Pion condensation in QCD at finite isospin density, the dilute Bose gas, and speedy Goldstone bosons

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We consider pion condensation in QCD at finite isospin density μ_I and zero temperature using two-flavor chiral perturbation theory (χPT). The pressure is calculated to next-to-leading order (NLO) in the low-energy expansion. In the nonrelativistic limit, we recover the classic result by Lee, Huang, and Yang for the energy density of a dilute Bose gas with an s-wave scattering length that includes loop corrections from χPT. In the chiral limit, higher-order calculations are tractable. We calculate the pressure to next-to-next-to-leading order (NNLO) in the low-energy expansion, which is an expansion in powers of $\mu_I^2/(4\pi)^2 f^2$, where f is the (bare) pion decay constant. The spontaneous breakdown of the global internal symmetry $U(1)_{I_3}$ gives rise to a massless Goldstone boson or phonon. We discuss the properties of the low-energy effective theory describing this mode. Finally, we compare our results for the pressure and the speed of sound with recent lattice simulations with $2 + 1$ flavors. The agreement is very good for isospin chemical potentials up to 180–200 MeV, depending on the physical quantity.

I. INTRODUCTION

Bose-Einstein condensation (BEC), spontaneous symmetry breaking and the subsequent appearance of massless Goldstone modes frequently occur in condensed matter and high-energy physics. In the past decades, there has been significant progress in our understanding of the properties of these massless excitations, their classification, and the construction of low-energy effective theories that describe their dynamics, see, e.g., Refs. [1–10].

The classic textbook example is the Bose-Einstein condensation of a dilute nonrelativistic Bose gas at zero and finite temperature [11]. The homogeneous Bose gas has been studied extensively for many decades beginning with the paper by Bogoliubov [12] in the 1940s. The nonrelativistic Lagrangian that describes the system at finite particle density is

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$$\mathcal{L} = \psi^\dagger(i\partial_0 + \mu_{\mathrm{NR}})\psi - \frac{1}{2m}\nabla\psi^\dagger \cdot \nabla\psi - \frac{1}{4}g(\psi^\dagger\psi)^2$$
$$- \frac{1}{36}g_3(\psi^\dagger\psi)^3 + \cdots, \tag{1}$$

where the quantum field $\psi^\dagger$ creates a particle, ψ destroys a particle, μ_{NR} is the nonrelativistic chemical potential, and g and g_3 are coupling constants. The term $(\psi^\dagger\psi)^2$ represents two-particle scattering and the coupling g is related to the s-wave scattering length a as $g = \frac{8\pi a}{m}$. The term $(\psi^\dagger\psi)^3$ represents $3 \to 3$ scattering. The dots indicate terms that are higher order in the fields ψ, $\psi^\dagger$ and/or their derivatives, and include all local terms that are consistent with the symmetries, for example, Galilean invariance and the phase symmetry $\psi \to e^{i\phi}\psi$. The latter continuous symmetry gives rise to the conservation of particle number. The coefficients g, g_3, and the coefficients that multiply the higher-order terms can in principle be determined from the n-body potentials that describe interatomic interactions [13].

At zero temperature, the expansion parameter of the dilute Bose gas is the so-called (dimensionless) gas parameter $\sqrt{na^3}$, where n is the number density. Bogoliubov [12] obtained the mean-field result for the energy density, $\mathcal{E}(n) = \frac{2\pi n^2 a}{m}$. The leading correction to Bogoliubov's result for the energy density was calculated by Lee, Huang, and

Yang (LHY) [14,15] for a hard-sphere potential. Later, part of the next-to-leading order correction was calculated by Wu [16], by Hugenholz and Pines [17], and by Sawada [18]. A complete next-to-leading result was obtained by Braaten and Nieto [13] using effective-field theory methods. The result depends not only on the scattering length a but also on an energy-independent term in the scattering amplitude for $3 \rightarrow 3$ scattering. The result is

$$
\begin{aligned}
\mathcal{E}(n) = \frac{2\pi n^2 a}{m} &\left[1 + \frac{128}{15\sqrt{\pi}} \sqrt{na^3} \right. \\
&\left. + \left(\frac{32\pi - 24\sqrt{3}}{3} \log(na^3) + C \right) na^3 \right],
\end{aligned} \quad (2)
$$

where C is a constant involving the coupling g_3. The dependence of physical results on quantities other than the s-wave scattering length was already realized by Hugenholz and Pines. These effects are referred to as nonuniversal effects and are mimicked by, e.g., the term $g_3(\psi^\dagger\psi)^3$ in Eq. (1). Similarly, the effective range r_s of the two-body potential can be included by adding the operator $\frac{1}{4}h[\nabla(\psi^\dagger\psi)]^2$. The coupling h is related to the effective range, $h = 2\pi a^2 r_s/m$. The leading correction to the results of LHY due to this term was calculated in Ref. [19] and is of higher order in the gas parameter than the terms shown in Eq. (2).

Another example of Bose-Einstein condensate is pion condensation in QCD at finite isospin density. Consider two-flavor QCD with two independent chemical potentials μ_u and μ_d. Instead of using the quark chemical potentials, we can express the Lagrangian in terms of the baryon and isospin chemical potentials $\mu_B = \frac{3}{2}(\mu_u + \mu_d)$ and $\mu_I = (\mu_u - \mu_d)$. Normally, the phase diagram is shown in the μ_B–T plane, but by allowing for nonzero μ_I, we can add a new axis, and the phases in the μ_I–T plane have received particular attention due to the fact that QCD is free of the sign problem for $\mu_B = 0$. The system is therefore amenable to lattice simulations employing importance-sampling techniques. In a series of papers, Brandt *et al.* have mapped out the phase diagram in the μ_I–T plane, calculated the equation of state, the isospin density, quark and pion condensates, and the speed of sound [20–25]. The EoS has been used to model pion stars [24], which are compact objects consisting of a pion condensate with leptons and neutrinos to ensure electric neutrality and weak equilibrium, first proposed in [26].

Chiral perturbation theory (χPT) [27–30] is a low-energy effective theory for the pseudo-Goldstone bosons that appear in QCD as a consequence of spontaneous symmetry breaking in the vacuum by the quark condensate. χPT at finite isospin chemical potential was first considered in Refs. [31,32]. The quark, pion, and axial condensates of QCD at finite isospin density and $T = 0$ have been calculated at next-to-leading order in [33,34]. The results

are generally in good agreement with lattice simulations with $2 + 1$ flavors for small values of the isospin chemical potential, $\mu_I \ll 4\pi f$, where χPT is expected to be valid. In this paper, we compare the pressure and the speed of sound including updated lattice simulations with $2 + 1$ flavor, that recently appeared in Ref. [25].

At first glance, the dilute Bose gas and Bose condensation in QCD may seem unrelated. However, it is known that a relativistic Bose condensate for low densities, i.e. near the onset of Bose condensation,[1] behaves as a dilute Bose gas. It was explicitly shown in Ref. [35] that the pions in the pion stars mentioned above, to a good approximation are non-relativistic, i.e. the pion density is very low. In this paper, we show that at the next-to-leading order, the pion condensate behaves as a dilute Bose for low densities: We reproduce the corrections to Bogoliubov's results in the appropriate limit, namely for μ_I just above the physical pion mass. In the final stages of this work, Ref. [36] appeared with some overlapping results, for example recovering the LHY correction for interacting scalars. We also consider the opposite limit, namely the ultrarelativistic or high-density regime. In this regime, higher-order calculations are in fact tractable and we calculate the pressure to next-to-next-to-leading order in the low-energy expansion. Finally, we use Son's construction of the effective field theory for the phonons to obtain a low-energy description of the system. The damping rate of the phonons scales as p^5 for small p, a result that was obtained by Beliaev long ago in the case of a weakly interacting Bose gas [37].

