Non-Gaussianity from Schwinger-Keldysh effective field theory

Shu Lin, [1,2,*] Yanyan Bu, [3,†] and Chang Lei [1,‡]

[1] *School of Physics and Astronomy, Sun Yat-Sen University, Zhuhai 519082, China*
[2] *Guangdong Provincial Key Laboratory of Quantum Metrology and Sensing, Sun Yat-Sen University, Zhuhai 519082, China*
[3] *School of Physics, Harbin Institute of Technology, Harbin 150001, China*

(Received 21 January 2023; revised 21 March 2023; accepted 8 January 2024; published 20 February 2024)

We present a systematic treatment of non-Gaussianity in stochastic systems using the Schwinger-Keldysh effective field theory framework, in which the non-Gaussianity is realized as nonlinear terms in the fluctuation field. We establish two stochastic formulations of the Schwinger-Keldysh effective field theory, with those nonlinear terms manifested as multiple non-Gaussian noises in the Langevin equation and as higher order diffusive terms in the Fokker-Planck equation. The equivalence of the stochastic formulations with the original Schwinger-Keldysh effective field theory is demonstrated with nontrivial examples for arbitrary non-Gaussian parameters. The stochastic formulations will be more flexible and effective in studying nonequilibrium dynamics. We also reveal an ambiguity when coarse-graining timescale and non-Gaussian parameters vanish simultaneously, which may be responsible for the unphysical divergence found in perturbative analysis.

DOI: 10.1103/PhysRevD.109.036018

I. INTRODUCTION

The Gaussian white noise has been widely used in modeling of stochastic dynamics. If thermodynamic limit strictly applies to the system in question, the Gaussian noise is a consequence of ensemble average due to the central limit theorem. On the other hand, the white noise follows from a coarse-grained description of the system: when the coarse-graining timescale is much longer than the microscopic timescale of the system, the white noise becomes accurate. In reality, deviations of both idealizations can occur. Non-Gaussian noises have wide applications in statistical physics [1], cosmology [2], condensed matter physics [3], and quantum optics [4]. Colored noises generically occur when one considers dynamics at timescale comparable to the coarse-graining scale. Most implementations of non-Gaussian colored noises are based on phenomenological models to date. Microscopic derivations of stochastic dynamics with non-Gaussian noises exist for a simple degree of freedom [5], but generalization to more complicated systems such as hydrodynamics is far from obvious.

*linshu8@mail.sysu.edu.cn
†yybu@hit.edu.cn
‡leich6@mail2.sysu.edu.cn

Modern description of a stochastic system uses Schwinger-Keldysh effective field theory (SKEFT) [6–10]. Thanks to doubling of degrees of freedom, the SKEFT incorporates fluctuations and dissipations systematically, going beyond the Martin-Siggia-Rose formalism for stochastic models with Gaussian noise [6]. The SKEFT follows from averaging out fast modes and governs stochastic evolution of slow modes. The SKEFT is defined with an implicit coarse-graining timescale separating the fast modes and slow modes. The SKEFT is organized as a systematic expansion in temporal gradient, which characterizes the slowness of the dynamics, as well as expansion in the fields. These expansions allow us to study deviations of Gaussian white noise discussed above systematically: expansion in temporal gradient allows one to access dynamics comparable to the coarse-graining scale and expansion in the fields characterizes the non-Gaussianity through nonlinear effect. Recently there have been extensive studies on nonlinear effect in dynamics of Brownian particle [11–16] and hydrodynamics [17–19].

In this paper, we use Brownian particle as an example to illustrate formulations of non-Gaussianity from nonlinear effect. We will establish three equivalent formulations of non-Gaussianity: SKEFT, Langevin equation, and Fokker-Planck (FP) equation. A crucial difference between this study and those on related subject is that we do not assume small non-Gaussian parameters and our formulations are exact in these parameters. This is in contrast to [12] where equivalence has been established at lowest order in the parameters. We will offer caveat to the perturbative analysis in the non-Gaussian parameters, which contains unphysical divergence. We suggest that the divergence is tied to the ambiguity when the coarse-graining scale and non-Gaussian parameters vanish simultaneously. The formulations in this paper can also be straightforwardly adapted to more interesting hydrodynamic systems.

The rest of this paper will be structured as follows. In Sec. II we present a simple Schwinger-Keldysh effective field theory (EFT) incorporating non-Gaussianity. We also review its two equivalent stochastic formulations when non-Gaussian terms in noise field are absent. In Sec. III we establish two stochastic formulations of the Schwinger-Keldysh EFT when generic nonlinear terms are present. In Sec. IV, we demonstrate the equivalence of the stochastic formulations with the original Schwinger-Keldysh EFT. In Sec. V we make a brief summary and outlook interesting future directions. Appendix provides further details on the equivalence demonstration.

II. SCHWINGER-KELDYSH EFFECTIVE FIELD THEORY

We begin with the following effective Lagrangian for a Brownian particle

$$L = iT\Delta_a^2 - \Delta_a\partial_t\Delta_r - m\Delta_a\Delta_r + i\epsilon_1\Delta_a^4 - \epsilon_2\Delta_a^3\Delta_r + i\epsilon_3\Delta_a^2\Delta_r^2 - \epsilon\Delta_a\Delta_r^3, \tag{1}$$

where $\Delta_r = \frac{1}{2}(\Delta_1 + \Delta_2)$ and $\Delta_a = \Delta_1 - \Delta_2$ with $\Delta_{1,2}$ being real scalar fields on the SK contour. Δ_r is identified with momentum of Brownian particle, and Δ_a encodes the fluctuation. We have only expanded L to the leading order in temporal gradient and to quartic order in the fields. Structure like (1) has been obtained from holographic model calculations [20,21]. The first three terms are Gaussian, which determine two-point correlation functions. With T identified as the temperature, the first two terms satisfy the Kubo-Martin-Schwinger (KMS) symmetry [8]

$$\Delta_a(t) \to -\Delta_a(-t) - iT^{-1}(\partial_t\Delta_r)(-t),$$
$$\Delta_r(t) \to -\Delta_r(-t) \tag{2}$$

Indeed, (2) leaves the third and last terms invariant up to a total derivative. The remaining non-Gaussian terms determine higher-point correlation functions. One may further constrain these terms using KMS symmetry, which amounts to choosing an equilibrium state [18]. Nevertheless, we choose to not impose the KMS symmetry for the non-Gaussian terms, which is applicable to a nonequilibrium state. Indeed, model calculations have shown violation of the KMS symmetry in higher-point correlation functions from non-Gaussian terms once the condition of equilibrium state is relaxed [15]. All nonlinear parameters $\epsilon_{1,2,3}$, ϵ are real by Z_2-reflection symmetry of SKEFT [8].

