Flipped $SU(6)$ unification of the sequential $SU(3)_c \times SU(3)_L \times U(1)_X$ model

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We propose to partially unify the sequential $SU(3)_c \times SU(3)_L \times U(1)_X$ model (with $\beta = 1/\sqrt{3}$) into the flipped $SU(6)$ model with the gauge group $SU(6) \times U(1)_K$. Gauge anomaly cancellation can easily be satisfied. We discuss the relevant Higgs sector, the low energy *331* model spectrum, and the unification of $SU(3)_c$ and $SU(3)_L$ gauge couplings. Neutrino mass generation and successful gauge coupling unification can set lower/upper bounds on the *331* breaking scale. The partial proton decay lifetime of various channels, for example, the $p \to e^+\pi^0$ channel, in flipped $SU(6)$ GUT are discussed. We find that certain parameter region with $M_{331} \sim 10^{15}$ GeV of case II (for case with M_{331} scale $\tilde{\mathbf{H}}_{\mathbf{3,8}}$ Higgs field) can predict a partial proton lifetime of order 10^{34} years for $p \to e^+\pi^0$ mode, which can be tested soon by future DUNE and Hyper-Kamiokande experiments.

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I. INTRODUCTION

The standard model (SM) of particle physics, based on the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge group, has been extremely successful in describing phenomena below the weak scale. However, the SM still leaves some theoretical and aesthetical questions unanswered, for example, the origin of charge quantization, the values of the low energy parameters and the origin of the flavor structures. Such questions can be answered in the framework of grand unified theory (GUT), such as $SU(5)$ [1] and $SO(10)$ [2] GUT. In the GUT framework, the matter fields of SM can be embedded into certain representations of the GUT group, indicating that the low energy Yukawa couplings can be obtained from a single Yukawa coupling (or few Yukawa couplings) at the GUT scale. The approximate unification of the SM couplings strongly indicate the existence of GUT. We know that the $SU(5)$ GUT model unifies the SM gauge group directly at the GUT scale without any intermediate partial unification step. If intermediate partial unification exists at a higher scale beyond M_Z, for example, a Pati-Salam $SU(4)_c \times SU(2)_L \times SU(2)_R$ partial unification step, genuine gauge coupling unification needs a larger GUT group, such as $SO(10)$ in this case. So, it is interesting to seek other unification model with some intermediate partial unification steps, such as the (partial) GUT model with an intermediate $SU(3)_c \times SU(3)_L \times U(1)_X$ partial unification [3–7] step.

The measured value of the electroweak mixing angle $\sin^2\theta_W(M_Z) = 0.23 \lesssim 0.25$ appears to obey an $SU(3)$ symmetry in such a way that $\sin^2\theta_W(\mu) = 1/4$ at some new fundamental energy scale μ upon TeV [8]. By introducing an extra $U(1)$ factor to accommodate quark sector, one can arrive at an $SU(3)_c \times SU(3)_L \times U(1)_X$ model (*331* model). Depending on different choices of the β value ($\beta = 1/\sqrt{3}$ [9,10] or $\beta = \sqrt{3}$ [9,11,12]) within the embedding of the electric charge, *331* models in the literatures need to introduce different electrically charged particles for the fitting of the $SU(3)_L$ representations. It is remarkable that the existence of three matter generations could be the consequence of gauge anomaly cancellation requirements. Besides, the heaviness of the top quark mass and the emergence of the Peccei-Quinn (PQ) symmetry can also possibly be explained in the *331* framework [13,14].

To understand the origin of charge quantization and the values of the low energy parameters, the intermediate *331* model needs to be unified into a true GUT theory. The unification of sequential *331* model into $SU(6)$ model had been proposed in [15,16] and studied in [17–19]. On the other hand, the genuine unification of *331* into $SU(6)$ needs the introduction of additional adjoint fermions and scalars etc at some intermediate scale between the 331 scale [at $\mathcal{O}$ (TeV)] and GUT scale [15], reducing the predicability of the GUT theory. So, it is interesting to seek alternative (partial) unification steps for the *331* models.

We propose to partially unify the *331* gauge group into flipped $SU(6)$ with the gauge group $SU(6) \times U(1)_K$. Similar to the flipped $SU(5)$ model [20–22], the flipped $SU(6)$ model can be well motivated from string theory

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models [23–25], which uses level-one Kac-Moody algebras and do not need adjoint Higgs fields for symmetry breaking [26]. We know that the flipped $SU(5)$ can adopt the economical missing-partner mechanism and possibly provide an unified cosmological scenario for inflation, dark matter and baryogenesis etc [27–29]. We anticipate that such virtues can also be present for flipped $SU(6)$. Flipped $SU(6)$ model can also be unified into SO(12) or E_6 GUT via intermediate $SU(6)\times SU(2)$ step.

This paper is organized as follows. In Sec. II, we briefly review the sequential *331* model and discuss the embedding of such *331* model into flipped $SU(6)$ GUT model. In Sec. III, we discuss various sub-scenarios of *331* model and the corresponding gauge coupling unification. In Sec. IV, we discuss the triggered proton decay modes and lifetimes in flipped $SU(6)$. Section V contains our conclusions.

II. $SU(3)_c\times SU(3)_L\times U(1)_X$ UNIFICATION INTO $SU(6)\times U(1)_K$ MODEL

A. Brief review of the *331* model with $\beta=1/\sqrt{3}$

Ordinary (nonsequential) *331* model with $\beta=1/\sqrt{3}$ assigns different $SU(3)_c\times SU(3)_L\times U(1)_X$ quantum numbers for the three generations. The filling of the matter fields is given as

$$Q_{iL}(\mathbf{3},\mathbf{3},\mathbf{0})\sim\begin{pmatrix}U_L\\ D_L\\ (XD)_L\end{pmatrix},\quad Q_{3L}\left(\mathbf{3},\bar{\mathbf{3}},\frac{\mathbf{1}}{\mathbf{3}}\right)\sim\begin{pmatrix}b_L\\ t_L\\ (XT)_L\end{pmatrix},$$
$$F_{aL}\left(\mathbf{1},\bar{\mathbf{3}},-\frac{\mathbf{1}}{\mathbf{3}}\right)\sim\begin{pmatrix}E_L\\ -\nu_L\\ N_L^s\end{pmatrix},\qquad E^c_{aL}\sim(\mathbf{1},\mathbf{1},\mathbf{1}),\tag{2.1}$$

and

$$U^c_{iL}\sim\left(\bar{\mathbf{3}},\mathbf{1},-\frac{\mathbf{2}}{\mathbf{3}}\right),\quad D^c_{iL}\sim\left(\bar{\mathbf{3}},\mathbf{1},\frac{\mathbf{1}}{\mathbf{3}}\right),\quad (XD)^c_{iL}\sim\left(\bar{\mathbf{3}},\mathbf{1},\frac{\mathbf{1}}{\mathbf{3}}\right),$$
$$t^c_{iL}\sim\left(\bar{\mathbf{3}},\mathbf{1},-\frac{\mathbf{2}}{\mathbf{3}}\right),\quad b^c_{iL}\sim\left(\bar{\mathbf{3}},\mathbf{1},\frac{\mathbf{1}}{\mathbf{3}}\right),\quad (XT)^c_{L}\sim\left(\bar{\mathbf{3}},\mathbf{1},-\frac{\mathbf{2}}{\mathbf{3}}\right),\tag{2.2}$$

with $a=1,2,3$ and $i=1,2$ the family indices. Here $(XD)_{iL},(XD)^c_{iL}$ and $(XT),(XT)^c_L$ denote some exotic vectorlike quarks with the SM quantum numbers

$$(XD)_{iL}:\ \left(\mathbf{3},\mathbf{1},-\frac{\mathbf{1}}{\mathbf{3}}\right),\qquad (XT)_L:\ \left(\mathbf{3},\mathbf{1},\frac{\mathbf{2}}{\mathbf{3}}\right),$$
$$(XD)^c_{iL}:\ \left(\bar{\mathbf{3}},\mathbf{1},\frac{\mathbf{1}}{\mathbf{3}}\right),\qquad (XT)^c_L:\ \left(\bar{\mathbf{3}},\mathbf{1},-\frac{\mathbf{2}}{\mathbf{3}}\right).\tag{2.3}$$

We adopt the notation $N^c_L\equiv(N^c)_L\equiv(N_R)^c$, where $\psi^c=C\bar{\psi}^T$ and C the charge conjugate matrix. The relation between the hypercharge and the $SU(3)_L\times U(1)_X$ generators is given by

$$Y=\frac{1}{\sqrt{3}}T_8+X,\tag{2.4}$$

with the choice of T_8 for fundamental representation $\mathbf{3}$ of $SU(3)_L$ as

$$T_8=\frac{1}{2\sqrt{3}}\mathrm{diag}(-1,-1,2).\tag{2.5}$$

It can be checked that the gauge anomaly can cancel only if we take into account the contributions from all the generations.

Ordinary *331* model contains a simple lepton sector and can be potentially tested in the TeV scale. There is an interesting variant *331* model called sequential *331* model [15], which, unlike ordinary *331* models, assigns identically the matter quantum numbers for the three generations. Therefore, the gauge anomalies are canceled for each generation separately. The filling of the matter fields in sequential $SU(3)_c\times SU(3)_L\times U(1)_X$ model with $\beta=1/\sqrt{3}$ is given by

$$F_L\left(\mathbf{1},\bar{\mathbf{3}},-\frac{\mathbf{1}}{\mathbf{3}}\right)\sim\begin{pmatrix}E_L\\ -\nu_L\\ N_L^s\end{pmatrix},Q_L(\mathbf{3},\mathbf{3},\mathbf{0})\sim\begin{pmatrix}U_L\\ D_L\\ (XD)_L\end{pmatrix},$$
$$\tilde{X}_L\left(\mathbf{1},\bar{\mathbf{3}},-\frac{\mathbf{1}}{\mathbf{3}}\right)\sim\begin{pmatrix}(XE)_L\\ (XN)_L\\ N_L\end{pmatrix},\tilde{Y}_L\left(\mathbf{1},\bar{\mathbf{3}},\frac{\mathbf{2}}{\mathbf{3}}\right)\sim\begin{pmatrix}(XN)^c_L\\ (XE)_L\\ E^c_L\end{pmatrix},\tag{2.6}$$

and

$$U^c_L\sim\left(\bar{\mathbf{3}},\mathbf{1},-\frac{\mathbf{2}}{\mathbf{3}}\right),\quad D^c_L\sim\left(\bar{\mathbf{3}},\mathbf{1},\frac{\mathbf{1}}{\mathbf{3}}\right),\quad (XD)^c_L\sim\left(\bar{\mathbf{3}},\mathbf{1},\frac{\mathbf{1}}{\mathbf{3}}\right),\tag{2.7}$$

with the SM quantum numbers for some exotic vectorlike quarks and leptons $(XL)_L$ and $(XL)^c_L$ given by

$$(XL)_L:\ \left(\mathbf{1},\bar{\mathbf{2}},-\frac{\mathbf{1}}{\mathbf{2}}\right),\quad (XD)_L:\ \left(\mathbf{3},\mathbf{1},\frac{\mathbf{1}}{\mathbf{3}}\right),$$
$$(XL)^c_L:\ \left(\mathbf{1},\bar{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}\right),\quad (XD)^c_L:\ \left(\bar{\mathbf{3}},\mathbf{1},-\frac{\mathbf{1}}{\mathbf{3}}\right).\tag{2.8}$$

