Toward an estimate of the amplitude $X(3872) \to \pi^0\chi_{c1}(1P)$

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The well-known model of the triangle diagrams with $D^*\bar{D}D^*$ and $\bar{D}^*D\bar{D}^*$ mesons in the loops is compared with the modern data on the amplitude of the $X(3872) \to \pi^0\chi_{c1}(1P)$ decay. Considering the $X(3872)$ object as a $\chi_{c1}(2P)$ charmonium state, we introduce a parameter ξ characterizing the scale of the isotopic symmetry violation in this decay and find a lower limit of $\xi \simeq 0.0916$. The model incorporates the only fitted parameter associated with the form factor. We analyze in detail the influence of the form factor on the amplitude $X(3872) \to \pi^0\chi_{c1}(1P)$ and on the parameter ξ. As the suppression of the amplitude by the form factor increases, ξ increases. Because the $X(3872)$ resonance is located practically at the threshold of the $D^0\bar{D}^{*0}$ channel, the amplitude of $X(3872) \to \pi^0\chi_{c1}(1P)$ turns out to be proportional to $\sqrt{m_d - m_u}$. Using the estimating values for the coupling constants $g_{XD\bar{D}^*}$, $g_{\chi_{c1}D\bar{D}^*}$, and $g_{D^{*0}D^{*0}\pi^0}$, we show that the model of the triangle loop diagrams is in reasonable agreement with the available data. Apart from the difference in the masses of neutral and charged charmed mesons, any additional exotic sources of isospin violation in $X(3872) \to \pi^0\chi_{c1}(1P)$ (such as a significant difference between the coupling constants $g_{XD^0\bar{D}^{*0}}$ and $g_{XD^+D^{*-}}$) are not required to interpret the data. This indirectly confirms the isotopic neutrality of the $X(3872)$, which is naturally realized for the $c\bar{c}$ state $\chi_{c1}(2P)$.

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I. INTRODUCTION

The state $X(3872)$ or $\chi_{c1}(3872)$ [1] was observed for the first time by the Belle Collaboration in 2003 in the process $B^\pm \to (X(3872) \to \pi^+\pi^- J/\psi)K^\pm$ [2]. Then it was observed in many other experiments in other processes and decay channels [1,3]. The $X(3872)$ is a very narrow resonance. Its visible width depends on the decay channel. In the $\pi^+\pi^- J/\psi$ channel, the width of the $X(3872)$ peak is approximately of 1 MeV [1,4,5] and in the $(D^{*0}\bar{D}^0 + \bar{D}^{*0}D^0) \to D^0\bar{D}^0\pi^0$ channel, it is of about 2–5 MeV [1,6–10]. Its mass coincides practically with the $D^{*0}\bar{D}^0$ threshold [1]. The $X(3872)$ has the quantum numbers $I^G(J^{PC}) = 0^+(1^{++})$ [1,4,11–13]. In addition to decays into $\pi^+\pi^- J/\psi$ [2,4,5,12,14] and $D^0\bar{D}^0\pi^0$ [6–10], the $X(3872)$ also decays into $\omega J/\psi$ [15–17], $\gamma J/\psi$ [15,18–21], $\gamma\psi(2S)$ [18–20], and $\pi^0\chi_{c1}(1P)$ [22,23]. The $X(3872)$ became the first candidate for exotic charmoniumlike states, and many hypotheses have been put forward about its nature; see Refs. [1–34] and references herein. For example, the $X(3872)$ is interpreted as a hadronic $D\bar{D}^*$ molecule [25,26], a compact tetraquark state [27], a conventional charmonium state $\chi_{c1}(2P)$ [28–31], a mixture of a molecule, and an excited charmonium state [32–34], etc. So far, none of these explanations have become generally accepted. But there is hope that new, more and more accurate experiments will allow us to make a definite choice between the different interpretations.

Of great interest are the $X(3872)$ decays that violate isospin: $X(3872) \to \pi^+\pi^- J/\psi$, $X(3872) \to \pi^0\chi_{c1}(1P)$, and $X(3872) \to \pi^0\pi^+\pi^-$ [15–17,22,23,32,35–52]. In what follows, we will discuss the $X(3872) \to \pi^0\chi_{c1}(1P)$ decay. Even before the appearance of the BESIII [22] and Belle [23] data (see also [1]), a number of model predictions were made for it [37,38,44]. Then this decay was studied in the works [46,47,50]. In Ref. [37], under the assumption that $X(3872)$ is a conventional $c\bar{c}$ state and that π^0 is produced in its decay via two-gluon mechanism, the value of $\simeq$0.06 keV was obtained for the width $\Gamma(X(3872) \to \pi^0\chi_{c1}(1P))$, which is several orders of magnitude less than what follows from the experiment [22]. In Ref. [38], the $X(3872)$ was considered as a loosely bound state of neutral charmed mesons $D^0\bar{D}^{*0} + \bar{D}^0D^{*0}$. If the decay of such a molecular quarkonium into $\pi^0\chi_{c1}(1P)$ results from the neutral charmed meson loop mechanism, then, according to the estimate [44], $\Gamma(X(3872) \to \pi^0\chi_{c1}(1P))$ turns out to be greater than the total $X(3872)$ width. To avoid contradictions with experiment, it was proposed [44] to take into

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account the coupling of the $X(3872)$ to charged charmed mesons $D^+D^{*-} + D^-D^{*+}$. In this case, the contributions of the triangle loops with neutral and charged $D^{(*)}$ mesons should partially compensate each other in the transition amplitude $X(3872) \to \pi^0\chi_{c1}(1P)$ [44], which is completely natural for the $X(3872)$ state with isospin $I = 0$. In Ref. [46], to describe the $X(3872)$, a scheme was used in which $D\bar{D}^*$ pairs were considered as the dominant components in its wave function, and it was obtained that $\Gamma(X(3872) \to \pi^0\chi_{c1}(1P))$ is an order of magnitude smaller than $\Gamma(X(3872) \to \pi^+\pi^-J/\psi)$. In Ref. [47], the molecular scenario for the $X(3872)$ was considered. It was assumed that the strong isospin violation in the decays $X(3872) \to \pi^+\pi^-J/\psi$, $X(3872) \to \pi^+\pi^-\pi^0J/\psi$, and $X(3872) \to \pi^0\chi_{cJ}(1P)$ comes from the different coupling strengths of the $X(3872)$ to its charged D^+D^{*-} and neutral $D^0\bar{D}^{*0}$ components as well as through the interference between the charged and neutral meson loops. In Ref. [47], the nonstandard normalizations were used for $\Gamma(X(3872) \to \pi^+\pi^-J/\psi)$ and $\Gamma(X(3872) \to \pi^+\pi^-\pi^0J/\psi)$ (see Ref. [53]), and therefore, the agreement with experiment obtained for the ratio $\Gamma(X(3872) \to \pi^0\chi_{c1}(1P))/\Gamma(X(3872) \to \pi^+\pi^-J/\psi)$ is doubtful. In Ref. [50], the $X(3872)$ was considered as a tetraquark state with the $I = 0$ and 1 isospin components, and its decays were analyzed via the QCD sum rules. In so doing, for $\Gamma(X(3872) \to \pi^0\chi_{c1}(1P))$ the value of ≈ 0.0016 MeV was obtained, which is approximately 20 times smaller in comparison with the experimental estimate [1].

