ideal mixing ($\alpha_i = 90°$), f' becomes pure $s\bar{s}$ and the f pure $u\bar{u} + d\bar{d}$.

Let us return to the C-parity transformation issue. Under the transformation of the generalized inherent C parity, $\hat{C}[O_1O_2]_x \to \xi_x[O_1O_2]_x$, where $x = 0, d, f$, that is $\xi_0 = +1$, $\xi_d = +1$, $\xi_f = -1$. In addition, under C-parity transformation, $\hat{C}O_i \to \eta_{O_i}O_i$, $(i = 1, 2)$, synthetically,

$$\hat{C}[O_1O_2]_x = \eta_{O_1}\eta_{O_2}\xi_x[O_1O_2]_x. \tag{A8}$$

At the same time for the initial state of ψ

$$\hat{C}\psi = \eta_\psi\psi. \tag{A9}$$

Therefore, the term $[O_1O_2]_x$ is allowed in the effective Hamiltonian as long as $\eta_\psi = -1 = \eta_{O_1}\eta_{O_2}\xi_x$. Otherwise, the term is forbidden. With this criterion, it is easy to figure out what kind of terms can be presented in the effective Hamiltonian for various kinds of final states; the results are summarized in Table XIII.

By virtue of Table XIII, two types of Hamiltonian forms exist. One type contains both $[O_1O_2]_0$ and $[O_1O_2]_d$ terms, while the other contains only $[O_1O_2]_f$ term, that is

$$\mathcal{H}_{\rm eff}^{O_1O_2} = g_0 \cdot [O_1O_2]_0 + g_m \cdot ([O_1O_2]_d)^3_3 + g_e \cdot ([O_1O_2]_d)^1_1, \tag{A10}$$

or

$$\mathcal{H}_{\rm eff}^{O_1O_2} = g_m \cdot ([O_1O_2]_f)^3_3 + g_e \cdot ([O_1O_2]_f)^1_1. \tag{A11}$$

Comparing with Eqs. (21) and (22), Eq. (A10) can be called the VP-type Hamiltonian, while Eq. (A11), the PP-type Hamiltonian. For the most general case of PP-type Hamiltonian, the mesons of final state may belong to distinctive octets, take PT mode as an example, by virtue of Eq. (A11), the parametrization form is obtained and displayed in Table XIV.

The second issue of this appendix is about another approach to derive the effective Hamiltonian.

Besides the nonet approach, the singlet component can be treated separately. Corresponding to the matrices in Eqs. (8) and (9), the singlet matrices are introduced as follows:

$$\mathbf{S}_V = \begin{pmatrix} \omega^1/\sqrt{3} & & \\ & \omega^1/\sqrt{3} & \\ & & \omega^1/\sqrt{3} \end{pmatrix}, \tag{A12}$$

and

$$\mathbf{S}_P = \begin{pmatrix} \eta^1/\sqrt{3} & & \\ & \eta^1/\sqrt{3} & \\ & & \eta^1/\sqrt{3} \end{pmatrix}. \tag{A13}$$

Therefore, besides the octet-octet effective Hamiltonian, it also has the singlet-singlet effective Hamiltonian and the octet-singlet effective Hamiltonian, as listed in the following equations:

$$\begin{aligned}
\mathcal{H}_{\rm eff}^{88} &= g_0^{88}[VP]_0 + g_m^{88}([VP]_d)^3_3 + g_e^{88}([VP]_d)^1_1,\\
\mathcal{H}_{\rm eff}^{11} &= g_0^{11}[S_VS_P]_0 + g_m^{11}([S_VS_P]_d)^3_3 + g_e^{11}([S_VS_P]_d)^1_1,\\
\mathcal{H}_{\rm eff}^{18} &= g_0^{18}[S_VP]_0 + g_m^{18}([S_VP]_d)^3_3 + g_e^{18}([S_VP]_d)^1_1,\\
\mathcal{H}_{\rm eff}^{81} &= g_0^{81}[VS_P]_0 + g_m^{81}([VS_P]_d)^3_3 + g_e^{81}([VS_P]_d)^1_1.
\end{aligned} \tag{A14}$$