II. THERMODYNAMICS

In the discussion below, the thermodynamic potential is a function of a chemical potential μ and a parameter that we for now denote by α, $\Omega = \Omega(\mu, \alpha)$. In the case of a dilute Bose gas, the parameter α is identified with the order parameter v, which is the condensate density. In chiral perturbation theory, the quark condensate is rotated into a pion condensate, specified by the rotation angle α, cf. Eq. (26) below. The Bose-condensed phase is characterized by a nonzero value of α.

The thermodynamic potential can be systematically expanded in powers of the gas parameter $\sqrt{na^3}$ in the dilute Bose gas or in powers of p/f in a low-energy expansion in χPT (see explanation in the beginning of Sec. IV). We can write this expansion as

$$
\Omega(\mu, \alpha) = \Omega_0(\mu, \alpha) + \Omega_1(\mu, \alpha) + \cdots, \quad (3)
$$

where the subscript n indicates the nth order contribution in the expansion. The value of α that minimizes $\Omega(\mu, \alpha)$ is denoted by α^* and is found by solving

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[1]The onset of BEC is exactly at $\mu_I = m_\pi$, where m_π is the physical pion mass.

$$\frac{\partial\Omega}{\partial\alpha} = 0. \tag{4}$$

The pressure $\mathcal{P}$ is given by minus the thermodynamic potential evaluated at its minimum as a function of α

$$\mathcal{P}(\mu) = -\Omega(\mu, \alpha^*). \tag{5}$$

The charge density n_Q associated with μ is given by

$$n_Q(\mu) = -\frac{\partial\Omega}{\partial\mu}\bigg|_{\alpha=\alpha^*} = \frac{d\mathcal{P}}{d\mu}, \tag{6}$$

where we in the last step have used Eq. (4). Finally, the energy density is given by a Legendre transform of the pressure,

$$\mathcal{E}(n_Q) = -\mathcal{P}(\mu) + \mu n_Q(\mu). \tag{7}$$

The solution α^* to Eq. (4) can also be written as a series

$$\alpha^* = \alpha_0 + \alpha_1 + \cdots. \tag{8}$$

We can find expressions for $\alpha_0, \alpha_1\ldots$ by expanding Eq. (4) around $\alpha = \alpha_0$. We find

$$\frac{\partial\Omega}{\partial\alpha}\bigg|_{\alpha=\alpha^*} = \frac{\partial\Omega_0}{\partial\alpha}\bigg|_{\alpha=\alpha_0} + \frac{\partial^2\Omega_0}{\partial\alpha^2}\bigg|_{\alpha=\alpha_0}\alpha_1 + \frac{\partial\Omega_1}{\partial\alpha}\bigg|_{\alpha=\alpha_0}$$
$$+ \cdots = 0. \tag{9}$$

α_0 is simply given by the solution to $\frac{\partial\Omega_0}{\partial\alpha} = 0$, so the first term in Eq. (9) vanishes. We can solve then solve Eq. (9) for α_1,

$$\alpha_1 = -\frac{\partial\Omega_1}{\partial\alpha}\bigg|_{\alpha=\alpha_0} \bigg/ \frac{\partial^2\Omega_0}{\partial\alpha^2}\bigg|_{\alpha=\alpha_0}. \tag{10}$$

Finally, the expansion of the pressure reads

$$\mathcal{P}(\mu) = -\Omega(\mu, \alpha^*)$$
$$= -\Omega_0(\mu, \alpha_0) - \frac{\partial\Omega_0}{\partial\alpha}\bigg|_{\alpha=\alpha_0}\alpha_1 - \Omega_1(\mu, \alpha_0) + \cdots$$
$$= -\Omega_0(\mu, \alpha_0) - \Omega_1(\mu, \alpha_0) + \cdots. \tag{11}$$

III. DILUTE BOSE GAS

We now derive the first two terms in the expansion Eq. (2). The first term is the mean-field result, while the second arises from a one-loop calculation. The complex field is written as $\psi = v + \tilde{\psi}$, where $v = \langle\psi\rangle$ is its expectation value and $\tilde{\psi}$ is a fluctuating quantum field. The fluctuating field is written as $\tilde{\psi} = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$. To second order in the fluctuations, one finds the different terms of the Lagrangian Eq. (1)

$$\mathcal{L}^{\text{static}} = \mu_{\text{NR}}v^2 - \frac{1}{4}gv^4, \tag{12}$$

$$\mathcal{L}^{\text{linear}} = \frac{vX}{\sqrt{2m}}\psi_1, \tag{13}$$

$$\mathcal{L}^{\text{quadratic}} = \frac{1}{2}(\dot{\psi}_1\psi_2 - \psi_1\dot{\psi}_2) + \frac{1}{4m}\psi_1(\nabla^2 + Y)\psi_1$$
$$+ \frac{1}{4m}\psi_2(\nabla^2 + X)\psi_2, \tag{14}$$

where $X = 2m(\mu_{\text{NR}} - \frac{1}{2}gv^2)$ and $Y = 2m(\mu_{\text{NR}} - \frac{3}{2}gv^2)$. The propagator matrix is given by the inverse of the quadratic terms in $\mathcal{L}$. In momentum space, one finds

$$D(P) = \frac{i}{p_0^2 - E^2(p) + i\epsilon}\begin{pmatrix} \frac{1}{2m}(p^2 - X) & -ip_0 \\ ip_0 & \frac{1}{2m}(p^2 - Y) \end{pmatrix}, \tag{15}$$

where P is the four momentum, $P = (p_0, \mathbf{p})$, $p = |\mathbf{p}|$, and the spectrum is

$$E(p) = \frac{1}{2m}\sqrt{(p^2 - X)(p^2 - Y)}. \tag{16}$$

The thermodynamic potential in the mean-field approximation is as usual given by minus the static part of the Lagrangian,

$$\Omega_0(\mu_{\text{NR}}, v) = -\mu_{\text{NR}}v^2 + \frac{1}{4}gv^4. \tag{17}$$

The linear term vanishes at the minimum $v_0 = \sqrt{\frac{2\mu_{\text{NR}}}{g}}$ of the thermodynamic potential $\Omega_0(\mu_{\text{NR}}, v)$. At the minimum, Eq. (16) reduces to the Bogoliubov spectrum $E_p = \frac{p}{2m}\sqrt{p^2 + 4m\mu_{\text{NR}}}$. The dispersion relation is gapless and linear for small momenta $p^2 \ll 4m\mu_{\text{NR}}$ and that of a free nonrelativistic particle for large momenta $p^2 \gg 4m\mu_{\text{NR}}$. The length scale $1/\sqrt{4m\mu_{\text{NR}}}$ is referred to as the coherence length. This is the Goldstone mode, which is the result of the spontaneous breaking of the $U(1)$ phase symmetry mentioned above. The NLO pressure is given by the thermodynamic potential evaluated at the classical minimum v_0 cf. Eq. (11). This is convenient since $X = 0$. The NLO pressure is

$$\mathcal{P}(\mu_{\text{NR}}) = -\Omega_0(\mu_{\text{NR}}, v_0) - \Omega_1(\mu_{\text{NR}}, v_0)$$
$$= \frac{\mu_{\text{NR}}^2}{g} - \frac{1}{2}\int_p E(p) = \frac{\mu_{\text{NR}}^2}{g} - \frac{1}{4m}I_{0,-1}(M^2)$$
$$= \frac{\mu_{\text{NR}}^2}{g}\left[1 - \frac{16(4m)^{\frac{3}{2}}\sqrt{\mu_{\text{NR}}g^2}}{15(4\pi)^2}\right]. \tag{18}$$

where the integrals $I_{m,n}(M^2)$ are defined in Eq. (A12), $I_{0,-1}(M^2)$ is defined in Eq. (A15), and $M^2 = 4m\mu_{NR}$. The density is then given by

$$n(\mu_{NR}) = \frac{dP}{d\mu} = \frac{2\mu_{NR}}{g} - \frac{1}{2}I_{1,1}(M^2)$$
$$= \frac{2\mu_{NR}}{g}\left[1 - \frac{4(4m)^{\frac{3}{2}}\sqrt{\mu_{NR}}g^2}{3(4\pi)^2}\right], \qquad (19)$$

where we have used Eq. (16) in the last line. We can invert Eq. (19) to obtain the chemical potential in terms of the number density. To the order we are calculating, we obtain

$$\mu_{NR}(n) = \frac{1}{2}gn + \frac{1}{4}gI_{1,1}(2mgn)$$
$$= \frac{4\pi na}{m}\left[1 + \frac{32}{3\sqrt{\pi}}\sqrt{na^3}\right]. \qquad (20)$$