In the present work, we consider the $X(3872)$ meson as a $\chi_{c1}(2P)$ charmonium state, which has the equal coupling constants with the $D^0\bar{D}^{*0}$ and D^+D^{*-} channels owing to the isotopic symmetry. Section II collects the available data on the $X(3872) \to \pi^0\chi_{c1}(1P)$ decay. In Sec. III, we calculate the transition amplitude $X(3872) \to \pi^0\chi_{c1}(1P)$ corresponding to the simplest $D^*\bar{D}D^* +$ c.c. loop mechanism [44,47], we pay attention to details that were not previously discussed, and introduce the parameter ξ characterizing the natural scale of isospin violation for the process under consideration. In Sec. IV, we analyze in detail the influence of the form factor on the magnitude of the amplitude $X(3872) \to \pi^0\chi_{c1}(1P)$ and on the parameter ξ. Using the evaluating values for coupling constants $g_{XD\bar{D}^*}$, $g_{\chi_{c1}D\bar{D}^*}$, and $g_{D^{*0}D^{*0}\pi^0}$, we show that the model of charmed meson loops explains the data on the absolute value of the amplitude of the $X(3872) \to \pi^0\chi_{c1}(1P)$ decay by a quite naturally way. Our conclusions from the presented analysis are given in Sec. V, together with a short comment regarding the molecular model of the $X(3872)$ state.

II. DATA ON THE $X(3872) \to \pi^0\chi_{c1}(1P)$ DECAY

Let us write the transition amplitude $X(3872) \to \pi^0\chi_{c1}(1P)$ in the form,

$$\mathcal{M}(X(3872) \to \pi^0\chi_{c1}(1P); s) \equiv \mathcal{M}_{\pi^0}(s) = \varepsilon^{\mu\nu\lambda\kappa}\epsilon^X_\mu(p_1)\epsilon^{*\chi_{c1}}_\nu(p_2)p_{1\lambda}p_{2\kappa}G_{\pi^0}(s), \quad (1)$$

where $\epsilon^X(p_1)$ and $\epsilon^{*\chi_{c1}}(p_2)$ are the polarization four-vectors of the $X(3872)$ and $\chi_{c1}(1P)$ mesons, respectively (helicity indices omitted), p_1, p_2 and $p_3 = p_1 - p_2$ are the four-momenta of $X(3872)$, $\chi_{c1}(1P)$, and π^0, respectively, $s = (p_2 + p_3)^2$ is the squared invariant mass of the $\pi^0\chi_{c1}(1P)$ system or of the virtual $X(3872)$ state, and $G_{\pi^0}(s)$ is the invariant amplitude. The energy-dependent width of the $X(3872) \to \pi^0\chi_{c1}(1P)$ decay in the rest frame of $X(3872)$ is expressed in terms of $G_{\pi^0}(s)$ as follows:

$$\Gamma(X(3872) \to \pi^0\chi_{c1}(1P); s) = \frac{|G_{\pi^0}(s)|^2}{12\pi}|\vec{p}_3|^3, \quad (2)$$

where $|\vec{p}_3| = \sqrt{s^2 - 2s(m^2_{\chi_{c1}} + m^2_{\pi^0}) + (m^2_{\chi_{c1}} - m^2_{\pi^0})^2}/(2\sqrt{s})$. The following information is available about the decay of $X(3872) \to \pi^0\chi_{c1}(1P)$. The BESIII Collaboration [22] observed this decay and determined the value of the ratio,

$$\frac{\mathcal{B}(X(3872) \to \pi^0\chi_{c1}(1P))}{\mathcal{B}(X(3872) \to \pi^+\pi^-J/\psi)} = 0.88^{+033}_{-027} \pm 0.10. \quad (3)$$

The Belle Collaboration [23] set an upper limit for this ratio,

$$\frac{\mathcal{B}(X(3872) \to \pi^0\chi_{c1}(1P))}{\mathcal{B}(X(3872) \to \pi^+\pi^-J/\psi)} < 0.97, \quad (4)$$

at the 90% confidence level. The Particle Data Group (PDG) [1] gives for the branching fraction of $X(3872) \to \pi^0\chi_{c1}(1P)$ the following value:

$$\mathcal{B}(X(3872) \to \pi^0\chi_{c1}(1P)) = (3.4 \pm 1.6)\%, \quad (5)$$

and also gives a constraint $\mathcal{B}(X(3872) \to \pi^0\chi_{c1}(1P)) < 4\%$ based on the Belle data. Moreover, according to the analysis presented in Ref. [54], $\mathcal{B}(X(3872) \to \pi^0\chi_{c1}(1P)) = (3.6^{+2.2}_{-1.6})\%$.

Using Eqs. (2) and (5) and the value of the $X(3872)$ total decay width presented by the PDG [1], $\Gamma^{\rm tot}_X = (1.19 \pm 0.21)$ MeV, we obtain the following approximate estimates for the absolute decay width of $X(3872) \to \pi^0\chi_{c1}(1P)$ and for the effective coupling constant $|G_{\pi^0}(m^2_X)|$:

$$\Gamma(X(3872) \to \pi^0\chi_{c1}(1P); m^2_X) = (0.04 \pm 0.02)\ \text{MeV},$$
$$|G_{\pi^0}(m^2_X)| = (0.216 \pm 0.054)\ \text{GeV}^{-1}. \quad (6)$$

III. LOOP MECHANISM OF $X(3872) \to \pi^0\chi_{c1}(1P)$

Let us consider the simplest model of triangle loop diagrams for the amplitude $\mathcal{M}_{\pi^0}(s)$ introduced in Eq. (1). It is graphically depicted in Fig. 1. The specific structure of the vertices in these diagrams is determined with use of the effective Lagrangian,

$$\mathcal{L} = ig_{XD\bar{D}^*}X^\mu\left(D^\dagger D^*_\mu - DD^{*\dagger}_\mu\right) + ig_{\chi_{c1}D\bar{D}^*}\chi^\mu_{c1}\left(D^\dagger D^*_\mu - DD^{*\dagger}_\mu\right) + g_{D^{*0}D^{*0}\pi^0}\varepsilon^{\mu\nu\lambda\kappa}\partial_\mu D^*_\nu\left(\hat{\vec{\tau}}\,\vec{\pi} + \eta_0\right)\partial_\lambda D^{*\dagger}_\kappa, \tag{7}$$