The relation between the hypercharge and the $SU(3)_L\times U(1)_X$ generators is the same as the nonsequential case. For later convenience, we show explicitly the $U(1)_Y$ charges for the three components within $(1,\mathbf{3},Q_X)$ representation of *331* model, which are given as

$$Q_Y[\Psi_{(1,\mathbf{3},Q_X)}] = \left(\frac{1}{6}+Q_X, \frac{1}{6}+Q_X, -\frac{1}{3}+Q_X\right). \tag{2.9}$$

B. The fitting of matter fields into flipped $SU(\mathbf{6})$

We propose to partially unify the *331* gauge group into $SU(6)\times U(1)_K$ gauge group. The normalized $U(1)_P$ generator within $SU(6)$, which is the (remaining) diagonal generator for $SU(6)$ other than the diagonal ones in $SU(3)_c$ and $SU(3)_L$, can be written as

$$T_P = \frac{1}{2\sqrt{3}}(-1,-1,-1,1,1,1). \tag{2.10}$$

We can embed the representations of *331* model with $\beta = 1/\sqrt{3}$ (denoted by their $SU(3)_c\times SU(3)_L\times U(1)_P\times U(1)_K$ quantum numbers) into flipped $SU(6)$ representations for each generation[1]

$$\begin{aligned}
\bar{\mathbf{6}}_{-\frac{1}{2}} &= U_L^c\left(\bar{\mathbf{3}},1,\frac{\mathbf{1}}{\mathbf{2}\sqrt{\mathbf{3}}}\right)_{-\frac{1}{2}} \oplus (L_L,N_L^s)\left(\mathbf{1},\bar{3},-\frac{1}{2\sqrt{3}}\right)_{-\frac{1}{2}},\\
\mathbf{15}_0 &= D_L^c\left(\bar{\mathbf{3}},\mathbf{1},-\frac{\mathbf{1}}{\sqrt{\mathbf{3}}}\right)_{\mathbf{0}} \oplus (Q_L,(XD)_L)(\mathbf{3},3,0)_{\mathbf{0}} \oplus ((XL)_L,N_L^c)\left(\mathbf{1},\bar{3},\frac{1}{\sqrt{3}}\right)_{\mathbf{0}},\\
\bar{\mathbf{6}}_{\frac{1}{2}} &= (XD)_L^c\left(\bar{\mathbf{3}},\mathbf{1},\frac{\mathbf{1}}{\mathbf{2}\sqrt{\mathbf{3}}}\right)_{\frac{1}{2}} \oplus ((XL)_L^c,E_L^c)\left(\mathbf{1},\bar{3},-\frac{1}{2\sqrt{3}}\right)_{\frac{1}{2}}.
\end{aligned} \tag{2.11}$$

The $U(1)_X$ quantum number is related to the corresponding $U(1)_K$ and $U(1)_P$ charges by

$$Q_X = -\frac{\sqrt{3}}{3}Q_P + Q_K, \tag{2.12}$$

after the breaking of $SU(6)\times U(1)_K$ into $SU(3)_c\times SU(3)_L\times U(1)_X$.

It is obvious that the $SU(6)-SU(6)-SU(6)$ gauge anomaly is canceled with two $\bar{\mathbf{6}}$ representation fields and one antisymmetric $\mathbf{15}$ representation field for each generation, as the anomaly coefficients for various $SU(6)$ representation fermions are given by

$$\mathcal{A}(\bar{\mathbf{6}}) = -1, \mathcal{A}(\mathbf{15}) = 2, \tag{2.13}$$

with $\mathrm{Tr}(\{T^a(R),T^b(R)\}T^c(R)) = A(R)d^{abc}/2$. The $U(1)_K$ related anomalies are canceled because the $U(1)_K$ quantum numbers for fermions within each generation satisfy

$$\begin{aligned}
6\left(\frac{1}{2}-\frac{1}{2}\right) &= 0, \quad \text{for } SU(6)-U(1)_K-U(1)_K,\\
6\left[\left(\frac{1}{2}\right)^3-\left(\frac{1}{2}\right)^3\right] &= 0, \quad \text{for } U(1)_K-U(1)_K-U(1)_K.
\end{aligned} \tag{2.14}$$

To avoid the gravitational violation of gauge symmetry, the anomaly related to gravity should vanish. It can be seen that the graviton-graviton-$U(1)$ anomaly is canceled in our model, because only the Abelian $U(1)_K$ generator is relevant and the $U(1)_K$ charges for the chiral fermions satisfy

$$6\left(\frac{1}{2}-\frac{1}{2}\right) = 0. \tag{2.15}$$

So, we have an anomaly-free fitting of matter fields in flipped $SU(6)$ representations

$$\bar{\mathbf{6}}_{-\frac{1}{2}} \supseteq [U_L^c, L_L], \quad \mathbf{15}_0 \supseteq [Q_L, D_L^c, N_L^c], \quad \bar{\mathbf{6}}_{\frac{1}{2}} \supseteq [(XD)_L^c, E_L^c], \tag{2.16}$$

with $E_L^C \in (1,\bar{3},\frac{2}{3})$ of $SU(3)_c\times SU(3)_L\times U(1)_X$ and $N_L^c \in (1,\bar{3},-\frac{1}{3})$.[2] We can see that the fillings of $\bar{\mathbf{6}}_{-\frac{1}{2}}$ (containing U_L^c and L_L), $\mathbf{15}_0$ (containing Q_L, D_L^c, N_L^c) and $\bar{\mathbf{6}}_{\frac{1}{2}}$ (containing E_L^c) are similar to that in flipped $SU(5)$.

The fact that the gauge anomaly cancelation in flipped $SU(6)$ holds for each generation is the reminiscent of the gauge anomaly cancelation conditions of sequential *331* model. Such generation by generation gauge anomaly

[1]The case for the partial unification of *331* (with $\beta = \sqrt{3}$) into flipped $SU(6)$ is rather tricky, especially the relevant anomaly cancellation conditions. We will discuss it in our subsequent study. It seems not possible for such a case to unify in ordinary $SU(6)$.

[2]We note that the fitting of $(XD)_L^c$ and D_L^C can be exchanged [also (L_L, N_L^s) and $((XL)_L, N_L^c)$]. To ensure the VEV of $\mathbf{15}_{0;H}$ is small, such a fitting

$$\bar{\mathbf{6}}_{-\frac{1}{2}} \supseteq [U_L^c, L_L], \quad \mathbf{15}_0 \supseteq [Q_L, (XD)_L^c, N_L^c], \quad \bar{\mathbf{6}}_{\frac{1}{2}} \supseteq [D_L^c, E_L^c], \tag{2.17}$$

with $E_L^c \in (1,\bar{3},\frac{2}{3})$ of $SU(3)_c\times SU(3)_L\times U(1)_X$ and $N_L^c \in (1,\bar{3},-\frac{1}{3})$ is not adopted here. We will discuss such alternative choices in our subsequent studies.

cancelation conditions will in general not hold for the fitting of nonsequential *331* model into flipped $SU(6)$ model.

We should briefly comment on the anomaly cancelation conditions in nonflipped versus flipped $SU(6)$ (partial) unification of (non)sequential *331* models. In the ordinary $SU(6)$ unification of the sequential *331* model, each generation will still be fitted into $\mathbf{15}\oplus\bar{\mathbf{6}}\oplus\bar{\mathbf{6}}$ representations so as that the gauge anomaly cancelation conditions are satisfied for each generation. In the ordinary $SU(6)$ unification of the nonsequential *331* model, the RH-charged leptons $E^c_{aL}\sim(\mathbf{1},\mathbf{1},\mathbf{1})$ need to be fitted into **20** representation of $SU(6)$. The quarks (including the exotic vectorlike quarks) and LH leptons can be fitted into $\mathbf{15}\oplus\bar{\mathbf{6}}\oplus\bar{\mathbf{6}}$ for the first two generations, while those of the third generation need to be fitted into $\mathbf{20}\oplus\mathbf{15}\oplus\mathbf{15}\oplus\bar{\mathbf{6}}$. We can see that the gauge anomaly cancelation conditions for the first two generations in ordinary $SU(6)$ unification of nonsequential *331* model are satisfied for each generation while the third generation needs additional exotic fermions in $\bar{\mathbf{6}}$ representations to cancel anomaly. Therefore, the anomaly cancelation for nonflipped $SU(6)$ unification of both sequential and nonsequential *331* model always hold generation by generation, even though such conditions for the low energy nonsequential *331* model are satisfied nontrivially unless contributions from all the three generations are included.

In the flipped $SU(6)$ partial unification of nonsequential *331* model, which will discussed in detail in our future work [30], the RH-charged leptons $E^c_{aL}\sim(\mathbf{1},\mathbf{1},\mathbf{1})$ can still be kept as $SU(6)$ singlet. The gauge anomaly for the first two generations, which are given by $15_0\oplus\bar{6}_{1/2}\oplus\bar{6}_{-1/2}\oplus 1_1$ will no-longer cancel unless we include the third generation. We need to introduce only one **20** representation of $SU(6)$ for the third generation, which needs fairly small additional exotic matter fields in contrast to ordinary nonflipped $SU(6)$ unification of nonsequential *331* model.