The energy density is then

$$\mathcal{E}(n) = -\mathcal{P}(\mu_{NR}) + \mu_{NR}(n)n,$$
$$= \frac{1}{4}gn^2 + \frac{1}{4m}I_{0,-1}(2mgn), \qquad (21)$$

where we have used the LO relation $2\mu_{NR} = gn$ in the integral, which is correct to the order we are calculating. Note that the terms involving $I_{1,1}(\mu_{NR})$ cancel in the final result for the energy density. We then obtain the result of Lee, Huang, and Yang [14,15],

$$\mathcal{E}(n) = \frac{2\pi n^2 a}{m}\left[1 + \frac{128}{15\sqrt{\pi}}\sqrt{na^3}\right]. \qquad (22)$$

It is amusing to note that all the divergent integrals appearing in this calculation are finite in dimensional regularization in the limit $d \to 3$. The reason is simply that the UV divergences are power divergences which are always set to zero. Using a more conventional three-dimensional cutoff Λ requires the renormalization of g and μ to obtain the finite result above [13].

The quantum field theory in Eq. (1) is nonrenormalizable. This implies that the UV divergences that show up in the calculations of e.g. the energy density at higher loop orders cannot be removed by renormalizing the couplings of the terms involved in the calculation. This is exactly what happens at NNLO in the low-density expansion. Some of the relevant two-loop vacuum graphs arising from the operator $(\psi^\dagger\psi)^2$ have logarithmic divergences proportional to a^4n^3. These divergences are canceled by the counterterm for g_3 that multiplies the operator $(\psi^\dagger\psi)^3$ [13].

IV. CHIRAL PERTURBATION THEORY

Chiral perturbation theory is a low-energy effective theory for QCD that describes the pseudo-Goldstone bosons [27–30], where the $SU(2)_L \times SU(2)_R$ global symmetry of QCD (for two flavors) is realized nonlinearly. The vacuum manifold is $SU(2)_L \times SU(2)_R/SU(2)_V \simeq SU(2)$ and parametrized as $\Sigma = e^{i\phi_a\tau_a/f}$, where ϕ_a are the Goldstone fields, f is the bare pion-decay constant, and τ_a are the broken generators.

The chiral Lagrangian has a systematic low-energy expansion. Each covariant derivative counts as one power of momentum p and each quark mass term counts as two powers of p. Using this power-counting scheme, one writes down all possible terms at each order in the expansion. At leading order, there are two terms in the chiral Lagrangian. For two flavors, it reads

$$\mathcal{L}_2 = \frac{1}{4}f^2\langle\nabla^\mu\Sigma^\dagger\nabla_\mu\Sigma\rangle + \frac{1}{4}f^2\langle\chi^\dagger\Sigma + \Sigma^\dagger\chi\rangle, \quad (23)$$

where $\langle A\rangle$ denotes the trace of a 2×2 matrix A and the covariant derivatives are

$$\nabla_\mu\Sigma = \partial_\mu\Sigma - i[v_\mu, \Sigma], \qquad (24)$$

$$\nabla_\mu\Sigma^\dagger = \partial_\mu\Sigma^\dagger - i[v_\mu, \Sigma^\dagger], \qquad (25)$$

where $v_\mu = \frac{1}{2}\mu_I\tau_3\delta_{\mu 0}$, μ_I is the isospin chemical potential, and $\chi = 2B_0\text{diag}(m_u, m_d)$. The constant B_0 is related to the tree-level values of the light quark condensates in the vacuum via $\langle\bar{d}d\rangle = \langle\bar{u}u\rangle = -f^2B_0$. In the remainder of this paper, we work in the isospin limit, $m_u = m_d$.

The ground state for two-flavor QCD is of the form $\Sigma = e^{i\phi_a\tau_a/f}$ for constant fields ϕ_a. It is convenient to reparametrize the constant fields defining $\hat{\phi}_a$ and α via $\phi_a = \hat{\phi}_a\alpha f$ with the constraint $\hat{\phi}_1^2 + \hat{\phi}_2^2 + \hat{\phi}_3^2 = 1$. The ground state is now denoted by $\Sigma_\alpha = e^{i\hat{\phi}_a\tau_a\alpha}$ and is properly normalized, $\Sigma_\alpha^\dagger\Sigma_\alpha = \mathbb{1}$. In the QCD vacuum, the ground state Σ_0 corresponds to $\phi_a = \alpha = 0$, which is simply the unit matrix, $\Sigma_0 = \mathbb{1}$. Using the normalization of $\hat{\phi}_a$ and the properties of the Pauli matrices, we can write the ground state as

$$\Sigma_\alpha = e^{i\hat{\phi}_a\tau_a\alpha} = \mathbb{1}\cos\alpha + i\hat{\phi}_a\tau_a\sin\alpha. \qquad (26)$$

The parameter α can be thought of as a rotation angle, where the QCD vacuum ($\alpha = 0$) with a quark condensate is rotated into a state with a nonzero pion condensate. We can further restrict the values of $\hat{\phi}_a$ by considering the static Hamiltonian $\mathcal{H}_{static}$ corresponding to the static part of the Lagrangian Eq. (23). It reads

$$\mathcal{H}_{static} = -\frac{1}{4}f^2\langle\chi^\dagger\Sigma + \Sigma^\dagger\chi\rangle + \frac{1}{16}f^2\mu_I^2\langle[\Sigma^\dagger, \tau_3][\Sigma, \tau_3]\rangle. \quad (27)$$

Substituting Eq. (26) into Eq. (27), we obtain the ground-state energy density

$$\langle \mathcal{H}_{\text{static}} \rangle = -f^2 m_{\pi,0}^2 \cos \alpha - \frac{1}{2} f^2 \mu_I^2 (\hat{\phi}_1^2 + \hat{\phi}_2^2) \sin^2 \alpha. \quad (28)$$

Here $m_{\pi,0}$ is the tree-level pion mass, which satisfies $m_{\pi,0}^2 = B_0(m_u + m_d)$. The first term is minimized for $\alpha = 0$. i.e. the QCD vacuum, while the second term is minimized for $\alpha = \frac{1}{2}\pi$. The second term is minimized for $\hat{\phi}_1^2 + \hat{\phi}_2^2 = 1$ and therefore $\hat{\phi}_3 = 0$. Without loss of generality we can choose $\hat{\phi}_2 = 1$, which we do henceforth. The ground state can now be compactly written as

$$\Sigma_\alpha = A_\alpha \Sigma_0 A_\alpha, \quad (29)$$

with $A_\alpha = e^{\frac{1}{2} i \tau_2 \alpha}$.