where D, $D^\dagger$, D^*, and $D^{*\dagger}$ are the charm meson isodoublets, $\hat{\vec{\tau}} = (\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3)$ are the Pauli matrices, $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ is the isotopic triplet of π mesons, the $\pi_3 = \pi^0$ state has the quark structure $(u\bar{u} - d\bar{d})/\sqrt{2}$, and η_0 denotes isosinglet pseudoscalar state with the quark structure $(u\bar{u} + d\bar{d})/\sqrt{2}$. The amplitude of the virtual η_0 state production, $\mathcal{M}_{\eta_0}(s)$, will be useful to us in the following. For the coupling constants indicated in Eq. (7), we introduce short notations: $g_{XD\bar{D}^*} = g_X$, $g_{\chi_{c1}D\bar{D}^*} = g_{\chi_{c1}}$, and $g_{D^{*0}D^{*0}\pi^0} = g_{\pi^0}$. In accordance with Fig. 1, we represent the amplitudes $\mathcal{M}_{\pi^0}(s)$ and $\mathcal{M}_{\eta_0}(s)$ in the following form:

$$\mathcal{M}_{\pi^0}(s) = \frac{g_X g_{\chi_{c1}} g_{\pi^0}}{16\pi}\varepsilon^{\mu\nu\lambda\kappa}\epsilon^X_\mu(p_1)\epsilon^{*\chi_{c1}}_\nu(p_2) \times \left[2C^n_\lambda(s) - 2C^c_\lambda(s)\right]p_{3\kappa}, \tag{8}$$

$$\mathcal{M}_{\eta_0}(s) = \frac{g_X g_{\chi_{c1}} g_{\eta_0}}{16\pi}\varepsilon^{\mu\nu\lambda\kappa}\epsilon^X_\mu(p_1)\epsilon^{*\chi_{c1}}_\nu(p_2) \times \left[2C^n_\lambda(s) + 2C^c_\lambda(s)\right]p_{3\kappa}, \tag{9}$$

where $g_{\eta_0} = g_{\pi^0}$, the amplitudes $C^n_\lambda(s)$ and $C^c_\lambda(s)$ correspond to the diagrams with neutral and charged particles in the loops, respectively, and the factor 2 in front of them takes into account that for each type of particles there are two such diagrams. The amplitudes $C^n_\lambda(s)$ and $C^c_\lambda(s)$ are converged separately and have the form,

$$C^n_\lambda(s) = \frac{i}{\pi^3}\int\frac{k_\lambda d^4k}{(k^2 - m^2_{D^{*0}} + i\varepsilon)((p_1 - k)^2 - m^2_{\bar{D}^0} + i\varepsilon)((k - p_3)^2 - m^2_{D^{*0}} + i\varepsilon)} = p_{1\lambda}C^n_{11}(s) + p_{3\lambda}C^n_{12}(s), \tag{10}$$

$$C^c_\lambda(s) = \frac{i}{\pi^3}\int\frac{k_\lambda d^4k}{(k^2 - m^2_{D^{*+}} + i\varepsilon)((p_1 - k)^2 - m^2_{D^-} + i\varepsilon)((k - p_3)^2 - m^2_{D^{*+}} + i\varepsilon)} = p_{1\lambda}C^c_{11}(s) + p_{3\lambda}C^c_{12}(s). \tag{11}$$

Substitution of Eqs. (10) and (11) into Eq. (8) and comparison the result with Eq. (1) give [the functions $C^n_{12}(s)$ and $C^c_{12}(s)$ do not contribute]

$$\mathcal{M}_{\pi^0}(s) = -\frac{g_X g_{\chi_{c1}} g_{\pi^0}}{16\pi}\varepsilon^{\mu\nu\lambda\kappa}\epsilon^X_\mu(p_1)\epsilon^{*\chi_{c1}}_\nu(p_2)p_{1\lambda}p_{2\kappa} \times \left[2C^n_{11}(s) - 2C^c_{11}(s)\right], \tag{12}$$

$$G_{\pi^0}(s) = -\frac{g_X g_{\chi_{c1}} g_{\pi^0}}{16\pi}\left[2C^n_{11}(s) - 2C^c_{11}(s)\right]. \tag{13}$$

The representation of invariant amplitudes $C^n_{11}(s)$ and $C^c_{11}(s)$ via dilogarithms is well known [55–58]. However, it will be convenient for us to calculate them using the dispersion method. To do this, we shall first find their imaginary parts. They are determined by the contributions of real intermediate states, i.e., contributions in which both charmed mesons outgoing from the vertex of the $X(3872)$ decay are on the mass shell. Applying the Kutkosky rule [59] to the amplitude $C^n_\lambda(s)$ [see diagram (a) in Fig. 1], we find

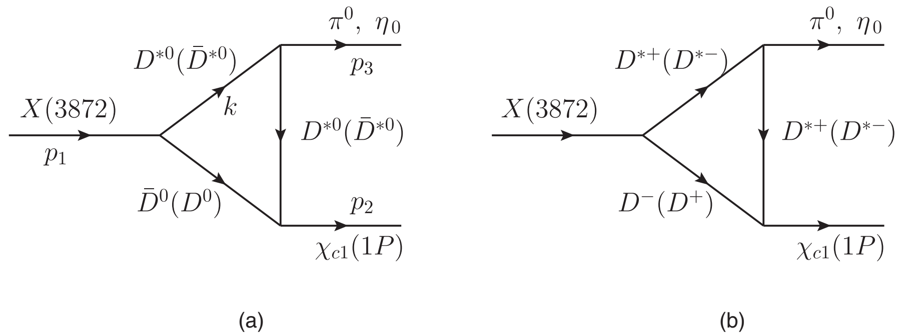

FIG. 1. The model of triangle loop diagrams for the transitions $X(3872) \to (D\bar{D}^* + \bar{D}D^*) \to (\pi^0, \eta_0)\chi_{c1}(1P)$.

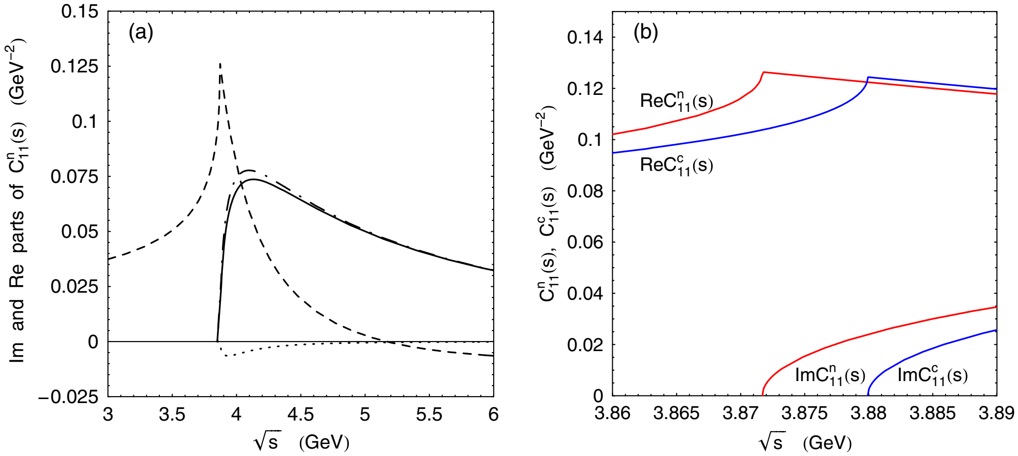

FIG. 2. (a) The solid and dashed curves show the imaginary and real parts of the amplitude $C^n_{11}(s)$ constructed using Eqs. (15) and (16), respectively, in a wide region of $\sqrt{s}$. The dash-dotted curve shows the contribution to $\mathrm{Im}C^n_{11}(s)$ from the first (dominant) term in Eq. (15) and the dotted curve shows the contribution from the term containing a logarithm. (b) The imaginary and real parts of the amplitudes $C^n_{11}(s)$ and $C^c_{11}(s)$ in the region of the $D\bar{D}^*$ thresholds.