C. The Higgs sector

The Higgs fields introduced in our model are responsible for the breaking of $SU(6)\times U(1)_K$ gauge group, the breaking of the $SU(3)_c\times SU(3)_L\times U(1)_X$ gauge group and the mass generation for the SM quarks and leptons, the exotic vectorlike fermions and the sterile neutrinos. The total Higgs sector in our flipped $SU(6)$ model contains the following Higgs fields

$$\mathbf{20}_{\frac{1}{2};H},\ \bar{\mathbf{6}}_{\frac{1}{2};H},\ \bar{\mathbf{6}}_{-\frac{1}{2};H},\ \mathbf{15}_{0;H},\ (\overline{\mathbf{105}})^s_{0;H},\ \mathbf{21}_{1;H}, \tag{2.18}$$

with the gauge symmetry broken by the corresponding VEVs

$$\begin{aligned}SU(6)\times U(1)_K &\xrightarrow{\mathbf{20}_{\frac{1}{2};H}} SU(3)_c\times SU(3)_L\times U(1)_X\\ &\xrightarrow{\bar{\mathbf{6}}_{-\frac{1}{2};H},(\overline{\mathbf{105}})^s_{0;H},\mathbf{21}_{1;H}} SU(3)_c\times SU(2)_L\times U(1)_Y\\ &\xrightarrow{\bar{\mathbf{6}}_{\frac{1}{2};H},\mathbf{15}_{0;H},\mathbf{21}_{1;H}} SU(3)_c\times U(1)_Y.\end{aligned} \tag{2.19}$$

To break the flipped $SU(6)$ into $SU(3)_c\times SU(3)_L\times U(1)_X$, we introduce a $\mathbf{20}_{\frac{1}{2}}$ representation Higgs with its decomposition in terms of $SU(3)_c\times SU(3)_L\times U(1)_P\times U(1)_K$ quantum numbers

$$\begin{aligned}\mathbf{20}_{\frac{1}{2};H}=&\left(1,1,-\frac{3}{2\sqrt{3}}\right)_{\frac{1}{2};H}\oplus\left(1,1,\frac{3}{2\sqrt{3}}\right)_{\frac{1}{2};H}\\ &\oplus\left(3,\bar{3},-\frac{1}{2\sqrt{3}}\right)_{\frac{1}{2};H}\oplus\left(\bar{3},3,\frac{1}{2\sqrt{3}}\right)_{\frac{1}{2};H},\end{aligned} \tag{2.20}$$

with

$$Q_X:\left(1,1,\frac{3}{2\sqrt{3}}\right)_{\frac{1}{2};H}=0,\quad Q_X:\left(1,1,-\frac{3}{2\sqrt{3}}\right)_{\frac{1}{2};H}=-1. \tag{2.21}$$

The $(1,1,\frac{3}{2\sqrt{3}})_{\frac{1}{2};H}$ component of $\mathbf{20}_{\frac{1}{2};H}$ can acquire a vacuum expectation value (VEV) $\langle\mathbf{20}_{\frac{1}{2};\mathbf{H}}\rangle=M_X$ to break the flipped $SU(6)$ into $SU(3)_c\times SU(3)_L\times U(1)_X$.

To break the residue $SU(3)_c\times SU(3)_L\times U(1)_X$ gauge symmetry into SM and generate masses for the SM matter fields, we need to introduce additional Higgs fields $\bar{\mathbf{6}}_{\frac{1}{2};H}$, $\mathbf{15}_{0;H}$ and $\bar{\mathbf{6}}_{-\frac{1}{2};H}$ with their decompositions in terms of $SU(3)_c\times SU(3)_L\times U(1)_P\times U(1)_K$ quantum numbers

$$\begin{aligned}\bar{\mathbf{6}}_{\frac{1}{2};H}&=\left(\bar{3},1,\frac{1}{2\sqrt{3}}\right)_{\frac{1}{2};H}\oplus\left(1,\bar{3},-\frac{1}{2\sqrt{3}}\right)_{\frac{1}{2};H},\\ \bar{\mathbf{6}}_{-\frac{1}{2};H}&=\left(\bar{3},1,\frac{1}{2\sqrt{3}}\right)_{-\frac{1}{2};H}\oplus\left(1,\bar{3},-\frac{1}{2\sqrt{3}}\right)_{-\frac{1}{2};H},\\ \mathbf{15}_{0;H}&=\left(\bar{3},1,-\frac{1}{\sqrt{3}}\right)_{0;H}\oplus(3,3,0)_{0;H}\oplus\left(1,\bar{3},\frac{1}{\sqrt{3}}\right)_{0;H}.\end{aligned} \tag{2.22}$$

The Yukawa couplings for the fermions can be written as

$$\begin{aligned}\mathcal{L}\supseteq&-\sum_{a,b=1}^{3}Y_{U;ab}\bar{\mathbf{6}}^a_{-\frac{1}{2}}\mathbf{15}^b_0\bar{\mathbf{6}}_{\frac{1}{2};H}-\sum_{a,b=1}^{3}Y_{E;ab}\bar{\mathbf{6}}^a_{-\frac{1}{2}}\bar{\mathbf{6}}^b_{\frac{1}{2}}\mathbf{15}_{0,H}\\ &-\sum_{a,b=1}^{3}Y_{D,N;ab}\mathbf{15}^a_0\mathbf{15}^b_0\mathbf{15}_{0;H}-\sum_{a,b=1}^{3}Y_{XD;ab}\bar{\mathbf{6}}^a_{\frac{1}{2}}\mathbf{15}^b_0\bar{\mathbf{6}}_{-\frac{1}{2};H},\end{aligned} \tag{2.23}$$

with "a, b" the family indices. The low energy Higgs fields in $SU(3)_c \times SU(3)_L \times U(1)_X$ models contain the $(1, \bar{3}, -\frac{1}{2\sqrt{3}})_{\frac{1}{2};H} \in \bar{\mathbf{6}}_{\frac{1}{2};H}$ Higgs field (corresponds to $H_1(1, \bar{3}, \frac{2}{3})$ that contains doublet H_u Higgs field in 2HDM), the $(1, \bar{3}, -\frac{1}{2\sqrt{3}})_{-\frac{1}{2};H} \in \bar{\mathbf{6}}_{-\frac{1}{2};H}$ Higgs field [corresponds to $H_2(1, \bar{3}, -\frac{1}{3})$] and the $(1, \bar{3}, \frac{1}{\sqrt{3}})_{0;H} \in \mathbf{15}_{0;H}$ Higgs field (corresponds to $H_3(1, \bar{3}, -\frac{1}{3})$ that contains doublet H_d Higgs field in 2HDM), which are just needed to generate properly the masses for matter fields. As the $\bar{6}_{-\frac{1}{2};H}$ Higgs is not responsible for the mass generation of SM matter fields, the simplest choice to break $SU(3)_c \times SU(3)_L \times U(1)_X$ to SM is to adopt the VEV for its triplet component $H_2(1, \bar{3}, -\frac{1}{3})$, which can be decomposed in terms of the SM gauge quantum numbers as

$$H_2\left(1, \bar{3}, -\frac{1}{3}\right) = H'\left(1, \bar{2}, \frac{2}{3}\right) \oplus N'_H(1, 1, 0),$$

Such a VEV along the (1, 1, 0) direction (in terms of SM gauge quantum number) can be denoted by $\langle H_2 \rangle = M_{331}$. The VEVs of the relevant Higgs fields can be written as

$$\langle H_1 \rangle = \sqrt{2}\begin{pmatrix} v_u \\ 0 \\ 0 \end{pmatrix}, \qquad \langle H_2 \rangle = \sqrt{2}\begin{pmatrix} 0 \\ 0 \\ M_{331} \end{pmatrix},$$
$$\langle H_3 \rangle = \sqrt{2}\begin{pmatrix} v_d \\ 0 \\ 0 \end{pmatrix}, \tag{2.24}$$

with the VEVs of H_1 and H_3 trigger the breaking of the SM electrweak symmetry into $U(1)_Q$.

It can be seen from the Yukawa couplings in Eq. (2.23) that the $\bar{\mathbf{6}}^a_{\frac{1}{2}}\mathbf{15}^b_0\bar{\mathbf{6}}_{-\frac{1}{2};H}$ term will leads to

$$\left[\left(1, \bar{3}, -\frac{1}{2\sqrt{3}}\right)_{\frac{1}{2}} \otimes \left(1, \bar{3}, \frac{1}{\sqrt{3}}\right)_0 \otimes \left(1, \bar{3}, -\frac{1}{2\sqrt{3}}\right)_{-\frac{1}{2};H}\right] \supseteq [(XL)_L \otimes (XL)^c_L \otimes N'_H(1, 1, 0)],$$
$$\left[(3, 3, 0)_0 \otimes \left(\bar{3}, 1, \frac{1}{2\sqrt{3}}\right)_{\frac{1}{2}} \otimes \left(1, \bar{3}, -\frac{1}{2\sqrt{3}}\right)_{-\frac{1}{2};H}\right] \supseteq [(XD)_L \otimes (XD)^c_L \otimes N'_H(1, 1, 0)],$$

which will generate Dirac mass $Y_{XD}M_{331}$ for vectorlike heavy extra leptons $(XL)_L, (XL)^c_L$ and vectorlike heavy quarks $(XD)_L, (XD)^c_L$.

Experimental measurements for the square of the mass differences [31] for neutrinos indicate that the heaviest neutrino mass should be of order 10^{-2} eV. Such tiny neutrino masses can either be Dirac type or be Majorana type from dim-5 Weinberg operator. We should note that it is not possible to adopt only the Dirac type masses for neutrinos because the Yukawa terms involving $Y_{U;ab}$ generate identical masses for both the up-type quark masses and Dirac-type neutrino masses at the flipped $SU(6)$ breaking scale M_X. Large hierarchy between the up-type quark masses and tiny Dirac type neutrino masses cannot be generated by pure renormalization group equation (RGE) effects, that is, by RGE evolution from M_X to M_Z. Tiny Majorana neutrino masses from dim-5 Weinberg operator can be UV completed to various mechanisms, for example, the type-I seesaw mechanism, which can be used to generate tiny neutrino masses after introducing additional Majorana mass terms for RH-neutrinos N^c_L. Bare Majorana mass terms are not allowed because the RH-neutrinos are fitted into nonsinglet $\mathbf{15}_0$ representations of $SU(6) \times U(1)_K$. So, such Majorana mass terms for RH neutrinos can only be generated by a new term involving certain new Higgs field that couples to RH neutrinos. From the production of $\mathbf{15}_0$ representation

$$\mathbf{15} \otimes \mathbf{15} = \overline{\mathbf{15}} \oplus \mathbf{105}^s \oplus \mathbf{105}^a, \tag{2.25}$$

we can see that the proper choice is $\overline{\mathbf{105}}^{\mathbf{s}}_0$, which is decomposed as

$$(\overline{\mathbf{105}})^s_{0;H} = \left(\mathbf{1}, \mathbf{6}, -\frac{\mathbf{2}}{\sqrt{\mathbf{3}}}\right)_{\mathbf{0}} \oplus \left(\mathbf{6}, \mathbf{1}, \frac{\mathbf{2}}{\sqrt{\mathbf{3}}}\right)_{\mathbf{0}} \oplus \left(\mathbf{8}, \bar{\mathbf{3}}, \frac{\mathbf{1}}{\sqrt{\mathbf{3}}}\right)_{\mathbf{0}}$$
$$\oplus \left(\bar{\mathbf{3}}, \mathbf{8}, -\frac{\mathbf{1}}{\sqrt{\mathbf{3}}}\right)_{\mathbf{0}} \oplus (\mathbf{6}, \mathbf{6}, \mathbf{0})_{\mathbf{0}} \oplus (\mathbf{3}, \mathbf{3}, \mathbf{0})_{\mathbf{0}},$$

in terms of $SU(3)_c \times SU(3)_L \times U(1)_P \times U(1)_K$ quantum numbers. The term responsible for the generation of Majorana neutrino masses can be written as

$$\mathcal{L} \supseteq Y^m_{ab}\mathbf{15}^a_0\mathbf{15}^b_0(\overline{\mathbf{105}})^s_{0;H}$$
$$\supseteq \left(1, \bar{3}, \frac{1}{\sqrt{3}}\right)^a_0 \otimes \left(1, \bar{3}, \frac{1}{\sqrt{3}}\right)^b_0 \otimes \left(1, 6, -\frac{2}{\sqrt{3}}\right)_{0;H}, \tag{2.26}$$

which will generate Majorana masses for RH neutrinos after the $(1, 6, -\frac{2}{\sqrt{3}})_{0;H}$ component of $\overline{\mathbf{105}}_s$ develops a VEV along the (1, 1, 0) direction (in terms of SM quantum numbers). As $\langle \overline{\mathbf{105}}_s \rangle = M_S$ will also break the $SU(3)_c \times SU(3)_L \times U(1)_X$ gauge symmetry, we require that $M_S \lesssim M_{331}$. The neutrino masses can be given by