Equation (1) can be inspected as a series expansion in Δ_a. At linear order, the action variation with respect to Δ_a gives the deterministic equation for Δ_r:

$$\partial_t\Delta_r = -m\Delta_r - \epsilon\Delta_r^3. \tag{3}$$

We readily identify (3) as a nonlinear damping equation of particle's momentum. Stability requires $m > 0$ and $\epsilon > 0$ so that the corresponding terms act like restoring force when Δ_r moves away from the origin. The quadratic terms in Δ_a encode stochastic property of the system, turning the deterministic equation (3) into a stochastic one with Gaussian noise. As we shall show, the remaining cubic and quartic terms in Δ_a give rise to non-Gaussian noises. For completeness, we will review the derivation of two well-known formulations of stochastic dynamics: Langevin equation and FP equation from the Gaussian terms (linear and quadratic in Δ_a). Then we will extend the analysis by including non-Gaussian terms (cubic and quartic in Δ_a).

Before proceeding, we remark that (1) also contains nonlinearity in Δ_r. However, this nonlinearity exists generically in interacting systems without stochasticity, thus not affecting the noise. It is well known how to treat this with standard perturbative method. So the non-Gaussianity inherent to stochastic systems arises from the nonlinear $\epsilon_{1,2}$ terms.

We start by converting the first term in (1) into a noise term added to (3). Following standard procedure, we rewrite the first term in the path integral as [6]

$$e^{-\int dtT\Delta_a^2} = \int \mathcal{D}\xi e^{-\int dt\left(\frac{\xi^2}{4T} - i\xi\Delta_a\right)}. \tag{4}$$

With ξ introduced, it is easy to integrate out Δ_a

$$\int \mathcal{D}\Delta_a e^{i\int dtL} = \int \mathcal{D}\xi e^{-\int dt\frac{\xi^2}{4T}}\delta(-\partial_t\Delta_r - m\Delta_r - \epsilon\Delta_r^3 + \xi),$$

which gives rise to the following Langevin equation with a nonlinear damping term

$$\partial_t\Delta_r = -m\Delta_r - \epsilon\Delta_r^3 + \xi. \tag{5}$$

ξ is identified as a Gaussian noise whose variance is determined by the exponent $e^{-\int dt\frac{\xi^2}{4T}}$ as

$$\langle\xi\rangle = 0, \qquad \langle\xi(t)\xi(t')\rangle = 2T\delta(t - t'). \tag{6}$$

The ϵ_3 term can be included straightforwardly, which turns (5) into a multiplicative form

$$\partial_t\Delta_r = -m\Delta_r - \epsilon\Delta_r^3 + (1 + \epsilon_3T^{-1}\Delta_r^2)^{1/2}\xi. \tag{7}$$

We require $\epsilon_3 > 0$ so that prefactor of the noise is real [22]. It is well known that (7) with (6) is ambiguous. We will adopt the Ito regularization [6]

$$\Delta_{r,i} - \Delta_{r,i-1} = \delta_t[-m\Delta_{r,i-1} - \epsilon\Delta_{r,i-1}^3 + (1+\epsilon_3 T^{-1}\Delta_{r,i-1}^2)^{1/2}\xi_i], \tag{8}$$

in which the multiplicative factor of noise at step i depends on the field at one step earlier. The noise is normalized as $\langle\xi_i\xi_j\rangle = 2T\delta_{ij}\delta_t^{-1}$ [23], with ξ_i being the discretized noise at step i. δ_t is the time step used in discretization, which is also the coarse-graining timescale.

It is convenient to pass from SKEFT to the FP equation, which governs the evolution of probability function $P(t,\Delta_r)$. The probability satisfies the following evolution equation from the discretized path integral

$$P(t_i,\Delta_{r,i}) = \int d\Delta_{r,i-1}d\Delta_{a,i}\exp[-\delta_t T\Delta_{a,i}^2 - i\Delta_{a,i}\delta_r - i\delta_t m\Delta_{a,i}\Delta_{r,i-1} - \delta_t\epsilon_3\Delta_{a,i}^2\Delta_{r,i-1}^2 - i\delta_t\epsilon\Delta_{a,i}\Delta_{r,i-1}^3] P(t_{i-1},\Delta_{r,i-1}), \tag{9}$$

with $\delta_r = \Delta_{r,i} - \Delta_{r,i-1}$. In the limit $\delta_t \to 0$, the exponent suggests $\Delta_{a,i} \sim \delta_t^{-1/2}$ and $\delta_r \sim \delta_t^{1/2}$. We may regard t and Δ_r as continuous variables and expand

$$P(t_{i-1},\Delta_{r,i-1}) = P(t_i,\Delta_{r,i}) - \delta_t\dot{P}(t_i,\Delta_{r,i}) - \delta_r P'(t_i,\Delta_{r,i}) + \frac{1}{2}\delta_r^2 P''(t_i,\Delta_{r,i}) + \cdots, \tag{10}$$

with dot and prime denoting derivatives with respect to t and Δ_r, respectively. Plugging (10) into (9) and making a change of field $d\Delta_{r,i-1} = d\delta_r$, we can perform the integrals easily to obtain the following FP equation from the coefficient of δ_t:

$$\partial_t P = T\partial_\Delta^2 P + m\partial_\Delta(\Delta P) + \epsilon_3\partial_\Delta^2(\Delta^2 P) + \epsilon\partial_\Delta(\Delta^3 P). \tag{11}$$

We have renamed $\Delta_r \to \Delta$ for notational simplicity. Here, in accord with (8), Ito regularization has been assumed so that ∂_Δ is always later than Δ.