$$\begin{aligned}\mathrm{Im}C^n_\lambda(s) &= \frac{-|\vec{k}|}{2\pi\sqrt{s}}\int\frac{k_\lambda d\cos\theta d\varphi}{(k-p_3)^2-m^2_{D^{*0}}}\\ &= \frac{-|\vec{k}|}{2\pi\sqrt{s}}\int\frac{k_\lambda d\cos\theta d\varphi}{m^2_{\pi^0}-2k_0p_{30}+2|\vec{k}||\vec{p}_3|\cos\theta}\\ &= p_{1\lambda}\mathrm{Im}C^n_{11}(s)+p_{3\lambda}\mathrm{Im}C^n_{12}(s), \end{aligned} \tag{14}$$

where k_λ are the components of the four-momentum $k=(k_0,\vec{k})$ of the intermediate D^{*0} meson [outgoing from the vertex of the $X(3872)$ decay] on its mass shell in the rest frame of $X(3872)$, the polar angle θ and the azimuthal angle φ determine the direction of the vector $\vec{k}$ in the reference frame with the z axis directed along the momentum $\vec{p}_3$; $k_0=(s+m^2_{D^{*0}}-m^2_{D^0})/(2\sqrt{s})$, $|\vec{k}|=\sqrt{s^2-2s(m^2_{D^{*0}}+m^2_{D^0})+(m^2_{D^{*0}}-m^2_{D^0})^2}/(2\sqrt{s})$, and $p_{30}=(s+m^2_{\pi^0}-m^2_{\chi_{c1}})/(2\sqrt{s})$. After calculating the scalar products $p^\lambda_1\mathrm{Im}C^n_\lambda(s)$ and $p^\lambda_3\mathrm{Im}C^n_\lambda(s)$, we get

$$\mathrm{Im}C^n_{11}(s)=\frac{1}{s|\vec{p}_3|^2}\left[|\vec{k}|p_{30}-\left(k_0-\frac{1}{2}p_{30}\right)\frac{m^2_{\pi^0}}{2|\vec{p}_3|}\times\ln\left(\frac{m^2_{D^{*0}}-t_-}{m^2_{D^{*0}}-t_+}\right)\right], \tag{15}$$

where $t_\pm=m^2_{D^{*0}}+m^2_{\pi^0}-2k_0p_{30}\pm2|\vec{k}||\vec{p}_3|$ are the boundary values of the variable $t=(k-p_3)^2$ at $\cos\theta=\pm1$. For $\sqrt{s}\gg4$ GeV, $\mathrm{Im}C^n_{11}(s)\sim1/s$. We determine the real part of the amplitude $C^n_{11}(s)$ numerically from the dispersion relation,

$$C^n_{11}(s)=\frac{1}{\pi}\int_{s_n}^\infty\frac{\mathrm{Im}C^n_{11}(s')}{s'-s-i\varepsilon}ds', \tag{16}$$

where $s_n=(m_{D^0}+m_{\bar{D}^{*0}})^2$. Figure 2(a) shows the result of calculating the imaginary and real parts of the amplitude $C^n_{11}(s)$ using Eqs. (15) and (16) in a wide region of $\sqrt{s}$. Of course, we will ultimately be interested a very narrow energy region near the $D^0\bar{D}^{*0}$ threshold where the $X(3872)$ object is located. The amplitude $C^c_{11}(s)$ is calculated in exactly the same way. In the region of the $D\bar{D}^*$ thresholds, the imaginary and real parts of the amplitudes $C^n_{11}(s)$ and $C^c_{11}(s)$ are shown in Fig. 2(b), and the modulus and imaginary part of the difference $2C^n_{11}(s)-2C^c_{11}(s)$ are shown in Fig. 3(a). The s dependence of the function $2C^n_{11}(s)-2C^c_{11}(s)$ in this region is well approximated by the difference between the rapidly changing threshold factors $\rho^n(s)$ and $\rho^c(s)$ [see the dotted curve in Fig. 3(a) as an example]:

$$2C^n_{11}(s)-2C^c_{11}(s)\simeq i[\rho^n(s)-\rho^c(s)]\times(0.692\ \mathrm{GeV}^{-2}), \tag{17}$$

where $\rho^n(s)=\sqrt{1-(m_{D^0}+m_{\bar{D}^{*0}})^2/s}$ and $\rho^c(s)=\sqrt{1-(m_{D^+}+m_{D^{*-}})^2/s}$ for $\sqrt{s}$ above the corresponding threshold, and below one $\rho^n(s)\to i|\rho^n(s)|$ and $\rho^c(s)\to i|\rho^c(s)|$. Note that at the $D^0\bar{D}^{*0}$ threshold $2C^n_{11}((m_{D^0}+m_{D^{*0}})^2)-2C^c_{11}((m_{D^0}+m_{D^{*0}})^2)\simeq|\rho^c((m_{D^0}+m_{D^{*0}})^2)|\times(0.692\ \mathrm{GeV}^{-2})$; i.e., as a result of compensation, this difference is determined by the remainder of the contribution of charged intermediate states $D^+D^{*-}+D^-D^{*+}$. For $\sqrt{s}$ between the $D\bar{D}^*$ thresholds, we have

$$|\rho^n(s)-\rho^c(s)|\simeq\sqrt{\frac{2(m_{D^+}+m_{D^{*-}}-m_{D^0}-m_{\bar{D}^{*0}})}{m_{D^0}+m_{\bar{D}^{*0}}}}\simeq0.0652. \tag{18}$$

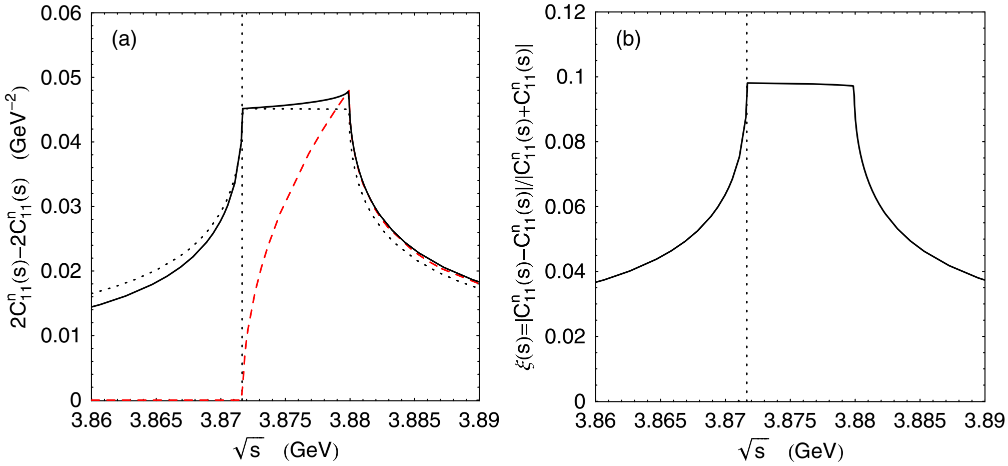

FIG. 3. (a) The solid and dashed curves show the magnitude and imaginary part of the amplitude $2C^n_{11}(s) - 2C^c_{11}(s)$ from Eq. (12); the dotted curve corresponds to the approximation of $|2C^n_{11}(s) - 2C^c_{11}(s)|$ using Eq. (17). (b) The energy-dependent isospin violation parameter $\xi(s) = |C^n_{11}(s) - C^c_{11}(s)|/|C^n_{11}(s) + C^c_{11}(s)|$. The vertical dashed lines in (a) and (b) mark the position of the $X(3872)$ resonance.