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & \mathcal{M}^T_{\nu;D} \\ \mathcal{M}_{\nu;D} & Y^m M_S \end{pmatrix}. \tag{2.27}$$

The natural up-type quark masses, which are also the typical Dirac-type neutrino masses, are given by $m_U \simeq Y^U v_u \simeq \mathcal{M}_{\nu;D} \sim \mathcal{O}(10^2)$ GeV. To obtain tiny neutrino masses of order 10^{-2} eV with the seesaw mechanism

$$M_\nu \simeq \frac{M_{\nu;D} M^T_{\nu;D}}{Y^m M_S} \sim 5 \times 10^{-2}\ \text{eV}, \tag{2.28}$$

the 331 breaking scale M_{331} is constrained to lie naturally at about 10^{14} GeV for $Y^m \sim \mathcal{O}(1)$, otherwise the generated neutrino masses should be much larger than 10^{-2} eV. The bounds on $M_{331} \sim 10^{14}$ GeV from neutrino masses can be relaxed to $M_{331} \gtrsim 10^{14}$ GeV if the coupling Y^m can be much smaller than identity. We should note that constraints on the scale of M_{331} from neutrino masses can be relaxed if a mixed type I + II seesaw mechanism is used for neutrino mass generations, within which a small VEV for an additional $\mathbf{21}_{1;H}$ representation Higgs field along the $SU(2)_L$ triplet direction is needed. We will discuss such a possibility shortly after. On the other hand, it will be clear soon that successful gauge coupling unification for g_{3c} and g_{3L} requires the M_{331} scale to be higher than 10^{16} GeV. Such a bound can be relaxed unless certain additional colored Higgs field lies of order M_{331} scale. Given the neutrino masses, the Yukawa coupling involved can be defined in terms of the physical neutrino parameters, up to an orthogonal complex matrix R [32],

$$Y^U \approx \sqrt{2}\frac{i}{v_u}\sqrt{\hat{M}_R} R \sqrt{\hat{m}_\nu} V^\dagger_{\text{PMNS}}, \tag{2.29}$$

where $\hat{m}_\nu$, $\hat{M}_R$ being the diagonal matrices for the light and heavy neutrino masses, and V_{PMNS} being the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) lepton mixing matrix. In our case, the neutrino hierarchical spectrum can either be normally ordered (NO) or inversely ordered (IO), depending on the Yukawa parameters introduced in the theory.

If we adopt nonrenormalizable Weinberg operator

$$\Delta\mathcal{L}_{\text{Weinberg}} = \frac{y^\nu_{ab}}{M}(\bar{\mathbf{6}}_{\mathbf{-1/2;a}}\bar{\mathbf{6}}_{\mathbf{-1/2;b}})(\bar{\mathbf{6}}_{\mathbf{-1/2,H}}\bar{\mathbf{6}}_{\mathbf{-1/2,H}})^\dagger, \tag{2.30}$$

to generate tiny neutrino mass for flipped $SU(6)$ without specifying its concrete UV completion model, the previous lower bound on M_{331} from neutrino mass generation can be relaxed. Here M denotes the scale of the heavy modes, which are integrated out and responsible for the generation of Weinberg operator. As the nonrenormalizability of the Weinberg operator requires M to be larger than the flipped $SU(6)$ breaking scale M_X, we thus obtain an upper bound for M_X with $M_X < 10^{14}$ GeV in this case. To be consistent, we need to ensure that such a constraint is satisfied for the choice of M_{331} scale and the Higgs contents of the low energy *331* model. We leave the numerical discussions of this possibility in our future work.

The new sterile neutrino component N^s_L within $\bar{\mathbf{6}}_{-\frac{1}{2}}$ can also obtain masses after EWSB, which couples to the $(XL)_L$ and $(XL)^c_L$ components via the Yukawa coupling terms involving Y_U and Y_E

$$\begin{aligned}
&\left[\left(1,\bar{3},-\frac{1}{2\sqrt{3}}\right)_{-\frac{1}{2}} \otimes \left(1,\bar{3},\frac{1}{\sqrt{3}}\right)_0 \otimes \left(1,\bar{3},-\frac{1}{2\sqrt{3}}\right)_{\frac{1}{2};H}\right] \supseteq [N^S_L \otimes (XL)_L \otimes H_u], \\
&\left[\left(1,\bar{3},-\frac{1}{2\sqrt{3}}\right)_{-\frac{1}{2}} \otimes \left(1,\bar{3},-\frac{1}{2\sqrt{3}}\right)_{\frac{1}{2}} \otimes \left(1,\bar{3},\frac{1}{\sqrt{3}}\right)_{0;H}\right] \supseteq [N^S_L \otimes (XL)^c_L \otimes H_d].
\end{aligned} \tag{2.31}$$

So the mass matrix for the new sterile neutrinos can be given by

$$M'_S \equiv \begin{pmatrix} 0 & \mathcal{M}^T_U & \mathcal{M}^T_E \\ \mathcal{M}_U & 0 & Y_{XD}M_{331} \\ \mathcal{M}_E & Y_{XD}M_{331} & 0 \end{pmatrix}, \tag{2.32}$$

in the basis of $N^s_L, N_{(XL)}, N_{(XL)^c}$. Here $N_{(XL)}, N_{(XL)^c}$ denote the neutral components within $(XL)_L$ and $(XL)^c_L$, respectively. The M_U, M_E scales lie typically at the up-type quark mass scales $\mathcal{O}(10^2)$ GeV and charged lepton mass scales $\mathcal{O}(1)$ GeV, respectively. After diagonalizing the mass matrix, we can obtain that the mass scale for the lightest new sterile neutrino is

$$m_S \sim \frac{\mathcal{M}_U \mathcal{M}_E}{Y_{XD} M_{331}} \sim 10^{-3}\ \text{eV}, \tag{2.33}$$

for $Y_{XD} \sim \mathcal{O}(1)$, which can contribute to additional light effective degrees of freedom Δg_* at the BBN era and cause cosmological difficulties. Therefore, we should try to push heavy such new sterile neutrinos, for example, by choosing unnaturally small Y_{XD}. An interesting solution to such a problem without unnatural parameters is to introduce additional Majorana type masses for N^s_L. We can introduce new $\mathbf{21}_{1;H}$ representation Higgs field, which has the following decomposition

$$\mathbf{21}_{1;H} = \left(6,1,-\frac{1}{\sqrt{3}}\right)_{1;H} \oplus (3,3,0)_{1;H} \oplus \left(1,6,\frac{1}{\sqrt{3}}\right)_{1;H}, \tag{2.34}$$

in terms of $SU(3)_c \times SU(3)_L \times U(1)_P \times U(1)_K$ quantum numbers and the relevant Yukawa coupling is

$$\mathcal{L} \supseteq -y_{S;ab}\bar{\mathbf{6}}^a_{-\frac{1}{2}}\bar{\mathbf{6}}^b_{-\frac{1}{2}}\mathbf{21}_{1;H}. \tag{2.35}$$

When the $(1,6,\frac{1}{\sqrt{3}})_{1;H}$ component of $\mathbf{21}_{1;H}$ develops a VEV with $\langle\mathbf{21}_{1;H}\rangle = M_{S'} \sim M_{331}$ along the (1, 1, 0) direction (in terms of SM quantum number), Majorana mass term can be generated for N_L^S. With new contribution $(M'_S)_{11} \sim M_{331}$ in Eq. (2.32), the eigenvalues of M'_S all lie at the M_{331} scale and will not cause cosmological difficulties. On the other hand, if $M_{331} \gg \langle\mathbf{21}_{1;H}\rangle \gtrsim 0.1$ keV (or choosing $Y_{XD} \lesssim 10^{-6}$ for the first solution), the lightest sterile neutrinos with masses of order $\langle\mathbf{21}_{1;H}\rangle$ can act as a fermionic dark matter candidate, which also satisfy the Tremaine-Gunn (TG) bound [33].

Besides, if the (1, 3, 1) direction (in terms of SM quantum number) of $(1,6,\frac{1}{\sqrt{3}})_{1;H}$ component within $\mathbf{21}_{1;H}$ Higgs field also develops a small triplet VEV (which also breaks the SM electroweak gauge symmetry), ordinary LH neutrinos of SM can also acquire Majorana masses so as that a mixed type I + II seesaw mechanism can be applied to the nonsterile neutrino sector. With a small nonvanishing $(\mathcal{M}_\nu)_{11}$ component for the mass matrix (2.27), the 331 breaking scale M_{331} can be much lower than 10^{14} GeV with large fine-tuning among the type-I and type-II seesaw contributions for the neutrino masses. For example, the choice of $M_{331} \sim 10^3$ GeV requires $\mathcal{O}(10^{-11})$ fine tuning for both contributions to get tiny neutrino masses of order 10^{-2} eV. However, our numerical results indicate that successful gauge coupling unification for g_{3c} and g_{3L} still requires large M_{331} scale (larger than 10^{12} GeV) in the case with small $SU(2)_L$ triplet VEV, unless we keep some additional Higgs field as light as M_{331} scale, for example, keeping the $(\bar{\mathbf{3}},\mathbf{8},-\frac{1}{\sqrt{3}})_\mathbf{0}$ Higgs field within $\overline{105}^s$ to lie at M_{331} scale.

It is worth to note that the Higgs field in **6**, **15**, **20** representation of $SU(6)$ can be generated at the Kac-Moody level one [34]. The large representation $\overline{\mathbf{105}}^s$ and **21** Higgs fields from higher Kac-Moody level can in fact be replaced by double **6** or $\overline{\mathbf{15}}$ Higgs fields, similar to that appeared in the nonrenormalizable Weinberg operators.