We have rotated the ground state according to Eq. (26), however, it turns out that the naive expression for Σ, $\Sigma = U\Sigma_\alpha U$, where $U = e^{i\phi_a \tau_a/2f}$, is no longer valid [38]. Instead of using $\Sigma = U\Sigma_\alpha U$, we must use

$$\Sigma = L_\alpha \Sigma_\alpha R_\alpha^\dagger, \quad (30)$$

with

$$L_\alpha = A_\alpha U A_\alpha^\dagger, \qquad R_\alpha = A_\alpha^\dagger U^\dagger A_\alpha. \quad (31)$$

The correct expression for Σ then reads

$$\Sigma = A_\alpha U^2 A_\alpha. \quad (32)$$

V. THERMODYNAMICS TO $\mathcal{O}(p^4)$

Using the parametrization Eq. (32), we can expand the LO chiral Lagrangian in powers of the fields ϕ_a,

$$\mathcal{L}_2 = \mathcal{L}_2^{(0)} + \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} + \mathcal{L}_2^{(3)} + \mathcal{L}_2^{(4)} + \cdots, \quad (33)$$

where the superscript indicates the number of fields and where

$$\mathcal{L}_2^{(0)} = f^2 m_{\pi,0}^2 \cos \alpha + \frac{1}{2} f^2 \mu_I^2 \sin^2 \alpha, \quad (34)$$

$$\mathcal{L}_2^{(1)} = -f m_{\pi,0}^2 \sin \alpha \phi_2 + f\mu_I^2 \sin \alpha \cos \alpha \phi_2 - f\mu_I \sin \alpha \partial_0 \phi_1, \quad (35)$$

$$\mathcal{L}_2^{(2)} = \frac{1}{2} \partial^\mu \phi_a \partial_\mu \phi_a + \frac{1}{2} m_{12}(\phi_1 \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1) - \frac{1}{2} m_a^2 \phi_a^2, \quad (36)$$

$$\mathcal{L}_2^{(3)} = \frac{m_{\pi,0}^2 \sin \alpha - 4\mu_I^2 \sin \alpha \cos \alpha}{6f} \phi_2 \phi_a \phi_a + \frac{\mu_I \sin \alpha}{f} \partial_0 \phi_1 (\phi_2^2 + \phi_3^2), \quad (37)$$

$$\mathcal{L}_2^{(4)} = \frac{1}{24f^2} \phi_a \phi_a [(m_{\pi,0}^2 \cos \alpha - 4\mu_I^2 \cos^2\alpha)\phi_1^2 + (m_{\pi,0}^2 \cos \alpha - 4\mu_I^2 \cos 2\alpha)\phi_2^2 + (m_{\pi,0}^2 \cos \alpha + 4\mu_I^2 \sin^2\alpha)\phi_3^2] - \frac{\mu_I \cos \alpha}{3f^2} \phi_a \phi_a (\phi_1 \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1) + \frac{1}{6f^2} [\phi_a \phi_b \partial^\mu \phi_a \partial_\mu \phi_b - \phi_a \phi_a \partial^\mu \phi_b \partial_\mu \phi_b], \quad (38)$$

where the masses are

$$m_1^2 = m_{\pi,0}^2 \cos \alpha - \mu_I^2 \cos^2 \alpha, \quad (39)$$

$$m_2^2 = m_{\pi,0}^2 \cos \alpha - \mu_I^2 \cos 2\alpha, \quad (40)$$

$$m_3^2 = m_{\pi,0}^2 \cos \alpha + \mu_I^2 \sin^2 \alpha, \quad (41)$$

$$m_{12} = 2\mu_I \cos \alpha. \quad (42)$$

From the quadratic term, the inverse propagator is

$$D^{-1}(P) = \begin{pmatrix} D_{12}^{-1}(P) & 0 \\ 0 & P^2 - m_3^2 \end{pmatrix}, \quad (43)$$

$$D_{12}^{-1}(P) = \begin{pmatrix} P^2 - m_1^2 & ip_0 m_{12} \\ -ip_0 m_{12} & P^2 - m_2^2 \end{pmatrix}. \quad (44)$$

The dispersion relations are found by solving $\det D^{-1}(P) = 0$. One finds

$$E_\pm^2(p) = p^2 + \frac{1}{2}(m_1^2 + m_2^2 + m_{12}^2) \pm \frac{1}{2}\sqrt{4p^2 m_{12}^2 + (m_1^2 + m_2^2 + m_{12}^2)^2 - 4m_1^2 m_2^2}, \quad (45)$$

$$E_3^2(p) = p^2 + m_3^2. \quad (46)$$

At LO, the thermodynamic potential is minus the static Lagrangian Eq. (34),

$$\Omega_0(\mu_I, \alpha) = -f^2 m_{\pi,0}^2 \cos \alpha - \frac{1}{2} f^2 \mu_I^2 \sin^2 \alpha. \quad (47)$$

The value α_0 that extremizes $\Omega_0(\mu_I, \alpha)$ satisfies $\cos \alpha_0 = m_{\pi,0}^2/\mu_I^2$. Note that the transition from the vacuum phase ($\alpha = 0$) to the pion-condensed phase ($\alpha > 0$) takes place at $\mu_I = m_{\pi,0}$, i.e. the tree-level value of the physical

pion mass m_π. The onset of BEC at the physical pion mass is expected to hold to all orders in the low-energy expansion. It follows from Eq. (47) that the pressure and the other thermodynamic quantities are independent of μ_I for $\mu_I \in [0, m_{\pi,0}]$, i. e. before the onset of Bose condensation. This is an example of the Silver-Blaze property [39]. Subtracting the constant pressure in the vacuum, the LO pressure in the BEC phase,

$$\mathcal{P}_0 = \frac{1}{2}f^2\mu_I^2 \left[1 - \frac{m_{\pi,0}^2}{\mu_I^2}\right]^2, \qquad \mu_I \geq m_{\pi,0}. \quad (48)$$

At $\mathcal{O}(p^4)$, the chiral Lagrangian at finite chemical potential is [28]

$$\mathcal{L}_4 = \frac{1}{4}l_1\langle\nabla_\mu\Sigma^\dagger\nabla^\mu\Sigma\rangle^2 + \frac{1}{4}l_2\langle\nabla_\mu\Sigma^\dagger\nabla_\nu\Sigma\rangle\langle\nabla^\mu\Sigma^\dagger\nabla^\nu\Sigma\rangle$$
$$+ \frac{1}{16}(l_3 + l_4)\langle\chi^\dagger\Sigma + \Sigma^\dagger\chi\rangle^2$$
$$+ \frac{1}{8}l_4\langle\nabla_\mu\Sigma^\dagger\nabla^\mu\Sigma\rangle\langle\chi^\dagger\Sigma + \Sigma^\dagger\chi\rangle + \frac{1}{2}h_1\langle\chi^\dagger\chi\rangle, \quad (49)$$

where l_1–l_4 and h_1 are bare coupling constants. In a next-to-leading order calculation, there are two contributions to the thermodynamic potential. The first one is the one-loop functional determinant of bosonic fluctuations and the second is the static part of $\mathcal{L}_4$. The functional determinant is regularized using dimensional regularization and contains poles in ϵ, where $d = 3 - 2\epsilon$. These ultraviolet divergences are removed by renormalizing the constants l_i and h_1. The relations between the bare couplings and their renormalized counterparts are

$$l_i = l_i^r - \frac{\gamma_i\Lambda^{-2\epsilon}}{2(4\pi)^2}\left[\frac{1}{\epsilon} + 1\right], \quad h_i = h_i^r - \frac{\delta_i\Lambda^{-2\epsilon}}{2(4\pi)^2}\left[\frac{1}{\epsilon} + 1\right], \quad (50)$$

where Λ is the renormalization scale in the $\overline{\text{MS}}$ scheme and γ_i and δ_i are pure numbers,

$$\gamma_1 = \frac{1}{3}, \qquad \gamma_2 = \frac{2}{3}, \qquad \gamma_3 = -\frac{1}{2}, \quad (51)$$

$$\gamma_4 = 2, \qquad \delta_1 = 0. \quad (52)$$

The bare quantities are independent of the scale Λ, which implies that the renormalized couplings satisfy simple renormalization group equations. In the two flavor-case, it is convention to introduce the quantities $\bar{l}_i$ via the solutions to these equations in the limit $\epsilon \to 0$

$$l_i^r(\Lambda) = \frac{\gamma_i}{2(4\pi)^2}\left[\bar{l}_i + \log\frac{m_{\pi,0}^2}{\Lambda^2}\right]. \quad (53)$$

Up to a prefactor, $\bar{l}_i$ equals the running coupling l_i^r evaluated at the scale $\Lambda = m_{\pi,0}$. Note that since $\delta_1 = 0$,

the coupling h_1^r does not depend on Λ. The contact term $\frac{1}{2}h_1\langle\chi^\dagger\chi\rangle$ yields a constant contribution $h_1 m_{\pi,0}^4 = h_1^r m_{\pi,0}^4$ to the pressure, which will be ignored in the following. In Sec. VII, we discuss the chiral limit, i.e. $m_{\pi,0} = 0$. In this case, we keep the running couplings $l_i^r(\Lambda)$.