Since the $X(3872)$ resonance is located almost at the threshold of the $D^0\bar{D}^{*0}$ channel [see Fig. 3(a)], then the amplitude of the isospin-violating decay $X(3872) \to \pi^0\chi_{c1}(1P)$, that is due to the considered loop mechanism, turns out to be proportional to $\sqrt{m_d - m_u}$ [see Eqs. (17) and (18)], rather than to $m_d - m_u$ [similar to the threshold effect of the $a_0(980) - f_0(980)$ mixing [60,61]].

As a dimensionless parameter characterizing the scale of isospin violation, it is natural to take the ratio of the production amplitudes of the π^0 and η_0 states [see. diagrams in Fig. 1 and Eqs. (8) and (9)], i.e., the quantity

$$\xi(s) = \frac{|C^n_{11}(s) - C^c_{11}(s)|}{|C^n_{11}(s) + C^c_{11}(s)|}. \tag{19}$$

The energy dependence of the parameter $\xi(s)$ is shown in Fig. 3(b). At $\sqrt{s} = m_X = 3871.65$ MeV [1], we have

$$\xi = \xi(m_X^2) \simeq 0.0916. \tag{20}$$

As we will see in the next section, this value is a lower limit for ξ in the considered model. If the above estimate of ξ being the relative quantity can be rated as sufficiently reasonable, then to estimates of the absolute values of the strong interaction amplitudes $C^n_{11}(s)$ and $C^c_{11}(s)$ [see Figs. 2 and 3(a)], we should treat with the extreme caution. Here, we mean the need to take into account the influence of the form factor on these amplitudes in order to obtain physically more meaningful estimates for them. We discuss this issue below.

IV. ESTIMATE OF THE AMPLITUDE $X(3872) \to \pi^0\chi_{c1}(1P)$

In order to take into account to some extent the internal structure and the off-mass-shell effect for the D^* meson, by which there is the exchange between the intermediate $D(\bar{D})$ and $\bar{D}^*(D^*)$ mesons in the triangle loops (see Fig. 1), it is necessary to introduce the form factor into each vertex of the D^* exchange,

$$\mathcal{F}(q^2, m^2_{D^*}) = \frac{\Lambda^2 - m^2_{D^*}}{\Lambda^2 - q^2}, \tag{21}$$

where Λ is the cutoff parameter, m_{D^*} and q are the mass and four-momentum of the exchanged D^* meson, respectively. Such a type of the monopole form factor was first used in [62,63] to calculate triangle loops when describing the annihilation process at rest $p\bar{p} \to \pi\phi$, introduced into use [64,65] for estimating rescattering effects in $B^- \to K^-\chi_{c0}$, $B^- \to K^-h_c$ decays, discussed in detail in calculations of final state interactions in various hadronic B meson decay channels [66], and is now widely used in describing loop mechanisms of heavy quarkonium decays; see, for example, [36,43,47,67–69] and references herein. The standard form of the parameter Λ is [66] $\Lambda = m_{D^*} + \alpha\Lambda_{\mathrm{QCD}}$, where $\Lambda_{\mathrm{QCD}} = 220$ MeV and *a priori* unknown value of α is found from fitting the data. Let us rewrite Eq. (21) as follows: $\mathcal{F}(q^2, m^2_{D^*}) = \frac{1}{1+(m^2_{D^*}-q^2)/(\Lambda^2-m^2_{D^*})}$. From here, it is clear that the parameter $1/(\Lambda^2 - m^2_{D^*})$ determines the rate of change of the form factor when the D^* meson leaves the mass shell.

Let us now write the expression for $\mathrm{Im}C^n_\lambda(s)$ [see Eq. (14)] taking into account the form factor,

$$\mathrm{Im}C^n_\lambda(s) = p_{1\lambda}\mathrm{Im}C^n_{11}(s) + p_{3\lambda}\mathrm{Im}C^n_{12}(s) = \frac{-|\vec{k}|}{2\pi\sqrt{s}}\int\frac{\mathcal{F}^2(q^2, m^2_{D^{*0}})k_\lambda d\cos\theta d\varphi}{(k-p_3)^2 - m^2_{D^{*0}}}, \quad (22)$$

where $q^2 = (k-p_3)^2$, and carry out the corresponding calculations. As a result, the first term in Eq. (15) is multiplied by

$$\frac{(\Lambda^2 - m^2_{D^{*0}})^2}{(\Lambda^2 - t_+)(\Lambda^2 - t_-)}, \quad (23)$$

and $\ln[(m^2_{D^{*0}} - t_-)/(m^2_{D^{*0}} - t_+)]$ is replaced by

$$\ln\left[\frac{(m^2_{D^{*0}} - t_-)(\Lambda^2 - t_+)}{(m^2_{D^{*0}} - t_+)(\Lambda^2 - t_-)}\right] - \frac{(\Lambda^2 - m^2_{D^{*0}})(t_+ - t_-)}{(\Lambda^2 - t_+)(\Lambda^2 - t_-)}. \quad (24)$$

Note that in the case under consideration, the virtuality of the D^{*0}-meson, i.e., $(m^2_{D^{*0}} - q^2)$ turns out to be greater than 1.373 GeV^2. At $\sqrt{s} \gg 4$ GeV, the amplitude $\mathrm{Im}C^n_{11}(s)$ taking into account the form factor falls as $1/s^2$. The real part of $C^n_{11}(s)$ is determined numerically from the dispersion relation (16). The amplitude $C^c_{11}(s)$ taking into account the form factor is calculated in exactly the same way. Figure 4(a) shows as an example the result of the calculation of the imaginary and real parts of the amplitude $C^n_{11}(s)$ taking into account the form factor (21) at $\alpha = 2.878$ ($\Lambda \simeq 2.64$ GeV) in a wide region of $\sqrt{s}$. In the region of the $D\bar{D}^*$ thresholds, the imaginary and real parts of the amplitudes $C^n_{11}(s)$ and $C^c_{11}(s)$ taking into account form factors at $\alpha = 2.878$ are shown in Fig. 4(b). Comparison of the curves in Fig. 4(b) with those in Fig. 2(b), which correspond to $\mathcal{F}^2(q^2, m^2_{D^*}) \equiv 1$ (i.e., $\alpha = \infty$), shows that the form factor with $\alpha = 2.878$ reduces the amplitudes near the $D\bar{D}^*$ thresholds by approximately 3.5 times.