III. NON-GAUSSIAN LANGEVIN AND FP EQUATIONS

So far what has been presented is textbook materials [6]. Now we wish to generalize the Langevin and FP equations to the non-Gaussian case. Naive application of the method outlined above encounters immediate difficulties. The derivation of the Langevin equation relies on Gaussian integration, which cannot treat cubic and quartic terms in Δ_a; the derivation of FP equation seems to involve potential divergence. If we still choose $\Delta_a \sim \delta_t^{-1/2}$ as in the Gaussian case, then the term $i\epsilon_1\Delta_a^4$ for example becomes singular in the continuum limit $\delta_t \to 0$.

The difficulties associated with the two equations are in fact related: the technical difficulty of the non-Gaussian integration is tied to the fact that there is no simple scaling of Δ_a with δ_t in the multiscale integral, thus we would not have a simple noise with $\xi \sim \delta_t^{-1/2}$ like in the Gaussian case. Similarly, if we were able to perform the non-Gaussian integral based on (1) in full, i.e., without assuming a simple scaling of Δ_a, we would not expect any divergence. Indeed the Lagrangian (1) is essentially a quantum mechanical one with all the couplings having mass dimension one, so the SKEFT is superrenormalizable.

The analysis above suggests that we should treat nonlinear term separately rather than assume a uniform scaling. Below we shall derive the non-Gaussian Langevin and FP equations and demonstrate their equivalence with the SKEFT formulation.

Let us begin with the non-Gaussian Langevin equation. Note that (4) for an infinitesimal time interval reads

$$e^{-\delta_t T\Delta_{a,i}^2} = \int d\xi_i e^{i\delta_t\xi_i\Delta_{a,i}} e^{-\delta_t\frac{\xi_i^2}{4T}}, \tag{12}$$

which allows us to trade $\Delta_{a,i}$ with ξ_i. Note that this is nothing but an inverse Fourier transform. We can apply the same transform to the cubic and quartic terms (omitting subscript i for notational simplicity)

$$e^{-\delta_t\epsilon_1\Delta_a^4} = \int d\eta e^{i\delta_t\eta\Delta_a} f(\eta),$$
$$e^{-i\delta_t\epsilon_2\Delta_a^3\Delta_r} = \int d\chi e^{i\delta_t\chi\Delta_a} g(\chi), \tag{13}$$

with

$$f(\eta) = \frac{1}{2\pi}\left[2\left(\frac{\delta_t^3}{\epsilon_1}\right)^{1/4}\Gamma\left(\frac{5}{4}\right){}_0F_2\left(\frac{1}{2},\frac{3}{4};\frac{\eta^4\delta_t^3}{256\epsilon_1}\right) - \left(\frac{\eta^2}{4}\right) \times\left(\frac{\delta_t^3}{\epsilon_1}\right)^{3/4}\Gamma\left(\frac{3}{4}\right){}_0F_2\left(\frac{3}{2},\frac{5}{4};\frac{\eta^4\delta_t^3}{256\epsilon_1}\right)\right],$$
$$g(\chi) = \left(\frac{\delta_t^2}{3\epsilon_2\Delta_r}\right)^{1/3} Ai\left(-\chi\left(\frac{\delta_t^2}{3\epsilon_2\Delta_r}\right)^{1/3}\right). \tag{14}$$

Here ${}_0F_2$ and Ai are the generalized hypergeometric function and Airy function, respectively. As in the Gaussian case, $f(\eta)$ and $g(\chi)$ are interpreted as weight of noises η and χ, respectively.

Note that the non-Gaussian parameters appear in the weight functions as δ_t^3/ϵ_1 and δ_t^2/ϵ_2, which are ambiguous in the limits $\delta_t \to 0$ and $\epsilon_{1,2} \to 0$. We suggest that the unphysical divergence found in earlier studies may be due to improper implementation of the limit in perturbative analysis in the continuum form [12,14]. Our derivation indicates that it is crucial to proceed in discretized form and not to use perturbation. We also note that η and χ scale differently with δ_t: $\eta \sim \delta_t^{-3/4}$, $\chi \sim \delta_t^{-2/3}$.

The weight functions are plotted in Fig. 1. Two comments of the weight functions are in order. First, they are not positive definite. This is unavoidable: by taking derivatives with respect to Δ_a in (13) and then setting $\Delta_a = 0$, we can show the first nonvanishing moments are $\langle \eta^4 \rangle$ and $\langle \chi^3 \rangle$, respectively [24]. Therefore, region with negative weight must be present such that $\langle \eta^2 \rangle = \langle \chi^2 \rangle = 0$. Second, $g(\chi)$ depends on Δ_r. In the Ito regularization scheme, the weight of χ_i depends on $\Delta_{r,i-1}$. In the special limit $\Delta_r \to 0$, we can see from the definition (13) that $g(\chi) \to \delta(\chi)$.

It is then straightforward to integrate out $\Delta_{a,i}$ to arrive at the Langevin equation with multiple noises

$$\begin{aligned}\Delta_{r,i} - \Delta_{r,i-1} &= \delta_t[-m\Delta_{r,i-1} - \epsilon\Delta_{r,i-1}^3 \\ &\quad + (1 + \epsilon_3 T^{-1}\Delta_{r,i-1}^2)^{1/2}\xi_i + \eta_i + \chi_i]. \end{aligned} \tag{15}$$

Now we turn to the derivation of the FP equation. Note that in (1) only Δ_r is the dynamical field and Δ_a is an auxiliary one that we wish to integrate out. By performing the Legendre transformation, we find Δ_a and Δ_r form a conjugate pair

$$p = \frac{\partial L}{\partial(\partial_t \Delta_r)} = -\Delta_a. \tag{16}$$

The corresponding Hamiltonian is expressed as

$$\begin{aligned} H = &-iT\Delta_a^2 + m\Delta_a\Delta_r - i\epsilon_1\Delta_a^4 + \epsilon_2\Delta_a^3\Delta_r \\ &- i\epsilon_3\Delta_a^2\Delta_r^2 + \epsilon\Delta_a\Delta_r^3. \end{aligned} \tag{17}$$

In order to describe evolution of probability distribution, we need to promote the classical Hamiltonian to a quantum