III. GAUGE COUPLING UNIFICATION

The GUT symmetry breaking chain is given by

$$\begin{aligned} SO(12)/E_6 &\xrightarrow{M_G} SU(6)\times U(1)_K \xrightarrow{M_X} SU(3)_c\times SU(3)_L\times U(1)_X \\ &\xrightarrow{M_{331}} SU(3)_c\times SU(2)_L\times U(1)_Y \xrightarrow{M_Z} SU(3)_c\times U(1)_Y, \end{aligned} \tag{3.1}$$

with the partial unification $SU(6)\times U(1)_K$ gauge group can be further unified into a simple $SO(12)$ group (or E_6 group via intermediate $SU(2)\times SU(6)$ step). The relations among the $U(1)$ generators

$$Q_X = -\frac{\sqrt{3}}{3}Q_P + Q_K, \qquad Q_Y = \frac{1}{\sqrt{3}}T_8 + Q_X, \tag{3.2}$$

lead to the relations for the relevant gauge couplings

$$\frac{1}{g_X^2} = \frac{1}{3}\frac{1}{g_P^2} + \frac{1}{g_K^2}, \qquad \frac{1}{g_Y^2} = \frac{1}{3}\frac{1}{g_{3L}^2} + \frac{1}{g_X^2}, \tag{3.3}$$

holding at the $SU(6)\times U(1)_K$ breaking scale M_X (for $1/g_X^2$) and the M_{331} scale (for $1/g_Y^2$), respectively. If we fit the flipped $SU(6)$ gauge couplings within $SO(12)$, the coupling g_K should be normalized into canonical $g_{\tilde{K}}$ with $g_K^2 = g_{\tilde{K}}^2/3$. It should be noted that the charge quantization can only be explained in the framework of $SO(12)$ or E_6 GUT instead of our partial unification scheme. The $U(1)_K$ charges in our intermediate partial unification $SU(6)\times U(1)_K$ model are still not quantized, which are constrained only by gauge anomaly cancellation conditions.

The one-loop beta function for the couplings are given by

$$\frac{d}{dt}g_i = \frac{1}{16\pi^2}b_i^0 g_i^3, \tag{3.4}$$

with

$$b_0^i = -\frac{11}{3}C(G_i) + \frac{2}{3}\sum_{\text{fermion}} T(r_f^i) + \frac{1}{3}\sum_{\text{scalar}} T(r_s^i), \tag{3.5}$$

for Weyl fermions in $\mathbf{r}_f^i$ representation and complex scalars in $\mathbf{r}_s^i$ representation.

In SUSY $SU(5)$ GUT model, the doublet Higgs field that responsible for electroweak symmetry breaking should be much lighter than the colored triplet Higgs field so as that the dim-5 operator induced proton decay mode suppressed by the triplet Higgs mass can still be consistent with current proton decay bounds. There are many proposals to deal with such doublet-triplet (D-T) splitting problem, such as the missing partner mechanism [35], complicated version of sliding singlet mechanism in $SU(6)$ extension [36], missing VEV in SO(10) [37], pseudo Nambu-Goldstone bosons [38] etc. In missing partner mechanism, the color-triplet Higgs fields can be coupled to other colored fields and acquire large masses, whereas the doublet Higgs fields lack such partners so as that they can still be light. Missing partner mechanism can be elegantly realized in flipped $SU(5)$ GUT model, which does not require adjoint or larger Higgs representations and can be seen as a virtue of flipped $SU(5)$. We should note that the missing partner mechanism can also be realized in our flipped $SU(6)$ model (see the discussions in the Appendix A 1).

In our following discussions, we assume that the splitting among the colored/uncolored Higgs fields can be successfully realized and we do not specify the origins of such splitting, for example, by missing partner mechanism or by orbifold projection (see Appendix A 2). Therefore, rendering the colored Higgs fields to lie near the $SU(6)$ breaking scale, the Higgs sector for the low energy *331* model contains

$$H_1\left(1,\bar{3},\frac{2}{3}\right) \ni H_u, \qquad H_2\left(1,\bar{3},-\frac{1}{3}\right) \ni H_d,$$
$$H_3\left(1,\bar{3},-\frac{1}{3}\right), \qquad H_S^i\left(1,6,\frac{2}{3}\right) \quad (i=1,2). \quad (3.6)$$

We have the following coefficients for *331* model

$$(b_3^c, b_3^L, b_1^X) = \left(-5,-\frac{17}{6},\frac{94}{9}\right). \quad (3.7)$$

Upon the M_X scale, where $\alpha_{3c}(M_X)=\alpha_{3L}(M_X)=\alpha_6(M_X)\equiv\alpha_X$, the flipped $SU(6)$ fields include the matter fields for three generations $\bar{\mathbf{6}}_{-\frac{1}{2};i}$,$\bar{\mathbf{6}}_{\frac{1}{2};i}$,$\mathbf{15}_{0;i}$ and the Higgs fields $\mathbf{20}_{-\frac{1}{2};H}$,$\mathbf{21}_{1;H}$,$\overline{\mathbf{105}}_{0;H}$,$\bar{\mathbf{6}}_{\frac{1}{2};H}$, $\mathbf{15}_{0,H}$ and $\bar{\mathbf{6}}_{-\frac{1}{2};H}$. The beta functions for $SU(6)$ and $U(1)_K$ are given by

$$(b_6, b_K) = \left(-\frac{38}{3},\frac{47}{3}\right), \quad (3.8)$$

which, after normalization into $SO(12)$, are given by

$$(b_6, b_{K'}) = \left(-\frac{38}{3},\frac{47}{9}\right). \quad (3.9)$$

After the breaking of *331* gauge group into $SU(3)_c\times SU(2)_L\times U(1)_Y$ at about the M_{331} scale, the theory reduce to two Higgs doublet model (requiring $M_{331}\gtrsim 10^{14}$ GeV) or two Higgs doublet model plus an $SU(2)_L$ triplet (with relaxed M_{331} scale, for example, at TeV scale, although large fine tuning is needed).

The relevant beta functions are given by

(i) Case I: two Higgs doublet model (2HDM)

$$(b_3, b_2, b_Y) = (-7,-3,7). \quad (3.10)$$

(ii) Case II: 2HDM plus an $SU(2)_L$ triplet

$$(b_3, b_2, b_Y) = \left(-7,-\frac{7}{3},8\right). \quad (3.11)$$

Note that we do not have the $\frac{3}{5}$ factor for b_Y because we do not normalize the g_Y couplings within $SU(5)$. We adopt the following inputs at M_Z scale [39]

$$\alpha_{em}^{-1}=127.951\pm0.009, \quad \sin^2\theta_W=0.23129\pm0.00033,$$
$$\alpha_s(M_Z)=0.1185\pm0.0016,$$

to obtain the central values [40]

$$\alpha_2^{-1}(M_Z)=29.594, \qquad \alpha_Y^{-1}(M_Z)=98.357,$$
$$g_3(m_t)=1.3075. \quad (3.12)$$

We can calculate the relevant $SU(6)\times U(1)_K$ breaking scale M_X for various low energy cases, after specifying the matter and Higgs contents of the low energy *331* models and their corresponding low energy models at the electroweak scale (in our case, the 2HDM or 2HDM plus an $SU(2)_L$ triplet). Given an M_{331} scale, the $SU(6)\times U(1)_K$ breaking scale and the corresponding $SU(6)$ gauge coupling can be obtained numerically, using the corresponding beta functions for the gauge couplings given in previous discussions. In our partial unification model, the M_X scale is defined as the intersection scale of the RGE evolution trajectories for $SU(3)_c$ and $SU(3)_L$ gauge couplings. The $U(1)_K$ coupling strength at M_X can be obtained by the combinations of $U(1)_X$ coupling and the coupling of gauge field corresponding to the diagonal $U(1)_P$ generator within $SU(6)$, which is just the $SU(6)$ gauge coupling strength at M_X. The RGE evolution trajectory of $U(1)_K$ gauge coupling upon M_X will eventually intersect/unify with that of $SU(6)$ gauge coupling at the $SO(12)/E_6$ unification scale (or at the string scale $M_{\rm str}$ with gravity).

We randomly scan the values of M_{331} within the ranges that are compatible with the lower bound from neutrino mass generation and the upper Planck scale bound. Our numerical results indicate that the flipped $SU(6)$ unification of *331* model can indeed be possible. In Fig. 1, we show the RGE evolutions of the gauge couplings for scenarios with the low energy theory as 2HDM below M_{331} scale (case I, left panel) and 2HDM plus $SU(2)_L$ triplet Higgs below M_{331} scale (case II, right panel), respectively. The corresponding *331* symmetry breaking scale for the benchmark scenarios are $M_{331}=10^{16}$ GeV (left panel) and $M_{331}=7.94\times10^{11}$ GeV (right panel), respectively. The corresponding flipped $SU(6)$ unification scales are $M_X=10^{19.03}$ GeV with $\alpha_6^{-1}(M_X)\approx 48.05$ (left panel) and $M_X=10^{19.13}$ GeV with $\alpha_6^{-1}(M_X)\approx 45.12$