Let us now trace with the help of Figs. 5 and 6(a) for the influence of the form factor on the modulus of the amplitude difference $2C^n_{11}(s) - 2C^c_{11}(s)$, the parameter $\xi(s)$ and its particular value $\xi = \xi(m^2_X)$ [see Eqs. (19) and (20)]. As can be seen from the examples shown in Fig. 5, $|2C^n_{11}(s) - 2C^c_{11}(s)|$ and $\xi(s)$ have opposite dependences on α. With increasing suppression of the $|2C^n_{11}(s) - 2C^c_{11}(s)|$ amplitude by the form factor (i.e., with decreasing α), $\xi(s)$ increases. For $\sqrt{s} = m_X$, $|2C^n_{11}(m^2_X) - 2C^c_{11}(m^2_X)|$ and $\xi = \xi(m^2_X)$ as functions of α are shown in Fig. 6(a). This figure also explains why there is an increase in isospin violation, i.e., increasing the parameter $\xi = \xi(m^2_X)$, with decreasing α. This behavior of ξ is due to different suppression rate of the amplitudes $|2C^n_{11}(m^2_X) - 2C^c_{11}(m^2_X)|$ and $|2C^n_{11}(m^2_X) + 2C^c_{11}(m^2_X)|$ with decreasing α (or Λ) in the form factor; see the dashed and dash-dotted curves in Fig. 6(a).

Now we are ready to estimate the absolute value of the $X(3872) \to \pi^0\chi_{c1}(1P)$ decay amplitude. First of all, we indicate those values of the product of coupling constants $|g_X g_{\chi_{c1}} g_{\pi^0}|/(16\pi)$ for which the considered model can be consistent with available data. Using Eqs. (6) and (13), we write

$$\frac{|g_X g_{\chi_{c1}} g_{\pi^0}|}{16\pi} = \frac{|G_{\pi^0}(m^2_X)|}{|2C^n_{11}(m^2_X) - 2C^c_{11}(m^2_X)|} = \frac{(0.216 \pm 0.054)\ \mathrm{GeV}^{-1}}{|2C^n_{11}(m^2_X) - 2C^c_{11}(m^2_X)|}. \quad (25)$$

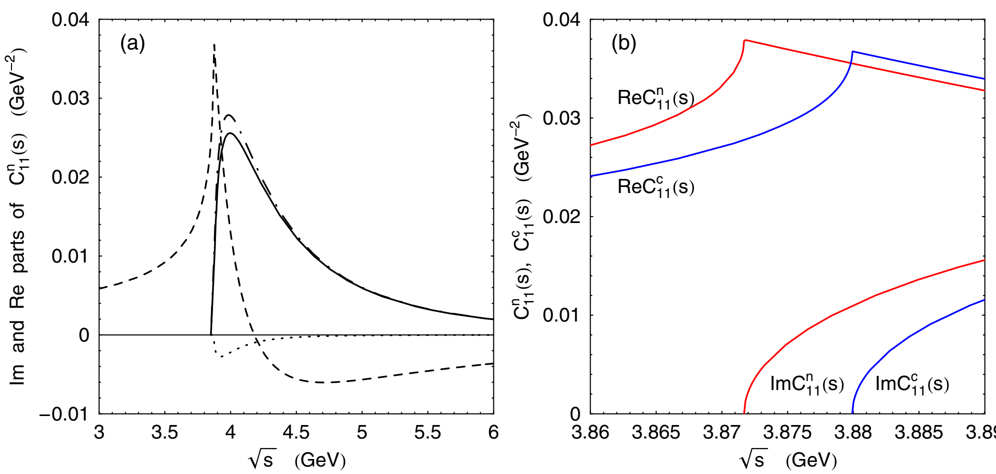

FIG. 4. (a) The solid and dashed curves show the imaginary and real parts of the amplitude $C^n_{11}(s)$, respectively, constructed in a wide region of $\sqrt{s}$ taking into account the form factor, see Eq. (21), at $\alpha = 2.878$ ($\Lambda \simeq 2.64$ GeV) according to Eqs. (16) and (22)–(24). The dash-dotted curve shows the contribution to $\mathrm{Im}C^n_{11}(s)$ from the first term in Eq. (15) modified according to Eq. (23), and the dotted curve is from the term containing logarithm modified according to Eq. (24). (b) Imaginary and real parts of the amplitudes $C^n_{11}(s)$ and $C^c_{11}(s)$ in the region of the $D\bar{D}^*$ thresholds taking into account the form factors at $\alpha = 2.878$.

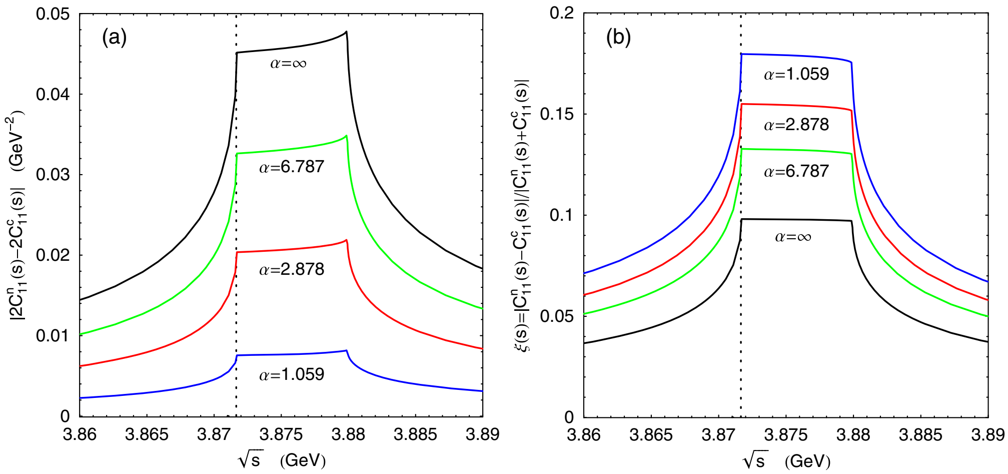

FIG. 5. (a) Modulus of the amplitude $2C^n_{11}(s) - 2C^c_{11}(s)$ for several values of the parameter α. (b) The energy-dependent isospin violation parameter $\xi(s) = |C^n_{11}(s) - C^c_{11}(s)|/|C^n_{11}(s) + C^c_{11}(s)|$ for the same values of α. The vertical dotted lines in (a) and (b) mark the position of the $X(3872)$ resonance.