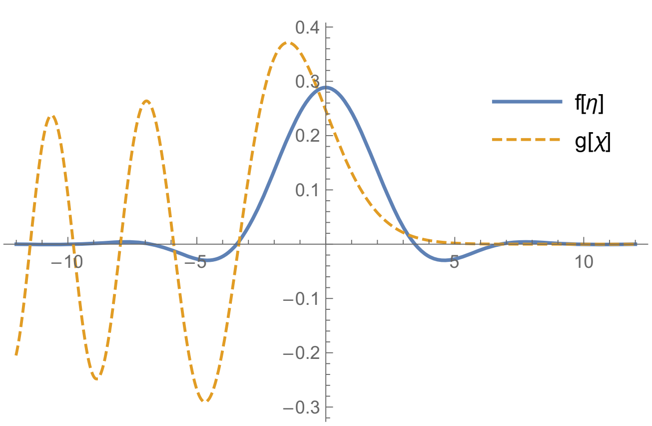

FIG. 1. Nonpositive definite weight functions $f(\eta)$ and $g(\chi)$, with $\epsilon_1/\delta_t^3 = \epsilon_2\Delta_r/\delta_t^2 = 1$.

one, in which the conjugate fields have the commutator $[p, \Delta_r] = -i$. This allows us to represent the operator $p = -i\partial_{\Delta_r}$. It is convenient to use $\tilde{p} = -ip$. Then, the Hamiltonian appears purely imaginary

$$H = i(T\tilde{p}^2 - m\tilde{p}\Delta_r - \epsilon_1\tilde{p}^4 + \epsilon_2\tilde{p}^3\Delta_r + \epsilon_3\tilde{p}^2\Delta_r^2 - \epsilon\tilde{p}\Delta_r^3). \tag{18}$$

Recall in the Gaussian case, the probability distribution P evolves according to the phase factor $e^{i\int dtL}$, which is equivalent to the Schrödinger equation $i\partial_t P = HP$. However, in the non-Gaussian case, the equivalence is lost due to cubic and quartic terms in $\tilde{p}$. We should resort to the Schrödinger equation for the evolution. Upon using $p = -i\partial_\Delta$, we have

$$\begin{aligned} \partial_t P = &T\partial_\Delta^2 P + m\partial_\Delta(\Delta P) - \epsilon_1\partial_\Delta^4 P - \epsilon_2\partial_\Delta^3(\Delta P) \\ &+ \epsilon_3\partial_\Delta^2(\Delta^2 P) + \epsilon\partial_\Delta(\Delta^3 P). \end{aligned} \tag{19}$$

We have renamed $\Delta_r \to \Delta$ for notational simplicity. Again the ordering of operator ∂_Δ and Δ matters. We have adopted the Ito regularization so that ∂_Δ is always later than Δ, to be consistent with the regularization in Langevin equation. Equation (19) generalizes (9) to the non-Gaussian case. Here the term $-\epsilon_1\partial_\Delta^4 P$ can be viewed as a higher order diffusive term in addition to the diffusive term $T\partial_\Delta^2 P$. Stability of the diffusion requires $\epsilon_1 > 0$. The sign of ϵ_2 is not constrained.

IV. EQUIVALENCE DEMONSTRATION

Now we demonstrate the equivalence of the three formulations by calculating a same set of equal-time correlation functions. As simple examples, we consider $\langle \Delta^2(t) \rangle$ and $\langle \Delta^4(t) \rangle_c \equiv \langle \Delta^4(t) \rangle - 3\langle \Delta^2(t) \rangle^2$, in which the latter is the connected part of the four-point correlation function. These correlation functions are directly comparable among the three formulations. We have argued below (3), (7), and (19) that all parameters except ϵ_2 are constrained to be positive.

Let us first calculate them by solving the FP equation (19). Multiplying $d\Delta\Delta^2$ on both sides of (19) and integrating by parts, we obtain the following equation for the second moment

$$\partial_t\langle \Delta^2 \rangle = 2T - 2m\langle \Delta^2 \rangle + 2\epsilon_3\langle \Delta^2 \rangle - 2\epsilon\langle \Delta^4 \rangle. \tag{20}$$

The equation for $\langle \Delta^2 \rangle$ does not close as it involves $\langle \Delta^4 \rangle$. This is of course allowed, reflecting effect of nonlinearity in Δ_r. Since our focus is on nonlinear terms in Δ_a, we will set $\epsilon = 0$ to simplify the comparison. Then, the equation for the fourth moment reads

$$\partial_t\langle\Delta^4\rangle = 12(T+2\epsilon_2)\langle\Delta^2\rangle - 24\epsilon_1 + 4(3\epsilon_3 - m)\langle\Delta^4\rangle. \quad (21)$$

From (20) and (21), we see that in order for the moments not to blow up, we need $m > \epsilon_3$ and $m > 3\epsilon_3$. In fact, we can obtain the condition for the $2n$th moment would be $m > (2n-1)\epsilon_3$, which will eventually fail for sufficient large n. The reason for the failure can be seen from the Langevin equation (15): for large Δ, the multiplicative noise always win over the m term, but it can be cured by the ϵ term we choose to turn off. Consequently to ensure stability of the FP equation, we should set $\epsilon_3 = 0$ in (20) and (21) as well. This leaves us with ϵ_1 and ϵ_2 being the only nonvanishing parameters. We stress that they are also the full non-Gaussian parameters in the SKEFT.

In fact, we can show ϵ_2 is also constrained to be positive from the equation for the $2n$th moment

$$\partial_t\langle\Delta^{2n}\rangle = 2n(2n-1)(T+\epsilon_2(2n-2))\langle\Delta^{2n-2}\rangle - 2nm\langle\Delta^{2n}\rangle - 2n(2n-1)(2n-2)(2n-3)\epsilon_1\langle\Delta^{2n-4}\rangle. \quad (22)$$

Since all even moments are positive, a negative ϵ_2 combined with the other two terms on the right-hand side implies a negative $\partial_t\langle\Delta^{2n}\rangle$ for large enough n. Consequently the final state can only have vanishing moments for large n. We do not consider this trivial possibility. Below we take all parameters to be positive.

To solve (20) and (21), we need to specify initial condition [25]. In fact, as $t\to+\infty$, stability of the FP equation ensures that any initial state will approach the unique steady state, with all the moments tending to constants. We stress that the steady state reached in the presence of non-Gaussian noise is nonequilibrium in the absence of KMS condition, see also [26,27] for examples in phenomenological models. With this in mind, we take initial condition $\langle\Delta^2(0)\rangle = \langle\Delta^4(0)\rangle = 0$ and easily obtain

$$\langle\Delta^2(t)\rangle = \frac{T}{m}(1-e^{-2mt}),$$
$$\langle\Delta^4(t)\rangle_c = -\frac{6\epsilon_1}{m}(1-e^{-4mt}) + \frac{6T\epsilon_2}{m^2}(1-e^{-2mt})^2. \quad (23)$$

It is worth pointing out that (23) is exact in the non-Gaussian parameters ϵ_1 and ϵ_2.