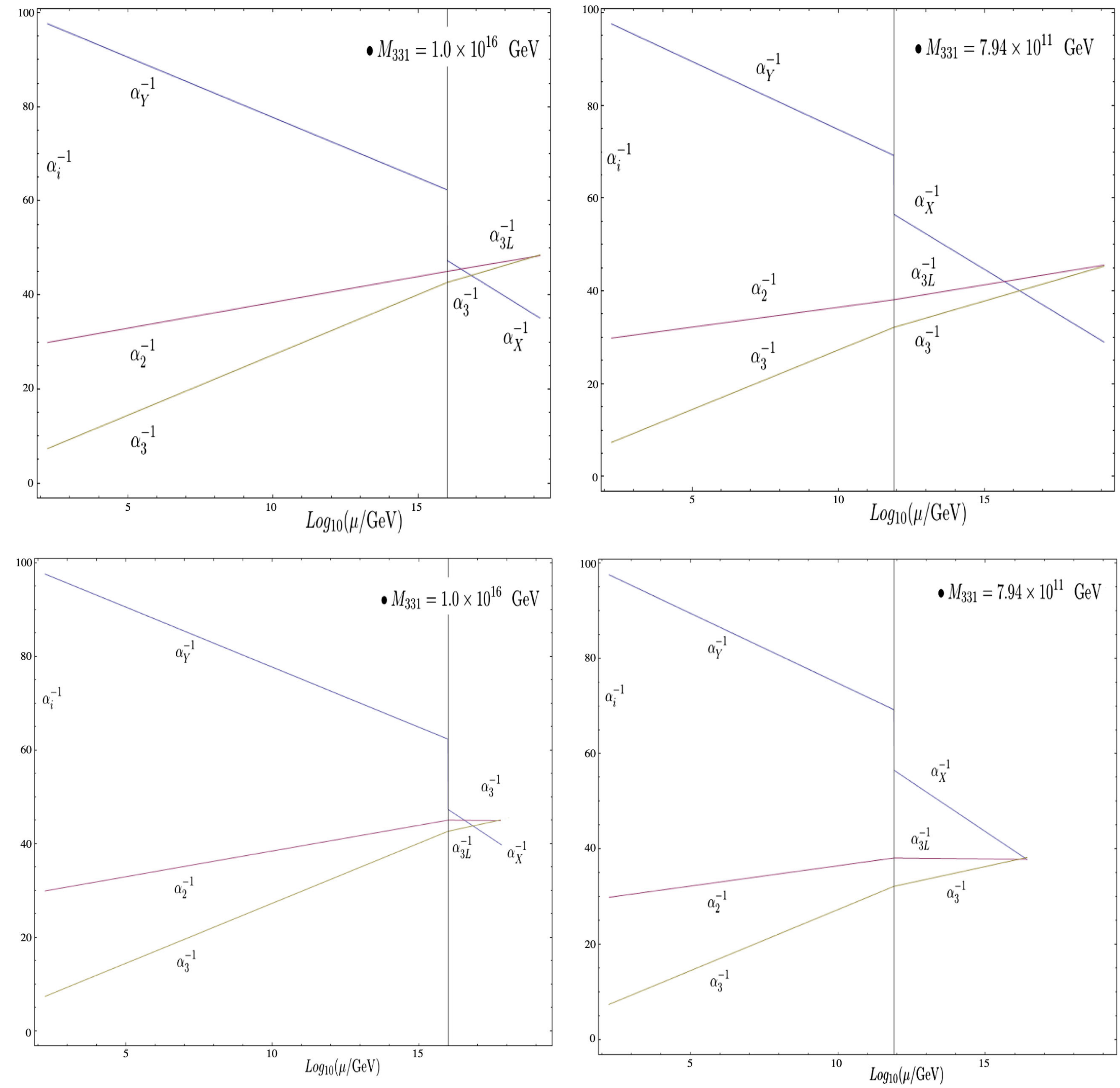

FIG. 1. The RGE evolutions of the gauge couplings for the *331* models are shown for scenarios with 2HDM below M_{331} (case I, left panels) and 2HDM plus $SU(2)_L$ triplet Higgs below M_{331} (case II, right panels) with/without light $\tilde{\mathbf{H}}_{\mathbf{3},\mathbf{8}}$ Higgs, respectively. With the *331* symmetry breaking scale $M_{331} = 10^{16}$ GeV (left panels) and $M_{331} = 7.94 \times 10^{11}$ GeV (right panels), the $SU(3)_L$ and $SU(3)_c$ gauge couplings can be unified into the flipped $SU(6)$ GUT model at the scale $M_X = 10^{19.03}$ GeV (left panel) and $M_X = 10^{19.13}$ GeV (right panel), respectively. The upper (and lower) panels correspond to the cases without (and with) the surviving M_{331} scale $\tilde{\mathbf{H}}_{\mathbf{3},\mathbf{8}}$ Higgs field, respectively.

(right panel). On the other hand, requiring the unification scales to lie below the Planck scale constrains $M_{331} \gtrsim 10^{15.9}$ GeV for case I and $M_{331} \gtrsim 10^{11.8}$ GeV for case II. Larger M_{331} scale will lead to smaller flipped $SU(6)$ GUT scale M_X. Besides, the requirement that the $SU(2)_L$ coupling (within $SU(3)_L$) and $SU(3)_c$ coupling should not intersect below M_{331} scale, that is, $M_X \gtrsim M_{331}$, sets an upper bound for M_{331} scale, which requires it to lie below $10^{17.6}$ GeV for case I and below $10^{15.3}$ GeV for case II.

We should note that it is possible to push down the M_{331} scale without spoiling successful flipped $SU(6)$ GUT by keeping the $(\bar{\mathbf{3}}, \mathbf{8}, -\frac{1}{\sqrt{3}})_{\mathbf{0}}$ Higgs field $\tilde{\mathbf{H}}_{\mathbf{3},\mathbf{8}}$ [corresponding to

TABLE I. We list the flipped $SU(6)$ GUT scales $\log_{10}(\frac{M_X}{\mathrm{GeV}})$ and corresponding $\alpha_6^{-1}(M_X)$ values for some benchmark points for various M_{331} values in case I and case II with/without light $\tilde{\mathbf{H}}_{\mathbf{3,8}}$ Higgs, respectively. The "\" symbols, followed by reasons that denoted by R_i, indicate that either the unification scale M_X lies upon the Planck scale (denoted by R_1), or g_{3L} and g_{3c} cannot unify (denoted by R_2), or the bound on M_{331} from neutrino mass generations is not satisfied for case I (denoted by R_3).

	M_{331}/GeV			
$(\log_{10}(\frac{M_X}{\mathrm{GeV}}), \alpha_6^{-1}(M_X))$	3.16×10^3	1.0×10^{11}	5.0×10^{14}	3.16×10^{16}
Case I without $\tilde{\mathbf{H}}_{\mathbf{3,8}}$	$\backslash; R_3$	$\backslash; R_3$	$\backslash; R_1$	(18.61, 47.77)
Case II without $\tilde{\mathbf{H}}_{\mathbf{3,8}}$	$\backslash; R_1$	$\backslash; R_1$	(16.18, 42.02)	$\backslash; R_2$
Case I with light $\tilde{\mathbf{H}}_{\mathbf{3,8}}$	$\backslash; R_3$	$\backslash; R_3$	(17.75, 43.41)	(17.70, 45.51)
Case II with light $\tilde{\mathbf{H}}_{\mathbf{3,8}}$	(17.98, 30.03)	(16.32, 36.88)	(15.52, 40.44)	$\backslash; R_2$

the *331* quantum numbers $(\bar{\mathbf{3}}, \mathbf{8}, \frac{\mathbf{1}}{\mathbf{3}})$] as light as the M_{331} scale, for example, by choosing proper boundary conditions in the orbifolding projection. In this case, the gauge beta functions change into

$$(b_3^c, b_3^L, b_1^X) = \left(-\frac{11}{3}, \frac{1}{6}, \frac{34}{3}\right), \tag{3.13}$$

upon the M_{331} scale with the relative running $b_{32} \equiv b_{3c} - b_{3L}$ increasing to $\frac{23}{6}$. The evolutions of the gauge couplings are also shown in the lower panels in Fig. 1. We show in Table I the flipped $SU(6)$ GUT scales and corresponding $\alpha_6^{-1}(M_X)$ values for some benchmark points in case I/case II with and without the surviving light (M_{331} scale) $\tilde{\mathbf{H}}_{\mathbf{3,8}}$ Higgs, respectively. The upper bounds $10^{17.6}$ GeV for case I (and $10^{15.3}$ GeV for case II) on M_{331} scale also apply here to guarantee $M_X \gtrsim M_{331}$. On the other hand, the requirement $M_{\mathrm{Pl}} \gtrsim M_X$, that is, the unification scale should lie below the Planck scale, will not give interesting constrains on M_{331} for both case I and case II with light (M_{331} scale) $\tilde{\mathbf{H}}_{\mathbf{3,8}}$ Higgs.

IV. PROTON DECAY IN FLIPPED $SU(6)$ GUT

The instability of proton is one of the most striking consequence of the GUT models, which can possibly be tested by various experiments. Current lower limits on the lifetimes for many possible proton decay modes can be used to constrain various GUT models [41–43]. The nucleon decay via the exchange of GUT-breaking heavy gauge bosons is strongly suppressed by M_{GUT}^4 while the nucleon decay via the exchange of color-triplet Higgs is suppressed by M_H^2, which is the dominant proton decay mode for many SUSY GUT models even though it has additional suppression factors with light fermion Yukawa couplings and a loop factor. Here we concentrate on the non-SUSY case and leave the SUSY case in our subsequent studies. So, proton decay in our nonsupersymmetric flipped $SU(6)$ model is mediated by dimension-6 operators, due to the exchanging of $SU(6)$ gauge bosons.

The relevant gauge interaction terms are given by

$$\begin{aligned}\mathcal{L} \supseteq -\sqrt{2} g_6 \big[&\big(\epsilon_{\alpha\beta} V_l (U_L^{c;A})^\dagger \gamma^\mu X_{\mu;A}^\beta L_L^\alpha\big) \\ &+ \epsilon_{ABC} V_{\mathrm{CKM}} (Q_L^A)^\dagger \gamma^\mu X_{\mu;B} D_L^{c;C} \\ &+ \epsilon_{\alpha\beta} V_N^\dagger (N_L^\alpha)^\dagger \gamma^\mu X_{\mu;B}^\beta D_L^{c;B} \big],\end{aligned} \tag{4.1}$$

with $X_{\mu;A}^\beta$ the heavy $SU(6)$ gauge bosons, the index $'\beta = 4, 5'$ (corresponds to the $SU(2)_L$ index within $SU(3)_L$), $A = 1,\ 2,\ 3$ the $SU(3)_c$ color indices and $V_{\mathrm{CKM}}, V_l, V_N$ being the mixing matrices for quarks, charged leptons, and neutrinos, respectively.

Similar to flipped $SU(5)$ [44], below the flipped $SU(6)$ breaking scale, the dim-6 operator from integrating out the heavy gauge bosons of $SU(6)$ can be written as

$$\begin{aligned}\mathcal{L} \supseteq \frac{g_6^2}{M_X^2} \epsilon_{ijk} \epsilon_{\alpha\beta} \big[&C_1 (\bar{d}_{iL}^c \gamma^\mu Q_{j\alpha L})(\bar{u}_{kL}^c \gamma_\mu L_{\beta L}) \\ &+ C_2 (\bar{d}_{iL}^c \gamma^\mu Q_{j\alpha L})(\bar{N}_L^c \gamma_\mu Q_{k\beta L})\big],\end{aligned} \tag{4.2}$$

with C_1, C_2 the relevant coefficients. As the RH neutrinos are much heavier than the proton, only the first operator will contribute to proton decay. The effective dim-6 operator that trigger the decay of protons takes the form

$$\mathcal{L}(p \to \pi^0 l_a^+) \supseteq \frac{g_6^2}{M_X^2} V_{CKM;11}^* (U_l)_{a1} \epsilon_{ijk} (u_R^i d_R^j)(l_{La} u_L^k), \tag{4.3}$$

below the EW scale. The proton decay rate in our flipped $SU(6)$ model can be calculated by generalizing the flipped $SU(5)$ case [45]

