Explicit no-π^2 renormalization schemes in QCD at five loops

J. A. Gracey

Theoretical Physics Division, Department of Mathematical Sciences, University of Liverpool, P.O. Box 147, Liverpool, L69 3BX, United Kingdom

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We examine a variety of renormalization schemes in QCD based on its 3-point vertices where the β-functions, gluon, ghost, quark and quark mass anomalous dimensions in each scheme do not depend on ζ_4 or ζ_6 in an arbitrary linear covariant gauge at five loops. We comment on the C-scheme.

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I. INTRODUCTION

Over the last decade or so remarkable advances have been made in computing the renormalization group functions of four dimensional quantum field theories to very high orders. Perhaps the most significant of these is the advancement of quantum chromodynamics (QCD) to five loops [1–22]. This has been made possible, for instance, due to the development of the FORCER package [23,24] written in the symbolic manipulation language FORM [25,26]. The package evaluates four loop massless Feynman integrals contributing to 2-point functions in d-dimensions as well as carrying out the expansion in ϵ where $d = 4 - 2\epsilon$. Its application to five loop computations has been possible through the use of the $R^\star$ operation [27–36] and the four loop FORCER package itself. While gauge theories are central to the Standard Model, progress in renormalizing scalar theories has advanced to an even higher loop order [37,38]. Such results are important too as they give insight into the number basis structure of the renormalization group functions that ought to have parallels in gauge theories when they are advanced to the next level. For instance, it is widely known that in the modified minimal subtraction ($\overline{\text{MS}}$) scheme [39,40] the renormalization group functions involve the Riemann ζ-function up to ζ_{11} at seven loops in the ϕ^4 β-function [38]. In addition the multiple zetas $\zeta_{5,3}$ and $\zeta_{5,3,3}$ first appear at six and seven loops respectively [37,38]. On top of this a new period, denoted by $P_{7,11}$ in [38], occurs which is believed to be inexpressible in terms of multiple zetas although it can be written in terms of multiple polylogarithms of the sixth roots of unity [38]. Beyond seven loops it is conjectured that other new periods will arise [38].

Knowledge of such potential structures is important in devising efficient computational tools to carry out future higher order renormalization. From the high loop order data that has been accumulated over many years several features have become apparent. For instance, at L loops no ζ_n can be present where $n > (2L - 3)$ beginning at $L = 3$. Earlier numbers in the ζ_n sequence, the Euler-Mascheroni constant γ and ζ_2, are absent. The former due to the choice of the $\overline{\text{MS}}$ scheme [39,40] over its predecessor the minimal subtraction scheme, and the latter as it actually cancels when the renormalization group functions are compiled from the renormalization constants. However we qualify this ζ_n structure by noting that it is primarily based on what occurs in a scalar theory. When symmetries are present the structure may be simpler. This is the case in the QCD β-function as ζ_3 first appears at four loops and ζ_5 at five loops in the $\overline{\text{MS}}$ scheme [11,16,18,19,21,22]. This is not the situation for the remaining core renormalization group functions which follow the ζ_{2L-3} pattern for their first occurrence. By core we will mean throughout the gluon, Faddeev-Popov ghost, quark and quark mass anomalous dimensions. The absence of ζ_3 at three loops in the β-function is due to gauge symmetry and the Slavnov-Taylor identity [41,42]. In schemes other than $\overline{\text{MS}}$, such as the minimal momentum (mMOM) scheme of [43], the ζ_{2L-3} pattern appears for the first time at L loops in the β-function for $L \geq 3$.

Another property of the number structure of the QCD β-function that is not unrelated to that for the odd zetas concerns the location of the even ones. Ordinarily ζ_{2L-4} would be expected to first appear at L loops for $L \geq 4$ but in the $\overline{\text{MS}}$ scheme this happens at $(L + 1)$ loops. We recall that ζ_{2n} is proportional to π^{2n} for $n \geq 1$. As before this even zeta structure is not mirrored in the other core renormalization group functions. Over the years the location or otherwise of the sequence of even zetas in renormalization group functions in various theories has been the subject of more than passing interest. Indeed a debate has ensued as to whether or not there is a natural way that $\zeta_4 = \frac{\pi^4}{90}$ and $\zeta_6 = \frac{\pi^6}{945}$ can be excluded and in what circumstances. One aspect of what is meant by this is whether there is an appropriate renormalization prescription that produces this scenario at a fundamental level. Another line of study is to examine whether there are so-called redefinitions of the odd zeta sequences that are universal across theories and schemes. Perhaps the pivotal instance where this was illuminated was provided in [44]. The focus there was on the perturbative structure of the Adler D-function in the $\overline{\text{MS}}$ scheme and the absence of ζ_4 was noted in the $O(a^4)$ term corresponding to the evaluation of five loop graphs, a property which is also shared with the R ratio at the same order. Prior to [44] it had been shown that ζ_4 appeared in the five loop quark mass anomalous dimension [20] as well as at four loops [12,13]. A detailed analysis of these observations in [44] suggested that for the renormalization group functions there was a systematic way of removing the even zetas by using an ϵ dependent transformation based on a connection with their odd partners. This was robustly examined for other QCD quantities as well as in other theories [44]. Moreover the observation was grounded from a renormalization group perspective. The idea was to concentrate on the location of the ζ_n sequence within the ϵ expansion of contributions to Feynman integrals that will affect the higher loop order counterterms. Aspects of this perspective were verified at very high orders in ϵ at several orders in the large N_f expansion [44,45] where N_f is the number of quarks. More recent developments have followed in the multicoupling context and for theories with supersymmetry for instance [46].

In field theory language the ideas of [44–46] should translate to changing to a different renormalization scheme. This is borne out by a second line of investigation which is to find a renormalization scheme that automatically produces a ζ_{2n} free set of renormalization group functions. There are already several pointers to such a scheme but in each case it appears that the picture is not complete. For instance, in [47] three schemes were introduced in QCD, and allocated the scheme label $\widetilde{\text{MOM}}$, with the low orders of the β-function determined in a specific gauge. The three schemes were based on each of the three 3-point vertices. The perturbative structure of these β-functions was extended to four loops in the Landau gauge in [48,49] as well as an additional scheme based on a different renormalization condition. That extra scheme had been introduced in [50] and the β-function constructed to three loops before being extended to four loops in [49]. The $\widetilde{\text{MOM}}$ scheme setup of [47,49] is a variant of the symmetric point momentum subtraction (MOM) ones of [51,52]. There each 3-point vertex was evaluated to the finite part and the finite part absorbed into the coupling renormalization constant with the same subtraction prescription used for the 2-point functions. In the $\widetilde{\text{MOM}}$ schemes of [47,49] the same subtraction is carried out but the 3-point vertices are first evaluated where one of the external momenta is set to zero. In other words in a situation where there is only one external momentum rather than two independent momenta as in the case of the MOM schemes of [51,52]. Throughout we will use the notation that $\widetilde{\text{MOM}}$ is the umbrella term for the momentum configuration used in [47,49] for 3-point vertices and use that term when referring to articles where the prescription was in fact employed but a different label, such as MOM, was given. This is to avoid confusion since MOM is more commonly used for the schemes of [51,52]. While [49] provided a four loop analysis the absence of ζ_4 was obvious though not surprising given it does not occur in the $\overline{\text{MS}}$ scheme at that order. Where the absence of ζ_4 became more significant was in the construction of the five loop β-function in the $\widetilde{\text{MOM}}$ scheme in quantum electrodynamics (QED) [53]. There the Ward-Takahashi identity was exploited to deduce the β-function from the photon renormalization constant with the finite part of the 2-point function removed at the subtraction point. The resulting β-function was devoid of ζ_4 and ζ_6. This example has since been classified as occurring in an anomalous dimension (AD) theory [46]. Such a theory is one where the β-function is deduced via a symmetry, which could be gauge symmetry for example, as it is directly related to the anomalous dimension of one or a combination of fields.

Another similar AD example is the Wess-Zumino theory [54] which is a four dimensional supersymmetric model. Its coupling constant renormalization is not independent since it is related to the field anomalous dimension via a supersymmetry Ward identity. In [55] the three loop $\widetilde{\text{MOM}}$ β-function was determined and subsequently extended to five loops in [56]. The four loop $\overline{\text{MS}}$ β-function was provided in [57] which corrected errors in earlier lower loop calculations. What was observed in the five loop $\widetilde{\text{MOM}}$ β-function [56] was the absence of ζ_4 and ζ_6 similar to [53]. Moreover the sector of the $\widetilde{\text{MOM}}$ β-function that corresponded to the iteration of the one loop bubble agreed precisely with the Hopf algebra $\widetilde{\text{MOM}}$ construction of that set of graphs presented in [58]. In fact iterating the Hopf algebra result well beyond the five loop order of [56] showed that there were no ζ_{2n} contributions to the very high order that was recorded in [58]. While such one loop bubble contributions are not the complete set of graphs it does provide strong evidence in a concrete example that the absence of even zetas may be a more fundamental property of the $\widetilde{\text{MOM}}$ scheme at least in the case of AD theories. One reason why the $\widetilde{\text{MOM}}$ scheme would offer a more satisfactory way to proceed, aside from being based on a Lagrangian and systematically implemented by a renormalization prescription, is that it is not clear what effect the detailed examination of the even zeta cancellation of [44,45] has on the remaining noneven zeta part of renormalization group functions. Ultimately it is the full renormalization group functions that are necessary for any application involving observables. Some progress in that direction has been provided in the C-scheme of [59,60]. This is a scheme that depends on a parameter C which is used as a measure of the scheme dependence of the coupling constant. One aspect of it is the claim that ζ_4 terms are not present in the C-scheme versions of the Adler D-function and several operator correlation functions including that of the field strength [60]. Moreover the mapping of the $\overline{\text{MS}}$ coupling constant to its C-scheme counterpart was discussed in depth in an analysis of the ζ_4 and ζ_6 dependence of the four loop anomalous dimensions of flavor nonsinglet and singlet twist-2 operators central to deep inelastic scattering for several low moments [61]. A variation of the C-scheme theme was explored in the $\hat{G}$-scheme provided in [44]. Another approach to the absence of π^2 contributions in correlation functions was examined in [62] through the application of the Landau-Khalatnikov-Fradkin transformation.

In other words based on the evidence discussed so far the finite subtraction approach is the most promising to pursue. However in compiling this overview for the $\widetilde{\text{MOM}}$ scheme and the zeta mapping analyses of [44–46] what appears to be absent for the former is a full renormalization of each Lagrangian and in particular that of QCD for a general color group and arbitrary linear covariant gauge. The main focus generally has been on the β-function compared with occasional interest in the core anomalous dimensions for non-AD theories or additionally in the case of a gauge theory only one specific gauge, the Landau one, has been considered [49]. The most recent $\widetilde{\text{MOM}}$ schemes recorded for QCD were at four loops [49], although the five loop $\overline{\text{MS}}$ renormalization group functions are all now available for an arbitrary linear covariant gauge [1–22]. More recently the determination of the five loop mMOM scheme core renormalization group functions has been completed [63,64] for an arbitrary linear gauge. From the currently available $\widetilde{\text{MOM}}$ expressions it has been indicated either explicitly or implicitly that there are no π^2 terms in the recorded β-functions. It is worth noting one case where the π^2 absence was indeed highlighted which was in the determination of the five loop QED $\widetilde{\text{MOM}}$ β-function [53]. Another reason to consider the core anomalous dimensions in a non-Abelian gauge theory is the fact that the first location of ζ_{2n} is different from the β-function. Therefore to be credible any $\widetilde{\text{MOM}}$ style prescription has to produce a universal π^2 absence across all renormalization group functions as well as all gauges and color groups for QCD. It is not clear how that would be effected at the level of the ϵ expansion of the ζ_n sequence within a suite of Feynman graphs. Having reviewed the background it is therefore the purpose of this article to balance both points of attack to understand the absence of π^2 in certain scenarios in non-Abelian gauge theories. To achieve this we will provide various renormalization schemes in QCD based on the three 3-point vertices of the linear covariant gauge fixed Lagrangian which extends previous work. This will be several more than those discussed in [47,49]. In each of these schemes we will demonstrate that none of the renormalization group functions depend on ζ_4 or ζ_6 to *five* loops. One aim is to present as full and complete an analysis as possible using all available data and techniques especially properties of the renormalization group equation. This has the added benefit that the results we present can be used in future to examine the effects using such explicit no-π^2 schemes[1] have on phenomenological predictions along similar lines to those of [59,60].

The article is organized as follows. We review the basic methods and techniques used for the analysis in Sec. II and en route define our notation, conventions and the schemes we will focus on. The way we constructed the relevant renormalization constants is touched upon too. Our results are summarized and presented in Sec. III as well as several checks. Having established our main goal we devote Sec. IV to a comparison of the QCD $\widetilde{\text{MOM}}$ schemes with the C-scheme. Subsequently in Sec. V we examine renormalization schemes in a larger context where other more general schemes are proposed and discussed. While the practical calculational study of such schemes is not as well advanced in terms of loop order, partly due to the absence of master integrals for n-point functions higher than three and general kinematics in analytic form, it is worth lighting the path ahead for future studies. Moreover once a deeper knowledge of the mathematical properties of such master integrals is known more insight would be available to understand the ζ_{2n} absence in the set of $\widetilde{\text{MOM}}$ schemes at the level considered here and beyond. We summarize our findings in Sec. VI. Finally two appendices follow with the first giving the Landau gauge $SU(3)$ five loop β-functions for the QCD $\widetilde{\text{MOM}}$ schemes. The second appendix gives the $SU(3)$ Landau gauge anomalous dimensions for the scheme that closely connects with the mMOM scheme.

II. FORMALISM

We devote this section to recalling the relevant formalism of the renormalization group that we exploit to determine the five loop renormalization group functions as well as the

[1]We will refer to schemes that have no ζ_{2n} dependence as no-π^2 ones in order to avoid confusion with what is termed the no-π theorem of [44].

method of computation to extract the four loop renormalization constants. First as a reference point for our conventions we recall the bare QCD Lagrangian is

$$L = -\frac{1}{4} G^a_{\text{o}\,\mu\nu} G^{a\,\mu\nu}_{\text{o}} - \frac{1}{2\alpha_{\text{o}}} (\partial^\mu A^a_{\text{o}\,\mu})^2 - \bar{c}^a_{\text{o}} (\partial_\mu D^\mu_{\text{o}} c_{\text{o}})^a + i \bar{\psi}^{iI}_{\text{o}} (\not{D}_{\text{o}} \psi_{\text{o}})^{iI} \quad (2.1)$$

with $_{\text{o}}$ denoting a bare object where the respective gluon, ghost and quark fields are A^a_μ, c^a and ψ^{iI}. We assume the fields lie in a general Lie group and the indices take the ranges $1 \leq a \leq N_A$, $1 \leq i \leq N_f$ and $1 \leq I \leq N_c$ where N_A is the adjoint representation dimension and N_c is the fundamental representation dimension. As in [63] we use the canonical linear covariant gauge fixing term with parameter α where $\alpha = 0$ is the Landau gauge. The mapping of bare variables to their renormalized counterparts is defined by

$$A^{a\,\mu}_{\text{o}} = \sqrt{Z_A} A^{a\,\mu}, \qquad c^a_{\text{o}} = \sqrt{Z_c} c^a, \qquad \psi_{\text{o}} = \sqrt{Z_\psi} \psi,$$
$$g_{\text{o}} = \mu^\epsilon Z_g\, g, \qquad \alpha_{\text{o}} = Z^{-1}_\alpha Z_A\, \alpha \quad (2.2)$$

and we have introduced the mass scale μ that arises when the Lagrangian is dimensionally regularized in $d = 4 - 2\epsilon$ dimensions, which we have used throughout, to ensure the coupling constant g remains dimensionless. Once the renormalization constants are determined to a particular loop order they are encoded in the renormalization functions given by

$$\gamma_\phi(a, \alpha) = \mu \frac{\partial}{\partial \mu} \ln Z_\phi, \qquad \beta(a, \alpha) = \mu \frac{\partial a}{\partial \mu},$$
$$\gamma_\alpha(a, \alpha) = \frac{\mu}{\alpha} \frac{\partial \alpha}{\partial \mu} \quad (2.3)$$

where ϕ is an element in the labelling set $\{A, c, \psi, m\}$ denoting the gluon, ghost, quark and quark mass renormalization respectively and

$$\mu \frac{\partial}{\partial \mu} = \beta(a, \alpha) \frac{\partial}{\partial a} + \alpha \gamma_\alpha(a, \alpha) \frac{\partial}{\partial \alpha} \quad (2.4)$$

with α dependence included in the β-function. While the $\overline{\text{MS}}$ scheme β-function and quark mass anomalous dimensions are α independent [39], this is not the case in general in schemes where a finite part is absorbed into the renormalization constant of the coupling. For completeness we note that (2.2) and (2.3) lead to

$$\gamma_A(a, \alpha) = \beta(a, \alpha) \frac{\partial}{\partial a} \ln Z_A + \alpha \gamma_\alpha(a, \alpha) \frac{\partial}{\partial \alpha} \ln Z_A$$
$$\gamma_\alpha(a, \alpha) = \left[\beta(a, \alpha) \frac{\partial}{\partial a} \ln Z_\alpha - \gamma_A(a, \alpha) \right] \left[1 - \alpha \frac{\partial}{\partial \alpha} \ln Z_\alpha \right]^{-1}$$
$$\gamma_c(a, \alpha) = \beta(a, \alpha) \frac{\partial}{\partial a} \ln Z_c + \alpha \gamma_\alpha(a, \alpha) \frac{\partial}{\partial \alpha} \ln Z_c$$
$$\gamma_\psi(a, \alpha) = \beta(a, \alpha) \frac{\partial}{\partial a} \ln Z_\psi + \alpha \gamma_\alpha(a, \alpha) \frac{\partial}{\partial \alpha} \ln Z_\psi \quad (2.5)$$

where $a = g^2/(16\pi^2)$. The relation between Z_α and $\gamma_\alpha(a, \alpha)$ is more general than one would expect in the canonical linear covariant gauge fixing. It is only when calculations in this gauge determine Z_α to be unity that the more widely known relation

$$\gamma_\alpha(a, \alpha) = -\gamma_A(a, \alpha) \quad (2.6)$$

results which will be the case for each of the new schemes introduced here.

As the renormalization of the parameters a and α will be carried out in several schemes one has to be able to map their running with μ from one scheme to another. This is achieved by realizing the bare coupling parameter can be expressed in terms of g in two different schemes producing

$$g_{\mathcal{S}}(\mu) = \frac{Z^{\overline{\text{MS}}}_g}{Z^{\mathcal{S}}_g} g_{\overline{\text{MS}}}(\mu), \qquad \alpha_{\mathcal{S}}(\mu) = \frac{Z^{\overline{\text{MS}}}_A Z^{\mathcal{S}}_\alpha}{Z^{\mathcal{S}}_A Z^{\overline{\text{MS}}}_\alpha} \alpha_{\overline{\text{MS}}}(\mu) \quad (2.7)$$

where the relation for α follows from similar reasoning for α_{o}. Throughout we will use $\mathcal{S}$ to indicate a scheme in general. For convenience it is simpler to use the $\overline{\text{MS}}$ scheme as the base or reference scheme for the discussion on the mapping of variables between schemes. For the fields and quark mass we can construct similar conversion functions which are given by

$$C^{\mathcal{S}}_\phi(a, \alpha) = \frac{Z^{\mathcal{S}}_\phi}{Z^{\overline{\text{MS}}}_\phi}, \qquad C^{\mathcal{S}}_\alpha(a, \alpha) = \frac{Z^{\mathcal{S}}_\alpha Z^{\overline{\text{MS}}}_A}{Z^{\overline{\text{MS}}}_\alpha Z^{\mathcal{S}}_A} \quad (2.8)$$

where the latter is included for completeness. In each of these definitions one has also to map the $\mathcal{S}$ scheme parameters to the $\overline{\text{MS}}$ ones by using, in this instance,

$$Z^{\mathcal{S}}_\phi = Z^{\mathcal{S}}_\phi(a_{\mathcal{S}}(a, \alpha), \alpha_{\mathcal{S}}(a, \alpha)),$$
$$Z^{\mathcal{S}}_\alpha = Z^{\mathcal{S}}_\alpha(a_{\mathcal{S}}(a, \alpha), \alpha_{\mathcal{S}}(a, \alpha)) \quad (2.9)$$

as otherwise expressions with poles in ϵ will be present. Once the parameter mappings and conversion functions have been explicitly established the renormalization group equations can be deduced via

$$\beta^{\mathcal{S}}(a_{\mathcal{S}},\alpha_{\mathcal{S}}) = \left[\beta^{\overline{\rm MS}}(a_{\overline{\rm MS}})\frac{\partial a_{\mathcal{S}}}{\partial a_{\overline{\rm MS}}} + \alpha_{\overline{\rm MS}}\gamma_\alpha^{\overline{\rm MS}}(a_{\overline{\rm MS}},\alpha_{\overline{\rm MS}})\frac{\partial a_{\mathcal{S}}}{\partial \alpha_{\overline{\rm MS}}}\right]_{\overline{\rm MS}\to\mathcal{S}}$$

$$\gamma_\phi^{\mathcal{S}}(a_{\mathcal{S}},\alpha_{\mathcal{S}}) = \left[\gamma_\phi^{\overline{\rm MS}}(a_{\overline{\rm MS}}) + \beta^{\overline{\rm MS}}(a_{\overline{\rm MS}})\frac{\partial}{\partial a_{\overline{\rm MS}}}\ln C_\phi^{\mathcal{S}}(a_{\overline{\rm MS}},\alpha_{\overline{\rm MS}}) + \alpha_{\overline{\rm MS}}\gamma_\alpha^{\overline{\rm MS}}(a_{\overline{\rm MS}},\alpha_{\overline{\rm MS}})\frac{\partial}{\partial \alpha_{\overline{\rm MS}}}\ln C_\phi^{\mathcal{S}}(a_{\overline{\rm MS}},\alpha_{\overline{\rm MS}})\right]_{\overline{\rm MS}\to\mathcal{S}}. \quad (2.10)$$

As the expressions on the left-hand side are dependent on the scheme $\mathcal{S}$ variables the final stage in their construction is to use the inverse relations to map $a_{\overline{\rm MS}}$ and $\alpha_{\overline{\rm MS}}$ to $a_{\mathcal{S}}$ and $\alpha_{\mathcal{S}}$ which is indicated by the restriction on each of the final brackets. One useful aspect of this formalism is that if the renormalization group functions are available in one scheme at $(L+1)$ loops then those of the other scheme need only to be explicitly evaluated at L loops in order to use the conversion functions to find the parallel $(L+1)$ loop expressions. This situation arises here since we will renormalize QCD in the set of $\widetilde{\rm MOM}_*$ schemes at four loops from which the $O(a^4)$ conversion functions can be deduced. We use $\widetilde{\rm MOM}_*$ to denote the QCD $\widetilde{\rm MOM}$ schemes to be introduced shortly. Given that the five loop $\overline{\rm MS}$ renormalization group functions are known [15–22] then (2.10) can be applied to determine the five loop $\widetilde{\rm MOM}_*$ renormalization group functions. This follows simply from (2.10) given that $\beta^{\overline{\rm MS}}(a_{\overline{\rm MS}})$ and $\gamma_\alpha^{\overline{\rm MS}}(a_{\overline{\rm MS}},\alpha_{\overline{\rm MS}})$ are $O(a^2)$ and $O(a)$ respectively. In this context it is worth remarking that the renormalization group functions are an encoding or representation of the renormalization constants computed explicitly from the field theory.