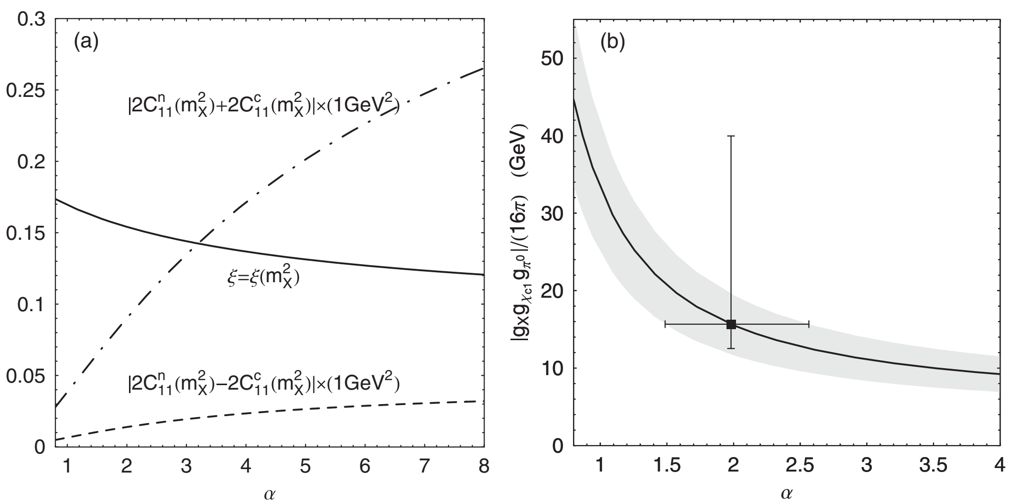

FIG. 6. (a) The isospin violation parameter $\xi = \xi(m_X^2)$ and dimensionless amplitudes $|2C^n_{11}(m_X^2) - 2C^c_{11}(m_X^2)| \times (1\ \mathrm{GeV}^2)$ and $|2C^n_{11}(m_X^2) + 2C^c_{11}(m_X^2)| \times (1\ \mathrm{GeV}^2)$ as functions of α [for $\alpha = \infty$ ($\Lambda = \infty$), i.e., when $\mathcal{F}^2(q^2, m^2_{D^*}) \equiv 1$, the asymptotics of these quantities are 0.0916, 0.0419, and 0.457, respectively]. (b) The shaded band shows the dependence on α of the values of the right-hand side of Eq. (25) lying within the uncertainty of the quantity $|G_{\pi^0}(m_X^2)|$; the solid curve inside the band corresponds to the central value of $|G_{\pi^0}(m_X^2)|$. The dot with vertical error bars shows the estimate presented in Eq. (28) for the left side of Eq. (25); the horizontal segment of the straight line marks the interval of α values at which the Eq. (25) is consistent.

From Eq. (25), it follows that the suitable values of $|g_X g_{\chi_{c1}} g_{\pi^0}|/(16\pi)$ (for reasonable values of α) lie in the shaded band shown in Fig. 6(b). The band is due to the uncertainty in the value of $|G_{\pi^0}(m_X^2)|$. The solid curve inside the band corresponds to the central value of $|G_{\pi^0}(m_X^2)|$. In the absence of the form factor, i.e., for $\alpha = \infty$, for $|g_X g_{\chi_{c1}} g_{\pi^0}|/(16\pi)$ is predicted the range of values from 3.87 to 6.45 GeV. If $|g_X g_{\chi_{c1}} g_{\pi^0}|/(16\pi) < 3.87$ GeV, then the model is unsatisfactory. Sources of information about the constants g_X, $g_{\chi_{c1}}$, and g_{π^0}, which determine the left side of Eq. (25), are the data on the $X(3872) \to (D^0\bar{D}^{*0} + \bar{D}^0 D^{*0}) \to D^0\bar{D}^0\pi^0$, and $X(3872) \to \pi^+\pi^- J/\psi$ decays and theoretical considerations. An approximate value of $g_X \equiv g_{XD\bar{D}^*} \equiv g_{XD^0\bar{D}^{*0}}$ [see Eq. (7)] we will take from the processing of the data on the $X(3872)$ decays obtained by the Belle [7] (for processing see Ref. [28]), LHCb [5], Belle [8,9], and BESIII [10] Collaborations. The coupling constant g_X in Ref. [28] was denoted as g_A. Let us note that the fitted parameter used in Refs. [5,8–10] was the coupling constant g, which is related to g_X by the relation $g = g_X^2/(4\pi m_X^2)$. Information about the values of g and g_X and their statistical errors are collected in Table I. The lower limits for g were also obtained in Refs. [8,9]: $g > 0.075$ ($g_X > 3.76$ GeV) and $g > 0.094$ ($g_X > 4.21$ GeV) at 95%

TABLE I. Information about the $X(3872)$ coupling to the $D^0\bar{D}^{*0}$ system.

Data analysis	AR [28]	LHCb [5]	Belle [8,9]	BESIII [10]
g	$0.181^{+0.647}_{-0.127}$	0.108 ± 0.003	$0.29^{+2.69}_{-0.15}$	0.16 ± 0.10
g_X (GeV)	$5.85^{+10.42}_{-2.04}$	4.51 ± 0.06	$7.39^{+34.28}_{-1.91}$	5.49 ± 1.72

and 90% confidence level, respectively. Some difficulties with determining the value of g (partly associated with limited statistics) and the estimates of systematic uncertainties are discussed in detail in Refs. [5,8–10]. Here, we only note that the sensitivity of g to the mass of $X(3872)$ (caused by its proximity to the $D^0\bar{D}^{*0}$ threshold) and weak dependence of the $X(3872)$ line shape in the $D^0\bar{D}^0\pi^0$ channel on g at large g generate significant positive uncertainties in this constant in the fits. In our opinion, a large positive error in g should not be given any decisive significance compared to the central value of g. New experiments with high statistics should clarify the situation. For our purposes, we will use the average value of $g_X = (5.81^{+8.97}_{-0.82})$ GeV found from the data in Table I.

To estimate the constants $g_{\chi_{c1}} \equiv g_{\chi_{c1}D\bar{D}^*}$ and $g_{\pi^0} \equiv g_{D^{*0}D^{*0}\pi^0}$ [see Eq. (7)], we use the results obtained in Refs. [36,47,64–66,70] in the framework of the heavy quark effective theory,

$$g_{\chi_{c1}D\bar{D}^*} = 2\sqrt{2}g_1\sqrt{m_D m_{D^*} m_{\chi_{c1}}}, \qquad g_1 = -\frac{\sqrt{m_{\chi_{c0}}/3}}{f_{\chi_{c0}}},$$
$$f_{\chi_{c0}} = (510 \pm 40)\ \text{MeV}, \tag{26}$$

$$g_{D^{*0}D^{*0}\pi^0} = \frac{g_{D^{*0}D^0\pi^0}}{\sqrt{m_D m_{D^*}}} = \frac{\sqrt{2}g}{f_\pi}, \qquad f_\pi = 132\ \text{MeV},$$
$$g = 0.59 \pm 0.07. \tag{27}$$

Thus we have $g_{\chi_{c1}} \equiv g_{\chi_{c1}D\bar{D}^*} = (-21.45 \pm 1.68)$ GeV, $g_{\pi^0} \equiv g_{D^{*0}D^{*0}\pi^0} = (6.32 \pm 0.75)\ \text{GeV}^{-1}$, and

$$\frac{|g_X g_{\chi_{c1}} g_{\pi^0}|}{16\pi} = \left(15.67^{+24.29}_{-3.14}\right)\ \text{GeV}. \tag{28}$$