The contribution from $\mathcal{L}_4^{\text{static}}$ to the thermodynamic potential is

$$\Omega_1^{\text{static}}(\mu_I, \alpha) = -(l_1 + l_2)\mu_I^4\sin^4\alpha - l_4 m_{\pi,0}^2\mu_I^2 \cos\alpha\sin^2\alpha$$
$$- (l_3 + l_4)m_{\pi,0}^4\cos^2\alpha, \quad (54)$$

The one-loop contribution to the thermodynamic potential follows directly from the inverse propagator. After going to Euclidean space the result is

$$\Omega_1^{\text{loop}}(\mu_I, \alpha) = \frac{1}{2}\int_P \log\left[(P^2 + m_1^2)(P^2 + m_2^2) + p_0^2 m_{12}^2\right]$$
$$+ \frac{1}{2}\int_P \log\left[P^2 + m_3^2\right]. \quad (55)$$

The pressure $\mathcal{P}$ at NLO is found by evaluating the thermodynamic potential at the LO minimum satisfying $\cos\alpha_0 = \frac{m_{\pi,0}^2}{\mu_I^2}$, as explained. This simplifies the calculations somewhat since $m_1^2 = 0$. We therefore need to evaluate

$$\mathcal{P}_1^{\text{loop}}(\mu_I) = -\frac{1}{2}\int_P \log\left[P^2(P^2 + m_2^2) + p_0^2 m_{12}^2\right]$$
$$- \frac{1}{2}\int_P \log\left[P^2 + m_3^2\right]$$
$$= -\frac{1}{2}\int_P \log\left[P^2 + m_2^2\right] - \frac{1}{2}\int_P \log\left[P^2 + m_3^2\right]$$
$$+ \frac{1}{2}\sum_{n=1}^\infty \frac{(-1)^n m_{12}^{2n}}{n}\int_P \frac{p_0^{2n}}{P^{2n}(P^2 + m_2^2)^n}, \quad (56)$$

where we in the last line have expanded the logarithm of the first term in powers of $z = m_{12}^2/m_2^2$. After integrating over angles in d dimensions, we can write

$$\mathcal{P}_1^{\text{loop}}(\mu_I) = \frac{1}{2}I_0'(m_2^2) + \frac{1}{2}I_0'(m_3^2) + \frac{\Gamma(2 - \epsilon)}{2\Gamma\left(\frac{1}{2}\right)}$$
$$\times \sum_{n=1}^\infty \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma(n + 2 - \epsilon)}\frac{(-1)^n m_{12}^{2n}}{n}I_n(m_2^2). \quad (57)$$

We single out the two divergent terms with $n = 1, 2$ and resum the rest of the series [40]. In the remaining finite terms ($n \geq 3$ in the sum), we can set $d = 3$. This yields

$$\mathcal{P}_1^{\text{loop}}(\mu_I) = \frac{1}{2}I_0'(m_2^2) + \frac{1}{2}I_0'(m_3^2) - \frac{m_{12}^2}{2(d+1)}I_1(m_2^2)$$
$$+ \frac{3m_{12}^4}{4(d+1)(d+3)}I_2(m_2^2) - \frac{5m_{12}^6}{768(4\pi)^2 m_2^2}$$
$$\times {}_3F_2\left[\begin{matrix} 1, & 1, & \frac{7}{2} \\ & 4,5 \end{matrix}\,\middle|\, -\frac{m_{12}^2}{m_2^2}\right], \tag{58}$$

$${}_3F_2\left[\begin{matrix} 1, & 1, & \frac{7}{2} \\ & 4,5 \end{matrix}\,\middle|\, z\right] = \frac{16}{5}\left[\frac{(3z^2 - 10z - 8)(1-\sqrt{1-z})}{z^4}\right.$$
$$+ \frac{z^2+4}{z^3} - 3\frac{z^2 - 4z + 8}{z^3}$$
$$\left.\times \log\frac{1+\sqrt{1-z}}{2}\right]. \tag{59}$$

where ${}_3F_2$ is a hypergeometric function [41], which is given by

We briefly discuss hypergeometric functions in the Appendix. Combining Eqs. (48), (54) and (58) together with Eqs. (50) and (53), the pressure is

$$\mathcal{P}_{0+1}(\mu_I) = \frac{1}{2}f^2\mu_I^2\left[1 - \frac{m_\pi^2}{\mu_I^2}\right] - \frac{1}{2}f^2\frac{m_{\pi,0}^4}{m_\pi^2}\left[1 - \frac{m_\pi^2}{\mu_I^2}\right] + \frac{m_{\pi,0}^8}{6(4\pi)^2\mu_I^4}\left[\bar{l}_1 + 2\bar{l}_2 - \frac{3}{2}\bar{l}_3 - \frac{5}{4}\right]$$
$$+ \frac{3}{2}\log\frac{m_{\pi,0}^2\mu_I^2}{\mu_I^4 - m_{\pi,0}^4}\right] + \frac{\mu_I^4}{6(4\pi)^2}\left[\bar{l}_1 + 2\bar{l}_2 + \frac{3}{2} + \frac{3}{2}\log\frac{m_{\pi,0}^4}{\mu_I^4 - m_{\pi,0}^4}\right] - \frac{5m_{\pi,0}^{12}}{12(4\pi)^2(\mu_I^4 - m_{\pi,0}^4)\mu_I^4}$$
$$\times {}_3F_2\left[\begin{matrix} 1, & 1, & \frac{7}{2} \\ & 4,5 \end{matrix}\,\middle|\, -\frac{4m_{\pi,0}^4}{\mu_I^4 - m_{\pi,0}^4}\right] - \frac{m_{\pi,0}^4}{3(4\pi)^2}\left[\bar{l}_1 + 2\bar{l}_2 - \frac{3}{4}\bar{l}_3 + \frac{9}{8}\right], \tag{60}$$

where we have written the argument of the hypergeometric function as $-m_{12}^2/m_2^2 = -4m_{\pi,0}^4/(\mu_I^4 - m_{\pi,0}^4)$ and added a constant such that the pressure vanishes at $\mu_I = m_\pi$ (in the loop corrections, $m_\pi^2 = m_{\pi,0}^2$ to the order we are calculating). From the pressure, one can calculate the isospin density n_I and the energy density $\mathcal{E}$ using the standard thermodynamic relations.

We close this section by making a comparison of our χPT result for the pressure with recent lattice data [25] using physical quark masses. The pion mass is $m_\pi = 135$ MeV and the pion decay constant was slightly different in the two simulations, $f_\pi = \frac{130\pm3}{\sqrt{2}}$ MeV and $f_\pi = \frac{136\pm4}{\sqrt{2}}$ MeV, respectively. The simulations are done with two different lattice spacings, $a \approx 0.22$ fm and $a \approx 0.15$ fm. The results were not continuum extrapolated.