This process can of course be reversed to a certain extent. For instance it is well known that in the $\overline{\rm MS}$ scheme given the renormalization group functions at L loops one can deduce the L loop renormalization constants correctly by integrating (2.5). However this needs to be qualified for schemes where the renormalization prescription involves the subtraction of a finite part of a Green's function such as the $\widetilde{\rm MOM}_*$ schemes considered here. This is because the final stage of extracting renormalization group functions from the renormalization constants is to lift the regularization. In dimensional regularization this would correspond to the limit $\epsilon \to 0$. For the $\overline{\rm MS}$ scheme the only ϵ dependence is in the $O(a)$ term of the β-function; there is no ϵ dependence in any other core $\overline{\rm MS}$ renormalization group functions. By contrast for schemes where there is a finite part in the subtraction to determine the renormalization constants, such as $\widetilde{\rm MOM}_*$, the coefficient of a in each term of the core renormalization group functions is linear in ϵ prior to removing the regularization. This ϵ-dependent coefficient corresponds directly to the finite part of the renormalization constant and the dependence is rarely recorded in articles. Therefore in integrating (2.5) to find the $\widetilde{\rm MOM}_*$ renormalization constants, for example, one would have to initiate the process using the ϵ-dependent renormalization group functions.

Having outlined the relevant formalism to construct the renormalization group functions of our $\widetilde{\rm MOM}_*$ schemes we now focus on details. First there are three 3-point vertices which in [51,52] led to three distinct symmetric point schemes denoted by MOMg, MOMc and MOMq based respectively on the triple gluon, ghost-gluon and quark-gluon vertices. In these symmetric point schemes the renormalization prescription is to remove the poles as well as the finite parts at the momentum configuration where the squared momenta of the three external legs of the 3-point vertices are equal. None of the underlying external momenta are nullified. Equally the 2-point functions are rendered finite by removing the finite part in addition to the poles. By contrast the $\widetilde{\rm MOM}_*$ schemes are defined from the 3-point vertices but where the momentum configuration has one external momentum nullified at the outset. Similar to the schemes of [51,52] the finite parts of both the 3-point vertices for this configuration and the 2-point functions are absorbed into the respective renormalization constants. One difference is that there are more $\widetilde{\rm MOM}_*$ schemes [49] than the three MOM schemes of [51,52]. In the first instance this is because for each of the ghost-gluon and quark-gluon vertices there are two possible external leg nullifications. These are either the gluon leg or one of the respective ghost or quark legs. One might suspect there is a third ghost-gluon vertex scheme given the asymmetric nature of the ghost-gluon interaction in (2.1). However it is trivial to see that the nullification of the $\bar{c}^a$ leg produces zero for each graph of the 3-point vertex in the linear covariant gauge. This is not the case for nonlinear covariant gauges for instance. For clarity it is worth recalling the tensor structure of the vertex functions for reference and to assist with the scheme definitions. Based on [18,47] we have

$$\left\langle A^a_\mu(p)A^b_\nu(-p)A^c_\sigma(0)\right\rangle = -igf^{abc}\left[(2\eta_{\mu\nu}p_\sigma - 2\eta_{\mu\sigma}p_\nu - 2\eta_{\nu\sigma}p_\mu)T_1(p^2) - \left[\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right]p_\sigma T_2(p^2)\right]$$
$$\left\langle c^a(p)\bar{c}^b(-p)A^c_\mu(0)\right\rangle = -igf^{abc}p_\mu\tilde{\Gamma}_g(p^2)$$
$$\left\langle c^a(0)\bar{c}^b(p)A^c_\mu(-p)\right\rangle = -igf^{abc}p_\mu\tilde{\Gamma}_c(p^2)$$
$$\langle\psi^{iI}(p)\bar{\psi}^{jJ}(0)A^c_\mu(-p)\rangle = -gT^c\left[\gamma_\mu\Lambda_q(p^2) + \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\gamma^\nu\Lambda^T_q(p^2)\right]$$
$$\langle\psi^{iI}(p)\bar{\psi}^{jJ}(-p)A^c_\mu(0)\rangle = -gT^c\left[\gamma_\mu\Lambda_g(p^2) + \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\gamma^\nu\Lambda^T_g(p^2)\right] \qquad (2.11)$$

where the various form factors are used for the different renormalization prescriptions and T^a and f^{abc} are the color group generators and structure constants respectively. We note that the external momentum configuration that defines $\tilde{\Gamma}_c(p^2)$ is the one that is the foundation for the mMOM scheme of [43].

To define the set of $\widetilde{\text{MOM}}_*$ schemes that we will determine to five loops we first recall those of [47] and introduce our syntax. In [47] the three $\widetilde{\text{MOM}}$ schemes were defined with respect to one nullification of each of the 3-point vertices. For each case we record the condition on the respective form factors and the $\widetilde{\text{MOM}}_*$ scheme label we will use as its notation. We have

$$T_1(\mu^2) = 1 \leftrightarrow \widetilde{\text{MOM}}_{ggg0g}$$
$$\tilde{\Gamma}_c(\mu^2) = 1 \leftrightarrow \widetilde{\text{MOM}}_{ccg0c}$$
$$\Lambda_q(\mu^2) = 1 \leftrightarrow \widetilde{\text{MOM}}_{qqg0q} \qquad (2.12)$$

where g, c and q denote the gluon, ghost and quark respectively with the letter after 0 in the subscript indicating which leg is nullified. In [49] the additional scheme introduced in [50] was also examined. In the syntax of (2.12) its defining condition and label is

$$T_1(\mu^2) - \frac{1}{2}T_2(\mu^2) = 1 \leftrightarrow \widetilde{\text{MOM}}_{ggg0gg}. \qquad (2.13)$$

Given that combinations of form factors can be used to form renormalization prescriptions we introduce the remaining schemes considered here. These are

$$\tilde{\Gamma}_g(\mu^2) = 1 \leftrightarrow \widetilde{\text{MOM}}_{ccg0g}$$
$$\Lambda_g(\mu^2) = 1 \leftrightarrow \widetilde{\text{MOM}}_{qqg0g}$$
$$\Lambda_q(\mu^2) + \Lambda^T_q(\mu^2) = 1 \leftrightarrow \widetilde{\text{MOM}}_{qqg0qT}$$
$$\Lambda_g(\mu^2) + \Lambda^T_g(\mu^2) = 1 \leftrightarrow \widetilde{\text{MOM}}_{qqg0gT} \qquad (2.14)$$

where the equality is the value of the form factors after renormalization. In essence these schemes differ from those of [51,52] in that there is only one independent momentum flowing through the Green's function instead of two. So in effect the computation of the vertex functions reduces to that of evaluating 2-point functions. With regard to the prescription for the field renormalization we first note that the propagators will have the form

$$\left\langle A^a_\mu(p)A^b_\nu(-p)\right\rangle = -\frac{\delta^{ab}}{p^2}\left[\left[\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right] \times \frac{1}{[1+\Pi_g(p^2)]} - \alpha\frac{p_\mu p_\nu}{p^2}\right]$$
$$\left\langle c^a(p)\bar{c}^b(-p)\right\rangle = \frac{\delta^{ab}}{p^2}\frac{1}{[1+\Pi_c(p^2)]}$$
$$\langle\psi^{iI}(p)\bar{\psi}^{jJ}(-p)\rangle = \frac{\delta^{ij}\delta^{IJ}}{p^2}\frac{\not{p}}{[1+\Sigma_q(p^2)]} \qquad (2.15)$$

which defines the various 2-point form factors. The $\widetilde{\text{MOM}}_*$ prescription is that the propagator denominators are unity at the subtraction point of $p^2 = \mu^2$. As we will also renormalize the quark mass operator, that immediately determines the quark mass renormalization constant, from the Green's function

$$\left\langle\psi^{iI}(p)\bar{\psi}^{jJ}(-p)[\bar{\psi}^{kK}\psi^{kK}](0)\right\rangle = \delta^{ij}\delta^{IJ}\Gamma_m(p^2) \qquad (2.16)$$

which defines $\Gamma_m(p^2)$ with the prescription that this has to be unity at $p^2 = \mu^2$. The quark mass renormalization can be treated this way using (2.16) as in effect this equates to the way the mass term of the quark 2-point function in the massive version of (2.1) is renormalized. For the massive Lagrangian the quark 2-point function can be expanded in powers of the quark mass m to $O(m)$ thence producing the same Feynman graphs that constitute (2.16). This is the reason why we will only consider the momentum routing in (2.16) rather than the one where a momentum passes through the operator itself. The key advantage of considering (2.16) is that FORCER can be exploited to evaluate the constituent Feynman integrals since the setup will then be a massless one.

One observation needs to be made concerning each scheme. If for instance $\langle A^a_\mu(p)A^b_\nu(-p)A^c_\sigma(0)\rangle$ is selected to construct the $\widetilde{\text{MOM}}_{ggg0g}$ scheme renormalization group functions this means that $T_1(\mu^2)$ is unity as noted in (2.12). One question then concerns what form do the remaining vertex functions of (2.11) take at their indicated momentum configuration in the $\widetilde{\rm MOM}_{ggg0g}$ scheme. By the Slavnov-Taylor identities each of the other form factors will be finite in the same way that using one vertex to find the $\overline{\rm MS}$ coupling renormalization constant means the other vertices are immediately finite. However what is generally the case is that each of the other vertex functions in the $\widetilde{\rm MOM}_{ggg0g}$ scheme will not be solely the renormalized coupling constant at $p^2 = \mu^2$, unless the Slavnov-Taylor identities produce this. Instead the form factors of each of the other vertex functions will be a perturbative series in the renormalized coupling constant of that $\widetilde{\rm MOM}_{ggg0g}$ scheme. The same observation applies when each of the other schemes is considered in turn and this summarizes the comments made in [51,52] for the symmetric point MOM schemes.

Having defined the suite of $\widetilde{\rm MOM}_*$ schemes in relation to the respective Green's functions we need to record the particular algorithm to deduce the renormalization constants. Assuming the renormalization has been carried out to L loops in one of the $\widetilde{\rm MOM}_*$ schemes then the 2-point functions are renormalized at $(L+1)$ loops by removing the poles in ϵ and the finite part. Once the 2-point functions are rendered finite to $(L+1)$ loops then only the form factor of the vertex function of (2.11) relevant to defining that specific $\widetilde{\rm MOM}_*$ scheme has its poles and finite part absorbed into the coupling constant renormalization constant. After this has been achieved the process is repeated up to and including four loops using the automatic process introduced in [10]. Then this procedure is repeated for the subsequent $\widetilde{\rm MOM}_*$ scheme until all of the schemes have been constructed at *four* loops. The final step is to determine the conversion functions and parameter maps before applying (2.10) to deduce the five loop renormalization group functions in each of the $\widetilde{\rm MOM}_*$ schemes. Comment needs to be made on the definition of the $\widetilde{\rm MOM}_{ccg0c}$ scheme in relation to the mMOM scheme of [43]. That scheme is also based on the vertex with an external ghost leg nullification [43] and leads to the natural question are both schemes equivalent. The brief answer is that they are different except in one special instance which is the Landau gauge. The general difference is because the coupling constant renormalization in the mMOM scheme is constructed from the renormalization of 2-point functions and the $\overline{\rm MS}$ coupling renormalization constant alone, where the finite parts of the 2-point functions in mMOM are absorbed into the ghost and gluon renormalization constants. Then the mMOM coupling constant renormalization constant is deduced from the relation [43]

$$Z_g^{\rm mMOM}\sqrt{Z_A^{\rm mMOM}}Z_c^{\rm mMOM} = Z_g^{\overline{\rm MS}}\sqrt{Z_A^{\overline{\rm MS}}}Z_c^{\overline{\rm MS}} \qquad (2.17)$$

which is motivated from Taylor's observation that the ghost-gluon vertex is finite in the Landau gauge [41]. For a linear covariant gauge this is straightforward to see. Irrespective of the renormalization scheme that is used, for the case where the external ghost nullification is nontrivial the vertex containing that ghost has the loop momentum flowing through the other ghost and gluon. From (2.1) the Feynman rule for the vertex involves the loop momentum alone with an index contracted with the gluon propagator. This immediately reduces the combination to a term proportional to α which clearly vanishes in the Landau gauge. The condition (2.17), which equates the overall ghost-gluon vertex renormalization constants of the two schemes, seeks to preserve that equivalence property within the renormalization for arbitrary α. One interesting feature of the four and five loop mMOM β-function and other renormalization group functions is that they have neither ζ_4 nor ζ_6 dependence in the Landau gauge[2] which is the special instance noted earlier. There is ζ_4 and ζ_6 dependence for nonzero α in the mMOM scheme. Indeed this was a clue to the fact that if the constraint for this external momentum configuration was ignored and an independent coupling renormalization constant was introduced then it should be the case that the ζ_4 and ζ_6 dependence is absent in the $\widetilde{\rm MOM}_{ccg0c}$ scheme for all α. In other words the residual ζ_4 and ζ_6 dependence in terms involving α, that the condition (2.17) omits through its aim of preserving the no-renormalization property of the vertex for nonzero α, is accommodated within the $\widetilde{\rm MOM}_{ccg0c}$ scheme for all α. This turned out to be indeed the case as will be evident in our results. In one sense the Landau gauge mMOM β-function could be regarded as another example of an AD theory in the classification of [46].

With the formalism and $\widetilde{\rm MOM}_*$ scheme definitions established the task of determining the renormalization group functions explicitly remains. To do so at five loops will therefore require a full four loop renormalization of the 2-point functions and the respective Green's functions with the momenta configurations of (2.11). Helpfully useable data is already available [63] which is readily extracted using FORCER [23,24]. In [63] the ϵ expansion of the three 2-point functions and the form factors of the three 3-point functions in (2.11) have been provided as a function of the bare coupling constant and gauge parameter. For the determination of the quark mass renormalization constant we use the analogous four loop Green's function that was computed in [65]. We note that those four loop expressions had all been established with the Feynman graph integration package FORCER. To extract the respective

[2]We are indebted to I. Jack for drawing our attention to this implicit feature.

renormalization constants from each bare Green's function we follow the process outlined in [10] where the counterterms are introduced automatically using the relations given in (2.2). Invaluable in carrying out the renormalization was the symbolic manipulation language FORM [25,26] which meant the extraction of the renormalization constants progressed efficiently and these were straightforwardly converted to the five loop $\widetilde{\text{MOM}}_*$ renormalization group functions.

III. RESULTS

One of the main tasks was to demonstrate that in the suite of $\widetilde{\text{MOM}}_*$ schemes no ζ_4 or ζ_6 appears in the renormalization group functions. In order to present that observation we have to record the full five loop expressions that allows one to verify that no such numbers are present. As the full expressions for nonzero α and arbitrary color group are rather lengthy we will record the renormalization group functions for one scheme as an example and focus on the case of $SU(3)$ and the Landau gauge. Except that to illustrate the absence of ζ_4 and ζ_6 for all α we will record one representative β-function. Electronic versions of the results for all α and a general color group are provided in the data file [66] available via the arXiv version of this paper. It is then a simple matter to check that there are neither ζ_4 nor ζ_6 terms in any of the renormalization group functions for all the schemes in that file by employing a search tool for instance. The particular scheme we present the results for is the $\widetilde{\text{MOM}}_{ggg0g}$ scheme. First the β-function for nonzero α is