The value (28) is shown in Fig. 6(b) in the form of a dot with vertical error bars. Agreement with the data on the amplitude $|G_{\pi^0}(m_X^2)|$ [see Eqs. (6) and (25)] is achieved when this point falls inside the shaded band. This occurs in the α interval from 1.487 to 2.565 marked in Fig. 6(b) by a segment of a horizontal straight line. At $\alpha = 1.98$, the central values of the left and right sides of Eq. (25) coincide. In the indicated interval of α, the average value of the isospin violation parameter ξ is of about 0.15; see Fig. 6(a). For comparison, we point out that the isospin violation parameter for the π^0 production mechanism due to the $\pi^0 - \eta$ mixing is an order of magnitude smaller [71]: $\Pi_{\pi^0\eta}/(m_\eta^2 - m_{\pi^0}^2) \simeq 0.014$, where $\Pi_{\pi^0\eta}$ is the $\pi^0 \leftrightarrow \eta$ transition amplitude having dimension of a mass squared. Taking into account the mechanism of the $\pi^0 - \eta$ mixing and the relation $\eta_0 = \eta\sin(\theta_i - \theta_p) + \eta'\cos(\theta_i - \theta_p)$, where η and η' are the physical states of the lightest pseudoscalar isoscalar mesons, Eq. (13) takes the form,

$$G_{\pi^0}(s) = -\frac{g_X g_{\chi_{c1}} g_{\pi^0}}{16\pi}\left[2C_{11}^n(s) - 2C_{11}^c(s) + \sin(\theta_i - \theta_p)\right.$$
$$\left.\times\frac{\Pi_{\pi^0\eta}}{m_\eta^2 - m_{\pi^0}^2}\left(2C_{11}^n(s) + 2C_{11}^c(s)\right)\right]. \tag{29}$$

Here $\theta_i = 35.3°$ is the so-called "ideal" mixing angle and $\theta_p = -11.3°$ is the mixing angle in the nonet of the light pseudoscalar mesons [1]. The result of analyzing Eq. (29) is shown in Fig. 7. This result is similar to that based on Eq. (25) and shown in Fig. 6(b). Now the permissible values of α lie in the range from 1.406 to 2.368, and the central value of α is equal to 1.853; i.e., changes in α turn out to be less than 10%. Note that the parameter α confirms its status as an useful fitting parameter with expected fitted values of the order of 1. Improving data accuracy on the width of the $X(3872) \to \pi^0\chi_{c1}(1P)$ decay is one of the great demand and essential task. Our conclusions from the present analysis are briefly formulated in the next section.

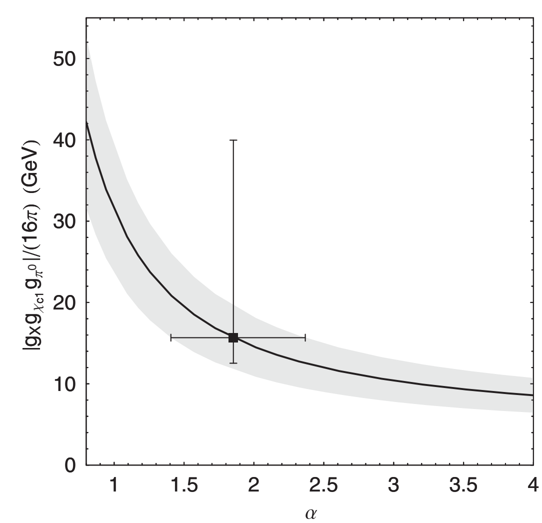

FIG. 7. The same as in Fig. 6(b) but with taking into account the $\pi^0 - \eta$ mixing, see the text.

V. CONCLUSION

Thus, we conclude that the considered model of triangle loops for the decay amplitude $X(3872) \to \pi^0\chi_{c1}(1P)$ is generally in reasonable agreement with the available data. Its distinctive feature is the convergence of diagrams with neutral and charged charmed mesons in the loops separately and without taking into account the form factor.

The significant amplitude of the process $X(3872) \to \pi^0\chi_{c1}(1P)$, which violates isospin, indicates the threshold nature of the origin of this effect. Due to incomplete compensation of the contributions of the $D^{*0}\bar{D}^0D^{*0}$ + c.c. and $D^{*+}D^-D^{*+}$ + c.c. loops, caused by the differences in the masses $m_{D^+} - m_{D^0}$ and $m_{D^{*+}} - m_{D^{*0}}$, the amplitude $X(3872) \to \pi^0\chi_{c1}(1P)$ near the $D^{*0}\bar{D}^0$ threshold turns out to be proportional to $\sqrt{m_d - m_u}$, and not $m_d - m_u$. That is, the mechanism of the charmed meson loops manifests itself at a qualitative level.

The product of the coupling constant $|g_X g_{\chi_{c1}} g_{\pi^0}|/(16\pi)$ and parameter α accumulate important information about the interactions of the $X(3872)$, $\chi_{c1}(1P)$, D, D^*, and π mesons and determine the loop mechanism of the process $X(3872) \to \pi^0\chi_{c1}(1P)$ in accordance with existing data.

Apart from the difference in the masses of neutral and charged charmed mesons, any additional exotic sources of isospin violation in $X(3872) \to \pi^0\chi_{c1}(1P)$ (such as a significant difference between the coupling constants $g_{XD^0\bar{D}^{*0}}$ and $g_{XD^+D^{*-}}$) are not required to interpret the data. This indirectly confirms the isotopic neutrality of the $X(3872)$, which is naturally realized for the $c\bar{c}$ state $\chi_{c1}(2P)$.

Increasing data accuracy about the $X(3872)$ in all directions [in particular, on the $X(3872) \to \pi^0\chi_{c1}(1P)$ decay] will certainly shed light on the mysterious nature of this extraordinary state.

Here, it would also be appropriate to note the importance of modern studies of the $X(3872)$ state in the molecular model. This model is significantly has evolved and extended its predictions to a large number of specific processes; see Refs. [25,26,44,47,69,72,73] and references herein. For example, recently in Ref. [73], using a molecular approach within the framework of the triangle diagram model, the large experimentally observed violation of the isospin symmetry in the $\mathcal{B}(B^+ \to X(3872)K^+)/\mathcal{B}(B^0 \to X(3872)K^0)$ ratio was explained. In the molecular model, the $X(3872)$ is formed by neutral $D^0\bar{D}^{*0} + \bar{D}^0D^{*0}$ and charged $D^+D^{*-} + D^-D^{*+}$ charmed meson pairs. Verification in different processes of model predictions based on the universality (i.e., independence from the process) of the couplings of $X(3872)$ to its neutral and charged constituents (the values of these couplings are different) seems to be extremely important for the molecular scenario.

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Correction: The value given for ξ in Eq. (20) and in the first sentence in the caption of Fig. 6 was incorrect and has been fixed. Reference [38] contained an error in the title and has been set right.