To make a reasonable comparison with the data, we choose $m_\pi = 135$ MeV and $f_\pi = \frac{133}{\sqrt{2}}$ MeV. At LO, we identify the parameters $m_{\pi,0}$ and f with the corresponding physical values. At NLO, we need the relations between the bare parameters $m_{\pi,0}$ and f and the physical observables at NLO,

$$m_\pi^2 = m_{\pi,0}^2\left[1 - \frac{m_{\pi,0}^2}{2(4\pi)^2 f^2}\bar{l}_3\right], \tag{61}$$

$$f_\pi^2 = f^2\left[1 + \frac{2m_{\pi,0}^2}{(4\pi)^2 f^2}\bar{l}_4\right]. \tag{62}$$

In order to invert these relations to solve for $m_{\pi,0}$ and f, we need the experimental values of $\bar{l}_3$ and $\bar{l}_4$. We also need the

numerical values for the couplings $\bar{l}_1$ and $\bar{l}_2$ appearing in Eq. (60). The numerical values for the couplings $\bar{l}_i$ are taken from Ref. [42] and read $\bar{l}_1 = -0.4 \pm 0.6$, $\bar{l}_2 = 4.3 \pm 0.1$, and $\bar{l}_3 = 2.9 \pm 2.4$, and $\bar{l}_4 = 4.4 \pm 0.2$. This yields $m_{\pi,0} = 136.50 \pm 1.2$ MeV and $f = 88.35 \pm 1.9$ MeV, respectively. The results for the pressure is shown in Fig. 1. The LO result is given by the black dashed line and the NLO result by the red solid curve, where we have used the central values of the parameters and couplings. Going from LO to
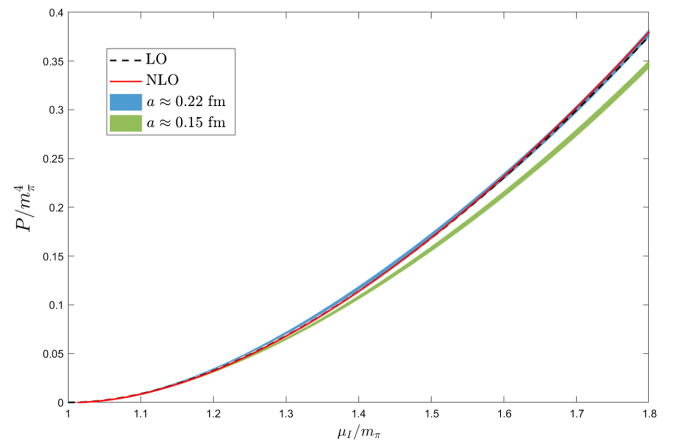

FIG. 1. The normalized pressure as a function of the normalized isospin chemical potential. The LO level and NLO level results are the black dashed and red solid lines, respectively. The blue and green bands are from two simulations with different lattice spacings [25]. See main text for details.

NLO in χPT hardly changes the result for the pressure. The agreement between the prediction between χPT and lattice simulations is very good up to $\mu_I/m_\pi \approx 1.5$. Of course, the comparison should be taken with a grain of salt since our results are for two flavors. A more detailed comparison with three-flavor χPT will be presented elsewhere [43].

VI. NONRELATIVISTIC LIMIT

In this section, we consider the nonrelativistic limit of the final expression for the pressure Eq. (60). In this limit, we will recover the classic results for thermodynamic quantities for the dilute Bose gas obtained over six decades ago, which we briefly discussed in Sec. III. It turns out that the nonrelativistic loop corrections are hidden in the hypergeometric function. It will be useful to express the results in terms of the physical pion mass m_π and the s-wave scattering length a, both calculated to one-loop and given in terms of the bare parameters $m_{\pi,0}$ and f, as well as low-energy constants $\bar{l}_1$, $\bar{l}_2$, and $\bar{l}_3$ [28,44,45]. The expression we need is

$$a_0^2 = -\frac{m_{\pi,0}^2}{4(4\pi)f^2}\left[1 - \frac{4m_{\pi,0}^2}{3(4\pi)^2 f^2}\left(\bar{l}_1 + 2\bar{l}_2 + \frac{3}{8}\right)\right], \quad (63)$$

where the s-wave scattering length is $a = -a_0^2/m_\pi$. We first write the isospin chemical potential $\mu_I = m_\pi + \mu_{\text{NR}}$, where m_π is the physical pion mass as given by Eq. (61) and μ_{NR} is the usual nonrelativistic chemical potential. Expanding the pressure in powers of μ_{NR} up to order $\mu_{\text{NR}}^{\frac{5}{2}}$, using Eqs. (61) and (63), we find

$$\mathcal{P} = \frac{m_\pi}{8\pi a}\mu_{\text{NR}}^2\left[1 - \frac{32}{15\pi}\sqrt{4m_\pi\mu_{\text{NR}}a^2}\right]. \quad (64)$$

The isospin density n_I is found using Eq. (6)

$$n_I = \frac{m_\pi}{4\pi a}\mu_{\text{NR}}\left[1 - \frac{8}{3\pi}\sqrt{4m_\pi\mu_{\text{NR}}a^2}\right]. \quad (65)$$

Inverting this equation to find the chemical potential as a function of n_I, we obtain

$$\mu_{\text{NR}} = \frac{4\pi n_I a}{m_\pi}\left[1 + \frac{32}{3\sqrt{\pi}}\sqrt{n_I a^3}\right]. \quad (66)$$

We can then finally calculate the nonrelativistic energy density using Eq. (7). This yields

$$\mathcal{E} = m_\pi n_I + \frac{2\pi n_I^2 a}{m_\pi}\left[1 + \frac{128}{15\sqrt{\pi}}\sqrt{n_I a^3}\right]. \quad (67)$$

The first term in Eq. (67) is the contribution to $\mathcal{E}$ associated with the rest mass m_π of the bosons. This term is absent in Eq. (2) since it is automatically removed by subtracting the rest mass energy in the nonrelativistic Lagrangian Eq. (1). The second term is the Bogoliubov mean-field term with loop corrections absorbed into the physical scattering length. The last term is the LHY correction term, where we have included these loop corrections as they are of higher orders. Similar results were recently obtained in Ref. [36] for interacting scalars.

VII. PRESSURE TO $\mathcal{O}(p^6)$ IN THE CHIRAL LIMIT

In the chiral limit, the calculations simplify significantly. The value of α that is a solution to the equation of motion is $\alpha = \frac{1}{2}\pi$ and the quark condensate is rotated into a pion condensate for any nonzero value of μ_I. The propagator becomes diagonal and the masses reduce to $m_1^2 = 0$, $m_2^2 = \mu_I^2$, and $m_3^2 = \mu_I^2$. Similarly, the counterterms as well as the cubic and quartic interaction terms are much simpler in this limit,

$$\mathcal{L}_2^{(3)} = \frac{\mu_I}{f}\partial_0\phi_1[\phi_2^2 + \phi_3^2], \quad (68)$$

$$\mathcal{L}_2^{(4)} = \frac{1}{6f^2}[\phi_a\phi_b(\partial_\mu\phi_a)(\partial^\mu\phi_b) - \phi_a\phi_a(\partial_\mu\phi_b)(\partial^\mu\phi_b)]$$
$$+ \frac{\mu_I^2}{6f^2}\phi_a\phi_a[\phi_2^2 + \phi_3^2], \quad (69)$$

$$\mathcal{L}_4^{(2)} = -(l_1 + l_2)\frac{2\mu_I^4}{f^2}[\phi_2^2 + \phi_3^2] + l_1\frac{2\mu_I^2}{f^2}\partial_\mu\phi_a\partial^\mu\phi_a$$
$$+ l_1\frac{4\mu_I^2}{f^2}(\partial_0\phi_1)^2 + l_2\frac{2\mu_I^2}{f^2}(\partial_0\phi_a)^2$$
$$+ l_2\frac{2\mu_I^2}{f^2}(\partial_0\phi_1)^2 + l_2\frac{2\mu_I^2}{f^2}\partial_\mu\phi_1\partial^\mu\phi_1. \quad (70)$$

The pressure at $\mathcal{O}(p^4)$ follows from Eq. (60) by taking the limit $m_{\pi,0}^2 \to 0$ and reinstating l_1^r and l_2^r. This yields

$$\mathcal{P}_{0+1} = \frac{1}{2}f^2\mu_I^2 + \mu_I^4\left[l_1^r + l_2^r + \frac{1}{4(4\pi)^2}\left(1 + 2\log\frac{\Lambda^2}{\mu_I^2}\right)\right], \quad (71)$$

where we notice that the term involving the hypergeometric function vanishes.