$$
\begin{aligned}
\beta^{SU(3)}_{\widetilde{\text{MOM}}_{ggg0g}}(a,\alpha) = {} & \left[\frac{2}{3}N_f - 11\right]a^2 + \left[-\frac{9}{4}\alpha^3 + \frac{38}{3}N_f + \frac{117}{4}\alpha - 102 - 3N_f\alpha - N_f\alpha^2 + 3\alpha^2\right]a^3 \\
& + \left[-\frac{58491}{16} - \frac{1575}{32}\alpha^3 - \frac{729}{8}\zeta_3\alpha - \frac{481}{27}N_f^2 - \frac{429}{4}N_f\alpha - \frac{243}{8}\zeta_3\alpha^2 \right. \\
& - \frac{165}{8}N_f\alpha^2 - \frac{119}{6}\zeta_3 N_f - \frac{63}{32}\alpha^4 - \frac{8}{9}\zeta_3 N_f^2 + \frac{3}{8}N_f\alpha^4 + \frac{15}{4}N_f\alpha^3 + \frac{1053}{16}\alpha^2 \\
& \left. + \frac{2277}{4}\zeta_3 + \frac{5643}{8}\alpha + \frac{15283}{24}N_f\right]a^4 \\
& + \left[-\frac{10982273}{64} - \frac{2677587}{128}\zeta_3\alpha - \frac{1075423}{144}\zeta_3 N_f - \frac{724445}{324}N_f^2 - \frac{506541}{64}N_f\alpha \right. \\
& - \frac{465651}{256}\alpha^3 - \frac{266373}{128}\zeta_3\alpha^2 - \frac{77441}{64}N_f\alpha^2 - \frac{60895}{9}\zeta_5 N_f - \frac{50481}{256}\alpha^4 \\
& - \frac{8935}{32}\zeta_5 N_f\alpha - \frac{5445}{128}\zeta_5\alpha^4 - \frac{1053}{128}\alpha^5 - \frac{945}{32}\zeta_5\alpha^5 - \frac{605}{4}\zeta_3 N_f^2\alpha \\
& - \frac{295}{32}\zeta_5 N_f\alpha^3 - \frac{105}{16}\zeta_5 N_f\alpha^4 - \frac{16}{3}\zeta_3 N_f^2\alpha^2 - \frac{16}{9}\zeta_3 N_f^3 - \frac{9}{8}N_f\alpha^5 - \frac{8}{9}N_f^3\alpha \\
& + \frac{32}{9}\zeta_3 N_f^3\alpha + \frac{39}{8}\zeta_3 N_f\alpha^4 + \frac{75}{8}N_f\alpha^4 + \frac{351}{16}\zeta_3\alpha^5 + \frac{445}{9}N_f^2\alpha^2 \\
& + \frac{529}{32}\zeta_3 N_f\alpha^3 + \frac{788}{27}N_f^3 + \frac{3709}{16}\zeta_3 N_f\alpha^2 + \frac{9280}{27}\zeta_5 N_f^2 + \frac{11903}{48}N_f^2\alpha \\
& + \frac{12237}{64}N_f\alpha^3 + \frac{12959}{54}\zeta_3 N_f^2 + \frac{17271}{128}\zeta_3\alpha^4 + \frac{20505}{128}\zeta_5\alpha^3 + \frac{36135}{2}\zeta_5 \\
& + \frac{53097}{128}\zeta_3\alpha^3 + \frac{82249}{32}\zeta_3 N_f\alpha + \frac{155205}{128}\zeta_5\alpha^2 + \frac{251793}{64}\alpha^2 + \frac{830955}{128}\zeta_5\alpha \\
& \left. + \frac{1425171}{32}\zeta_3 + \frac{2540673}{64}\alpha + \frac{3830167}{96}N_f\right]a^5 \\
& + \left[-\frac{65313445615}{27648}\zeta_5 N_f - \frac{25790811345}{2048} - \frac{18051846813}{4096}\zeta_7 \right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{6588808297}{10368}\zeta_3 N_f-\frac{3473740893}{8192}\zeta_7\alpha-\frac{3264898519}{10368}N_f^2+\frac{5165077893}{2048}\alpha\\
&+\frac{15665964165}{2048}\zeta_5+\frac{22912517957}{18432}\zeta_7 N_f+\frac{910029320399}{248832}N_f\\
&-\frac{1356316467}{1024}\zeta_3\alpha-\frac{650074537}{1024}N_f\alpha-\frac{303580683}{8192}\zeta_7\alpha^2\\
&-\frac{242578035}{4096}\zeta_5\alpha^2-\frac{224554437}{2048}\alpha^3-\frac{128139255}{4096}\zeta_5\alpha^3-\frac{51105929}{864}\zeta_7 N_f^2\\
&-\frac{31554195}{4096}\zeta_5\alpha^4-\frac{29533401}{1024}\zeta_3^2\alpha-\frac{19040275}{256}N_f\alpha^2-\frac{17251731}{4096}\zeta_7\alpha^3\\
&-\frac{8416485}{8192}\zeta_7\alpha^5-\frac{3996279}{2048}\zeta_7\alpha^4-\frac{3120849}{512}\alpha^4-\frac{2401857}{256}\zeta_3^2 N_f\\
&-\frac{1232293}{128}\zeta_3 N_f\alpha^3-\frac{1096745}{384}\zeta_3 N_f\alpha^2-\frac{721445}{576}\zeta_5 N_f^2\alpha^2-\frac{586025}{108}\zeta_5 N_f^3\\
&-\frac{436751}{24}\zeta_3 N_f^2\alpha-\frac{314631}{1024}\alpha^5-\frac{279765}{1024}\zeta_5\alpha^5-\frac{234087}{2048}\zeta_7 N_f\alpha^4\\
&-\frac{137619}{1024}\zeta_3\alpha^6-\frac{120423}{256}\zeta_3 N_f\alpha^4-\frac{103743}{1024}\zeta_3\alpha^5-\frac{80865}{1024}\zeta_5\alpha^6\\
&-\frac{53411}{36}\zeta_3^2 N_f^2-\frac{18823}{36}N_f^3\alpha-\frac{16627}{24}N_f^2\alpha^3-\frac{13455}{8}\zeta_5 N_f^2\alpha\\
&-\frac{7653}{128}\zeta_3 N_f\alpha^5-\frac{7065}{64}N_f\alpha^5-\frac{4263}{4}\zeta_7 N_f^2\alpha-\frac{2565}{128}\alpha^6-\frac{2482}{27}N_f^4\\
&-\frac{806}{9}N_f^3\alpha^2-\frac{710}{9}\zeta_5 N_f^3\alpha-\frac{409}{8}N_f^2\alpha^4-\frac{304}{27}\zeta_3 N_f^4-\frac{245}{4}\zeta_3^2 N_f^2\alpha\\
&-\frac{143}{3}\zeta_3 N_f^3\alpha^2-\frac{117}{32}\zeta_3 N_f\alpha^6-\frac{32}{3}\zeta_3 N_f^3\alpha^3-\frac{15}{2}\zeta_5 N_f^2\alpha^3-\frac{9}{16}N_f\alpha^6\\
&+\frac{5}{3}\zeta_5 N_f^3\alpha^2+\frac{8}{3}N_f^3\alpha^3+\frac{315}{64}\zeta_5 N_f\alpha^6+\frac{423}{8}\zeta_3^2 N_f\alpha^3+\frac{621}{64}\zeta_3^2 N_f\alpha^5\\
&+\frac{1141}{2}\zeta_3 N_f^2\alpha^3+\frac{1225}{3}\zeta_3 N_f^3\alpha+\frac{1760}{27}\zeta_5 N_f^4+\frac{2240}{27}\zeta_3^2 N_f^3+\frac{3549}{1024}\zeta_7 N_f\alpha^5\\
&+\frac{3615}{128}\zeta_5 N_f\alpha^5+\frac{6027}{128}\zeta_3^2 N_f\alpha^4+\frac{27945}{512}\zeta_3^2\alpha^6+\frac{28805}{256}\zeta_5 N_f\alpha^4\\
&+\frac{46845}{256}\zeta_3^2 N_f\alpha^2+\frac{61715}{64}\zeta_5 N_f\alpha^3+\frac{90297}{128}N_f\alpha^4+\frac{159705}{8192}\zeta_7\alpha^6\\
&+\frac{172179}{512}\zeta_3^2\alpha^5+\frac{351521}{288}\zeta_3 N_f^2\alpha^2+\frac{410595}{128}\zeta_3^2 N_f\alpha+\frac{441851}{648}\zeta_3 N_f^3\\
&+\frac{470223}{1024}\zeta_3^2\alpha^4+\frac{605583}{1024}\zeta_3^2\alpha^3+\frac{1673385}{1024}\zeta_7 N_f\alpha^2+\frac{2574009}{1024}\zeta_3^2\alpha^2\\
&+\frac{5911227}{1024}\zeta_3\alpha^4+\frac{10147923}{512}N_f\alpha^3+\frac{16512415}{512}\zeta_5 N_f\alpha+\frac{25235989}{2592}N_f^3\\
&+\frac{30395757}{1024}\zeta_3\alpha^2+\frac{36466719}{128}\zeta_3 N_f\alpha+\frac{39704059}{6912}N_f^2\alpha^2+\frac{42976673}{1152}N_f^2\alpha\\
&+\frac{43537571}{1024}\zeta_7 N_f\alpha+\frac{73833111}{1024}\zeta_3\alpha^3+\frac{74629145}{3072}\zeta_5 N_f\alpha^2\\
&+\frac{87682327}{5184}\zeta_3 N_f^2+\frac{95256513}{512}\zeta_3^2+\frac{384102657}{128}\zeta_3+\frac{399957783}{2048}\alpha^2\\
&+\frac{985401955}{5184}\zeta_5 N_f^2+\frac{1171309995}{4096}\zeta_5\alpha-5\zeta_3^2 N_f^2\alpha^2+15\zeta_3 N_f^2\alpha^4\Bigg]a^6+O(a^7). \qquad (3.1)
\end{aligned}
$$

Our convention throughout will be that the variables a and α are in the scheme attached to the renormalization group function itself. The Landau gauge anomalous dimensions of the gluon, ghost, quark and quark mass in the same scheme are

$$\begin{aligned}
\gamma^{SU(3)}_{A,\widetilde{\mathrm{MOM}_{ggg0g}}}(a,0) = {} & \left[-\frac{13}{2}+\frac{2}{3}N_f\right]a + \left[-\frac{255}{4}+\frac{67}{6}N_f\right]a^2 \\
& + \left[-\frac{68433}{32}-\frac{373}{27}N_f^2-\frac{83}{6}\zeta_3N_f-\frac{8}{9}\zeta_3N_f^2+\frac{4365}{16}\zeta_3+\frac{11293}{24}N_f\right]a^3 \\
& + \left[-\frac{6789623}{64}-\frac{1895585}{288}\zeta_5N_f-\frac{1683719}{288}\zeta_3N_f-\frac{1140955}{648}N_f^2-\frac{16}{9}\zeta_3N_f^3\right. \\
& \left.+\frac{671}{27}N_f^3+\frac{9280}{27}\zeta_5N_f^2+\frac{20707}{108}\zeta_3N_f^2+\frac{1090305}{64}\zeta_5+\frac{2793877}{96}N_f+\frac{7071297}{256}\zeta_3\right]a^4 \\
& + \left[-\frac{74174308299}{16384}\zeta_7-\frac{67523852455}{27648}\zeta_5N_f-\frac{33044274229}{4096}\right. \\
& -\frac{7482447221}{20736}\zeta_3N_f+\frac{2602029675}{2048}\zeta_3+\frac{7832693295}{1024}\zeta_5 \\
& +\frac{24152007425}{18432}\zeta_7N_f+\frac{658685409635}{248832}N_f-\frac{311150303}{1296}N_f^2 \\
& -\frac{51407573}{864}\zeta_7N_f^2-\frac{3552969}{256}\zeta_3^2N_f-\frac{590285}{108}\zeta_5N_f^3-\frac{54527}{36}\zeta_3^2N_f^2 \\
& -\frac{1906}{27}N_f^4-\frac{304}{27}\zeta_3N_f^4+\frac{1760}{27}\zeta_5N_f^4+\frac{2240}{27}\zeta_3^2N_f^3+\frac{545495}{648}\zeta_3N_f^3 \\
& +\frac{14365339}{5184}\zeta_3N_f^2+\frac{19696993}{2592}N_f^3+\frac{660827493}{2048}\zeta_3^2 \\
& \left.+\frac{1002897415}{5184}\zeta_5N_f^2\right]a^5+O(a^6)
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\gamma^{SU(3)}_{c,\widetilde{\mathrm{MOM}_{ggg0g}}}(a,0) = {} & -\frac{9}{4}a + \left[-\frac{153}{8}+\frac{3}{4}N_f\right]a^2 \\
& + \left[-\frac{46305}{64}-\frac{5}{2}N_f^2+\frac{9}{4}\zeta_3N_f+\frac{357}{4}N_f+\frac{1971}{32}\zeta_3\right]a^3 \\
& + \left[-\frac{8486505}{256}-\frac{13205}{48}N_f^2-\frac{5895}{16}\zeta_5N_f-\frac{1977}{4}\zeta_3N_f+\frac{5}{2}N_f^3+\frac{162405}{32}\zeta_5\right. \\
& \left.+\frac{748593}{128}N_f+\frac{1824363}{512}\zeta_3+26\zeta_3N_f^2\right]a^4 \\
& + \left[-\frac{18930657429}{8192}-\frac{17060330889}{32768}\zeta_7+\frac{5463760635}{4096}\zeta_5-\frac{435383505}{4096}\zeta_3^2\right. \\
& -\frac{98927995}{2304}N_f^2-\frac{74407755}{512}\zeta_5N_f-\frac{18046313}{512}\zeta_3N_f-\frac{3969}{2}\zeta_7N_f^2 \\
& -\frac{219}{2}\zeta_3^2N_f^2+\frac{7747}{6}N_f^3+\frac{109901}{48}\zeta_3N_f^2+\frac{282185}{48}\zeta_5N_f^2 \\
& +\frac{846441}{128}\zeta_3^2N_f+\frac{38085719}{1024}\zeta_7N_f+\frac{421898185}{768}N_f \\
& \left.+\frac{735148629}{4096}\zeta_3-65\zeta_5N_f^3-14N_f^4+\zeta_3N_f^3\right]a^5+O(a^6)
\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
\gamma^{SU(3)}_{\psi,\widetilde{\mathrm{MOM}_{ggg0g}}}(a,0) = {} & \left[-\frac{4}{3}N_f+\frac{67}{3}\right]a^2+\left[-\frac{706}{9}N_f-\frac{607}{2}\zeta_3+\frac{8}{9}N_f^2+\frac{29675}{36}+16\zeta_3N_f\right]a^3 \\
& +\left[-\frac{21455183}{648}\zeta_3-\frac{2401655}{324}N_f-\frac{272}{9}\zeta_3N_f^2-\frac{40}{9}N_f^3+\frac{2879}{9}N_f^2\right. \\
& \left.+\frac{73873}{27}\zeta_3N_f+\frac{7727771}{162}+\frac{15846715}{1296}\zeta_5-830\zeta_5N_f\right]a^4 \\
& +\left[-\frac{93917679073}{31104}\zeta_3-\frac{26588447977}{27648}\zeta_7+\frac{2239174289}{10368}\zeta_3^2+\frac{9255603625}{3888}\zeta_5\right. \\
& +\frac{14692571119}{5184}-\frac{333206965}{972}\zeta_5N_f-\frac{252766199}{432}N_f-\frac{4731481}{243}\zeta_3N_f^2 \\
& -\frac{576437}{54}\zeta_3^2N_f-\frac{80840}{81}N_f^3-\frac{6811}{2}\zeta_7N_f^2-\frac{3520}{27}\zeta_5N_f^3-\frac{128}{9}\zeta_3^2N_f^2 \\
& +\frac{160}{27}N_f^4+\frac{24064}{81}\zeta_3N_f^3+\frac{1069795}{27}N_f^2+\frac{3085750}{243}\zeta_5N_f^2+\frac{4429579}{36}\zeta_7N_f \\
& \left.+\frac{1741317151}{3888}\zeta_3N_f\right]a^5+O(a^6)
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
\gamma^{SU(3)}_{m,\widetilde{\mathrm{MOM}_{ggg0g}}}(a,0) = {} & -4a+\left[-\frac{209}{3}+\frac{4}{3}N_f\right]a^2 \\
& +\left[-\frac{23731}{9}-\frac{176}{9}\zeta_3N_f-\frac{8}{3}N_f^2+\frac{2723}{3}\zeta_3+\frac{4823}{27}N_f\right]a^3 \\
& +\left[-\frac{364717295}{2592}-\frac{318905}{54}\zeta_3N_f-\frac{16015}{3}\zeta_5-\frac{13741}{27}N_f^2-\frac{3200}{9}\zeta_5N_f\right. \\
& \left.+\frac{8}{3}N_f^3+\frac{1552}{9}\zeta_3N_f^2+\frac{5306821}{324}N_f+\frac{30602221}{432}\zeta_3\right]a^4 \\
& +\left[-\frac{1206873493973}{124416}-\frac{2433059707}{3456}\zeta_3^2+\frac{22567052305}{3888}\zeta_3\right. \\
& +\frac{77544762803}{46656}N_f+\frac{105673656865}{124416}\zeta_5-\frac{2023786561}{2592}\zeta_3N_f \\
& -\frac{870630095}{7776}\zeta_5N_f-\frac{362933429}{3888}N_f^2-\frac{68062141}{1296}\zeta_7N_f-\frac{60928}{81}\zeta_3^2N_f^2 \\
& -\frac{28096}{81}\zeta_3N_f^3-\frac{1600}{9}\zeta_5N_f^3-\frac{352}{27}N_f^4+\frac{1372}{3}\zeta_7N_f^2+\frac{468142}{243}N_f^3 \\
& \left.+\frac{3766661}{108}\zeta_3N_f^2+\frac{3825215}{486}\zeta_5N_f^2+\frac{6552685}{162}\zeta_3^2N_f+\frac{657118063}{27648}\zeta_7\right]a^5+O(a^6)
\end{aligned}
\tag{3.5}
$$

respectively. As an alternative perspective where the ζ_n structure is equally evident, it is instructive to view the Landau gauge pure Yang-Mills theory results for an arbitrary color group. We have

$$\begin{aligned}
\beta_{\widetilde{\mathrm{MOM}}_{ggg0g}}(a,0)\Big|_{N_f=0} &= -\frac{11}{3}C_Aa^2 - \frac{34}{3}C_A^2a^3 + \left[-\frac{6499}{48}C_A^3 + \frac{253}{12}\zeta_3C_A^3\right]a^4 \\
&\quad + \left[-\frac{10981313}{5184}C_A^4 - \frac{3707}{8}\zeta_3\frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{8}{9}\frac{d_A^{abcd}d_A^{abcd}}{N_A} \right. \\
&\quad \left. + \frac{6215}{24}\zeta_5\frac{d_A^{abcd}d_A^{abcd}}{N_A} + \frac{97405}{576}\zeta_5C_A^4 + \frac{1116929}{1728}\zeta_3C_A^4\right]a^5 \\
&\quad + \left[-\frac{8598255605}{165888}C_A^5 - \frac{1161130663}{73728}\zeta_7C_A^5 - \frac{35208635}{3072}\zeta_7C_A\frac{d_A^{abcd}d_A^{abcd}}{N_A} \right. \\
&\quad - \frac{28905223}{2304}\zeta_3C_A\frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{15922907}{9216}\zeta_3^2C_A^5 \\
&\quad + \frac{131849}{3456}C_A\frac{d_A^{abcd}d_A^{abcd}}{N_A} + \frac{4595789}{384}\zeta_3^2C_A\frac{d_A^{abcd}d_A^{abcd}}{N_A} \\
&\quad + \frac{7284505}{1152}\zeta_5C_A\frac{d_A^{abcd}d_A^{abcd}}{N_A} + \frac{30643529}{2048}\zeta_3C_A^5 \\
&\quad \left. + \frac{1667817635}{55296}\zeta_5C_A^5\right]a^6 + O(a^7)
\end{aligned}$$

$$\begin{aligned}
\gamma_{A,\widetilde{\mathrm{MOM}}_{ggg0g}}(a,0)\Big|_{N_f=0} &= -\frac{13}{6}C_Aa - \frac{85}{12}C_A^2a^2 + \left[-\frac{22811}{288}C_A^3 + \frac{485}{48}\zeta_3C_A^3\right]a^3 \\
&\quad + \left[-\frac{13595371}{10368}C_A^4 - \frac{45245}{192}\zeta_5\frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{475}{32}\zeta_3\frac{d_A^{abcd}d_A^{abcd}}{N_A} \right. \\
&\quad \left. + \frac{1075}{144}\frac{d_A^{abcd}d_A^{abcd}}{N_A} + \frac{1189237}{3456}\zeta_3C_A^4 + \frac{1195385}{4608}\zeta_5C_A^4\right]a^4 \\
&\quad + \left[-\frac{33074782019}{995328}C_A^5 - \frac{15073026227}{884736}\zeta_7C_A^5 + \frac{2673449615}{82944}\zeta_5C_A^5 \right. \\
&\quad - \frac{282030679}{36864}\zeta_7C_A\frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{129543941}{110592}\zeta_3^2C_A^5 \\
&\quad - \frac{6255185}{1728}\zeta_5C_A\frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{4876861}{768}\zeta_3C_A\frac{d_A^{abcd}d_A^{abcd}}{N_A} \\
&\quad + \frac{3050779}{20736}C_A\frac{d_A^{abcd}d_A^{abcd}}{N_A} + \frac{55278899}{4608}\zeta_3^2C_A\frac{d_A^{abcd}d_A^{abcd}}{N_A} \\
&\quad \left. + \frac{543400985}{82944}\zeta_3C_A^5\right]a^5 + O(a^6)
\end{aligned}$$

$$\begin{aligned}
\gamma_{c,\widetilde{\mathrm{MOM}}_{ggg0g}}(a,0)\Big|_{N_f=0} &= -\frac{3}{4}C_Aa - \frac{17}{8}C_A^2a^2 + \left[-\frac{1715}{64}C_A^3 + \frac{73}{32}\zeta_3C_A^3\right]a^3 \\
&\quad + \left[-\frac{26235}{64}C_A^4 - \frac{3375}{64}\zeta_3\frac{d_A^{abcd}d_A^{abcd}}{N_A} + \frac{101}{32}\frac{d_A^{abcd}d_A^{abcd}}{N_A} \right. \\
&\quad \left. + \frac{6795}{128}\zeta_5\frac{d_A^{abcd}d_A^{abcd}}{N_A} + \frac{7037}{128}\zeta_3C_A^4 + \frac{52835}{1024}\zeta_5C_A^4\right]a^4 \\
&\quad + \left[-\frac{2110649851}{221184}C_A^5 - \frac{161581091}{65536}\zeta_7C_A^5 - \frac{4327735}{24576}\zeta_3^2C_A^5\right.
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1979089}{384}\zeta_3 C_A \frac{d_A^{abcd} d_A^{abcd}}{N_A} - \frac{1284495}{1024}\zeta_3^2 C_A \frac{d_A^{abcd} d_A^{abcd}}{N_A} \\
& + \frac{80483}{512} C_A \frac{d_A^{abcd} d_A^{abcd}}{N_A} + \frac{4450325}{768}\zeta_5 C_A \frac{d_A^{abcd} d_A^{abcd}}{N_A} \\
& + \frac{12700107}{8192}\zeta_7 C_A \frac{d_A^{abcd} d_A^{abcd}}{N_A} + \frac{66809507}{36864}\zeta_3 C_A^5 \\
& + \frac{157858255}{36864}\zeta_5 C_A^5 \Bigg] a^5 + O(a^6)
\end{aligned} \tag{3.6}
$$

where higher order color Casimirs arise with the fully symmetric rank 4 tensor defined by [67]

$$
d_R^{abcd} = \frac{1}{6}\mathrm{Tr}\left(T^a T^{(b} T^c T^{d)}\right) \tag{3.7}
$$

in representation R. In expressions with these Casimirs either here or in the associated data file [66] we have implemented the identity $N_c C_F = T_F N_A$. We note that the pure Yang-Mills expressions for the other schemes have a similar structure.