The calculation results are displayed in Table XV. If the mixing between octet and singlet are taken into account according to Eqs. (24) and (26), the results are changed into

TABLE XIII. The determination of interaction terms in the effective Hamiltonian. The symbol $[O_1O_2]_x$ is shorthand for $[O_1O_2]_0$, $[O_1O_2]_d$, and $[O_1O_2]_f$. Herein there are essentially two types of Hamiltonian forms, that is "yyn," which means both $[O_1O_2]_0$ and $[O_1O_2]_d$ terms are allowed in the effective Hamiltonian, and "nny," which means only $[O_1O_2]_f$ term is allowed. The symbol "y" indicates the allowed term, while "n" indicates forbidden. O_1 and O_2 denote octets S, P, V, T, A^+, and A^-, which are shown in Table XII. The superscript of symbol indicates the generalized inherent C-parity of corresponding octet.

$[O_1O_2]_x$	S^+	P^+	V^-	T^+	A^+	A^-
S^+	nny	nny	yyn	nny	nny	yyn
P^+		nny	yyn	nny	nny	yyn
V^-			nny	yyn	yyn	nny
T^+				nny	nny	yyn
A^+					nny	yyn
A^-						nny

TABLE XIV. Amplitude parametrization form for decays of the ψ' or J/ψ into PT final states. The table can also be used for other similar decays by appropriate change in labeling.

Final state	Parametrization form	
$\pi^+a_2^-/\pi^-a_2^+$		$\pm g_e$
$K^+K_2^{\star -}/K^-K_2^{\star +}$	$\mp g_m$	$\pm g_e$
$K^0\bar{K}_2^{\star 0}/\bar{K}^0K_2^{\star 0}$	$\mp g_m$	

TABLE XV. Amplitude parametrization form for decays of the ψ' or J/ψ into VP final states. The results are based on the effective Hamiltonian in Eq. (A14). With the assumptions in the last three columns, the results are the same as those due to the nonet approach.

States	g_0^{88}	g_m^{88}	g_e^{88}	g_0^{11}	g_m^{11}	g_e^{11}	g_0^{18}	g_m^{18}	g_e^{18}	g_0^{81}	g_m^{81}	g_e^{81}	$g_0^{88}=g_0^{11}$	$g_m^{88}=g_m^{18}=g_m^{81}$	$g_e^{88}=g_e^{18}=g_e^{81}$
$\rho^0\pi^0$	1	$-2/3$	$1/3$	0	0	0	0	0	0	0	0	0	1	$-2/3$	$1/3$
$\rho^+\pi^-$	1	$-2/3$	$1/3$	0	0	0	0	0	0	0	0	0	1	$-2/3$	$1/3$
$\rho^+\pi^-$	1	$-2/3$	$1/3$	0	0	0	0	0	0	0	0	0	1	$-2/3$	$1/3$
$K^{*+}K^-$	1	$1/3$	$1/3$	0	0	0	0	0	0	0	0	0	1	$1/3$	$1/3$
$K^{*-}K^+$	1	$1/3$	$1/3$	0	0	0	0	0	0	0	0	0	1	$1/3$	$1/3$
$K^{*0}\bar{K}^0$	1	$1/3$	$-2/3$	0	0	0	0	0	0	0	0	0	1	$1/3$	$-2/3$
$\bar{K}^{*0}K^0$	1	$1/3$	$-2/3$	0	0	0	0	0	0	0	0	0	1	$1/3$	$-2/3$
$\omega^8\eta^8$	1	$2/3$	$-1/3$	0	0	0	0	0	0	0	0	0	1	$2/3$	$-1/3$
$\omega^8\pi^0$	0	0	$1/\sqrt{3}$	0	0	0	0	0	0	0	0	0	0	0	$1/\sqrt{3}$
$\rho^0\eta^8$	0	0	$1/\sqrt{3}$	0	0	0	0	0	0	0	0	0	0	0	$1/\sqrt{3}$
$\omega^1\eta^1$	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0
$\omega^1\eta^8$	0	0	0	0	0	0	0	$-2\sqrt{2}/3$	$\sqrt{2}/3$	0	0	0	0	$-2\sqrt{2}/3$	$\sqrt{2}/3$
$\omega^1\pi^0$	0	0	0	0	0	0	0	0	$\sqrt{2/3}$	0	0	0	0	0	$\sqrt{2/3}$
$\omega^8\eta^1$	0	0	0	0	0	0	0	0	0	0	$-2\sqrt{2}/3$	$\sqrt{2}/3$	0	$-2\sqrt{2}/3$	$\sqrt{2}/3$
$\rho^0\eta^1$	0	0	0	0	0	0	0	0	0	0	0	$\sqrt{2/3}$	0	0	$\sqrt{2/3}$