The Feynman diagrams contributing to the thermodynamic potential at order $\mathcal{O}(p^6)$ are shown in Fig. 2. The corresponding contribution to the pressure reads
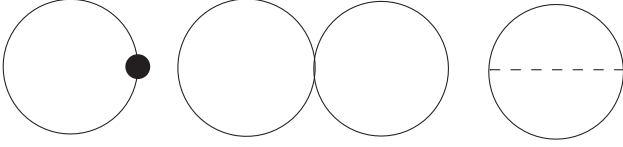
FIG. 2.   One and two-loop Feynman graphs contributing to the thermodynamic potential at $\mathcal{O}(p^6)$. Dashed lines represent the massless Goldstone boson and solid lines represent mesons with masses $m_2 = m_3 = \mu_I$. The black dot represents counterterm insertions from Eq. (70).

$$\mathcal{P}_2^{\text{loops}} = \frac{\mu_I^2}{6f^2}[3I_1^2(m_2^2) + 3I_1^2(m_3^2) + 2I_1(m_2^2)I_1(m_3^2)]$$

$$- \frac{1}{6f^2}[m_2^2 + m_3^2]I_1(m_2^2)I_1(m_3^2) - \frac{\mu_I^2}{f^2}[J(m_2^2) + J(m_3^2)]$$

$$- (l_1 + l_2)\frac{2\mu_I^4}{f^2}[I_1(m_2^2) + I_1(m_3^2)] + l_1 \frac{2\mu_I^2}{f^2}[m_2^2 I_1(m_2^2)$$

$$+ m_3^2 I_1(m_3^2)] + l_2 \frac{2\mu_I^2}{f^2}\left[\frac{m_2^2}{d+1}I_1(m_2^2) + \frac{m_3^2}{d+1}I_1(m_3^2)\right], \tag{72}$$

where the integral $J(m^2)$ is defined in Eq. (A9). The corresponding Feynman graphs are shown in Fig. 2. The integral $J(m^2)$ stems from the setting-sun diagram. Note also that the double-bubble graph with one massless particle and the massless bubble with counterterm insertions vanish in dimensional regularization since there is no mass scale in the integrals, thus the corresponding terms are not included in Eq. (72). Using $m_2 = m_3 = \mu_I$ and the expression for $J(m^2)$, Eq. (A11), the contribution reduces to

$$\mathcal{P}_2^{\text{loops}} = \frac{d-1}{d+1}\frac{\mu_I^2}{f^2}I_1^2(\mu_I^2) - l_2\frac{4d}{d+1}\frac{\mu_I^4}{f^2}I_1(\mu_I^2). \tag{73}$$

Note that the l_1-dependence drops out.

The $\mathcal{O}(p^6)$ Lagrangian contains a larger number of terms, 57 for $SU(2)$ and 94 for $SU(3)$ [30,46]. Most of them vanish in the chiral limit and for two flavors the NNLO Lagrangian reduces to

$$\mathcal{L}_6 = C_{24}\langle(\nabla_\mu\Sigma^\dagger\nabla^\mu\Sigma)^3\rangle$$

$$+ C_{25}\langle\nabla_\rho\Sigma^\dagger\nabla^\rho\Sigma\nabla_\mu\Sigma^\dagger\nabla_\nu\Sigma\nabla^\mu\Sigma^\dagger\nabla^\nu\Sigma\rangle$$

$$+ C_{26}\langle\nabla_\mu\Sigma^\dagger\nabla_\nu\Sigma\nabla_\rho\Sigma^\dagger\nabla^\mu\Sigma\nabla^\nu\Sigma^\dagger\nabla^\rho\Sigma\rangle. \tag{74}$$

The contribution from the static part of $\mathcal{L}_6$ to the pressure is

$$\mathcal{P}_2^{\text{static}} = 2(C_{24} + C_{25} + C_{26})\mu_I^6. \tag{75}$$

The relation between the bare couplings C_i and the renormalized couplings C_i^r is defined as

$$C_i = \frac{C_i^r\Lambda^{-4\epsilon}}{f^2} - \frac{\gamma_i^{(2)}\Lambda^{-4\epsilon}}{4(4\pi)^4 f^2}\left[\frac{1}{\epsilon} + 1\right]^2$$

$$+ \frac{(\gamma_i^{(1)}\Lambda^{-2\epsilon}(1+\epsilon) + \gamma_i^{(L)})\Lambda^{-2\epsilon}}{2(4\pi)^2 f^2}\left[\frac{1}{\epsilon} + 1\right]. \tag{76}$$

The coefficients needed are

$$\gamma_{24}^{(1)} = -\frac{1}{(4\pi)^2}\frac{9}{32}, \qquad \gamma_{25}^{(1)} = -\frac{1}{(4\pi)^2}\frac{67}{432}, \tag{77}$$

$$\gamma_{26}^{(1)} = \frac{1}{(4\pi)^2}\frac{449}{864}, \qquad \gamma_{24}^{(2)} = -\frac{137}{72}, \tag{78}$$

$$\gamma_{25}^{(2)} = \frac{5}{36}, \qquad \gamma_{26}^{(2)} = \frac{55}{72}, \tag{79}$$

$$\gamma_{24}^{(L)} = -2l_1^r - \frac{16}{3}l_2^r - \frac{5}{4}l_6^r, \quad \gamma_{25}^{(L)} = 2l_1^r - \frac{1}{3}l_2^r + \frac{1}{2}l_6^r, \tag{80}$$

$$\gamma_{26}^{(L)} = \frac{8}{3}l_2^r + \frac{3}{4}l_6^r. \tag{81}$$

We note that in the sum $C = C_{24} + C_{25} + C_{26}$ only l_2^r remains, which is needed to cancel the part of the divergences associated with l_2 in Eq. (73). Adding Eqs. (73) and (75), and renormalizing the couplings according to Eqs. (50) and (76), we obtain $\mathcal{P}_2 = \mathcal{P}_2^{\text{loops}} + \mathcal{P}_2^{\text{static}}$

$$\mathcal{P}_2 = 2C_r\frac{\mu_I^6}{f^2} - \frac{l_2^r\mu_I^6}{2(4\pi)^2 f^2}\left[1 - 6\log\frac{\Lambda^2}{\mu_I^2}\right]$$

$$- \frac{\mu_I^6}{24(4\pi)^4 f^2}\left[1 + 8\log\frac{\Lambda^2}{\mu_I^2} - 12\log^2\frac{\Lambda^2}{\mu_I^2}\right]. \tag{82}$$

The pressure can now be written as

$$\mathcal{P}_{0+1+2\ldots} = \frac{1}{2}f^2\mu_I^2 + a_1\mu_I^4 + a_2\frac{\mu_I^6}{f^2} + \cdots. \tag{83}$$

where a_1 can be read off Eq. (71) and a_2 from the expression for $\mathcal{P}_2$. This yields

$$a_1 = \left[l_1^r + l_2^r + \frac{1}{4(4\pi)^2}\left(1 + 2\log\frac{\Lambda^2}{\mu_I^2}\right)\right], \tag{84}$$

$$a_2 = 2C_r - \frac{l_2^r}{2(4\pi)^2}\left[1 - 6\log\frac{\Lambda^2}{\mu_I^2}\right]$$

$$- \frac{1}{24(4\pi)^4}\left[1 + 8\log\frac{\Lambda^2}{\mu_I^2} - 12\log^2\frac{\Lambda^2}{\mu_I^2}\right]. \tag{85}$$

The coefficient a_1 is independent of the scale Λ, which follows from the running of l_1^r and l_2^r, cf. Eq. (53). The coefficients a_2 does also not run, which follows from

the running of l_2^r and C_r, where the latter satisfies the renormalization group equation in the limit $\epsilon \to 0$

$$\Lambda \frac{dC^r}{d\Lambda} = \frac{1}{6(4\pi)^4} - \frac{3l_2^r}{(4\pi)^2}. \tag{86}$$

The independence of a_i on Λ guarantees that the pressure $\mathcal{P}$ is independent of the renormalization scale as well. Finally, we note that the effective expansion parameter is $\frac{\mu_I^2}{(4\pi)^2 f^2}$ and the chiral limit should therefore be a good approximation for $m_\pi \ll \mu_I \ll 4\pi f_\pi$.