There are several checks on the results. The main one is that we have verified that the Landau gauge four loop β-functions of the $\widetilde{\mathrm{MOM}}_{ccg0c}$, $\widetilde{\mathrm{MOM}}_{qqg0q}$, $\widetilde{\mathrm{MOM}}_{ggg0g}$ and $\widetilde{\mathrm{MOM}}_{ggg0gg}$ schemes agree with the $SU(3)$ expressions given in [49]. In addition we have reproduced the α dependent $O(a^4)$ coupling constant mappings for the same schemes as those recorded in [49] for the same color group at four loops. There is another check on our computations which is the five loop QED β-function provided in [53] for the $\widetilde{\mathrm{MOM}}$ scheme although it was termed the MOM scheme there. In [53] QED was renormalized at five loops in the $\overline{\mathrm{MS}}$ scheme. The $\widetilde{\mathrm{MOM}}$ β-function was produced as a corollary via the Ward-Takahashi identity. This implies that the coupling constant and photon renormalization constants are not independent placing the theory in the AD class of [46]. So the $\widetilde{\mathrm{MOM}}$ scheme β-function of [53] follows immediately by ensuring the photon 2-point function has its finite part absorbed into its renormalization constant. Taking the QED limit of the $\widetilde{\mathrm{MOM}}_{qqg0gT}$ scheme renormalization group functions reproduces the β-function of [53]. We note this gives

$$
\begin{aligned}
\beta^{\mathrm{QED}}_{\widetilde{\mathrm{MOM}}_{eep0pT}}(a,\alpha) &= \frac{4N_f}{3}a^2 + 4N_f a^3 + [96\zeta_3 N_f - 92N_f - 9]\frac{2N_f a^4}{9} \\
&+ [-128\zeta_3 N_f^2 + 192N_f^2 + 256\zeta_3 N_f - 640\zeta_5 N_f + 156N_f - 69]\frac{2N_f a^5}{3} \\
&+ [21504\zeta_3 N_f^3 + 30720\zeta_5 N_f^3 - 51456N_f^3 + 55296\zeta_3^2 N_f^2 - 157440\zeta_3 N_f^2 \\
&+ 138240\zeta_5 N_f^2 - 54128N_f^2 - 52992\zeta_3 N_f - 311040\zeta_5 N_f + 483840\zeta_7 N_f \\
&- 54216N_f + 6912\zeta_3 + 37413]\frac{N_f a^6}{54} + O(a^7) \\
\gamma^{\mathrm{QED}}_{A,\widetilde{\mathrm{MOM}}_{eep0pT}}(a,\alpha) &= \frac{4N_f}{3}a + 4N_f a^2 + [96\zeta_3 N_f - 92N_f - 9]\frac{2N_f a^3}{9} \\
&+ [-128\zeta_3 N_f^2 + 192N_f^2 + 256\zeta_3 N_f - 640\zeta_5 N_f + 156N_f - 69]\frac{2N_f a^4}{3} \\
&+ [21504\zeta_3 N_f^3 + 30720\zeta_5 N_f^3 - 51456N_f^3 + 55296\zeta_3^2 N_f^2 - 157440\zeta_3 N_f^2 \\
&+ 138240\zeta_5 N_f^2 - 54128N_f^2 - 52992\zeta_3 N_f - 311040\zeta_5 N_f + 483840\zeta_7 N_f \\
&- 54216N_f + 6912\zeta_3 + 37413]\frac{N_f a^5}{54} + O(a^6)
\end{aligned}
$$

$$\gamma^{\rm QED}_{\psi,\widetilde{\rm MOM}_{eep0pT}}(a,\alpha) = \alpha a - [4N_f+3]\frac{a^2}{2} + [16N_f^2 - 12N_f + 9]\frac{a^3}{6} + [-128\alpha N_f^2 - 96\alpha N_f - 640N_f^3 - 768\zeta_3 N_f^2 + 1200N_f^2 - 384\zeta_3 N_f + 3304N_f - 9600\zeta_3 + 15360\zeta_5 - 3081]\frac{a^4}{24} + [-768\zeta_3\alpha^2 N_f^2 + 384\alpha^2 N_f^2 - 576\zeta_3\alpha^2 N_f + 288\alpha^2 N_f - 3072\zeta_3\alpha N_f^3 + 2304\alpha N_f^3 - 2304\zeta_3\alpha N_f^2 - 576\alpha N_f^2 - 432\alpha N_f + 5120N_f^4 + 28672\zeta_3 N_f^3 - 55424N_f^3 + 147456\zeta_3^2 N_f^2 - 398336\zeta_3 N_f^2 + 496640\zeta_5 N_f^2 - 162576N_f^2 + 317952\zeta_3^2 N_f - 184128\zeta_3 N_f + 1674240\zeta_5 N_f - 1979712\zeta_7 N_f - 20568N_f + 179712\zeta_3^2 + 1152000\zeta_3 + 1627200\zeta_5 - 3429216\zeta_7 + 44793]\frac{a^5}{72} + O(a^6) \quad (3.8)$$

for an arbitrary gauge parameter where we use e and p in the scheme label to denote the electron and photon respectively in order to be clear which external leg was nullified. The β-function of (3.8) does indeed agree with the $\widetilde{\rm MOM}$ β-function of [53] and none of the expressions involve ζ_4 or ζ_6. Therefore we confirm that the vertex subtraction of [53] corresponds to nullifying the photon of the QED vertex. We have included the electron anomalous dimension (3.8) as it was not present in [53]. Unlike the QCD case the QED $\widetilde{\rm MOM}_{eep0pT}$ β-function is α independent. We have also reproduced (3.8) directly in order to find the parameter mappings which are

$$a^{\rm QED}_{\widetilde{\rm MOM}_{eep0pT}} = a_{\overline{\rm MS}} - \frac{20N_f}{9}a^2_{\overline{\rm MS}} + [400N_f + 1296\zeta_3 - 1485]\frac{N_f}{81}a^3_{\overline{\rm MS}} + 2[-4000N_f^2 - 50544\zeta_3 N_f + 63009N_f + 35964\zeta_3 - 58320\zeta_5 + 11583]\frac{N_f}{729}a^4_{\overline{\rm MS}} + [320000N_f^3 + 11244096\zeta_3 N_f^2 + 1866240\zeta_5 N_f^2 - 15967908N_f^2 + 8957952\zeta_3^2 N_f - 33195744\zeta_3 N_f + 454896\zeta_4 N_f + 28460160\zeta_5 N_f - 186381N_f - 1364688\zeta_3 - 25719120\zeta_5 + 29393280\zeta_7 + 135594]\frac{N_f}{13122}a^5_{\overline{\rm MS}} + O(a^6_{\overline{\rm MS}})$$

$$\alpha^{\rm QED}_{\widetilde{\rm MOM}_{eep0pT}} = \alpha_{\overline{\rm MS}} + \frac{20N_f}{9}\alpha_{\overline{\rm MS}}a_{\overline{\rm MS}} + \alpha_{\overline{\rm MS}}[-48\zeta_3 + 55]\frac{N_f}{3}a^2_{\overline{\rm MS}} + 2\alpha_{\overline{\rm MS}}[2736\zeta_3 N_f - 3701N_f - 3996\zeta_3 + 6480\zeta_5 - 1287]\frac{N_f}{81}a^3_{\overline{\rm MS}} + \alpha_{\overline{\rm MS}}[-465984\zeta_3 N_f^2 - 207360\zeta_5 N_f^2 + 786052N_f^2 - 622080\zeta_3^2 N_f + 2193696\zeta_3 N_f - 50544\zeta_4 N_f - 2125440\zeta_5 N_f + 304839N_f + 151632\zeta_3 + 2857680\zeta_5 - 3265920\zeta_7 - 15066]\frac{N_f}{1458}a^4_{\overline{\rm MS}} + O(a^5_{\overline{\rm MS}}). \quad (3.9)$$

While we have concentrated on the structure of the renormalization group functions of the $\widetilde{\rm MOM}_*$ schemes the conversion functions for the gluon, ghost, quark and quark mass share an interesting property which is

$$C_\phi^{\widetilde{\rm MOM}_{ccg0c}}(a,\alpha) = C_\phi^{\widetilde{\rm MOM}_{ccg0g}}(a,\alpha) = C_\phi^{\widetilde{\rm MOM}_{ggg0g}}(a,\alpha) = C_\phi^{\widetilde{\rm MOM}_{qqg0q}}(a,\alpha) = C_\phi^{\widetilde{\rm MOM}_{qqg0g}}(a,\alpha) \quad (3.10)$$

for each ϕ in the same labelling set as previously. These are all derived from the renormalization of 2-point functions with the same subtraction condition. This equivalence property equally occurs in the regularization invariant (RI′) and mMOM schemes [64,67] as well as those associated with the symmetric point MOM schemes of [51,52] that were provided in [68,69]. The common underlying property of all the $C_\phi^{\widetilde{\rm MOM}_*}(a,\alpha)$ conversion functions and the corresponding RI′, mMOM and MOM ones is that the prescription to define the respective wave function and quark mass renormalization constants in each of the schemes is the same. In other words the finite part of each 2-point function is absorbed into the renormalization constant. Moreover while the expressions for say Z_A constructed with this prescription in two different schemes will be formally different, in the determination of their conversion functions with respect to the reference $\overline{\rm MS}$ scheme the effect of the different coupling constant and gauge parameter mappings wash out. What is not the case is that there is a parallel equivalence for $C_g^{\mathcal{S}}(a,\alpha)$ as is evident from the data associated with the arXiv version of this article [66]. Moreover they ought not to be since the prescription to define Z_g for each $\widetilde{\rm MOM}_*$ is different. While (3.10) provides an interesting property of the conversion functions it could in principle ease future compilations of renormalization group functions for the wave function and quark mass anomalous dimensions. In other words for schemes where such 2-point subtractions are to be implemented one in effect only requires the coupling constant map to be computed explicitly. That for the gauge parameter is not independent of $C_A^{\mathcal{S}}(a,\alpha)$ in a linear covariant gauge. Furthermore one could have schemes which are hybrid in the sense that some 2-point functions are renormalized with an $\overline{\rm MS}$ prescription whereas the remaining ones are rendered finite with a finite subtraction too. In this sense the RI′ scheme of [70,71] could be regarded as a hybrid scheme since the coupling constant is renormalized with an $\overline{\rm MS}$ prescription, meaning $C_g^{\rm RI'}(a,\alpha)$ is trivially and obviously unity, but the 2-point functions have their finite parts subtracted [47,49,65] and satisfy (3.10). Finally as a side comment the fact that (3.10) was observed at five loops by direct explicit computation provides in part a reassuring consistency check on our overall approach.

IV. *C*-SCHEME MAPPING

Having established that the $\widetilde{\rm MOM}_*$ scheme renormalization group functions do not have any ζ_4 or ζ_6 dependence one question that arises is whether one of these schemes is in fact equivalent to the C-scheme of [59,60]. One claim of [59,60] is that ζ_4 is absent for certain physical quantities. One way to test whether there is a connection with the C-scheme is to compare the $\widetilde{\rm MOM}_*$ coupling constant maps with the map given in Eq. (7) of [59]. While that depends on the parameter C in the order L polynomial of the $O(a^{L+1})$ term of the mapping it might be possible to find a particular value of C that exactly matches the mapping of a $\widetilde{\rm MOM}_*$ scheme. Therefore in order to facilitate a comparison with [59,60] we note that for $N_f=3$ the $SU(3)$ mappings in the Landau gauge are

$$\begin{aligned}
a_{\widetilde{\rm MOM}_{ccg0c}}\Big|^{SU(3)}_{\alpha=0,N_f=3} &= a+\frac{43}{4}a^2+\left[\frac{15685}{48}-\frac{383}{8}\zeta_3\right]a^3 \\
&+\left[\frac{20589011}{1728}-\frac{408251}{144}\zeta_3-\frac{62255}{192}\zeta_5\right]a^4 \\
&+\left[\frac{2446354687}{36864}\zeta_7-\frac{683706835}{4608}\zeta_3-\frac{606373645}{4608}\zeta_5+\frac{3911}{32}\zeta_3^2\right. \\
&\left.+\frac{627809683}{1152}+1335\zeta_4\right]a^5+O(a^6) \\
a_{\widetilde{\rm MOM}_{ccg0g}}\Big|^{SU(3)}_{\alpha=0,N_f=3} &= a+\frac{61}{4}a^2+\left[\frac{22597}{48}-\frac{383}{8}\zeta_3\right]a^3 \\
&+\left[\frac{15762289}{864}-\frac{970483}{288}\zeta_3-\frac{74405}{192}\zeta_5\right]a^4 \\
&+\left[\frac{15799466317}{18432}+\frac{1876891639}{36864}\zeta_7-\frac{2039209463}{9216}\zeta_3-\frac{1934713015}{18432}\zeta_5\right. \\
&\left.+\frac{766871}{512}\zeta_3^2+1335\zeta_4\right]a^5+O(a^6)
\end{aligned}$$

$$a_{\widetilde{\mathrm{MOM}}_{ggg0g}}\Big|_{\alpha=0,N_f=3}^{SU(3)} = a + \frac{43}{4}a^2 + \left[\frac{7973}{24} - \frac{223}{4}\zeta_3\right]a^3 + \left[\frac{41856073}{3456} - \frac{1925333}{576}\zeta_3 - \frac{575}{12}\zeta_5\right]a^4 + \left[\frac{10225015489}{18432} - \frac{1369710205}{18432}\zeta_5 - \frac{110437973}{576}\zeta_3 - \frac{131329}{512}\zeta_3^2 + \frac{1652451493}{36864}\zeta_7 + 1335\zeta_4\right]a^5 + O(a^6)$$

$$a_{\widetilde{\mathrm{MOM}}_{ggg0gg}}\Big|_{\alpha=0,N_f=3}^{SU(3)} = a + 16a^2 + \left[\frac{93427}{192} - \frac{169}{4}\zeta_3\right]a^3 + \left[\frac{129114635}{6912} - \frac{1822913}{576}\zeta_3 - \frac{124835}{192}\zeta_5\right]a^4 + \left[\frac{4050665663}{4608} - \frac{393488663}{2304}\zeta_3 + \frac{980775}{512}\zeta_3^2 + \frac{1055749471}{36864}\zeta_7 - \frac{1387483355}{9216}\zeta_5 + 1335\zeta_4\right]a^5 + O(a^6)$$

$$a_{\widetilde{\mathrm{MOM}}_{qqg0g}}\Big|_{\alpha=0,N_f=3}^{SU(3)} = a + \frac{25}{4}a^2 + \left[\frac{725}{4} - 85\zeta_3\right]a^3 + \left[\frac{127615}{64}\zeta_5 - \frac{542609}{144}\zeta_3 + \frac{12018703}{3456}\right]a^4 + \left[\frac{5829675395}{82944} + \frac{2035638385}{41472}\zeta_5 - \frac{523779403}{18432}\zeta_7 - \frac{457075871}{5184}\zeta_3 + \frac{895703}{128}\zeta_3^2 + 1335\zeta_4\right]a^5 + O(a^6)$$

$$a_{\widetilde{\mathrm{MOM}}_{qqg0gT}}\Big|_{\alpha=0,N_f=3}^{SU(3)} = a + \frac{37}{4}a^2 + \left[\frac{1843}{6} - 98\zeta_3\right]a^3 + \left[\frac{36955015}{3456} - \frac{705631}{144}\zeta_3 + \frac{199895}{576}\zeta_5\right]a^4 + \left[\frac{13618908001}{27648} + \frac{2145762283}{55296}\zeta_7 - \frac{1176358325}{13824}\zeta_5 - \frac{296853959}{1152}\zeta_3 + \frac{28000843}{1152}\zeta_3^2 + 1335\zeta_4\right]a^5 + O(a^6)$$

$$a_{\widetilde{\mathrm{MOM}}_{qqg0q}}\Big|_{\alpha=0,N_f=3}^{SU(3)} = a + \frac{43}{4}a^2 + \left[\frac{16009}{48} - 58\zeta_3\right]a^3 + \left[\frac{21116969}{1728} - \frac{241291}{72}\zeta_3 - \frac{51815}{192}\zeta_5\right]a^4 + \left[\frac{1298610053}{2304} - \frac{829799785}{4608}\zeta_3 - \frac{560109325}{4608}\zeta_5 + \frac{1068169}{256}\zeta_3^2 + \frac{1047188135}{18432}\zeta_7 + 1335\zeta_4\right]a^5 + O(a^6)$$

$$a_{\widetilde{\mathrm{MOM}}_{qqg0qT}}\Big|_{\alpha=0,N_f=3}^{SU(3)} = a + \frac{227}{12}a^2 + \left[\frac{42365}{72} - \frac{653}{8}\zeta_3\right]a^3 + \left[-\frac{7398635}{1296}\zeta_3 - \frac{3660955}{5184}\zeta_5 + \frac{62323649}{2592}\right]a^4 + \left[\frac{73210930375}{62208} - \frac{51236375219}{124416}\zeta_3 - \frac{28655962325}{124416}\zeta_5 + \frac{12942575401}{110592}\zeta_7 + \frac{922617353}{20736}\zeta_3^2 + 1335\zeta_4\right]a^5 + O(a^6) \tag{4.1}$$

where a on the right-hand side is in the $\overline{\rm MS}$ scheme. In order to quantify the behavior of the mappings the numerical values of (4.1) are

$$\begin{aligned}
a_{\widetilde{\rm MOM}_{ccg0c}}\Big|^{SU(3)}_{\alpha=0,N_f=3} &= a + 10.7500a^2 + 269.2224a^3 + 8170.7954a^4 + 298706.1459a^5 + O(a^6) \\
a_{\widetilde{\rm MOM}_{ccg0g}}\Big|^{SU(3)}_{\alpha=0,N_f=3} &= a + 15.2500a^2 + 413.22236a^3 + 13790.9432a^4 + 537305.8623a^5 + O(a^6) \\
a_{\widetilde{\rm MOM}_{ggg0g}}\Big|^{SU(3)}_{\alpha=0,N_f=3} &= a + 10.7500a^2 + 265.1937a^3 + 8043.4603a^4 + 293487.5638a^5 + O(a^6) \\
a_{\widetilde{\rm MOM}_{ggg0gg}}\Big|^{SU(3)}_{\alpha=0,N_f=3} &= a + 16.0000a^2 + 435.8121a^3 + 14201.3422a^4 + 550737.2450a^5 + O(a^6) \\
a_{\widetilde{\rm MOM}_{qqg0g}}\Big|^{SU(3)}_{\alpha=0,N_f=3} &= a + 6.2500a^2 + 79.0752a^3 + 1015.7594a^4 - 1902.2308a^5 + O(a^6) \\
a_{\widetilde{\rm MOM}_{qqg0gT}}\Big|^{SU(3)}_{\alpha=0,N_f=3} &= a + 9.2500a^2 + 189.3651a^3 + 5162.5198a^4 + 170286.4367a^5 + O(a^6) \\
a_{\widetilde{\rm MOM}_{qqg0q}}\Big|^{SU(3)}_{\alpha=0,N_f=3} &= a + 10.7500a^2 + 263.8015a^3 + 7912.2228a^4 + 285890.5240a^5 + O(a^6) \\
a_{\widetilde{\rm MOM}_{qqg0qT}}\Big|^{SU(3)}_{\alpha=0,N_f=3} &= a + 18.9167a^2 + 490.2849a^3 + 16450.0060a^4 + 626761.5038a^5 + O(a^6).
\end{aligned} \tag{4.2}$$

All, bar the mapping for $\widetilde{\rm MOM}_{qqg0g}$, have a similar form in the sense that all the corrections are positive. The four loop coefficient of the $\widetilde{\rm MOM}_{qqg0g}$ scheme mapping is negative. Although the gauge used in [59,60] is not specified we have chosen the Landau gauge to also indicate some general properties of the mappings first.

For completeness we provide an example of the gauge parameter mapping. Again choosing $N_f = 3$ for $\alpha \neq 0$ we have

$$\begin{aligned}
\alpha_{\widetilde{\rm MOM}_{ggg0g}}\Big|^{SU(3)}_{N_f=3} = \alpha &+ \left[-\frac{19}{4}\alpha - \frac{3}{2}\alpha^2 - \frac{3}{4}\alpha^3 \right] a \\
&+ \left[-\frac{11537}{96}\alpha - \frac{9}{16}\alpha^4 + \frac{9}{16}\alpha^5 + \frac{15}{16}\alpha^3 + \frac{303}{32}\alpha^2 - 18\zeta_3\alpha^2 + 31\zeta_3\alpha \right] a^2 \\
&+ \left[-\frac{12861817}{3456}\alpha - \frac{26517}{32}\zeta_3\alpha^2 - \frac{8919}{128}\alpha^4 - \frac{6011}{128}\alpha^3 - \frac{1713}{32}\zeta_3\alpha^3 \right. \\
&- \frac{1341}{32}\zeta_4\alpha - \frac{315}{64}\zeta_5\alpha^5 - \frac{171}{16}\alpha^5 - \frac{45}{8}\zeta_5\alpha^4 - \frac{27}{64}\alpha^7 + \frac{27}{16}\alpha^6 + \frac{81}{8}\zeta_4\alpha^2 \\
&+ \frac{81}{32}\zeta_4\alpha^3 + \frac{117}{32}\zeta_3\alpha^5 + \frac{1341}{32}\zeta_3\alpha^4 + \frac{3105}{8}\zeta_5\alpha^2 + \frac{3465}{32}\zeta_5\alpha^3 + \frac{16567}{128}\alpha^2 \\
&\left. + \frac{41195}{192}\zeta_5\alpha + \frac{82711}{72}\zeta_3\alpha \right] a^3 \\
&+ \left[-\frac{1163178749}{6912}\alpha - \frac{851518199}{18432}\zeta_7\alpha - \frac{87691285}{3072}\zeta_3\alpha^2 - \frac{60649407}{4096}\zeta_7\alpha^2 \right. \\
&- \frac{12605197}{4608}\alpha^3 - \frac{4220519}{1536}\zeta_3\alpha^3 - \frac{2566431}{2048}\zeta_7\alpha^3 - \frac{1210101}{512}\alpha^4 \\
&- \frac{1075923}{2048}\alpha^5 - \frac{935271}{1024}\zeta_5\alpha^4 - \frac{851175}{1024}\zeta_6\alpha^2 - \frac{808011}{2048}\zeta_4\alpha \\
&- \frac{209169}{512}\zeta_3^2\alpha^3 - \frac{203949}{512}\zeta_3^2\alpha^2 - \frac{137781}{1024}\zeta_7\alpha^5 - \frac{103875}{32}\zeta_6\alpha
\end{aligned}$$

$$-\frac{53865}{256}\zeta_5\alpha^5-\frac{52569}{512}\zeta_3^2\alpha^4-\frac{23247}{1024}\zeta_4\alpha^4-\frac{19755}{512}\zeta_5\alpha^6-\frac{17631}{512}\zeta_3\alpha^6$$
$$-\frac{13041}{2048}\zeta_4\alpha^5-\frac{567}{256}\alpha^8-\frac{351}{64}\zeta_3\alpha^7+\frac{81}{256}\alpha^9+\frac{945}{128}\zeta_5\alpha^7+\frac{1863}{256}\zeta_3^2\alpha^6$$
$$+\frac{2349}{128}\alpha^7+\frac{4725}{1024}\zeta_6\alpha^5+\frac{5373}{512}\alpha^6+\frac{10647}{4096}\zeta_7\alpha^6+\frac{19737}{512}\zeta_3^2\alpha^5$$
$$+\frac{22275}{1024}\zeta_6\alpha^3+\frac{24975}{1024}\zeta_6\alpha^4+\frac{78327}{512}\zeta_4\alpha^3+\frac{132615}{512}\zeta_7\alpha^4+\frac{205719}{32}\zeta_3^2\alpha$$
$$+\frac{558927}{2048}\zeta_3\alpha^5+\frac{663687}{1024}\zeta_4\alpha^2+\frac{1674621}{1024}\zeta_3\alpha^4+\frac{5561003}{1024}\zeta_5\alpha^3$$
$$+\frac{8440805}{18432}\alpha^2+\frac{25777469}{1024}\zeta_5\alpha^2+\frac{663132857}{9216}\zeta_5\alpha+\frac{891656237}{18432}\zeta_3\alpha\Bigg]a^4+O(a^5) \qquad (4.3)$$

for the $\widetilde{\mathrm{MOM}}_{ggg0g}$ scheme. Unlike the coupling constant map ζ_4 first appears at $O(a^3)$ and ζ_6 is present at $O(a^4)$. The gauge parameter maps for the other $\widetilde{\mathrm{MOM}}_*$ schemes are formally the same for all color groups from (3.10).