Now we attempt to solve the Langevin equation (15). The usual method is to generate an ensemble of solutions to Langevin equation with random noises and then take the ensemble average. There is no conceptual difficulty with this method. The issue of negative weight can be treated with technique like in [28]. However, the introduction of the multiple noises makes the practical computation expensive. For the purpose of demonstrating the equivalence, we will use a hybrid method, in which the ensemble average is taken analytically and we only simulate the moments equation derived from the Langevin equation (15).

From (15), we easily obtain

$$\langle\Delta_{i+1}^2\rangle = (1-m\delta_t)^2\langle\Delta_i^2\rangle + \langle\xi_i^2\rangle\delta_t^2,$$
$$\langle\Delta_{i+1}^4\rangle = (1-m\delta_t)^4\langle\Delta_i^4\rangle + 6(1-m\delta_t)^2\delta_t^2\langle\xi_i^2\rangle\langle\Delta_i^2\rangle + \delta_t^4\langle\xi_i^4\rangle + 4(1-m\delta_t)\delta_t^3\langle\Delta_i\chi_i^3\rangle + \langle\eta_i^4\rangle\delta_t^4, \quad (24)$$

where various noise averages are [29]

$$\langle\xi_i^2\rangle = 2T/\delta_t, \qquad \langle\xi_i^4\rangle = 12T^2/\delta_t^2, \qquad \langle\eta_i^4\rangle = -24\epsilon_1/\delta_t^3$$
$$\langle\chi_i^3\rangle = 6\epsilon_2\Delta_i/\delta_t^2 \Rightarrow \langle\Delta_i\chi_i^3\rangle = 6\epsilon_2\langle\Delta_i^2\rangle/\delta_t^2. \quad (25)$$

Then, it is easy to show that (24) leads to

$$\frac{\langle\Delta_{i+1}^2\rangle - \langle\Delta_i^2\rangle}{\delta_t} = -2m\langle\Delta_i^2\rangle + 2T + O(\delta_t),$$
$$\frac{\langle\Delta_{i+1}^4\rangle - \langle\Delta_i^4\rangle}{\delta_t} = -4m\langle\Delta_i^2\rangle + 12T\langle\Delta_i^2\rangle + 2\epsilon_2\langle\Delta_i^2\rangle - 24\epsilon_1 + O(\delta_t). \quad (26)$$

which are obviously discretized forms of (20) and (21) (with $\epsilon_3 = \epsilon = 0$ as we have reasoned). Therefore, we are guaranteed to arrive at the same results as (23) in the limit $\delta_t \to 0$.

Finally, we calculate the same quantities within the SKEFT. Note that the effective Lagrangian in the absence of external sources corresponds to the steady state discussed above, which is reached by evolving the Langevin and FP equations till the limit $t\to\infty$. Then, $\langle\Delta(t)^2\rangle$ and $\langle\Delta(t)^4\rangle_c$ are represented by $\langle\Delta_r(t)^2\rangle$ and $\langle\Delta_r(t)^4\rangle_c$, which are most conveniently calculated diagrammatically in the ra basis. We treat the first three terms in (1) as the free part, giving the following propagators:

$$D_{rr}(t,t') = \frac{T}{m}e^{-m|t-t'|}, \qquad D_{ra}(t,t') = -ie^{-m(t-t')}\theta(t-t'),$$
$$D_{ar}(t,t') = D_{ra}(t',t). \quad (27)$$

The remaining terms give two vertices $-4!\epsilon_1$ and $-4!i\epsilon_2$.

The two-point correlation function is trivially given by the symmetric propagator $\langle\Delta_r(t)\Delta_r(t)\rangle = \frac{T}{m}$. The connected four-point correlation function receives contribution from diagrams in Fig. 2. Note that possible loop diagrams

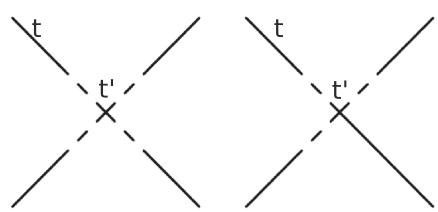

FIG. 2. Feynman diagrams for $\langle\Delta_r(t)^4\rangle_c$ when $\epsilon_3 = \epsilon = 0$. The ends of propagator can be of either dashed type or solid type, corresponding to a and r indices, respectively.

vanish identically because a loop involves only retarded (or advanced) propagators. The diagrams in Fig. 2 are easily evaluated to give

$$\begin{aligned}\langle\Delta_r(t)^4\rangle_c &= \int_{-\infty}^{t} dt' D_{ra}(t,t')^4(-4!\epsilon_1) \\ &\quad+\int_{-\infty}^{t} dt' D_{ra}(t,t')^3 D_{rr}(t,t')(-4!i\epsilon_2), \\ &= -\frac{6\epsilon_1}{m}+\frac{6T\epsilon_2}{m^2}. \end{aligned} \tag{28}$$

Equation (28) is in full agreement with (23) in the limit $t\to\infty$.

To further corroborate the equivalence, we have also performed a more sophisticated demonstration in the Appendix, in which we turn on both ϵ and ϵ_3. Performing calculations perturbatively in these two parameters, we still find perfect agreement among three formulations with

$$\begin{aligned}\langle\Delta_r(t)^2\rangle &= \frac{T}{m}-\frac{3T^2\epsilon}{m^3}+\frac{T\epsilon_3}{m^2}+\frac{6\epsilon_1\epsilon}{m^2}-\frac{6T\epsilon_2\epsilon}{m^3}, \\ \langle\Delta_r(t)^4\rangle_c &= -\frac{6\epsilon_1}{m}+\frac{6T\epsilon_2}{m^2}+\frac{6(T^2-3m\epsilon_1+4T\epsilon_2)\epsilon_3}{m^3}-\frac{6(T^3-12Tm\epsilon_1+15T^2\epsilon_2-26m\epsilon_1\epsilon_2+26T\epsilon_2^2)\epsilon}{m^4}. \end{aligned} \tag{29}$$

Equation (29) contains extra contributions at $O(\epsilon)$ and $O(\epsilon_3)$. In field theoretic approach, those contributions arise from about 30 loop diagrams, while in stochastic approaches, they are obtained simply by solving the FP and Langevin equations perturbatively in ϵ and ϵ_3. Thus the stochastic formulations provide a very efficient method of resumming diagrams, a notable virtue of the stochastic approaches.