$$\begin{aligned}\Gamma(p \to \pi^0 l_i^+) &= \frac{m_p}{32\pi} \left(1 - \frac{m_\pi^2}{m_p^2}\right)^2 |\mathcal{A}(p \to \pi^0 \bar{l}_i^+)|^2, \\ &= \frac{g_6^4}{32\pi M_X^4} m_p |V_{ud}|^2 |(U_l)_{i1}|^2 \left(1 - \frac{m_\pi^2}{m_p^2}\right)^2 \\ &\quad \times \mathcal{A}_L^2 \mathcal{A}_{S1}^2 (\langle \pi^0 | (ud)_R u_L | p \rangle_{l_i})^2,\end{aligned} \tag{4.4}$$

with the two-loop perturbative QCD renormalization factor $\mathcal{A}_L = 1.247$. The values $\mathcal{A}_{S1}$ of the renormalization factors for the dim-6 proton decay operator between the M_X scale and the m_Z scale are given with the following one-loop coefficients

$$\mathcal{A}_{S1} \approx \left[\frac{\alpha_3(M_{331})}{\alpha_3(M_X)}\right]^{\frac{2}{5}} \left[\frac{\alpha_3(M_Z)}{\alpha_3(M_{331})}\right]^{\frac{2}{7}} \left[\frac{\alpha_2(M_{331})}{\alpha_2(M_X)}\right]^{\frac{34}{27}} \left[\frac{\alpha_2(M_Z)}{\alpha_2(M_{331})}\right]^{\frac{3}{4}} \left[\frac{\alpha_Y(M_{331})}{\alpha_Y(M_X)}\right]^{-\frac{33}{362}} \left[\frac{\alpha_Y(M_Z)}{\alpha_Y(M_{331})}\right]^{-\frac{11}{84}}, \tag{4.5}$$

for case I without light $\tilde{\mathbf{H}}_{\mathbf{3,8}}$ Higgs and

$$\mathcal{A}_{S1} \approx \left[\frac{\alpha_3(M_{331})}{\alpha_3(M_X)}\right]^{\frac{6}{11}} \left[\frac{\alpha_3(M_Z)}{\alpha_3(M_{331})}\right]^{\frac{2}{7}} \left[\frac{\alpha_2(M_{331})}{\alpha_2(M_X)}\right]^{-\frac{27}{2}} \left[\frac{\alpha_2(M_Z)}{\alpha_2(M_{331})}\right]^{\frac{27}{28}} \left[\frac{\alpha_Y(M_{331})}{\alpha_Y(M_X)}\right]^{-\frac{33}{410}} \left[\frac{\alpha_Y(M_Z)}{\alpha_Y(M_{331})}\right]^{-\frac{11}{96}}, \tag{4.6}$$

for case II with light $\tilde{\mathbf{H}}_{\mathbf{3,8}}$ Higgs. Expressions of $\mathcal{A}_{S1}$ for other scenarios can be straightforwardly obtained. The hadronic matrix elements can be obtained by lattice calculations [46]

$$\langle \pi^0 | (ud)_R u_L | p \rangle_{l_i} = \begin{cases} -0.131, & (l_1 = e) \\ -0.118, & (l_2 = \mu) \end{cases}, \qquad \langle K^0 | (us)_R u_L | p \rangle_{l_i} = \begin{cases} 0.103, & (l_1 = e) \\ 0.099, & (l_2 = \mu) \end{cases},$$
$$\langle \pi^+ | (ud)_R d_L | p \rangle = -0.186, \tag{4.7}$$

within which the subscripts "e" and "μ" indicate that the matrix elements are evaluated at the corresponding lepton kinematic points. Using Eq. (4.4), we can calculate the partial lifetime of the $p \to e^+\pi^0$ mode in flipped $SU(6)$ GUT models. The decay widths for other proton decay channels, such as $p \to \pi^0\mu^+$ and $p \to K^0 e^+$ can be similarly obtained after taking into account proper CKM and PMNS matrix elements. The leptonic PMNS mixing matrix is given by [47,48]:

$$V_{\mathrm{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha_2}{2}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_3}{2}} \end{pmatrix}, \tag{4.8}$$

where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$ with the mixing angles $\theta_{ij} = [0, \pi/2]$, the Dirac CP phase $\delta \in [0, 2\pi]$ and the Majorana phases α_2 and α_3 being set to vanish. For normally ordered (NO) or inversely ordered (IO) neutrino hierarchical spectrum, the matrix elements of $U_l = V^*_{\mathrm{PMNS}} V_\nu$ can be given as

$$(U_l)_{11} \simeq \begin{cases} (V^*_{\mathrm{PMNS}})_{11} = c_{12}c_{13} & (\mathrm{NO}) \\ (V^*_{\mathrm{PMNS}})_{13} = s_{13}e^{i\delta - i\frac{\alpha_3}{2}} & (\mathrm{IO}) \end{cases}, \tag{4.9}$$

$$(U_l)_{21} \simeq \begin{cases} (V^*_{\mathrm{PMNS}})_{21} = -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta} & (\mathrm{NO}) \\ (V^*_{\mathrm{PMNS}})_{23} = s_{23}c_{12}e^{-i\frac{\alpha_3}{2}} & (\mathrm{IO}) \end{cases}. \tag{4.10}$$

We use the following best-fit value of the PMNS mixing and phase [49]

$$\theta_{12} = 33.82^{+0.78}_{-0.76}, \quad \theta_{23} = 48.3^{+1.2}_{-1.9}, \quad \theta_{13} = 8.61^{+0.13}_{-0.13}, \quad \delta = 222^{+38}_{-28}, \quad (\mathrm{NO}),$$
$$\theta_{12} = 33.82^{+0.78}_{-0.76}, \quad \theta_{23} = 48.6^{+1.1}_{-1.5}, \quad \theta_{13} = 8.65^{+0.13}_{-0.12}, \quad \delta = 285^{+24}_{-26}, \quad (\mathrm{IO}), \tag{4.11}$$

for IO and NO cases, respectively. Similar to ordinary flipped $SU(5)$ case [50], we can calculate the relations for partial decay widths

$$
\begin{aligned}
\frac{\Gamma(p\to\pi^0\mu^+)}{\Gamma(p\to\pi^0e^+)} &\simeq \frac{\langle\pi^0|(ud)_R u_L|p\rangle_\mu)^2}{(\langle\pi^0|(ud)_R u_L|p\rangle_e)^2}\frac{|(U_l)_{21}|^2}{|(U_l)_{11}|^2}\approx\begin{cases}0.114, & (\mathrm{NO})\\ 19.727, & (\mathrm{IO})\end{cases},\\
\frac{\Gamma(p\to K^0e^+)}{\Gamma(p\to\pi^0e^+)} &\simeq \frac{(\langle K^0|(us)_R u_L|p\rangle_e)^2}{(\langle\pi^0|(ud)_R u_L|p\rangle_e)^2}\frac{|V_{us}|^2}{|V_{ud}|^2}\approx 0.018,\\
\frac{\Gamma(p\to K^0\mu^+)}{\Gamma(p\to\pi^0e^+)} &\simeq \frac{(\langle K^0|(us)_R u_L|p\rangle_\mu)^2}{(\langle\pi^0|(ud)_R u_L|p\rangle_e)^2}\frac{|V_{us}|^2|(U_l)_{21}|^2}{|V_{ud}|^2|(U_l)_{11}|^2}\approx\begin{cases}0.004, & (\mathrm{NO})\\ 0.737, & (\mathrm{IO})\end{cases},\\
\frac{\Gamma(p\to\pi^+\bar\nu_i)}{\Gamma(p\to\pi^0e^+)} &\simeq \frac{(\langle\pi^+|(ud)_R d_L|p\rangle)^2}{(\langle\pi^0|(ud)_R u_L|p\rangle_e)^2}\frac{1}{|V_{ud}|^2|(U_l)_{11}|^2}\approx\begin{cases}3.151, & (\mathrm{NO})\\ 93.999, & (\mathrm{IO})\end{cases},
\end{aligned}
\tag{4.12}
$$

with $\Gamma(p\to\pi^+\bar\nu)=\sum_i\Gamma(p\to\pi^+\bar\nu_i)$. The $p\to K^+\bar\nu_i$ channels can be proven to be forbidden by the unitary property of the CKM matrix. We also have the relation $\Gamma(n\to\pi^-l_i^+)=\Gamma(p\to\pi^0l_i^+)/2$. By taking the ratio between the two partial decay widths, many of the factors in the expressions (such as the M_X scale, the $SU(6)$ gauge coupling g_6 and the $\mathcal{A}_{L;S}$ factors) can be canceled, making the comparison of those ratios in various GUT models meaningful.

We show in Table II the numerical results of $p\to e^+\pi^0$ partial life time for some benchmark points in case I/case II with and without M_{331} scale $\tilde{\mathbf{H}}_{\mathbf{3,8}}$ Higgs for NO neutrino mass hierarchy case. The partial proton decay lifetimes of $p\to e^+\pi^0$ for IO case is related to that of NO case by $\tau_{\mathrm{IO}}/\tau_{\mathrm{NO}}\approx 30.1$. We can see that without light $\tilde{\mathbf{H}}_{\mathbf{3,8}}$ Higgs, the partial proton decay lifetimes of the $p\to e^+\pi^0$ mode are rather large. Although the current proton decay bound $\tau(p\to e^+\pi^0)>1.6\times10^{34}$ year [41] can easily be satisfied, such proton decay mode cannot be observed in the forthcoming experiments. On the other hand, with M_{331} scale $\tilde{\mathbf{H}}_{\mathbf{3,8}}$ Higgs, the partial proton decay lifetimes $p\to e^+\pi^0$ can be as low as 2.80×10^{34} years for $M_{331}\approx5.0\times10^{14}$ GeV (of case II with NO neutrino hierarchy), although such a M_{331} value lies near the upper bound on M_{331}. It can be seen that such parameter regions for either cases will soon be tested by future DUNE [42] and Hyper-Kamiokande [51] experiments, which can reach as large as 1.3×10^{35} years for $p\to e^+\pi^0$ channel in future Hyper-Kamiokande.

For other proton decay channels, the only worrisome ones are $\tau(p\to\pi^0\mu^+)$ and $\tau(p\to\pi^+\bar\nu_i)$, with current 90% CL lower bounds

$$
\begin{aligned}
\tau(p\to\pi^0\mu^+) &\gtrsim 7.7\times10^{33}\ \mathrm{yr},\\
\tau(p\to\pi^+\bar\nu_i) &\gtrsim 0.39\times10^{33}\ \mathrm{yr}.
\end{aligned}
\tag{4.13}
$$

DG13870From the relations in (4.12), we can see that the benchmark parameter point with $\tau_{\mathrm{IO}}(p\to e^+\pi^0)=8.4\times10^{35}$ years for $M_{331}\approx5\times10^{14}$ GeV (of case II with IO neutrino hierarchy), corresponding to 2.8×10^{34} years for NO neutrino hierarchy, can be ruled out by the future proton decay constraints for $p\to\pi^0\mu^+$ channel in Hyper-Kamiokande, which can reach the sensitivities 6.9×10^{34} yr for this channel [51]. So, we can see that, for $M_{331}\sim10^{15}$ GeV of case II, both NO or IO neutrino hierarchy predictions can be tested in the future proton decay experiments.