One main observation from (4.1) is that ζ_4 appears in each of the $O(a^5)$ terms but ζ_6 is absent. The latter does not arise when $\alpha \neq 0$ nor for any color group. In the $O(a^3)$ terms in (4.1) ζ_3 is present but in the mapping of [59] there is no ζ_3 at the same order. Instead there are only rationals. While ζ_3 could in principle be introduced by a choice of C that would then mean ζ_3 is present in the $O(a^2)$ term which none of the mappings in (4.1) have. Equally if C is determined from the $O(a^2)$ term to match that of one of the $O(a^2)$ terms of the $\widetilde{\mathrm{MOM}}_*$ mappings, then that choice could not introduce a ζ_3 term at $O(a^3)$. At $O(a^4)$ ζ_5 is present in (4.1) but is absent at the corresponding order in the map of [59]. By contrast for the $N_f = 3$ expression provided in [59] there is a ζ_4 term at $O(a^5)$. Moreover its coefficient is *precisely* the same as that of ζ_4 at the same order in each of the $\widetilde{\mathrm{MOM}}_*$ schemes when $N_f = 3$, after allowing for a factor of 4 for differing coupling constant conventions as is evident in (4.1). It transpires that this equality occurs for all N_f and a general color group. Moreover it suggests that the coupling constant map of [59,60] does have the same underlying ζ_4 cancellation property whatever the renormalization prescription that underlies it is. In some sense the universality of this particular ζ_4 term in all the mappings reinforces the observations of [44–46] that the ζ_4 absence could be traced to a unique ϵ dependent transformation of ζ_3. That ϵ dependence would affect the counterterms in the underlying renormalization group functions. The fact that there is no other universal connection in any of the mappings for odd zetas merely reflects the different prescriptions defining those schemes. We have also examined the situation for $\alpha \neq 0$. While the extra parameter could in principle be exploited to find a suitable value for C to achieve a match at low order, this does not persist at higher order. So it would appear that none of our schemes have an immediate connection to the C-scheme aside from the ζ_4 one at $O(a^5)$.

We can examine the situation from another point of view. It is worth recalling the origin of the coupling constant map at a more formal level to see if it sheds light on the relation of the $\widetilde{\mathrm{MOM}}_*$ schemes to the C-scheme. In general the coupling renormalization constants for two schemes take the following forms

$$Z_g = 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{n} z_{g\,nm}\frac{a^n}{\epsilon^m}, \qquad Z_g^{\mathcal{S}} = 1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n} z_{g\,nm}^{\mathcal{S}}\frac{a_{\mathcal{S}}^n}{\epsilon^m} \qquad (4.4)$$

where $z_{g\,nm}$ and $z_{g\,nm}^{\mathcal{S}}$ are the residues of the poles in ϵ in the $\overline{\mathrm{MS}}$ scheme and a general scheme $\mathcal{S}$ respectively. Included in the $\mathcal{S}$ scheme definition are the finite parts with $z_{g\,n0}^{\mathcal{S}} \neq 0$ but there are no corresponding $z_{g\,n0}$ terms in keeping with the definition of the $\overline{\mathrm{MS}}$ scheme. We will assume there are no other parameters, such as a gauge parameter, in this formal analysis. So our focus is on the Landau gauge. If we define the relation between the two coupling constants in perturbation theory as

$$a_{\mathcal{S}} = \sum_{n=0}^{\infty} c_n a^{n+1} \qquad (4.5)$$

in the same notation as [59] and recall the definition of (2.7) which determines the mapping from the coupling renormalization constant, then it is a straightforward exercise to deduce

$$
\begin{aligned}
c_0 &= 1, \\
c_1 &= -2z^{\mathcal{S}}_{g\,10}, \\
c_2 &= 7(z^{\mathcal{S}}_{g\,10})^2 - 2z^{\mathcal{S}}_{g\,20} \\
c_3 &= -30(z^{\mathcal{S}}_{g\,10})^3 + 18z^{\mathcal{S}}_{g\,10}z^{\mathcal{S}}_{g\,20} - 2z^{\mathcal{S}}_{g\,30} \\
c_4 &= 143(z^{\mathcal{S}}_{g\,10})^4 - 132(z^{\mathcal{S}}_{g\,10})^2 z^{\mathcal{S}}_{g\,20} + 22z^{\mathcal{S}}_{g\,10}z^{\mathcal{S}}_{g\,30} \\
&\quad + 11(z^{\mathcal{S}}_{g\,20})^2 - 2z^{\mathcal{S}}_{g\,40} \\
c_5 &= -728(z^{\mathcal{S}}_{g\,10})^5 + 910(z^{\mathcal{S}}_{g\,10})^3 z^{\mathcal{S}}_{g\,20} - 182(z^{\mathcal{S}}_{g\,10})^2 z^{\mathcal{S}}_{g\,30} \\
&\quad - 182z^{\mathcal{S}}_{g\,10}(z^{\mathcal{S}}_{g\,20})^2 + 26z^{\mathcal{S}}_{g\,10}z^{\mathcal{S}}_{g\,40} + 26z^{\mathcal{S}}_{g\,20}z^{\mathcal{S}}_{g\,30} - 2z^{\mathcal{S}}_{g\,50}.
\end{aligned} \tag{4.6}
$$

We note that $z_{g\,nm}$ and $z^{\mathcal{S}}_{g\,nm}$ are predetermined when $m \geq 2$ from the simple pole residues and finite parts of the lower loop terms at each order n for each renormalization constant. Having formally derived (4.6) we have checked that all the Landau gauge mappings of (4.1) are reproduced from the finite parts computed to four loops. In examining the structure of the respective finite parts in each $\widetilde{\text{MOM}}_*$ scheme we note that for example $z^{\mathcal{S}}_{g\,20}$ involves ζ_3 but $z^{\mathcal{S}}_{g\,10}$ has only rationals for the $\widetilde{\text{MOM}}_*$ schemes. From (4.6) it is clear that for each n $z^{\mathcal{S}}_{g\,n0}$ appears for the first time in c_n in addition to all the lower order finite parts.

If we assume for the moment that the C-scheme satisfies these $\mathcal{S}$ scheme properties we can examine it in more detail. The C-scheme involves the parameter C which was motivated by the observation that the ratio of Λ-parameters between a scheme and the $\overline{\text{MS}}$ scheme is determined exactly by c_1/β_1 where c_1 is the one loop term of (4.5) and β_1 is the one loop coefficient of the $O(a^2)$ term in (3.1). Therefore making this connection with the formal origin of C in [59,60], where $\beta_1 = -9$ for $N_f = N_c = 3$, the power series dependence of $z^{\mathcal{S}}_{g\,10}$ in c_n in (4.6) parallels that of the parameter C in the coupling constant mapping of [59,60]. Specifically one can check that C is c_1 which is related to $z^{\mathcal{S}}_{g\,10}$. However, the very assumption in [59,60] that c_1 is nonzero in the Λ ratio immediately implies that whatever the renormalization prescription is that defines the C-scheme, at the level of subtracting divergences of a vertex function, it is one where $Z^{\mathcal{S}}_g$ has a nonzero finite contribution at each loop order. Therefore there ought to be the equivalent of $z^{\mathcal{S}}_{g\,n0}$ dependence in the coupling constant map of [59,60] from the C-scheme to the $\overline{\text{MS}}$ scheme for $n \geq 2$. Such dependence appears to be absent as the mapping of [59,60] depends on only one parameter and therefore only $z^{\mathcal{S}}_{g\,10}$. Unless the explicit values of $z^{\mathcal{S}}_{g\,n0}$ are all zero when computed for all $n \geq 2$, which would be peculiar, then it would appear that it is not possible to connect the C-scheme to any of the $\widetilde{\text{MOM}}_*$ schemes using the renormalization group based argument that led to (4.6). Indeed we took the values of c_n given in [59,60] and solved for $z^{\mathcal{S}}_{gn0}$ for each $\widetilde{\text{MOM}}_*$ scheme. After matching C to c_1 for each scheme the remaining $z^{\mathcal{S}}_{g\,n0}$ for $n \geq 2$ are not in agreement with the finite parts determined from each $\widetilde{\text{MOM}}_*$ renormalization.

Returning to the more general scheme $\mathcal{S}$ when there is a finite part in the coupling renormalization constant of (4.4), it is instructive to record the form of the β-function for nonzero ϵ and therefore clarify earlier comments. Using (2.3) and (4.4) we have the formal ϵ dependent β-function

$$
\begin{aligned}
\beta_{\mathcal{S}}(a,\epsilon) &= 2z^{\mathcal{S}}_{g\,11}a^2 + 4[-3z^{\mathcal{S}}_{g\,11}z^{\mathcal{S}}_{g\,10} + z^{\mathcal{S}}_{g\,21}]a^3 \\
&\quad + 2[27z^{\mathcal{S}}_{g\,11}(z^{\mathcal{S}}_{g\,10})^2 - 11z^{\mathcal{S}}_{g\,11}z^{\mathcal{S}}_{g\,20} - 11z^{\mathcal{S}}_{g\,10}z^{\mathcal{S}}_{g\,21} + 3z^{\mathcal{S}}_{g\,31}]a^4 \\
&\quad + 8[-27z^{\mathcal{S}}_{g\,11}(z^{\mathcal{S}}_{g\,10})^3 + 24z^{\mathcal{S}}_{g\,11}z^{\mathcal{S}}_{g\,10}z^{\mathcal{S}}_{g\,20} - 4z^{\mathcal{S}}_{g\,11}z^{\mathcal{S}}_{g\,30} + 12(z^{\mathcal{S}}_{g\,10})^2 z^{\mathcal{S}}_{g\,21} \\
&\quad -4z^{\mathcal{S}}_{g\,10}z^{\mathcal{S}}_{g\,31} - 5z^{\mathcal{S}}_{g\,20}z^{\mathcal{S}}_{g\,21} + z^{\mathcal{S}}_{g\,41}]a^5 \\
&\quad + 2[405z^{\mathcal{S}}_{g\,11}(z^{\mathcal{S}}_{g\,10})^4 - 567z^{\mathcal{S}}_{g\,11}(z^{\mathcal{S}}_{g\,10})^2 z^{\mathcal{S}}_{g\,20} + 138z^{\mathcal{S}}_{g\,11}z^{\mathcal{S}}_{g\,10}z^{\mathcal{S}}_{g\,30} + 85z^{\mathcal{S}}_{g\,11}(z^{\mathcal{S}}_{g\,20})^2 \\
&\quad -21z^{\mathcal{S}}_{g\,11}z^{\mathcal{S}}_{g\,40} - 189(z^{\mathcal{S}}_{g\,10})^3 z^{\mathcal{S}}_{g\,21} + 69(z^{\mathcal{S}}_{g\,10})^2 z^{\mathcal{S}}_{g\,31} + 170z^{\mathcal{S}}_{g\,10}z^{\mathcal{S}}_{g\,20}z^{\mathcal{S}}_{g\,21} \\
&\quad -21z^{\mathcal{S}}_{g\,10}z^{\mathcal{S}}_{g\,41} - 29z^{\mathcal{S}}_{g\,20}z^{\mathcal{S}}_{g\,31} - 29z^{\mathcal{S}}_{g\,21}z^{\mathcal{S}}_{g\,30} + 5z^{\mathcal{S}}_{g\,51}]a^6 \\
&\quad + [-a + 2z^{\mathcal{S}}_{g\,10}a^2 + 2[-3(z^{\mathcal{S}}_{g\,10})^2 + 2z^{\mathcal{S}}_{g\,20}]a^3 \\
&\quad +2[9(z^{\mathcal{S}}_{g\,10})^3 - 11z^{\mathcal{S}}_{g\,10}z^{\mathcal{S}}_{g\,20} + 3z^{\mathcal{S}}_{g\,30}]a^4 \\
&\quad +2[-27(z^{\mathcal{S}}_{g\,10})^4 + 48(z^{\mathcal{S}}_{g\,10})^2 z^{\mathcal{S}}_{g\,20} - 16z^{\mathcal{S}}_{g\,10}z^{\mathcal{S}}_{g\,30} - 10(z^{\mathcal{S}}_{g\,20})^2 + 4z^{\mathcal{S}}_{g\,40}]a^5 \\
&\quad + 2[81(z^{\mathcal{S}}_{g\,10})^5 - 189(z^{\mathcal{S}}_{g\,10})^3 z^{\mathcal{S}}_{g\,20} + 69(z^{\mathcal{S}}_{g\,10})^2 z^{\mathcal{S}}_{g\,30} + 85z^{\mathcal{S}}_{g\,10}(z^{\mathcal{S}}_{g\,20})^2 \\
&\quad -21z^{\mathcal{S}}_{g\,10}z^{\mathcal{S}}_{g\,40} - 29z^{\mathcal{S}}_{g\,20}z^{\mathcal{S}}_{g\,30} + 5z^{\mathcal{S}}_{g\,50}]a^6]\epsilon + O(a^7)
\end{aligned} \tag{4.7}
$$

where the ϵ dependent contributions follow the part that survives when the regularization is lifted. A similar expression can be constructed for the anomalous dimension of the fields and mass. In each case the coefficients will depend not only on the residues and finite parts of the respective renormalization constants but also on $Z_g^{\mathcal{S}}$. We recall that in a gauge theory the corresponding construction will be more involved for a nonzero covariant gauge parameter. Like (4.6) the $O(\epsilon)$ term of $\beta_{\mathcal{S}}(a)$ depends solely on $z_{g\,n0}^{\mathcal{S}}$ for $n \geq 1$. So knowledge of the coefficients of a in either of these means the coefficients of a in the other can be determined.

The necessity of the $O(\epsilon)$ piece is central to another aspect of the renormalization group properties. This concerns critical exponents which are renormalization group invariants and given by the evaluation of the anomalous dimensions at zeros of the β-function. In the case of the latter the relevant exponent is the slope of the β-function at criticality. Amongst the widely studied suite of exponents are those derived from the Wilson-Fisher fixed point [72] defined as the critical point closest to the origin for nonzero ϵ. For the $\overline{\text{MS}}$ scheme, where there are no ϵ terms in the β-function aside from the $O(a)$ one, which itself reflects the dimensionlessness of the d-dimensional coupling constant, the exponent $\omega = \beta'_{\mathcal{S}}(a,\epsilon)$ where the derivative acts on a, will only depend on the residues of the simple poles of ϵ in $Z_g^{\overline{\text{MS}}}$ as is clear from (4.7). Equally (4.7) suggests that evaluating ω for the generic scheme $\mathcal{S}$ would involve $z_{g\,n0}^{\mathcal{S}}$ as well. This might seem to imply that ω would be different in different schemes and hence contradict the renormalization group invariance of the exponents at the Wilson-Fisher fixed point. We have checked this is not the case to $O(\epsilon^5)$ in each of the $\widetilde{\text{MOM}}_*$ schemes considered here. This was for the Landau gauge as that is a fixed point of $\gamma_\alpha(a,\alpha)$. Moreover, the agreement has also been verified in [73] for the MOM schemes of [51,52]. In other words for a generic scheme the invariance of the exponents actually provides relations between $z_{g\,n1}$ and $z_{g\,n1}^{\mathcal{S}}$. In particular we record

$$\begin{aligned}
z_{g\,21}^{\mathcal{S}} &= 3z_{g\,11}z_{g\,10}^{\mathcal{S}} + z_{g\,21}, \quad z_{g\,31}^{\mathcal{S}} = 3z_{g\,11}(z_{g10}^{\mathcal{S}})^2 + 3z_{g\,11}z_{g\,20}^{\mathcal{S}} + 5z_{g\,21}z_{g\,10}^{\mathcal{S}} + z_{g\,31} \\
z_{g41}^{\mathcal{S}} &= z_{g\,11}(z_{g\,10}^{\mathcal{S}})^3 + 6z_{g\,11}z_{g\,10}^{\mathcal{S}}z_{g\,20}^{\mathcal{S}} + 3z_{g\,11}z_{g\,30}^{\mathcal{S}} + 10z_{g\,21}(z_{g\,10}^{\mathcal{S}})^2 + 5z_{g\,21}z_{g\,20}^{\mathcal{S}} + 7z_{g\,31}z_{g\,10}^{\mathcal{S}} + z_{g\,41} \\
z_{g\,51}^{\mathcal{S}} &= 3z_{g\,11}(z_{g\,10}^{\mathcal{S}})^2z_{g\,20}^{\mathcal{S}} + 6z_{g\,11}z_{g\,10}^{\mathcal{S}}z_{g\,30}^{\mathcal{S}} + 3z_{g\,11}(z_{g\,20}^{\mathcal{S}})^2 + 3z_{g\,11}z_{g\,40}^{\mathcal{S}} + 10z_{g\,21}(z_{g\,10}^{\mathcal{S}})^3 \\
&\quad + 20z_{g\,21}z_{g\,10}^{\mathcal{S}}z_{g\,20}^{\mathcal{S}} + 5z_{g\,21}z_{g\,30}^{\mathcal{S}} + 21z_{g\,31}(z_{g\,10}^{\mathcal{S}})^2 + 7z_{g\,31}z_{g\,20}^{\mathcal{S}} + 9z_{g\,41}z_{g\,10}^{\mathcal{S}} + z_{g\,51}
\end{aligned} \tag{4.8}$$

where we have assumed $z_{g\,11}^{\mathcal{S}} = z_{g\,11}$. While the two loop term of the β-function in a single coupling theory is scheme independent this does not imply $z_{g\,21}$ and $z_{g\,21}^{\mathcal{S}}$ are equal. Although we have checked these relations are satisfied in the $\widetilde{\text{MOM}}_*$ schemes we cannot do the same for the C-scheme β-function as only the purely four dimensional expression is available and not the ϵ dependent one. For the wave function renormalization Z_ϕ similar relations hold between the terms of the respective renormalization constants. If we define

$$Z_\phi = 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{n} z_{\phi\,nm}\frac{a^n}{\epsilon^m}, \qquad Z_\phi^{\mathcal{S}} = 1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n} z_{\phi\,nm}^{\mathcal{S}}\frac{a_{\mathcal{S}}^n}{\epsilon^m} \tag{4.9}$$

it is straightforward to deduce

$$\begin{aligned}
z_{\phi\,21}^{\mathcal{S}} &= 2z_{g\,10}^{\mathcal{S}}z_{\phi\,11} + z_{\phi\,11}z_{\phi\,10}^{\mathcal{S}} + z_{\phi\,21} \\
z_{\phi\,31}^{\mathcal{S}} &= (z_{g\,10}^{\mathcal{S}})^2z_{\phi\,11} + 2z_{g\,10}^{\mathcal{S}}z_{\phi\,11}z_{\phi\,10}^{\mathcal{S}} + 4z_{g\,10}^{\mathcal{S}}z_{\phi\,21} + 2z_{g\,20}^{\mathcal{S}}z_{\phi\,11} + z_{\phi11}z_{\phi\,20}^{\mathcal{S}} + z_{\phi\,21}z_{\phi\,10}^{\mathcal{S}} + z_{\phi\,31} \\
z_{\phi\,41}^{\mathcal{S}} &= (z_{g\,10}^{\mathcal{S}})^2z_{\phi\,11}z_{\phi\,10}^{\mathcal{S}} + 6(z_{g\,10}^{\mathcal{S}})^2z_{\phi\,21} + 2z_{g\,10}^{\mathcal{S}}z_{g\,20}^{\mathcal{S}}z_{\phi\,11} + 2z_{g\,10}^{\mathcal{S}}z_{\phi\,11}z_{\phi\,20}^{\mathcal{S}} + 4z_{g\,10}^{\mathcal{S}}z_{\phi\,21}z_{\phi\,10}^{\mathcal{S}} \\
&\quad + 6z_{g\,10}^{\mathcal{S}}z_{\phi\,31} + 2z_{g\,20}^{\mathcal{S}}z_{\phi\,11}z_{\phi\,10}^{\mathcal{S}} + 4z_{g\,20}^{\mathcal{S}}z_{\phi\,21} + 2z_{g\,30}^{\mathcal{S}}z_{\phi\,11} + z_{\phi\,11}z_{\phi\,30}^{\mathcal{S}} + z_{\phi\,21}z_{\phi\,20}^{\mathcal{S}} \\
&\quad + z_{\phi\,31}z_{\phi\,10}^{\mathcal{S}} + z_{\phi\,41} \\
z_{\phi\,51}^{\mathcal{S}} &= 4(z_{g\,10}^{\mathcal{S}})^3z_{\phi\,21} + (z_{g\,10}^{\mathcal{S}})^2z_{\phi\,11}z_{\phi\,20}^{\mathcal{S}} + 6(z_{g\,10}^{\mathcal{S}})^2z_{\phi\,21}z_{\phi\,10}^{\mathcal{S}} + 15(z_{g\,10}^{\mathcal{S}})^2z_{\phi\,31} \\
&\quad + 2z_{g\,10}^{\mathcal{S}}z_{g20}^{\mathcal{S}}z_{\phi\,11}z_{\phi\,10}^{\mathcal{S}} + 12z_{g\,10}^{\mathcal{S}}z_{g\,20}^{\mathcal{S}}z_{\phi\,21} + 2z_{g\,10}^{\mathcal{S}}z_{g\,30}^{\mathcal{S}}z_{\phi\,11} + 2z_{g\,10}^{\mathcal{S}}z_{\phi\,11}z_{\phi30}^{\mathcal{S}} \\
&\quad + 4z_{g\,10}^{\mathcal{S}}z_{\phi21}z_{\phi\,20}^{\mathcal{S}} + 6z_{g\,10}^{\mathcal{S}}z_{\phi\,31}z_{\phi\,10}^{\mathcal{S}} + 8z_{g\,10}^{\mathcal{S}}z_{\phi\,41} + (z_{g\,20}^{\mathcal{S}})^2z_{\phi\,11} + 2z_{g\,20}^{\mathcal{S}}z_{\phi\,11}z_{\phi\,20}^{\mathcal{S}} \\
&\quad + 4z_{g\,20}^{\mathcal{S}}z_{\phi\,21}z_{\phi\,10}^{\mathcal{S}} + 6z_{g\,20}^{\mathcal{S}}z_{\phi\,31} + 2z_{g30}^{\mathcal{S}}z_{\phi\,11}z_{\phi\,10}^{\mathcal{S}} + 4z_{g\,30}^{\mathcal{S}}z_{\phi\,21} + 2z_{g40}^{\mathcal{S}}z_{\phi\,11} + z_{\phi\,11}z_{\phi\,40}^{\mathcal{S}} \\
&\quad + z_{\phi\,21}z_{\phi\,30}^{\mathcal{S}} + z_{\phi\,31}z_{\phi\,20}^{\mathcal{S}} + z_{\phi\,41}z_{\phi\,10}^{\mathcal{S}} + z_{\phi\,51}
\end{aligned} \tag{4.10}$$

where $z_{\phi\,11}^{\mathcal{S}} = z_{\phi\,11}$ has been assumed.