V. SUMMARY AND OUTLOOK

We studied effect of nonlinear terms in a SKEFT expanded up to quartic order in the fields. There are nonlinearities in both the mean field Δ_r and the fluctuation field Δ_a. The nonlinearity in Δ_r is not specific to stochastic system and is known how to treat perturbatively. Our emphasis is on the cubic and quartic terms of Δ_a, which are generically present in a complete theory but usually ignored in literature. We established two stochastic formulations of the SKEFT: non-Gaussian Langevin equation and FP equation. In the former case, the cubic and quartic terms of Δ_a are manifested as two non-Gaussian noises in addition to the Gaussian noise corresponding to the quadratic terms of Δ_a. These two non-Gaussian noises have distinct scalings with the time step δ_t of the discrete Langevin equation. In the latter case, the nonlinear terms are manifested as higher order diffusive terms in the FP equation. Our formulations reveal an ambiguity as δ_t and non-Gaussian parameters tend to zero simultaneously, shedding light on the origin of unphysical divergence found in early studies. We demonstrated the equivalence among the three formulations for arbitrary non-Gaussian parameters subject to stability conditions.

The stochastic formulations established in this work will be found useful in addressing nonequilibrium dynamics with more flexibility and efficiency. On the one hand, it allows one to study nonequilibrium states by simply changing initial conditions. On the other hand, numerical implementation will help to efficiently resum diagrams in field theoretic approach. Moreover, by solving the non-Gaussian Langevin or FP equations, we could obtain unequal-time correlators [11,12,30], which would contain more important information about nonequilibrium physics.

The method outlined in this work can be generalized to more interesting theories such as hydrodynamics. Rapid progresses have been made in studying the effect of nonlinear Gaussian terms (quadratic in fluctuation field) [17,18]. Effect of non-Gaussian terms has also been discussed recently [19]. It would be useful to implement stochastic hydrodynamics with non-Gaussian noise, which would allow us to study a complete hydrodynamics out-of-equilibrium.

ACKNOWLEDGMENTS

This work is in part supported by NSFC under Grants No. 12075328 and No. 11735007 (S. L.).

APPENDIX: EQUIVALENCE WITH NONVANISHING ϵ_3 AND ϵ

We extend the equivalence demonstration to the case with nonvanishing ϵ_3 and ϵ. As we remarked in the main text, these parameters capture nonlinear effect in Δ_r. We will work perturbatively to $O(\epsilon_3\epsilon^0)$ and $O(\epsilon_3^0\epsilon)$, respectively. We begin with the FP equation. Following the same method as described for the case with $\epsilon_3=\epsilon=0$, we can derive the following moment equations up to $\langle\Delta^6\rangle$:

$$\begin{aligned}\partial_t\langle\Delta^2\rangle &= 2T-2m\langle\Delta^2\rangle+2\epsilon_3\langle\Delta^2\rangle-2\epsilon\langle\Delta^4\rangle, \\ \partial_t\langle\Delta^4\rangle &= 12T\langle\Delta^2\rangle-4m\langle\Delta^4\rangle-24\epsilon_1+24\epsilon_2\langle\Delta^2\rangle \\ &\quad+12\epsilon_3\langle\Delta^4\rangle-4\epsilon\langle\Delta^6\rangle, \\ \partial_t\langle\Delta^6\rangle &= 30T\langle\Delta^4\rangle-6m\langle\Delta^6\rangle-360\epsilon_1\langle\Delta^2\rangle+120\epsilon_2\langle\Delta^4\rangle \\ &\quad+30\epsilon_3\langle\Delta^6\rangle-6\epsilon\langle\Delta^8\rangle. \end{aligned} \tag{A1}$$

For the steady state solution approached at $t \to \infty$, we simply set the left-hand side of (A1) to zero. We will solve the moments equation (A1) perturbatively by the expansion

$$\langle \Delta^{2n} \rangle = \langle \Delta^{2n} \rangle_{\rm LO} + \langle \Delta^{2n} \rangle_{\rm NLO} + \cdots, \tag{A2}$$

with the leading order (LO) solution $\langle \Delta^{2n} \rangle_{\rm LO} \sim O(\epsilon_3^0 \epsilon^0)$ and the next-to-leading order (NLO) solution $\langle \Delta^{2n} \rangle_{\rm NLO}$ includes both $O(\epsilon_3 \epsilon^0)$ and $O(\epsilon_3^0 \epsilon)$. By setting $\epsilon_3 = \epsilon = 0$ in (A1), we easily obtain the LO solution:

$$\langle \Delta^2 \rangle_{\rm LO} = \frac{T}{m}, \qquad \langle \Delta^4 \rangle_{\rm LO} = \frac{3T(T^2 - 2m\epsilon_1 + 2T\epsilon_2)}{m^2},$$
$$\langle \Delta^6 \rangle_{\rm LO} = \frac{15(T^3 + 6T^2\epsilon_2 - T(6m\epsilon_1 - 8\epsilon_2^2)) - 8m\epsilon_1\epsilon_2}{m^3}. \tag{A3}$$

Now we proceed to the NLO solution, which satisfies

$$0 = -2m\langle \Delta^2 \rangle_{\rm NLO} + 2\epsilon_3 \langle \Delta^2 \rangle_{\rm LO} - 2\epsilon \langle \Delta^4 \rangle_{\rm LO},$$
$$0 = 12T\langle \Delta^2 \rangle_{\rm NLO} - 4m\langle \Delta^4 \rangle_{\rm NLO} + 24\epsilon_2 \langle \Delta^2 \rangle_{\rm NLO} + 12\epsilon_3 \langle \Delta^4 \rangle_{\rm LO} - 4\epsilon \langle \Delta^6 \rangle_{\rm LO}. \tag{A4}$$