It is worthy to be noted that an additional effective dim-6 operator other than that given in (4.3) can contribute to the decay of proton in ordinary $SU(6)$ GUT for *331* model. The Wilson coefficient for the dim-6 operator in (4.3) is also different to that of ordinary $SU(6)$ GUT. Therefore, the predictions for the ratios of various proton decay modes in our flipped $SU(6)$ model are different from that of ordinary $SU(6)$ GUT for *331* model, leading to different predictions for both models.

TABLE II. The partial proton decay lifetimes of the $(p\to e^+\pi^0)$ mode in flipped $SU(6)$ GUT for NO neutrino masses hierarchy. The partial proton decay lifetimes of this channel for IO case is related to that of NO case by $\tau_{\mathrm{IO}}/\tau_{\mathrm{NO}}\approx30.1$. We calculate such partial life time for some benchmark points for various M_{331} values in case I/case II with and without M_{331} scale $\tilde{\mathbf{H}}_{\mathbf{3,8}}$ Higgs, respectively. The definitions of "\" symbol and the corresponding reasons R_i are the same as that in Table I.

$\tau(p\to e^+\pi^0)_{\mathrm{flipped}}/\mathrm{yrs}$	M_{331}/GeV			
τ_{NO}	3.16×10^3	1.0×10^{13}	5.0×10^{14}	3.16×10^{16}
Case I without $\tilde{\mathbf{H}}_{\mathbf{3,8}}$	$\backslash;R_3$	$\backslash;R_3$	$\backslash;R_3$	6.92×10^{46}
Case II without $\tilde{\mathbf{H}}_{\mathbf{3,8}}$	$\backslash;R_1$	1.04×10^{45}	1.21×10^{37}	$\backslash;R_2$
Case I with $\tilde{\mathbf{H}}_{\mathbf{3,8}}$	$\backslash;R_3$	3.88×10^{43}	2.29×10^{43}	1.61×10^{43}
Case II with $\tilde{\mathbf{H}}_{\mathbf{3,8}}$	5.60×10^{43}	8.21×10^{35}	2.80×10^{34}	$\backslash;R_2$

V. CONCLUSIONS

We propose to partially unify the sequential $SU(3)_c \times SU(3)_L \times U(1)_X$ model (with $\beta = 1/\sqrt{3}$) into a flipped $SU(6)$ GUT model. It can checked that the gauge anomaly cancelation can be satisfied for each generation. We discuss the relevant Higgs sector, the low energy *331* model spectrum and the unification of $SU(3)_c$ and $SU(3)_L$ gauge couplings. Neutrino mass generation and successful gauge coupling unification can set lower/upper bounds on the *331* breaking scale. The partial proton decay lifetime of various channels, for example, the $p \to e^+\pi^0$ channel, in flipped $SU(6)$ GUT are discussed. We find that certain parameter region with $M_{331} \sim 10^{15}$ GeV of case II (for case with M_{331} scale $\tilde{\mathbf{H}}_{\mathbf{3,8}}$ Higgs field) can predict a partial proton lifetime of order 10^{34} years for $p \to e^+\pi^0$ mode, which can be tested soon by future DUNE and Hyper-Kamiokande experiments. The predictions for the ratios of various proton decay modes in our flipped $SU(6)$ model are different from that of ordinary $SU(6)$ GUT for *331* model.

It is known that there is a factor of approximately 20 to 25 between the unification scale for MSSM $M_{\rm GUT}$ and the string scale $M_{\rm str}$ in the weakly coupled heterotic string theory, which is given by $M_{\rm str} \approx 5.27 \times 10^{17}$ GeV for the string coupling constant $g_{\rm str} \sim \mathcal{O}(1)$. As string theory predicts "a prior" a unification of gauge couplings at $M_{\rm str}$, such an inconsistency may indicates the appearance of (partial) unification at the intermediate scale $M_{\rm GUT}$, which would then unify with gravity and any other "hidden-sector" gauge symmetries at $M_{\rm str}$ [34]. In our model, for most benchmark points, the $SU(6) \times U(1)_K$ breaking scale M_X can be calculated to lie in the range $\mathcal{O}(10^{15}$–$10^{18})$ GeV. We should note that the calculated M_X scale is sensitive to the additional Higgs contents (other than the minimal necessary $H_1(1, \bar{3}, \frac{2}{3})$, $H_2(1, \bar{3}, -\frac{1}{3})$, $H_3(1, \bar{3}, -\frac{1}{3})$ Higgs fields) adopted in the low energy *331* model. Taking into account the heavy string threshold corrections from the infinite towers of Planck-scale string states to RGE, the intermediate step $SU(6) \times U(1)_K$ partial unification can be welcome to realize string scale unification of flipped $SU(6)$ (or $SO(12)/E_6$ GUT group) with gravity.

We should note that nonsequential *331* model contains less exotic matter fields than the sequential *331* model. When the nonsequential $SU(3)_c \times SU(3)_L \times U(1)_X$ model is embedded into (partial) unification model, the flipped $SU(6)$ gauge group can be advantageous over ordinary $SU(6)$ gauge group in several aspects, for example, requiring much smaller representations for the embedding (thus much less exotic matters) and simpler anomaly cancelation constraints. The flipped $SU(6)$ partial unification of nonsequential *331* model will be given in our future works [30].

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APPENDIX: DOUBLET SPLITTING PROBLEM IN FLIPPED $SU(6)$ MODEL

1. Missing partner mechanism in flipped $SU(6)$

Below the flipped $SU(6)$ breaking scale M_X, the colored Higgs fields can acquire masses of order M_X while the uncolored ones can still be as light as M_{331} scale. Such doublet-triplet (D-T) like splitting can be realized by the missing partner mechanism or by orbifold projection with proper boundary conditions.

In the SUSY flipped $SU(6)$ model, the missing partner mechanism can be applied by adopting the following type of superpotential

$$
\begin{aligned}
W \supseteq\ & \epsilon^{ijklmn}\lambda(\mathbf{20}_{\frac{1}{2};\mathbf{H}})_{ijk}(\mathbf{15}'_{\mathbf{0};\mathbf{H}})_{lm}(\mathbf{6}_{-\frac{1}{2};\mathbf{H}})_n \\
& + \epsilon^{ijklmn}\lambda'(\overline{\mathbf{20}}_{-\frac{1}{2};\mathbf{H}})_{ijk}(\overline{\mathbf{15}}'_{\mathbf{0};\mathbf{H}})_{lm}(\bar{\mathbf{6}}_{\frac{1}{2};\mathbf{H}})_n \\
& + M_{\mathbf{15}'}\overline{\mathbf{15}}'_{\mathbf{0};\mathbf{H}}15'_{\mathbf{0};\mathbf{H}} + X(\mathbf{20}_{\frac{1}{2};\mathbf{H}}\overline{\mathbf{20}}_{-\frac{1}{2};\mathbf{H}} - M_X^2), \qquad \text{(A1)}
\end{aligned}
$$

for $M_{\mathbf{15}'} \sim M_X$.

The F-flat and D-flat conditions can render the VEVs $\langle \mathbf{20}_{\frac{1}{2};\mathbf{H}}\rangle = \langle \overline{\mathbf{20}}_{-\frac{1}{2};\mathbf{H}}\rangle = M_X$ (along the $(1, 1, \frac{3}{2\sqrt{3}})_{\frac{1}{2};H}$ directions) and break the flipped $SU(6)$ into *331* gauge group. The $(\bar{3}, 1, \frac{1}{2\sqrt{3}})_{\frac{1}{2};H}$ component within $\bar{\mathbf{6}}_{\frac{1}{2};\mathbf{H}}$ pairs to the $(3, 1, \frac{1}{\sqrt{3}})_{0;H'}$ components within $\overline{\mathbf{15}}'_{0;H}$ while the $(1, \bar{3}, -\frac{1}{2\sqrt{3}})_{\frac{1}{2};H}$ component cannot find any partner. Therefore, the colored component within $\bar{\mathbf{6}}_{\frac{1}{2};\mathbf{H}}$ is heavy while the uncolored one is much lighter (at or below the M_{331} scale). Missing partner mechanism of similar settings can also push heavy the colored components within $\bar{\mathbf{6}}_{-\frac{1}{2};\mathbf{H}}$ and $\mathbf{15}_{\mathbf{0};\mathbf{H}}$.

2. D-T splitting from orbifold projection

Boundary conditions [52–54] can also be used to split the colored and uncolored Higgs components. For example, we can adopt S^1/Z_2 orbifold by identifying the fifth coordinate y under the two operations

$$
\mathcal{Z}\colon\ y \to -y, \qquad \mathcal{T}\colon\ y \to y + 2\pi R. \qquad \text{(A2)}
$$

Two inequivalent 3-branes located at $y = 0$ (O-brane) and $y = \pi R$ (O'-brane). We can choose the boundary condition with $P = \mathrm{diag}(-1, -1, -1, 1, 1, 1)$ (under $\mathcal{Z}$ reflection) and $P' = \mathrm{diag}(1, 1, 1, 1, 1, 1)$ (under the reflection $\mathcal{Z}' = \mathcal{Z}T$) so as that the Higgs satisfy

$$\begin{aligned}
\bar{\mathbf{6}}_{\frac{1}{2};H} &= \left(\bar{3},1,\frac{1}{2\sqrt{3}}\right)^{(-,+)}_{\frac{1}{2};H} \oplus \left(1,\bar{3},-\frac{1}{2\sqrt{3}}\right)^{(+,+)}_{\frac{1}{2};H},\\
\bar{\mathbf{6}}_{-\frac{1}{2};H} &= \left(\bar{3},1,\frac{1}{2\sqrt{3}}\right)^{(-,+)}_{-\frac{1}{2};H} \oplus \left(1,\bar{3},-\frac{1}{2\sqrt{3}}\right)^{(+,+)}_{-\frac{1}{2};H},\\
\mathbf{15}_{0;H} &= \left(\bar{3},1,-\frac{1}{\sqrt{3}}\right)^{(-,+)}_{0;H} \oplus (3,3,0)^{(-,+)}_{0;H} \oplus \left(1,\bar{3},\frac{1}{\sqrt{3}}\right)^{(+,+)}_{0;H},\\
\mathbf{21}_{1;H} &= \left(6,1,-\frac{1}{\sqrt{3}}\right)^{(-,+)}_{1;H} \oplus (3,3,0)^{(-,+)}_{1;H} \oplus \left(1,6,\frac{1}{\sqrt{3}}\right)^{(+,+)}_{1;H},
\end{aligned} \tag{A3}$$

with proper brane localized terms to change some of the boundary conditions from Neumann to Dirichlet.

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