V. PERSPECTIVE ON SCHEMES

Having completed the explicit construction of the $\widetilde{\rm MOM}_*$ schemes at five loops in QCD, it is worth pausing to consider the position of such schemes in a more general context. The discussion, however, will be for massless theories so that particle masses do not feature in the underlying renormalization. First for the moment we will focus on a theory with a single field and an n-point interaction. Although initially we will consider a 3-point interaction as it will provide a simple introduction to classes of schemes. For instance, we will suggest that for such an interaction there are two classes of schemes which will be termed 1- and 3-variable. By 1-variable we mean those schemes where there is only one independent invariant or equivalently variable, which for the $\widetilde{\rm MOM}$ prescription is the external momentum of the 2- and 3-point functions that are used to determine the renormalization constants and recorded as an example in (2.11) for QCD. For the vertex function the momentum configuration is an exceptional one in that there are fewer independent momenta than the maximum permitted for a 3-point function; for an n-point function this is $(n-1)$. While the structure of the Feynman rules for each of the three vertices in QCD ensures that infrared rearrangement trivially implies there are no infrared issues, it also means that the number basis of the nullified 3-point vertices is ζ_n or multiple zetas up to at least six loops aside from rationals. For ϕ^3 theory in six dimensions one can nullify the external momentum of the cubic vertex since that is automatically infrared safe unlike four dimensions. By contrast with this 1-variable notion of scheme the 3-variable scheme for a cubic theory corresponds to schemes where the vertex momentum configuration is nonexceptional. In this instance the number basis is known to be different from that of the 1-variable case. For such 3-point function configurations there will be more invariants and hence they will depend on several variables. The reason for this is that there are now two independent momenta. For instance, for a 3-point vertex with nonzero external momenta p_1, p_2 and p_3, one of these is not independent, say p_3, via energy-momentum conservation. From the two independent momenta there are three scalar products p_1^2, p_2^2 and $p_1.p_2$ which can be regarded as two scales and essentially an angle. Alternatively one could take p_i^2 for $i = 1$, 2 and 3 as the independent set of variables. For the 3-point function one can form two dimensionless variables $x = \frac{p_1^2}{p_3^2}$ and $y = \frac{p_2^2}{p_3^2}$, say, leaving one variable p_3^2 as the dimensionful one. The overall scale will be common to all the Feynman graphs comprising a 3-point vertex meaning that the remaining vertex function, prior to renormalization, will depend on x and y which are not renormalized. By contrast, in a 1-variable scheme the variable itself, which is the square of the external momentum, does not feature in the actual renormalization constants purely as it is dimensionful but it will be present in the finite renormalized Green's function. So there is no remnant of the kinematics of the subtraction configuration in the renormalization group functions in a 1-variable scheme unlike the 3-variable one in this cubic theory example.

An example of such a scenario is the well-established symmetric point momentum subtraction scheme defined in QCD in [51,52]. For instance the full renormalization group functions of QCD in the three MOM schemes are available at three loops in [51,52,68,69,74] in an arbitrary linear covariant gauge and at four loops in the Landau gauge in [69]. In other words these three MOM schemes have $x = y = 1$. However there is no *a priori* reason why x and y should take these values. In principle they can be left as free variables although restricted to configurations where there are no collinear or infrared singularities for example. While schemes with x and y both free have not been studied as such at the Lagrangian level, a subset of nonunit x and y values have been, not only for the Lagrangian but also for operator renormalization in what is termed the interpolating MOM scheme [75–77]. This is the case where x and y are related to one common parameter $\hat{\omega}$.[3] Considering a 3-variable scheme which depends on two variables may seem an irrelevant exercise but it could have the advantage of tuning the convergence for the perturbative series of an observable to minimize theory uncertainties or alternatively provide a more informed method of estimating theory errors for instance. Equally having schemes depend on variables may not be aesthetically pleasing. On the other hand there is no *a priori* reason why the symmetric point configuration $x = y = 1$ should be singled out for special significance. A related issue to tuning is the situation of taking mathematical limits. One such limit would be that which should produce the lower variable scheme or schemes which for a cubic theory would be the 1-variable one. Therefore one could regard the development of the $\widetilde{\rm MOM}_*$ schemes in QCD as both the starting point as well as an endpoint check for building such a suite of schemes.

Of course in the QCD example the Lagrangian also possesses a quartic gluon vertex which in this vision would lead to another class of schemes. The pattern for this is now clear and would be a set of 6-variable schemes with five dimensionless variables. More generally for an n-point function the nonexceptional scheme would be an $\frac{1}{2}n(n-1)$-variable one. In the case of $n = 4$ with momenta p_1, p_2, p_3 and p_4 we can take the first three as the independent ones by energy-momentum conservation which can be used to construct six invariants. For example, these could either be the lengths of the six possible momenta, p_1, p_2, p_3, $(p_1 + p_2)$, $(p_1 + p_3)$ and $(p_2 + p_3)$, or the

[3] In [75–77] the variable is actually ω but we use $\hat{\omega}$ briefly here to avoid confusion with the use of ω for a different entity in the previous section and later in this one.

squares of the first three and in effect the three so-called angles derived from $p_1.p_2$, $p_1.p_3$ and $p_2.p_3$ or some other set. An explicit example of the dependence is provided in [78,79] where the analytic expression for the one loop box integral is provided in four dimensions. There, by contrast, the lengths of the external momenta and two Mandelstam variables were used as the invariants. It was shown in [78,79] that the planar box master with an arbitrary number of rungs is related to the master 3-point planar triangle with the equivalent number of rungs. In practice that master integral is actually a function with three arguments where each argument depends on combinations of the six underlying variables. Either way five dimensionless ratios plus one overall scale act as the independent variables for the renormalization of 4-point functions at a subtraction point. For instance a specific example of this in QCD was provided in [80] where the quartic gluon vertex was studied at one loop at the fully symmetric point and latterly in [81]. For a 4-point function aside from the 6-variable scheme there are in principle subvariable schemes such as those where one of the external momenta is initially nullified corresponding to an exceptional configuration to produce a 3-variable scheme. As noted in [46] provided one can carry out the renormalization in an infrared safe fashion using infrared rearrangement there would additionally be a set of 1-variable schemes. The number basis in that instance should be the same as that of the $\widetilde{\mathrm{MOM}}$ schemes and in a similar way the 3-variable schemes should involve the same suite of polylogarithm functions as those of an x and y dependent MOM scheme. A hint of the appearance of more involved mathematical functions in schemes is already available from [38] with the presence of an apparent nonmultiple zeta number $P_{7,11}$ in the seven loop $\overline{\mathrm{MS}}$ renormalization of ϕ^4 theory. In discussing the potential ordering of schemes in this way we qualify the situation by mentioning that the actual number of distinct $\frac{1}{2}n(n-1)$-variable schemes for n-point functions is dependent on the field content of the underlying Lagrangian. As the QCD situation shows there are several 1-variable schemes for each 3-point vertex, as constructed earlier, and in a linear covariant gauge fixing there is only one 6-variable scheme together with various lower variable schemes derived from it. This would complete the classification of all possible massless kinematic based schemes in QCD. By contrast in ϕ^3 theory there is only one 1-variable and one 3-variable scheme. The discussion of the n-variable scheme classification has rested on the vertex function. Within each particular n-variable scheme there is of course the further subdivision into the actual prescription to determine the renormalization constants themselves such as whether to include finite parts as in the $\widetilde{\mathrm{MOM}}$ set or not in the $\overline{\mathrm{MS}}$ case aside from a hybrid mixture akin to the RI$'$ scheme.

Returning to the theme of this article, which is the absence of ζ_4 and ζ_6 in $\widetilde{\mathrm{MOM}}_*$ schemes, in light of the previous remarks it would seem that this property may be specifically confined to 1-variable schemes. This is because the treatment of the 3-point renormalization by one external momentum nullification immediately reduces those vertices to the 1-variable case. From the higher variable scheme point of view if one has the general 3-variable scheme with the finite part subtraction then the source of the ζ_4 cancellation at four loops in the β-function could in principle be investigated, say, in the limit to the 1-variable case. At present the necessary three loop 3-point master integrals are not known for nonzero x and y; only the corresponding two loop masters are available [82–86]. If such a three loop arbitrary x and y renormalization could be carried out, it should be the case that taking the limit to the momentum configuration corresponding to one nullified external momentum produces a $\widetilde{\mathrm{MOM}}_*$ scheme as the endpoint. In that case the mathematical relations between the various types of polylogarithms, that ought to be the function basis for the three loop masters [87] may prove important in seeing how the ζ_4 cancellation emerges in all the renormalization group functions. Similar comments would equally apply to the next loop order to understand the relations that ensure the absence of ζ_6. An additional observation based on the 1-variable scheme situation is that there will be parallel $\widetilde{\mathrm{MOM}}$ schemes for the $\frac{1}{2}n(n-1)$-variable schemes for n-point functions. In those cases it would be interesting to ascertain if there is an analogous set of functions that arises in the finite part of the Green's functions at a particular loop order but does not contribute to the renormalization group functions at the next loop in such a $\widetilde{\mathrm{MOM}}$ prescription.

Some of the points made in this section can be illustrated by an example. The possibility that ζ_4 was perturbatively absent in a situation which had physical significance was illuminated in [44]. It centred on the Adler D-function in the $\overline{\mathrm{MS}}$ scheme. Subsequently this property was formulated in a no-π theorem in [44]. Aside from the absence of ζ_4 the theorem specified several conditions that involved what was termed p-integrals [44]. For the present discussion the relevant ones are that for a massless correlator evaluated in a π-safe class [44] using p-integrals then it is π-free in a renormalization scheme that is free of π [44]. Another way of expressing this is that the theorem only applied to massless correlation functions determined in 1-variable schemes. It is straightforward to see that for 3-variable schemes a different number basis structure is present. For instance if we define the perturbative expansion of the Adler D-function by

$$D(Q^2) = d_R C^{\mathrm{Adl}}(a,\alpha) \tag{5.1}$$

in the same notation as [88], where $d_R = 3$ for $SU(3)$, then in the MOMq scheme of [51,52] for the same group we have

$$
\begin{aligned}
C^{\mathrm{Adl}}_{\mathrm{MOMq}}(a,0)\Big|^{SU(3)} &= 1+4a+\left[-\frac{340}{81}\pi^2-\frac{92}{9}N_f+\frac{32}{3}\zeta_3N_f+\frac{170}{27}\psi^{(1)}\left(\frac{1}{3}\right)+\frac{463}{3}-176\zeta_3\right]a^2\\
&+\left[-\frac{1050461}{108}\zeta_3-\frac{228443}{486}\pi^2-\frac{53455}{2187}\pi^2\psi^{(1)}\left(\frac{1}{3}\right)-\frac{30722}{27}N_f\right.\\
&-\frac{14960}{27}\zeta_3\psi^{(1)}\left(\frac{1}{3}\right)-\frac{12968}{243}\psi^{(1)}\left(\frac{1}{3}\right)N_f-\frac{5440}{243}\zeta_3\pi^2N_f-\frac{1600}{9}\zeta_5N_f\\
&-\frac{64}{3}\zeta_3N_f^2-\frac{16}{27}\pi^4N_f+\frac{2}{9}\psi^{(3)}\left(\frac{1}{3}\right)N_f+\frac{505}{1944}\psi^{(3)}\left(\frac{1}{3}\right)\\
&+\frac{2720}{81}\zeta_3\psi^{(1)}\left(\frac{1}{3}\right)N_f+\frac{3032}{3}\zeta_3N_f+\frac{8800}{3}\zeta_5+\frac{25936}{729}\pi^2N_f+\frac{29920}{81}\zeta_3\pi^2\\
&\left.+\frac{48910}{6561}\pi^4+\frac{53455}{2916}\psi^{(1)}\left(\frac{1}{3}\right)^2+\frac{228443}{324}\psi^{(1)}\left(\frac{1}{3}\right)+32N_f^2+8679\right]a^3\\
&+\left[-\frac{44274159239}{77760}\zeta_3-\frac{15865609679}{349920}\pi^2-\frac{14446746791}{104976}\zeta_5\right.\\
&+\frac{15865609679}{233280}\psi^{(1)}\left(\frac{1}{3}\right)-\frac{342716743}{1119744}\psi^{(3)}\left(\frac{1}{3}\right)-\frac{141906113}{14580}\psi^{(1)}\left(\frac{1}{3}\right)N_f\\
&-\frac{86603095}{23328}\pi^2\psi^{(1)}\left(\frac{1}{3}\right)-\frac{56330612}{10935}\zeta_3\pi^2N_f-\frac{39752681}{324}N_f\\
&-\frac{15012587}{52488}\pi^2\psi^{(1)}\left(\frac{1}{3}\right)^2-\frac{14454071}{196830}\pi^6-\frac{6626701}{2834352}\pi^2\psi^{(3)}\left(\frac{1}{3}\right)\\
&-\frac{3164155}{13122}\pi^4N_f-\frac{1196852}{81}\zeta_3^2N_f-\frac{981344}{243}\zeta_3N_f^2-\frac{748000}{81}\zeta_5\pi^2\\
&-\frac{619520}{729}\zeta_3\pi^4-\frac{117040}{3}\zeta_7-\frac{76621}{486}\psi^{(1)}\left(\frac{1}{3}\right)^2N_f-\frac{68000}{81}\zeta_5\psi^{(1)}\left(\frac{1}{3}\right)N_f\\
&-\frac{55330}{27}\zeta_3\psi^{(1)}\left(\frac{1}{3}\right)^2-\frac{46684}{2187}\pi^4\psi^{(1)}\left(\frac{1}{3}\right)N_f-\frac{43130}{729}H_5\\
&-\frac{40240}{243}\zeta_3\pi^2\psi^{(1)}\left(\frac{1}{3}\right)N_f-\frac{27520}{9}\zeta_5N_f^2-\frac{16928}{81}\zeta_3\psi^{(1)}\left(\frac{1}{3}\right)N_f^2\\
&-\frac{15248}{81}\pi^2N_f^2-\frac{13246}{729}\zeta_3\psi^{(3)}\left(\frac{1}{3}\right)N_f-\frac{12473}{87480}\psi^{(5)}\left(\frac{1}{3}\right)N_f\\
&-\frac{9376}{729}\psi^{(1)}\left(\frac{1}{3}\right)^3N_f-\frac{5555}{243}\zeta_3\psi^{(3)}\left(\frac{1}{3}\right)-\frac{4736}{729}H_5N_f-\frac{3313}{2187}\psi^{(3)}\left(\frac{1}{3}\right)N_f^2\\
&-\frac{1072}{9}N_f^3-\frac{256}{81}\zeta_3\pi^4N_f^2-\frac{164}{229635}H_6-\frac{85}{81}\pi^2\psi^{(3)}\left(\frac{1}{3}\right)N_f\\
&-\frac{32}{27}\pi^2\psi^{(1)}\left(\frac{1}{3}\right)N_f^2+\frac{8}{9}\psi^{(1)}\left(\frac{1}{3}\right)^2N_f^2+\frac{32}{27}\zeta_3\psi^{(3)}\left(\frac{1}{3}\right)N_f^2\\
&+\frac{85}{54}\psi^{(1)}\left(\frac{1}{3}\right)\psi^{(3)}\left(\frac{1}{3}\right)N_f+\frac{448}{9}\zeta_3N_f^3+\frac{640}{9}\zeta_5N_f^3+\frac{2560}{9}\zeta_3^2N_f^2\\
&+\frac{7624}{27}\psi^{(1)}\left(\frac{1}{3}\right)N_f^2+\frac{10060}{81}\zeta_3\psi^{(1)}\left(\frac{1}{3}\right)^2N_f+\frac{18752}{729}\pi^2\psi^{(1)}\left(\frac{1}{3}\right)^2N_f\\
&+\frac{21280}{9}\zeta_7N_f+\frac{29096}{6561}\pi^4N_f^2+\frac{33856}{243}\zeta_3\pi^2N_f^2+\frac{63707}{9}N_f^2
\end{aligned}
$$

$$+\frac{136000}{243}\zeta_5\pi^2N_f+\frac{153242}{729}\pi^2\psi^{(1)}\left(\frac{1}{3}\right)N_f+\frac{221320}{81}\zeta_3\pi^2\psi^{(1)}\left(\frac{1}{3}\right)$$
$$+\frac{226688}{2187}\zeta_3\pi^4N_f+\frac{374000}{27}\zeta_5\psi^{(1)}\left(\frac{1}{3}\right)+\frac{433012}{32805}\pi^6N_f$$
$$+\frac{2244703}{34992}\psi^{(3)}\left(\frac{1}{3}\right)N_f+\frac{4491806}{27}\zeta_3^2+\frac{6626701}{1889568}\psi^{(1)}\left(\frac{1}{3}\right)\psi^{(3)}\left(\frac{1}{3}\right)$$
$$+\frac{15012587}{104976}\psi^{(1)}\left(\frac{1}{3}\right)^3+\frac{15246509}{18895680}\psi^{(5)}\left(\frac{1}{3}\right)+\frac{28165306}{3645}\zeta_3\psi^{(1)}\left(\frac{1}{3}\right)N_f$$
$$+\frac{64243291}{354294}\pi^4\psi^{(1)}\left(\frac{1}{3}\right)+\frac{86603095}{31104}\psi^{(1)}\left(\frac{1}{3}\right)^2+\frac{141906113}{21870}\pi^2N_f$$
$$+\frac{214548299}{2430}\zeta_3N_f+\frac{257743792}{6561}\zeta_5N_f+\frac{274741727}{432}+\frac{862335313}{419904}\pi^4$$
$$+\frac{1968292019}{43740}\zeta_3\pi^2-\frac{1968292019}{29160}\zeta_3\psi^{(1)}\left(\frac{1}{3}\right)\Bigg]a^4+O(a^5) \qquad (5.2)$$

where H_5 and H_6 are defined in the Supplemental Material of [69] and are shorthand for different combinations of harmonic or generalized polylogarithms. We have derived (5.2) using the Landau gauge coupling constant mapping between the $\overline{\text{MS}}$ and MOMq schemes computed in [51,52,68,69] which was applied to the $O(a^4)$ $\overline{\text{MS}}$ D-function of [88]. As indicated earlier the MOMq scheme is a 3-variable scheme but in this case the two dimensionless variables x and y are both unity. What is evident in (5.2) is that $\zeta_{2n}\propto\pi^{2n}$ appears in the $O(a^{n+1})$ term for $n\geq 1$. This includes ζ_2 which is absent in the $\overline{\text{MS}}$ and $\widetilde{\text{MOM}}$ schemes. However (5.2) does not violate the no-π theorem because the master integrals underlying the MOM schemes are not p-integrals. In fact in this MOM configuration the ζ_{2n} contributions are each connected to $\psi^{(2n-1)}(\frac{1}{3})$ in the master integrals. More specifically if one makes the redefinitions to $\hat{\psi}^{(n)}(\frac{1}{3})$ via

$$\psi^{(1)}\left(\frac{1}{3}\right)=\hat{\psi}^{(1)}\left(\frac{1}{3}\right)+4\zeta_2,\quad \psi^{(3)}\left(\frac{1}{3}\right)=\hat{\psi}^{(3)}\left(\frac{1}{3}\right)+240\zeta_4$$
$$\psi^{(5)}\left(\frac{1}{3}\right)=\hat{\psi}^{(5)}\left(\frac{1}{3}\right)+43680\zeta_6 \qquad (5.3)$$

then ζ_2, ζ_4 and ζ_6 are effectively hidden but not absent. By this we mean that (5.3) is a shorthand for combinations that appear in the MOM external momentum configuration. It ought not to be interpreted as the same as the ϵ dependent redefinition of the zeta sequence in the same manner as that introduced in [44] which is connected to the $\widetilde{\text{MOM}}$ scheme exclusion of even zetas.