Here we only keep the moments we need. The equations can be solved as

$$\langle \Delta^2 \rangle_{\rm NLO} = \frac{T\epsilon_3}{m^2} - \frac{(3T^2 - 6m\epsilon_1 + 6T\epsilon_2)\epsilon}{m^3},$$
$$\langle \Delta^4 \rangle_{\rm NLO} = -\frac{3(-4T^2 + 6m\epsilon_1 - 8T\epsilon_2)\epsilon_3}{m^3} - \frac{3(8T^3 - 36Tm\epsilon_1 + 42T^2\epsilon_2 - 52m\epsilon_1\epsilon_2 + 52T\epsilon_2^2)\epsilon}{m^4}. \tag{A5}$$

Immediately, (A3) and (A5) give

$$\langle \Delta^4 \rangle^c_{\rm LO} = -\frac{6\epsilon_1}{m} + \frac{6T\epsilon_2}{m^2},$$
$$\langle \Delta^4 \rangle^c_{\rm NLO} = \frac{6(T^2 - 3m\epsilon_1 + 4T\epsilon_2)\epsilon_3}{m^3} - \frac{6(T^3 - 12Tm\epsilon_1 + 15T^2\epsilon_2 - 26m\epsilon_1\epsilon_2 + 26T\epsilon_2^2)\epsilon}{m^4}. \tag{A6}$$

Then we solve the Langevin equation. Similar to the simple example in the main text, we derive the following evolution equations for moments

$$\langle \Delta_{j+1}^2 \rangle = (1 - 2m\delta_t)\langle \Delta_j^2 \rangle + (1 + \epsilon_3 T^{-1}\langle \Delta_j^2 \rangle)\langle \xi_i^2 \rangle \delta_t^2 - 2\epsilon \langle \Delta_j^4 \rangle \delta_t,$$
$$\langle \Delta_{j+1}^4 \rangle = (1 - 4m\delta_t)\langle \Delta_j^4 \rangle + 6(\langle \Delta_j^2 \rangle + \epsilon_3 T^{-1}\langle \Delta_j^4 \rangle)\langle \xi_i^2 \rangle \delta_t^2 + 4\langle \chi_j^3 \Delta_j \rangle \delta_t^2 + \langle \eta_j^4 \rangle \delta_t^4 - 4\epsilon \langle \Delta_j^6 \rangle \delta_t,$$
$$\langle \Delta_{j+1}^6 \rangle = (1 - 6m\delta_t)\langle \Delta_j^6 \rangle + 15(\langle \Delta_j^4 \rangle + \epsilon_3 T^{-1}\langle \Delta_j^6 \rangle)\langle \xi_j^2 \rangle \delta_t^2 + 15\langle \Delta_j^2 \rangle \langle \eta_j^4 \rangle d_t^4 + 20\langle \Delta_j^3 \xi_j^3 \rangle \delta_t^3 - 6\epsilon \langle \Delta_j^8 \rangle \delta_t. \tag{A7}$$

In the above we have used the scaling properties of noises with δ_t and kept only terms up to $O(\delta_t)$. Treating $\langle \Delta_j^{2n} \rangle$ as a continuous variables and using the expansion $\langle \Delta_{j+1}^{2n} \rangle = \langle \Delta_j^{2n} \rangle + \delta_t \partial_t \langle \Delta_j^{2n} \rangle$, we find the coefficients of δ_t give nothing but the discretized version of (A1). It follows that (A7) give the same steady state solution as (A3) and (A5).

Finally, we turn to diagrammatic computations of $\langle \Delta(t)^2 \rangle$ and $\langle \Delta(t)^4 \rangle_c$. The LO results have been obtained in the main text. For the NLO results, we need diagrams with one vertex of either *aarr* type or *aaar* type and arbitrary number of other vertices. We first look at $\langle \Delta(t)^2 \rangle_{\rm NLO}$, which receives contributions from one-loop and two-loop diagrams shown, respectively, in Figs. 3 and 4. The evaluations of them are straightforward. We obtain in the end

$$\langle \Delta(t)^2 \rangle_{\rm NLO} = -\frac{3T^2\epsilon}{m^3} + \frac{T\epsilon_3}{m^2} + \frac{6\epsilon_1\epsilon}{m^2} - \frac{6T\epsilon_2\epsilon}{m^3}, \tag{A8}$$

which is in perfect agreement with (A5). The situation of $\langle \Delta(t)^4 \rangle^c_{\rm NLO}$ is more complicated: it receive contributions from both tree-level diagrams and loop diagrams; moreover, the latter contain both reducible and irreducible ones.

FIG. 3. One-loop Feynman diagrams for $\langle \Delta_r(t)^2 \rangle_c$.

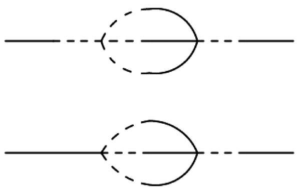

FIG. 4. Two-loop Feynman diagrams for $\langle\Delta_r(t)^2\rangle_c$.

The full tree-level diagrams are shown in Fig. 5. The first two are shown already in Fig. 3 of the main text and give the LO result (27) of the main text. The last two are easily evaluated to give the NLO result

$$\langle\Delta(t)^4\rangle^{c,\text{tree}}_{\text{NLO}} = \frac{6T^2\epsilon_3}{m^3} - \frac{6T^3\epsilon}{m^4}. \tag{A9}$$

The reducible and irreducible one-loop diagrams are shown, respectively, in Figs. 6 and 7. The reducible diagrams (Fig. 6) are evaluated to give

$$\langle\Delta(t)^4\rangle^{c,1\,\text{loop-red}}_{\text{NLO}} = \frac{18T\epsilon_1\epsilon}{m^3} - \frac{36T^2\epsilon_2\epsilon}{m^4} + \frac{6T\epsilon_2\epsilon_3}{m^3}. \tag{A10}$$

The irreducible diagrams (Fig. 7) give rise to the following results:

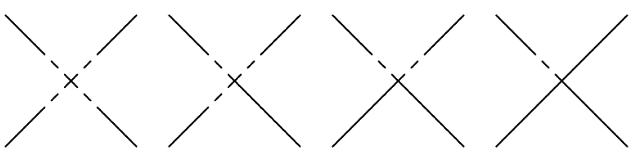

FIG. 5. Tree-level Feynman diagrams for $\langle\Delta_r(t)^4\rangle_c$.