TABLE XVI. Amplitude parametrization form for decays of the ψ' or J/ψ into VP final states. The mixing between octet and singlet are taken into account according to Eqs. (24) and (26). The shorthand symbols are defined as $s_\alpha = \sin\theta_\alpha$ and $c_\alpha = \cos\theta_\alpha$ ($\alpha = V, P$).

Decay mode	Coupling constant			
$\psi \to X$	g_0^{88}	g_0^{11}	g_m	g_e
$\rho^0\pi^0$	1	0	$-2/3$	$1/3$
$\rho^+\pi^-$	1	0	$-2/3$	$1/3$
$\rho^+\pi^-$	1	0	$-2/3$	$1/3$
$K^{*+}K^-$	1	0	$1/3$	$1/3$
$K^{*-}K^+$	1	0	$1/3$	$1/3$
$K^{*0}\bar{K}^0$	1	0	$1/3$	$-2/3$
$\bar{K}^{*0}K^0$	1	0	$1/3$	$-2/3$
$\phi\eta$	$c_V c_P$	$s_V s_P$	$\frac{2}{3}c_V c_P + \frac{2\sqrt{2}}{3}(c_V s_P + s_V c_P)$	$-\frac{1}{3}c_V c_P - \frac{\sqrt{2}}{3}(c_V s_P + s_V c_P)$
$\phi\eta'$	$c_V s_P$	$-s_V c_P$	$\frac{2}{3}c_V s_P - \frac{2\sqrt{2}}{3}(c_V c_P - s_V s_P)$	$-\frac{1}{3}c_V s_P + \frac{\sqrt{2}}{3}(c_V c_P - s_V s_P)$
$\omega\eta$	$s_V c_P$	$-c_V s_P$	$\frac{2}{3}s_V c_P - \frac{2\sqrt{2}}{3}(c_V c_P - s_V s_P)$	$-\frac{1}{3}s_V c_P + \frac{\sqrt{2}}{3}(c_V c_P - s_V s_P)$
$\omega\eta'$	$s_V s_P$	$c_V c_P$	$\frac{2}{3}s_V s_P - \frac{2\sqrt{2}}{3}(c_V s_P + s_V c_P)$	$-\frac{1}{3}s_V s_P + \frac{\sqrt{2}}{3}(c_V s_P + s_V c_P)$
$\rho^0\eta$	0	0	0	$\sqrt{\frac{1}{3}}c_P - \sqrt{\frac{2}{3}}s_P$
$\rho^0\eta'$	0	0	0	$\sqrt{\frac{1}{3}}s_P + \sqrt{\frac{2}{3}}c_P$
$\phi\pi^0$	0	0	0	$\sqrt{\frac{1}{3}}c_V - \sqrt{\frac{2}{3}}s_V$
$\omega\pi^0$	0	0	0	$\sqrt{\frac{1}{3}}s_V + \sqrt{\frac{2}{3}}c_V$

Table XVI. If replacing s_V and c_V with $\sqrt{1/3}$ and $\sqrt{2/3}$, respectively, the Table II in Ref. [5] will be recovered. It is should be noted that for g_m, there is an overall normalized factor $-2/\sqrt{3}$ between our calculation and that of Ref. [5]. Furthermore, if we introduce a new angle and define $\sin\theta_\gamma \equiv \sqrt{1/3}$ and $\cos\theta_\gamma \equiv \sqrt{2/3}$, the more compact form of Table XVI can be obtained as shown in Table III.

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