There is one aspect worth highlighting in this example. The Adler D-function is constructed from the derivative of the 2-point correlation function of the vector current. As such it is a Green's function depending on one variable which is clearly the magnitude of the momentum transfer. However the coupling constant renormalization can be carried out in schemes other than those we have designated as 1-variable ones. For instance, the MOMq scheme is in the class of 3-variable schemes but it clearly has π^2 contributions as well as derivatives of the Euler Γ-function. As this situation lies outside the conditions of the no-π theorem there is no contradiction with it. Instead it merely illustrates how the situation changes with regard to the appearance of π^2 for the same physical quantity but in a different class of schemes. However what is not immediately clear concerns the ζ_{2n} dependence of a more general situation. The no-π theorem makes no mention of whether the massless correlation function is restricted to that for two operators. For instance, in the case of a 3-point gauge invariant operator correlation function the situation is more involved. This is assuming none of the three operators have a zero momentum flow which would reduce the correlation function to an effective 2-point computation. For the 3-point correlator the finite expression after renormalization should depend on a similar number basis, or its generalization for the non-unit values of the x and y variables, as that given in the β-functions of [51,52,68,69]. It is therefore not clear if there is a scheme which would transform the finite part to the ζ_n and rational number basis of a 1-variable scheme for a purely 3-point operator correlation function. This may be the next task to study to understand the absence of ζ_{2n} in observable quantities.

One of the main reasons why we reviewed the renormalization group invariance of critical exponents in the previous section is that it adds to the understanding of the expression of the no-π theorem. Recalling that the ϵ expansions of critical exponents at the Wilson-Fisher fixed point are scheme independent, the $O(\epsilon^5)$ expansion of ω for $SU(3)$ in QCD is

$$\omega|^{SU(3)} = \epsilon + 6[19N_f - 153]\frac{\epsilon^2}{[2N_f - 33]^2}$$
$$+ [-650N_f^3 + 14931N_f^2 - 233937N_f + 860139]\frac{\epsilon^3}{[2N_f - 33]^4}$$
$$+ [-8744N_f^5 - 465984\zeta_3 N_f^4 + 264612N_f^4 + 16783200\zeta_3 N_f^3 + 14077854N_f^3$$
$$- 194038416\zeta_3 N_f^2 - 304090713N_f^2 + 1068622632\zeta_3 N_f + 2677244157N_f$$
$$- 5658783768\zeta_3 - 6752933307]\frac{\epsilon^4}{6[2N_f - 33]^6}$$
$$+ [175104\zeta_3 N_f^7 - 38560N_f^7 + 10041600\zeta_3 N_f^6 - 5591808\zeta_4 N_f^6 - 4769280\zeta_5 N_f^6$$
$$+ 13696280N_f^6 - 1091992320\zeta_3 N_f^5 + 385928064\zeta_4 N_f^5 + 675866880\zeta_5 N_f^5$$
$$- 804202392N_f^5 + 38019806208\zeta_3 N_f^4 - 10496977920\zeta_4 N_f^4 - 30361478400\zeta_5 N_f^4$$
$$+ 17650466742N_f^4 - 694400454720\zeta_3 N_f^3 + 144493398720\zeta_4 N_f^3$$
$$+ 639988905600\zeta_5 N_f^3 - 280199346390N_f^3 + 6778342959696\zeta_3 N_f^2$$
$$- 1125003472560\zeta_4 N_f^2 - 7142270968800\zeta_5 N_f^2 + 2651536463832N_f^2$$
$$- 33418251568944\zeta_3 N_f + 5732068510872\zeta_4 N_f + 43054184851920\zeta_5 N_f$$
$$- 13351743621324N_f + 81853049696616\zeta_3 - 18487246570056\zeta_4$$
$$- 120758445609120\zeta_5 + 24360811371837]\frac{\epsilon^5}{12[2N_f - 33]^8} + O(\epsilon^6) \quad (5.4)$$

with the exponents for the gluon, ghost, quark and quark mass taking a similar form. Clearly (5.4) depends on ζ_4 at $O(\epsilon^5)$ as do the other exponents. Endeavoring to remove such a contribution is not possible as (5.4) is independent of the scheme. Indeed we computed ω directly for each of the $\widetilde{\text{MOM}}_*$ schemes and verified that the same expression as (5.4) resulted. So for example the ϵ dependent mapping of the zetas of [44] cannot be applied. That in effect is related to a scheme change and such a change has already been incorporated within the $\widetilde{\text{MOM}}_*$ construction. Moreover the presence of ζ_4 does not contradict the criteria of the no-π theorem of [44]. While the computation that leads to (5.4) was also carried out in the $\overline{\text{MS}}$ scheme using p-integrals, it can equally well be carried out in any of the MOM schemes of [51,52] which do not use p-integrals. The same result is obtained.

As a final comment on (5.4) one can isolate the ζ_4 contribution at $O(\epsilon^5)$ for an arbitrary color group. Denoting this by $\omega|^{\zeta_4}_{\epsilon^5}$ we have

$$\omega|^{\zeta_4}_{\epsilon^5} = [11C_A^4 N_A - 102C_A^3 N_A N_f T_F + 164C_A^2 C_F N_A N_f T_F - 56C_A^2 N_A N_f^2 T_F^2$$
$$- 88C_A C_F^2 N_A N_f T_F - 112C_A C_F N_A N_f^2 T_F^2 + 176C_F^2 N_A N_f^2 T_F^2 - 528 d_A^{abcd} d_A^{abcd}$$
$$+ 1248 d_A^{abcd} d_F^{abcd} N_f - 384 d_F^{abcd} d_F^{abcd} N_f^2]\frac{162\zeta_4}{[11C_A - 4N_f T_F]^4 N_A}. \quad (5.5)$$

We noted that in $SU(3)$ the $O(a^5)$ coefficient of ζ_4 in the coupling constant maps of (4.1) was the same in all the $\widetilde{\text{MOM}}_*$ schemes when $N_f = 3$. Aside from ζ_4 there was no commonality for the odd zetas across all the scheme maps. This allows us to track down some aspects of how the ζ_4 contribution in the $\overline{\text{MS}}$ β-function at five loops leads to (5.5) as well as how the nonappearance of ζ_4 in $\beta_{\widetilde{\text{MOM}}_*}(a, \alpha)$ still results in its presence in this scheme independent exponent. It transpires that the route to (5.5) for these distinct schemes is different. First we isolated the ζ_4 coefficient at $O(a^5)$ of (4.1) for a general color group in the Landau gauge and found that it is *precisely* proportional to the numerator of (5.5) for all N_f and $\widetilde{\mathrm{MOM}}_*$ schemes. The constant of proportionality can be accounted for by the one loop coefficient of the β-function which is scheme independent. That term in the mappings can be traced back to the $O(\epsilon)$ contribution to the *four* loop term of $\beta_{\widetilde{\mathrm{MOM}}_*}(a,\alpha)$. That means it is present in the finite part of the vertex function for bare variables. This can be verified by noting that the coefficient of ζ_4 at five loops in the $\overline{\mathrm{MS}}$ β-function is proportional to the numerator of (5.5). The actual coefficient is the product of the numerator and the one loop coefficient of the β-function as well as a rational. In other words within the derivation of (5.5) the ζ_4 contribution emerges via two different routes depending on which scheme is used. In the $\overline{\mathrm{MS}}$ case it appears in ω directly from the five loop term in the β-function. By contrast it is absent in the $\widetilde{\mathrm{MOM}}_*$ β-function in four dimensions but present for nonzero ϵ at four loops. In carrying out this analysis what we are basically summarizing is the same process that the ϵ dependent redefinition of the zeta series of [44,45] is effecting but using the scheme independent ω as the pivot point to trace the details. We close the section by remarking that with respect to the classification introduced here one could regard ω and other such exponents as being determined in a 0-variable scheme. This is partly because exponents are dimensionless quantities as there is no scale at a critical point. In turn this means that there is no underlying single momentum invariant similar to that which is present in the 1-variable ones.

VI. DISCUSSION

One of the main aims of this study was to ascertain whether the extension of the so-called $\widetilde{\mathrm{MOM}}$ schemes that were examined in earlier articles at lower loop order retained the property of having no explicit π^2 terms in the renormalization group functions at five loops in QCD for all values of α in a linear covariant gauge fixing. By exploiting the available FORCER data on the bare 2- and 3-point functions [63], we were indeed able to demonstrate that this is the case for the various single scale 3-point vertices of (2.11). Moreover this observation in one sense both confirms and extends the study of [46] which centred on what was termed there as AD theories. These are ones which have symmetries, such as gauge symmetry or supersymmetry, that means the coupling renormalization constant is determined by a Ward identity that relates it to the renormalization of the fields. In QCD one such AD scheme was already known about which was the mMOM scheme based on Taylor's theorem that the ghost-gluon vertex function is finite in the Landau gauge. So the renormalization of the coupling is constructed purely from the ghost and gluon renormalization constants. From the available five loop renormalization group functions [43,63,64] this is implicitly evident but only in the Landau gauge as can be verified by examining the $\alpha = 0$ $\widetilde{\mathrm{MOM}}_{ccg0c}$ scheme expressions in Appendix B. This observation in effect became a focal point for realizing that if the defining mMOM constraint on the coupling constant renormalization was removed then the absence of π^2 in the $\widetilde{\mathrm{MOM}}_{ccg0c}$ scheme and the remaining $\widetilde{\mathrm{MOM}}_*$ schemes should follow for all α. In other words it should be possible to extend the groundwork analysis of [46] to non-AD theories. In one way supportive of that possibility is the fact that in QCD the gauge parameter could be regarded as a second coupling and [46] examined the $\widetilde{\mathrm{MOM}}$ construction in a multicoupling theory indicating that the ζ_4 cancellation would persist to six loops in some theories. Although what we have examined here has to be qualified by noting that a perturbative expansion is not carried out in the gauge parameter itself.

To construct an all orders proof of the ζ_{2n} absence may not be straightforward and the Hopf algebra approach of [58] in the Wess-Zumino model, motivated by the earlier work of [89,90] might allow for deeper understanding. For instance one feature of the $\widetilde{\mathrm{MOM}}$ scheme that seems to lie at its heart is that in the defining prescription the 3-point functions are quotiented by the 2-point functions of the relevant fields of that vertex prior to the remainder being removed from the 3-point vertex. This tallies with the approach of [44,45]. In graphs that involve a simple pole in ϵ and a residue that depends on ζ_3 the finite part will contain ζ_4 with its predetermined coefficient [44,45]. The removal of the finite part in a $\widetilde{\mathrm{MOM}}$ prescription means that it will contribute at the next order via the counterterms and thereby affect the coefficient of ζ_4. Indeed we were able to verify this in dissecting the passage of ζ_4 from the ϵ-dependent β-functions to how it appears in a scheme independent quantity. We need to qualify the absence of π^2 dependence at any higher loop order in a renormalization group function by noting that one would also have to prove that there are no single scale master Feynman graphs whose leading term in its ϵ expansion is proportional to π^2. From the high order four dimensional examples that are already available in, for instance [38,91,92], no such cases appear to be known.

Perhaps one place to examine these ideas in further detail would be in other spacetime dimensions. For instance, ϕ^3 theory in six dimensions has already been noted as a potential laboratory to study n-variable schemes. As alluded to earlier, examining ϕ^3 theory in principle would first require the construction and analysis of the 3-point Schwinger-Dyson equation. In the single field case the coupling constant renormalization is related to the renormalization of the mass of the ϕ field equating it to a 2-point function renormalization. Such a relationship however does not extend to the cubic theory with a symmetry. Another instance of where using a $\widetilde{\mathrm{MOM}}$ scheme might produce an interesting structure for a β-function is the two dimensional nonlinear σ model with $\mathcal{N} = 1$ supersymmetry. We note that the location of ζ_n in the β-function of these two dimensional supersymmetric models is that ζ_{L-1} appears for the first time at L loops starting with $L = 4$. Such supersymmetric models have been renormalized to four loops in the $\overline{\text{MS}}$ scheme on a general manifold [93–95] and revealed that there are no contributions after the one loop one until four loops. At that order the coefficient of the β-function involves only ζ_3. For nonlinear σ models with $\mathcal{N} = 2$ supersymmetry the four loop term has the same property. The situation beyond four loops has been probed in several ways. Restricting the geometry to the N dimensional sphere one can compute the $1/N$ corrections to the β-function in the large N expansion [96,97]. Aside from being an independent confirmation of the β-function of the general geometry of [93,94], the result indicated that the five loop term would involve ζ_4 in the $\overline{\text{MS}}$ scheme. In the $\mathcal{N} = 2$ supersymmetric case the β-function was examined at five loops explicitly for Kähler manifolds in [98]. Interestingly in that general geometry the five loop term can be made to vanish by a particular scheme choice [98]. Whether that scheme has a connection with the $\widetilde{\text{MOM}}$ prescription would be interesting to ascertain. If so it would tally with the absence of ζ_4 terms in that set of renormalization prescriptions. At six loop order it was argued in [99] that there would be a nonzero contribution, solely involving ζ_5, which remained even after any field redefinition. In the case of the N-sphere the five, six and seven loop coefficients depended only on zetas [96]. By this we specifically mean that ζ_4 and the pairs $\{\zeta_3, \zeta_5\}$ and $\{\zeta_3^2, \zeta_6\}$ appear respectively at $O(1/N^2)$. Other ζ_n contributions could of course arise at higher orders in $1/N$. The field anomalous dimension had a similar structure [97,100].

Given these observations it might be of interest to see whether the five loop ζ_4 and seven loop ζ_6 contributions are absent in a direct $\widetilde{\text{MOM}}$ scheme computation. The work of [98] suggests this might be the case for the former. Equally it would be interesting to see how a scheme transformation similar to the one for $\mathcal{N} = 2$ supersymmetry discussed in [98] would affect the zeta structure of an $\mathcal{N} = 1$ theory. For instance from the results in the $\mathcal{N} = 2$ σ model it appears that only zetas appear at L loops whose weight is $(L - 1)$ for $L \geq 4$. At seven loops in the $\mathcal{N} = 1$ case it is known that ζ_3^2 and ζ_6 are present [96]. Therefore for a transformation to a $\widetilde{\text{MOM}}$ scheme similar to that discussed here, the ζ_3^2 contribution for the N-sphere should remain but ζ_6 ought to be absent. In the $\mathcal{N} = 2$ case the latter ought also be absent but the status of ζ_3^2 in the β-function after a transformation is not clear. While renormalizing these two dimensional supersymmetric σ models is highly nontrivial beyond one loop [93–95,99] and accessing the various higher large N orders is equally computationally demanding, the coefficients of the β-function would appear to be only zetas and multiple zetas with no rationals [97,98,100]. It would therefore seem that these supersymmetric models might offer a future testing ground for analyzing even zeta cancellation in the $\widetilde{\text{MOM}}$ prescription at a deeper order beyond the five loop one considered here. However, the situation with regard to supersymmetric gauge theories is more intricate from the point of view of the C-scheme concept. For instance prior to [59,60] the exact NSVZ (Novikov-Shifman-Vainshtein-Zakharov) β-function [101–103] was examined in [104]. That β-function has similarities with the C-scheme β-function of [59,60]. Indeed a comparison between the $\mathcal{N} = 0$ and 1 β-functions in gauge theories was carried out in [105]. However as yet a detailed analysis of the ζ_{2n} dependence of all the renormalization group functions of supersymmetric gauge theories at high loop order has not been considered to the same depth as the nonsupersymmetic ones.

Finally, having established the $\widetilde{\text{MOM}}_*$ schemes have the property that there are no even zetas to five loops in QCD, there is scope now to apply these schemes to explore what effect they have on phenomenological precision and whether they can be employed for estimating theory errors. For instance the C-scheme was used [59] to study e^+e^- scattering and τ decays into hadrons. It was suggested that this scheme produces a scheme invariant scale running. Therefore it would seem appropriate to employ the $\widetilde{\text{MOM}}_*$ data now to complement that study to ascertain what effect the absence of ζ_4 and ζ_6 has and to see if there is a similar reduction in scale dependence. Equally the other question of what structures are absent in the analogous concept of $\widetilde{\text{MOM}}$ scheme renormalization of n-variable schemes would be an interesting avenue to pursue.

The data representing the main results here are accessible from [66].

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APPENDIX A: $SU(3)$ LANDAU GAUGE β-FUNCTIONS

In this appendix we record each of the $\widetilde{\text{MOM}}_*$ β-functions in the $SU(3)$ color group for the Landau gauge in order to clearly show that the only ζ_n dependence is ζ_3, ζ_5 and ζ_7 to five loops. First for the two ghost-gluon vertex schemes we have

$$\beta^{SU(3)}_{\widetilde{\mathrm{MOM}}_{ccg0c}}(a,0) = \left[\frac{2}{3}N_f - 11\right]a^2 + \left[\frac{38}{3}N_f - 102\right]a^3 + \left[\frac{3861}{8}\zeta_3 - \frac{28965}{8} - \frac{989}{54}N_f^2 - \frac{175}{12}\zeta_3 N_f - \frac{8}{9}\zeta_3 N_f^2 + \frac{7715}{12}N_f\right]a^4 + \left[-\frac{1380469}{8} - \frac{1027375}{144}\zeta_5 N_f - \frac{736541}{324}N_f^2 - \frac{516881}{72}\zeta_3 N_f - \frac{16}{9}\zeta_3 N_f^3 + \frac{800}{27}N_f^3 + \frac{6547}{27}\zeta_3 N_f^2 + \frac{9280}{27}\zeta_5 N_f^2 + \frac{625317}{16}\zeta_3 + \frac{772695}{32}\zeta_5 + \frac{970819}{24}N_f\right]a^5 + \left[-\frac{21619456551}{4096}\zeta_7 - \frac{18219328375}{6912}\zeta_5 N_f - \frac{10327103555}{20736}\zeta_3 N_f - \frac{3248220045}{256} + \frac{4922799165}{512}\zeta_5 + \frac{24870449471}{18432}\zeta_7 N_f + \frac{115659378547}{31104}N_f - \frac{833934985}{2592}N_f^2 - \frac{26952037}{432}\zeta_7 N_f^2 - \frac{299875}{54}\zeta_5 N_f^3 - \frac{82869}{32}\zeta_3^2 N_f - \frac{59531}{36}\zeta_3^2 N_f^2 - \frac{2617}{27}N_f^4 - \frac{304}{27}\zeta_3 N_f^4 + \frac{1760}{27}\zeta_5 N_f^4 + \frac{2240}{27}\zeta_3^2 N_f^3 + \frac{129869}{162}\zeta_3 N_f^3 + \frac{3249767}{324}N_f^3 + \frac{7696161}{64}\zeta_3^2 + \frac{13019053}{1296}\zeta_3 N_f^2 + \frac{65264845}{324}\zeta_5 N_f^2 + \frac{1064190195}{512}\zeta_3\right]a^6 + O(a^7) \tag{A1}$$

and

$$\beta^{SU(3)}_{\widetilde{\mathrm{MOM}}_{ccg0g}}(a,0) = \left[\frac{2}{3}N_f - 11\right]a^2 + \left[\frac{38}{3}N_f - 102\right]a^3 + \left[-\frac{28263}{8} - \frac{535}{27}N_f^2 - \frac{175}{12}\zeta_3 N_f - \frac{8}{9}\zeta_3 N_f^2 + \frac{3799}{6}N_f + \frac{3861}{8}\zeta_3\right]a^4 + \left[-\frac{5516125}{32} - \frac{1039525}{144}\zeta_5 N_f - \frac{198992}{81}N_f^2 - \frac{64411}{9}\zeta_3 N_f - \frac{16}{9}\zeta_3 N_f^3 + \frac{980}{27}N_f^3 + \frac{9280}{27}\zeta_5 N_f^2 + \frac{13445}{54}\zeta_3 N_f^2 + \frac{295581}{8}\zeta_3 + \frac{817245}{32}\zeta_5 + \frac{1953167}{48}N_f\right]a^5 + \left[-\frac{67000185565}{27648}\zeta_5 N_f - \frac{25922636709}{2048} - \frac{22470285835}{41472}\zeta_3 N_f - \frac{18877588191}{4096}\zeta_7 - \frac{3503317141}{10368}N_f^2 + \frac{2815217703}{1024}\zeta_3 + \frac{16486752015}{2048}\zeta_5 + \frac{23141911415}{18432}\zeta_7 N_f + \frac{929972881523}{248832}N_f - \frac{12779459}{216}\zeta_7 N_f^2 - \frac{612845}{108}\zeta_5 N_f^3 - \frac{267689}{144}\zeta_3^2 N_f^2 - \frac{3022}{27}N_f^4 - \frac{304}{27}\zeta_3 N_f^4 + \frac{1760}{27}\zeta_5 N_f^4 + \frac{2240}{27}\zeta_3^2 N_f^3 + \frac{491045}{648}\zeta_3 N_f^3 + \frac{1066263}{256}\zeta_3^2 N_f + \frac{28965085}{2592}N_f^3 + \frac{33124113}{512}\zeta_3^2 + \frac{62058733}{5184}\zeta_3 N_f^2 + \frac{1017487675}{5184}\zeta_5 N_f^2\right]a^6 + O(a^7). \tag{A2}$$