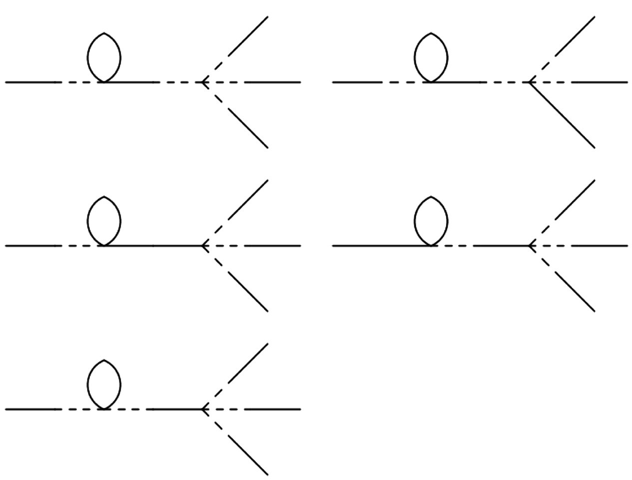

FIG. 6. Reducible one-loop Feynman diagrams for $\langle\Delta_r(t)^4\rangle_c$.

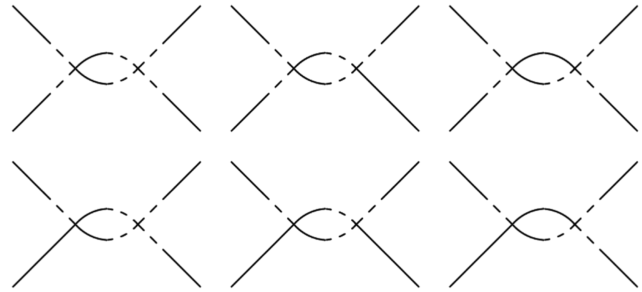

FIG. 7. Irreducible one-loop Feynman diagrams for $\langle\Delta_r(t)^4\rangle_c$.

$$\langle\Delta(t)^4\rangle^{c,1\,\text{loop-irred}}_{\text{NLO}} = -\frac{18\epsilon_1\epsilon_3}{m^2} + \frac{18T\epsilon_2\epsilon_3}{m^3} + \frac{54T\epsilon_1\epsilon}{m^3} - \frac{54T^2\epsilon_2\epsilon}{m^4}. \tag{A11}$$

At two-loop level, the reducible and irreducible diagrams are shown, respectively, in Figs. 8 and 9. Two-loop irreducible diagrams of a different topology are excluded by the presence of loop containing only *ra* type propagators. Being careful with the combinatoric factors, we arrive at

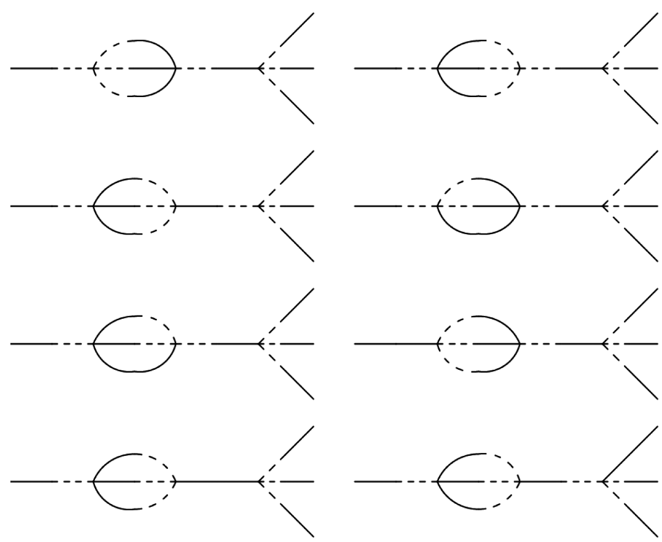

FIG. 8. Reducible two-loop Feynman diagrams for $\langle\Delta_r(t)^4\rangle_c$.

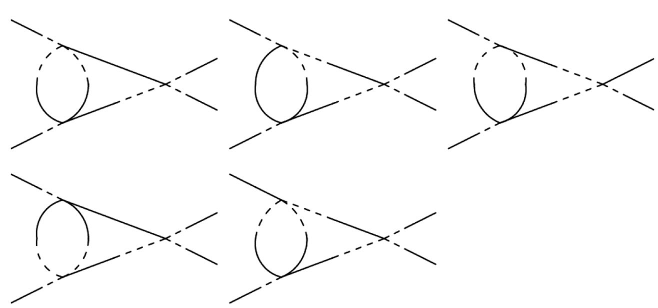

FIG. 9. Irreducible two-loop Feynman diagrams for $\langle\Delta_r(t)^4\rangle_c$.

$$\langle \Delta(t)^4 \rangle_{\rm NLO}^{c,2\,\text{loop-red}} = \frac{48\epsilon_1\epsilon_2\epsilon}{m^3} - \frac{48T\epsilon_2^2\epsilon}{m^4}. \qquad \text{(A12)}$$

for the reducible diagrams (Fig. 8) and

$$\langle \Delta(t)^4 \rangle_{\rm NLO}^{c,2\text{loop-irred}} = \frac{108\epsilon_1\epsilon_2\epsilon}{m^3} - \frac{108T\epsilon_2^2\epsilon}{m^4}, \qquad \text{(A13)}$$

for the irreducible diagrams (Fig. 9). The final result for $\langle \Delta(t)^4 \rangle_{\rm NLO}^{c}$ is the sum of (A9), (A10), (A11), (A12), and (A13), which reads

$$\langle \Delta(t)^4 \rangle_{\rm NLO}^{c} = \frac{6T^2\epsilon_3}{m^3} - \frac{6T^3\epsilon}{m^4} + \frac{72T\epsilon_1\epsilon}{m^3} - \frac{18\epsilon_1\epsilon_3}{m^2} + \frac{24T\epsilon_2\epsilon_3}{m^3} - \frac{90T^2\epsilon_2\epsilon}{m^4} + \frac{156\epsilon_1\epsilon_2\epsilon}{m^3} - \frac{156T\epsilon_2^2\epsilon}{m^4}. \qquad \text{(A14)}$$

This is also in full agreement with (A6). The agreement serves as a nontrivial demonstration of the equivalence among the three formulations.

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