The other scheme based on the triple gluon vertex produces

$$\beta^{SU(3)}_{\widetilde{\mathrm{MOM}_{ggg0gg}}}(a,0) = \left[\frac{2}{3}N_f - 11\right]a^2 + \left[\frac{38}{3}N_f - 102\right]a^3$$
$$+ \left[-\frac{186747}{64} - \frac{65}{6}\zeta_3 N_f - \frac{8}{9}N_f^3 - \frac{8}{9}\zeta_3 N_f^2 + \frac{829}{54}N_f^2 + \frac{1683}{4}\zeta_3 + \frac{35473}{96}N_f\right]a^4$$
$$+ \left[-\frac{20783939}{128} - \frac{1464379}{648}N_f^2 - \frac{1323259}{144}\zeta_3 N_f - \frac{908995}{144}\zeta_5 N_f - \frac{320}{81}N_f^4\right.$$
$$\left. - \frac{64}{9}\zeta_3 N_f^3 + \frac{3164}{27}N_f^3 + \frac{7540}{27}\zeta_5 N_f^2 + \frac{12058}{27}\zeta_3 N_f^2 + \frac{900075}{32}\zeta_5 + \frac{1300563}{32}\zeta_3 + \frac{2410799}{64}N_f\right]a^5$$
$$+ \left[-\frac{46418845041}{4096} - \frac{26430396425}{13824}\zeta_5 N_f - \frac{4685253111}{4096}\zeta_7\right.$$
$$+ \frac{2800824887}{18432}\zeta_7 N_f + \frac{7708557555}{1024}\zeta_5 + \frac{1422229465579}{497664}N_f$$
$$- \frac{1924634167}{10368}\zeta_3 N_f - \frac{421916191}{2592}N_f^2 - \frac{54502201}{2592}\zeta_3 N_f^2$$
$$- \frac{36167831}{3456}\zeta_7 N_f^2 - \frac{29633175}{512}\zeta_3^2 - \frac{433589}{648}N_f^3 - \frac{373877}{144}\zeta_3^2 N_f^2$$
$$- \frac{313255}{54}\zeta_5 N_f^3 - \frac{832}{27}\zeta_3 N_f^4 - \frac{416}{27}N_f^5 + \frac{998}{27}\zeta_3^2 N_f^3 + \frac{1323}{4}\zeta_7 N_f^3$$
$$+ \frac{1760}{27}\zeta_5 N_f^4 + \frac{33547}{81}N_f^4 + \frac{645595}{324}\zeta_3 N_f^3 + \frac{9288863}{256}\zeta_3^2 N_f$$
$$\left. + \frac{196487181}{256}\zeta_3 + \frac{437857925}{2592}\zeta_5 N_f^2\right]a^6 + O(a^7). \tag{A3}$$

Finally the schemes derived from the quark-gluon vertex lead to

$$\beta^{SU(3)}_{\widetilde{\mathrm{MOM}_{qqg0g}}}(a,0) = \left[\frac{2}{3}N_f - 11\right]a^2 + \left[+\frac{38}{3}N_f - 102\right]a^3$$
$$+ \left[-\frac{150931}{48} - \frac{797}{54}N_f^2 - \frac{118}{3}\zeta_3 N_f - \frac{8}{9}\zeta_3 N_f^2 + \frac{42089}{72}N_f + 891\zeta_3\right]a^4$$
$$+ \left[-\frac{44627671}{576} - \frac{3718495}{96}\zeta_5 - \frac{959761}{648}N_f^2 - \frac{110887}{12}\zeta_3 N_f - \frac{16}{9}\zeta_3 N_f^3\right.$$
$$\left. + \frac{164}{9}N_f^3 + \frac{2800}{27}\zeta_5 N_f^2 + \frac{7771}{27}\zeta_3 N_f^2 + \frac{91645}{144}\zeta_5 N_f + \frac{1458193}{24}\zeta_3 + \frac{22417595}{864}N_f\right]a^5$$
$$+ \left[-\frac{110207309485}{27648} - \frac{37281587675}{20736}\zeta_5 N_f + \frac{6054470821}{9216}\zeta_7 N_f\right.$$
$$+ \frac{55672020805}{13824}\zeta_5 + \frac{244350005119}{124416}N_f - \frac{2027774803}{10368}N_f^2$$
$$- \frac{1790283341}{2048}\zeta_7 - \frac{249619937}{432}\zeta_3 - \frac{63316501}{2592}\zeta_3 N_f^2 - \frac{31626203}{864}\zeta_7 N_f^2$$
$$- \frac{5338625}{64}\zeta_3^2 N_f - \frac{159320}{27}\zeta_5 N_f^3 - \frac{1657}{27}N_f^4 - \frac{1648}{27}\zeta_3^2 N_f^3 - \frac{304}{27}\zeta_3 N_f^4$$
$$+ \frac{1760}{27}\zeta_5 N_f^4 + \frac{127765}{81}\zeta_3 N_f^3 + \frac{172645}{36}\zeta_3^2 N_f^2 + \frac{230059}{36}N_f^3$$
$$\left. + \frac{13628215}{1728}\zeta_3 N_f + \frac{44150073}{128}\zeta_3^2 + \frac{458425445}{2592}\zeta_5 N_f^2\right]a^6 + O(a^7)$$

$$\beta^{SU(3)}_{\widetilde{\text{MOM}}_{qqg0gT}}(a,0) = \left[\frac{2}{3}N_f - 11\right]a^2 + \left[\frac{38}{3}N_f - 102\right]a^3$$
$$+ \left[-\frac{185039}{48} - \frac{953}{54}N_f^2 - \frac{8}{9}\zeta_3 N_f^2 + \frac{47221}{72}N_f - 48\zeta_3 N_f + 1034\zeta_3\right]a^4$$
$$+ \left[-\frac{32456317}{192} - \frac{3369385}{432}\zeta_5 N_f - \frac{1412065}{648}N_f^2 - \frac{851009}{108}\zeta_3 N_f - \frac{16}{9}\zeta_3 N_f^3\right.$$
$$\left. + \frac{740}{27}N_f^3 + \frac{5875}{27}\zeta_3 N_f^2 + \frac{11440}{27}\zeta_5 N_f^2 + \frac{1269361}{32}N_f + \frac{3841475}{288}\zeta_5 + \frac{4134361}{72}\zeta_3\right]a^5$$
$$+ \left[-\frac{118778905711}{9216} - \frac{83114846969}{18432}\zeta_7 - \frac{19658419625}{6912}\zeta_5 N_f\right.$$
$$- \frac{3287597531}{10368}N_f^2 + \frac{37311626107}{27648}\zeta_7 N_f + \frac{43800203815}{4608}\zeta_5$$
$$+ \frac{463743395407}{124416}N_f - \frac{827828681}{1296}\zeta_3 N_f - \frac{139465865}{384}\zeta_3^2$$
$$- \frac{56355551}{864}\zeta_7 N_f^2 - \frac{158060}{27}\zeta_5 N_f^3 - \frac{52295}{12}\zeta_3^2 N_f^2 - \frac{2437}{27}N_f^4 - \frac{304}{27}\zeta_3 N_f^4$$
$$+ \frac{1760}{27}\zeta_5 N_f^4 + \frac{3536}{27}\zeta_3^2 N_f^3 + \frac{55165}{81}\zeta_3 N_f^3 + \frac{782282}{81}N_f^3$$
$$\left. + \frac{33559267}{576}\zeta_3^2 N_f + \frac{41202899}{2592}\zeta_3 N_f^2 + \frac{570243785}{2592}\zeta_5 N_f^2 + \frac{1248094123}{384}\zeta_3\right]a^6 + O(a^7)$$

$$\beta^{SU(3)}_{\widetilde{\text{MOM}}_{qqg0q}}(a,0) = \left[\frac{2}{3}N_f - 11\right]a^2 + \left[\frac{38}{3}N_f - 102\right]a^3$$
$$+ \left[-\frac{29559}{8} - \frac{989}{54}N_f^2 - \frac{64}{3}\zeta_3 N_f - \frac{8}{9}\zeta_3 N_f^2 + \frac{7769}{12}N_f + 594\zeta_3\right]a^4$$
$$+ \left[-\frac{2795027}{16} - \frac{1016935}{144}\zeta_5 N_f - \frac{737837}{324}N_f^2 - \frac{67939}{9}\zeta_3 N_f - \frac{16}{9}\zeta_3 N_f^3\right.$$
$$\left. + \frac{800}{27}N_f^3 + \frac{6709}{27}\zeta_3 N_f^2 + \frac{9280}{27}\zeta_5 N_f^2 + \frac{174207}{4}\zeta_3 + \frac{487751}{12}N_f + \frac{734415}{32}\zeta_5\right]a^5$$
$$+ \left[-\frac{18215306455}{6912}\zeta_5 N_f - \frac{10920152021}{20736}\zeta_3 N_f - \frac{9939037647}{2048}\zeta_7\right.$$
$$+ \frac{4803715005}{512}\zeta_5 + \frac{11859668359}{9216}\zeta_7 N_f + \frac{58653420029}{15552}N_f$$
$$- \frac{421368587}{1296}N_f^2 - \frac{51878571}{4} - \frac{25991539}{432}\zeta_7 N_f^2 - \frac{152570}{27}\zeta_5 N_f^3$$
$$- \frac{15776}{9}\zeta_3^2 N_f^2 - \frac{2617}{27}N_f^4 - \frac{304}{27}\zeta_3 N_f^4 + \frac{1760}{27}\zeta_5 N_f^4 + \frac{2240}{27}\zeta_3^2 N_f^3$$
$$+ \frac{127439}{162}\zeta_3 N_f^3 + \frac{495753}{128}\zeta_3^2 N_f + \frac{3273095}{324}N_f^3 + \frac{10403415}{256}\zeta_3^2$$
$$\left. + \frac{28286855}{2592}\zeta_3 N_f^2 + \frac{66058375}{324}\zeta_5 N_f^2 + \frac{1222682025}{512}\zeta_3\right]a^6 + O(a^7) \quad \text{(A4)}$$

and

$$\begin{aligned}
\beta^{SU(3)}_{\widetilde{\mathrm{MOM}}_{qqg0qT}}(a,0) &= \left[\frac{2}{3}N_f - 11\right]a^2 + \left[\frac{38}{3}N_f - 102\right]a^3 \\
&+ \left[-\frac{142793}{48} - \frac{961}{54}N_f^2 - \frac{80}{9}\zeta_3 N_f^2 + \frac{1427}{12}\zeta_3 N_f + \frac{3663}{8}\zeta_3 + \frac{39043}{72}N_f\right]a^4 \\
&+ \left[-\frac{242516239}{1296} - \frac{60944155}{3888}\zeta_5 N_f - \frac{11155843}{1944}\zeta_3 N_f - \frac{2700601}{972}N_f^2\right. \\
&- \frac{112}{9}\zeta_3 N_f^3 + \frac{3920}{81}N_f^3 + \frac{6800}{9}\zeta_5 N_f^2 + \frac{29501}{81}\zeta_3 N_f^2 + \frac{39000269}{1296}\zeta_3 \\
&\left. + \frac{41806447}{972}N_f + \frac{137211305}{2592}\zeta_5\right]a^5 \\
&+ \left[-\frac{251314718599}{20736} - \frac{206090400131}{36864}\zeta_7 - \frac{78145127179}{62208}\zeta_3 N_f\right. \\
&- \frac{27622282655}{20736}\zeta_5 N_f - \frac{10412288095}{6912}\zeta_3^2 + \frac{2490660389}{10368}\zeta_3^2 N_f \\
&+ \frac{26119820537}{4608}\zeta_3 + \frac{47880587209}{55296}\zeta_7 N_f + \frac{107580934405}{31104}N_f \\
&+ \frac{351316606415}{41472}\zeta_5 - \frac{1235249951}{3888}N_f^2 - \frac{4484669}{288}\zeta_7 N_f^2 - \frac{987613}{72}\zeta_3^2 N_f^2 \\
&- \frac{341953}{162}\zeta_3 N_f^3 - \frac{3969}{4}\zeta_7 N_f^3 - \frac{2885}{27}N_f^4 - \frac{640}{27}\zeta_5 N_f^4 + \frac{464}{27}\zeta_3 N_f^4 \\
&+ \frac{1465}{9}\zeta_5 N_f^3 + \frac{7676}{27}\zeta_3^2 N_f^3 + \frac{10511585}{972}N_f^3 + \frac{54873371}{648}\zeta_3 N_f^2 \\
&\left. + \frac{75156965}{1296}\zeta_5 N_f^2\right]a^6 + O(a^7). \qquad \text{(A5)}
\end{aligned}$$

The expressions for an arbitrary color group have the same ζ_n dependence as is evident in the data file associated with the article [66].

APPENDIX B: LANDAU GAUGE $\widetilde{\mathrm{MOM}}_{ccg0c}$ RESULTS

In order to make contact with previous results we provide the Landau gauge $SU(3)$ anomalous dimensions for the $\widetilde{\mathrm{MOM}}_{ccg0c}$ scheme as they can be compared directly with the five loop mMOM results of [43,63,64]. This equivalence serves in part as a check on our symbolic manipulation code but also emphasizes that the Landau gauge sector of the mMOM scheme involves neither ζ_4 nor ζ_6. In addition to the β-function of the previous appendix we have

$$\begin{aligned}
\gamma^{SU(3)}_{A,\widetilde{\mathrm{MOM}}_{ccg0c}}(a,0) &= \left[-\frac{13}{2} + \frac{2}{3}N_f\right]a + \left[-\frac{255}{4} + \frac{67}{6}N_f\right]a^2 \\
&+ \left[-\frac{8637}{4} - \frac{719}{54}N_f^2 - \frac{229}{12}\zeta_3 N_f - \frac{8}{9}\zeta_3 N_f^2 + \frac{11227}{24}N_f + 324\zeta_3\right]a^3 \\
&+ \left[-\frac{27189875}{256} - \frac{1118977}{648}N_f^2 - \frac{921265}{144}\zeta_5 N_f - \frac{889231}{144}\zeta_3 N_f\right. \\
&- \frac{16}{9}\zeta_3 N_f^3 + \frac{665}{27}N_f^3 + \frac{5143}{27}\zeta_3 N_f^2 + \frac{9280}{27}\zeta_5 N_f^2 + \frac{1950705}{128}\zeta_5 \\
&\left. + \frac{5549393}{192}N_f + \frac{7740879}{256}\zeta_3\right]a^4
\end{aligned}$$

$$
\begin{aligned}
&+\left[-\frac{71363464263}{16384}\zeta_7-\frac{16520894997}{2048}-\frac{16359945025}{6912}\zeta_5N_f\right.\\
&-\frac{8539017539}{20736}\zeta_3N_f+\frac{3150668061}{2048}\zeta_3+\frac{23633674547}{18432}\zeta_7N_f\\
&+\frac{29623505625}{4096}\zeta_5+\frac{327291152977}{124416}N_f-\frac{613783375}{2592}N_f^2\\
&-\frac{25237429}{432}\zeta_7N_f^2-\frac{979887}{64}\zeta_3^2N_f-\frac{292855}{54}\zeta_5N_f^3\\
&-\frac{51647}{36}\zeta_3^2N_f^2-\frac{1861}{27}N_f^4-\frac{304}{27}\zeta_3N_f^4+\frac{1760}{27}\zeta_5N_f^4+\frac{2240}{27}\zeta_3^2N_f^3\\
&+\frac{129545}{162}\zeta_3N_f^3+\frac{2420705}{324}N_f^3+\frac{13148441}{2592}\zeta_3N_f^2\\
&\left.+\frac{246191965}{1296}\zeta_5N_f^2+\frac{671260095}{2048}\zeta_3^2\right]a^5+O(a^6)
\end{aligned}
$$

$$
\begin{aligned}
\gamma^{SU(3)}_{\widetilde{c,\mathrm{MOM}_{ccg0c}}}(a,0)=&-\frac{9}{4}a+\left[-\frac{153}{8}+\frac{3}{4}N_f\right]a^2\\
&+\left[-\frac{11691}{16}-\frac{5}{2}N_f^2+\frac{9}{4}\zeta_3N_f+\frac{1269}{16}\zeta_3+\frac{1401}{16}N_f\right]a^3\\
&+\left[-\frac{16985133}{512}-\frac{16059}{32}\zeta_3N_f-\frac{13115}{48}N_f^2-\frac{5895}{16}\zeta_5N_f+\frac{5}{2}N_f^3\right.\\
&\left.+\frac{739053}{128}N_f+\frac{1140075}{256}\zeta_5+\frac{2264193}{512}\zeta_3+26\zeta_3N_f^2\right]a^4\\
&+\left[-\frac{15114361941}{32768}\zeta_7-\frac{9464865363}{4096}+\frac{9758887695}{8192}\zeta_5\right.\\
&-\frac{424982943}{4096}\zeta_3^2-\frac{34433025}{256}\zeta_5N_f-\frac{12230645}{288}N_f^2\\
&-\frac{86231}{2}\zeta_3N_f-\frac{3969}{2}\zeta_7N_f^2-\frac{219}{2}\zeta_3^2N_f^2+\frac{15353}{12}N_f^3\\
&+\frac{477395}{192}\zeta_3N_f^2+\frac{550645}{96}\zeta_5N_f^2+\frac{814149}{128}\zeta_3^2N_f+\frac{34354859}{1024}\zeta_7N_f\\
&\left.+\frac{1106092719}{4096}\zeta_3+\frac{1670942731}{3072}N_f-65\zeta_5N_f^3-14N_f^4+\zeta_3N_f^3\right]a^5+O(a^6) \qquad \text{(B1)}
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma^{SU(3)}_{\widetilde{\psi,\mathrm{MOM}_{ccg0c}}}(a,0)=&\left[-\frac{4}{3}N_f+\frac{67}{3}\right]a^2+\left[-\frac{706}{9}N_f-\frac{607}{2}\zeta_3+\frac{8}{9}N_f^2+\frac{29675}{36}+16\zeta_3N_f\right]a^3\\
&+\left[-\frac{21683117}{648}\zeta_3-\frac{2393555}{324}N_f-\frac{272}{9}\zeta_3N_f^2-\frac{40}{9}N_f^3+\frac{2861}{9}N_f^2\right.\\
&\left.+\frac{74440}{27}\zeta_3N_f+\frac{15846715}{1296}\zeta_5+\frac{31003343}{648}-830\zeta_5N_f\right]a^4\\
&+\left[-\frac{94958116621}{31104}\zeta_3-\frac{26588447977}{27648}\zeta_7+\frac{2313514793}{10368}\zeta_3^2\right.\\
&+\frac{14723323093}{5184}+\frac{18607183745}{7776}\zeta_5-\frac{667846415}{1944}\zeta_5N_f
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{251804567}{432}N_f - \frac{4726621}{243}\zeta_3 N_f^2 - \frac{596849}{54}\zeta_3^2 N_f - \frac{80606}{81}N_f^3 \\
& -\frac{6811}{2}\zeta_7 N_f^2 - \frac{3520}{27}\zeta_5 N_f^3 - \frac{128}{9}\zeta_3^2 N_f^2 + \frac{160}{27}N_f^4 + \frac{24064}{81}\zeta_3 N_f^3 \\
& + \frac{1063237}{27}N_f^2 + \frac{3085750}{243}\zeta_5 N_f^2 + \frac{4429579}{36}\zeta_7 N_f + \frac{1748707267}{3888}\zeta_3 N_f \Big] a^5 + O(a^6).
\end{aligned} \tag{B2}
$$

The quark mass dimension is

$$
\begin{aligned}
\gamma^{\widetilde{SU(3)}}_{m,\mathrm{MOM}_{ccg0c}}(a,0) = & -4a + \left[-\frac{209}{3} + \frac{4}{3}N_f\right]a^2 \\
& + \left[-\frac{95383}{36} - \frac{176}{9}\zeta_3 N_f - \frac{8}{3}N_f^2 + \frac{4742}{27}N_f + \frac{5635}{6}\zeta_3\right]a^3 \\
& + \Big[-\frac{182707879}{1296} - \frac{309295}{48}\zeta_5 - \frac{159817}{27}\zeta_3 N_f - \frac{13651}{27}N_f^2 \\
& - \frac{3200}{9}\zeta_5 N_f + \frac{8}{3}N_f^3 + \frac{1552}{9}\zeta_3 N_f^2 + \frac{5246557}{324}N_f + \frac{15752321}{216}\zeta_3\Big]a^4 \\
& + \Big[-\frac{75504232175}{7776} + \frac{3576071485}{27648}\zeta_7 + \frac{9610932889}{5832}N_f \\
& + \frac{17917034005}{31104}\zeta_5 + \frac{187324052147}{31104}\zeta_3 - \frac{310328447}{432}\zeta_3^2 \\
& - \frac{257106335}{324}\zeta_3 N_f - \frac{180251015}{1944}\zeta_5 N_f - \frac{22459484}{243}N_f^2 \\
& - \frac{4778536}{81}\zeta_7 N_f - \frac{60928}{81}\zeta_3^2 N_f^2 - \frac{28096}{81}\zeta_3 N_f^3 - \frac{1600}{9}\zeta_5 N_f^3 \\
& - \frac{352}{27}N_f^4 + \frac{1372}{3}\zeta_7 N_f^2 + \frac{464038}{243}N_f^3 + \frac{948548}{27}\zeta_3 N_f^2 \\
& + \frac{1850845}{243}\zeta_5 N_f^2 + \frac{6570181}{162}\zeta_3^2 N_f\Big]a^5 + O(a^6).
\end{aligned} \tag{B3}
$$

It is straightforward to verify that these expressions tally with those in [43,63,64].

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