VIII. DISCUSSION

So far in this paper, we have calculated the pressure to the next-to-leading order in chiral perturbation theory. We have taken the nonrelativistic limit of our result by writing $\mu_I = m_\pi + \mu_{\text{NR}}$ and shown that we recover Lee, Huang, and Yang's classic result for the energy density. In the ultrarelativistic limit, we have calculated the pressure to the next-to-next-to-leading order.

Assume now we are interested in the low-energy dynamics of the Goldstone bosons, i.e. in momenta p much smaller than the inverse coherence length in dilute Bose gases or more generally for momenta where the linear dispersion relation is a good approximation.[2] In this case, Son [47] derived a low-energy effective theory for the superfluid phonons. It is given in terms of the pressure $\mathcal{P}$ of the system as a function of the chemical potential (and possibly other quantities such as the pion mass or the quark mass) making the substitution $\mu \to \sqrt{(\partial_0 \phi - \mu)^2 + (\partial_i \phi)(\partial^i \phi)}$, or

$$\mathcal{L}_{\text{phonon}} = \mathcal{P}(\sqrt{\nabla_\mu \phi \nabla^\mu \phi}), \tag{87}$$

where the covariant derivative is $\nabla_\mu \phi = \partial_\mu \phi - \delta_{0\mu} \mu_I$. The only approximation that was made in the derivation, is that there are as many powers of the field ϕ as there are derivatives. Thus the dispersion relation will be linear and the effective theory will break down once there are non-negligible corrections to this [47].

The conventional view of Lagrangians such as Eq. (87) is that Lorentz invariance is explicitly broken by introducing a chemical potential as the zeroth component of a gauge field. However, in the case of a broken charge associated with μ, there is another equivalent view [5,10]. Since Q is broken, the ground state of the system is not an eigenstate of Q. It is not an eigenstate of the original Hamiltonian of the system either, but it is an eigenstate of the grand-canonical Hamiltonian $\mathcal{H}_Q = \mathcal{H} - \mu Q$. Since Q is broken, so are the original time translations generated by

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$\mathcal{H}$. However, the new generator of time translations defined by $\mathcal{H}_Q$ is unbroken. The ground state of the system is now time-dependent $\sim e^{-i\mu Q t}$, which one expands about. Using a time-dependent ground state, the system is described by the original Lorentz invariant Lagrangian. Clearly, the ground state breaks the original Lorentz invariance (as well as boosts), so instead of breaking this symmetry explicitly, one may consider it being broken spontaneously by the ground state. Lorentz invariance is then realized non-linearly as is the $U(1)$-symmetry, where the phase transforms as $\phi \to \phi + a$. The three broken boost generators and the broken internal $U(1)_{I_3}$ symmetry now appear, seemingly on the same footing, in the coset construction, each multiplied by a field. However, as pointed out in e.g. Ref. [8], the four fields do not represent four truly independent physical fluctuations. In fact some of them represent gauge redundancies and are removed by a gauge choice. In the present case, there is only one physical fluctuation, as expected.

We will use Son's prescription to derive an effective low-energy theory for the massless mode in dense QCD at finite isospin. This low-energy effective theory interpolates between the nonrelativistic regime and the ultrarelativistic regime dependent of the value of the dimensionless ratio $\mu_I/m_{\pi,0}$. Even at the tree level, the Lagrangian contains some interesting physics. Making the substitution $\mu_I \to \sqrt{(\partial_0 \phi - \mu_I)^2 + (\partial_i \phi)(\partial^i \phi)}$ in the LO pressure Eq. (48) and expanding the Lagrangian in powers of derivatives and rescaling the field, we obtain

$$\mathcal{L} = \frac{1}{2}(\partial_0 \phi)^2 - \frac{1}{2}c_s^2(\nabla\phi)^2 + c_1(\partial_0 \phi)^3 + \cdots, \tag{88}$$

where the speed of sound or phonon speed c_s and the coupling c_1 are

$$c_s = \sqrt{\frac{\mu_I^4 - m_{\pi,0}^4}{3m_{\pi,0}^4 + \mu_I^4}}, \tag{89}$$

$$c_1 = \frac{2m_{\pi,0}^4 \mu_I}{f} \frac{1}{(3m_{\pi,0}^4 + \mu_I^4)^{\frac{3}{2}}}. \tag{90}$$

Note that in the chiral limit, the Lagrangian Eq. (88) describes a free theory. The phonon speed interpolates between $c_s = 1$ in the ultrarelativistic limit (hence "speedy Goldstone bosons" in the title) $m_{\pi,0} \to 0$ and the nonrelativistic limit $c_s = \sqrt{\frac{\mu_{\text{NR}}}{m_{\pi,0}}}$, where the latter agrees with the Bogoliubov spectrum for small p. Moreover, the phonon speed c_s and the coefficients c_i are all subject to renormalization. The leading corrections to the coefficients can be

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[2]This includes the color-flavor locked (CFL) phase of dense QCD, where the momenta must be much smaller than the superconducting gap.
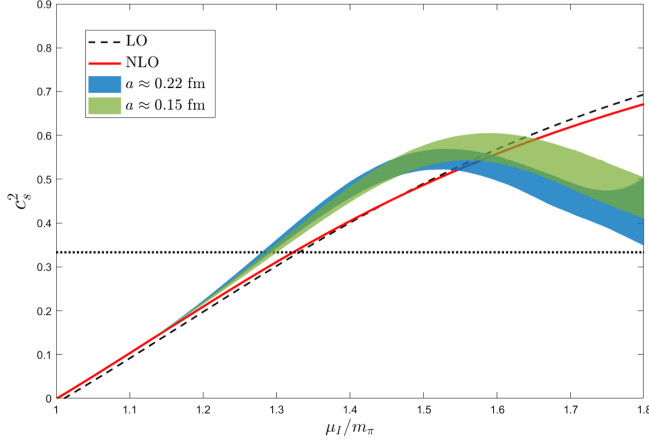
FIG. 3. Speed of sound squared c_s^2 as a function of the normalized isospin chemical potential. See. main text for details.

found by expanding the NLO effective Lagrangian using the NLO pressure. The speed of sound can also be found by using the thermodynamic relation

$$c_s^2 = \frac{d\mathcal{P}}{d\mathcal{E}}. \tag{91}$$

In Fig. 3, we show the χPT results for the speed of sound squared c_s^2 at LO (dashed black line) and NLO (solid red line) in the low-energy expansion. The red line corresponds to the central values of the parameters $m_{\pi,0}$, f, and $\bar{l}_i$ as before. The correction to the LO result is small in the entire region shown. The curves approach unity in the limit $\mu_I \to \infty$. The lattice data are of from different simulations using two different lattice spacings ($a \approx 0.22$ fm and $a \approx 0.15$ fm), respectively [25]. The bands indicate the uncertainty of the simulations. The agreement between the predictions from χPT and the lattice results is good for μ_I up to approximately $1.3m_\pi$. At μ_I around $1.55m_\pi$, the simulations show a peak for c_s and it thereafter decreases. At asymptotic values of μ_I, c_s is expected to approach the conformal limit of QCD, $c_s^2 = \frac{1}{3}$. This value is shown as the horizontal dotted line in the figure. We note that χPT fails to reproduce the behavior found on the lattice for larger values of the isospin chemical potential. This is expected since χPT has the wrong degrees of freedom: In QCD at large isospin, one expects loosely bound Cooper pairs due to the attractive color-singlet channel in one-gluon exchange [31,32].[3]

Let us finally briefly return to the effective Lagrangian and the damping of phonons. Son's prescription was employed in Refs. [48,49] in the CFL phase of QCD, including scale-breaking effects due to the finite mass m_s of the s-quark. Including the leading effects, the phonon speed

was found to be $c_s = \sqrt{\frac{1}{3} - \frac{m_s^2}{9\mu_B^2}}$, where μ_B is the baryon chemical potential. Without these effects, the phonons propagate with the speed $c_s = 1/\sqrt{3}$, which is the conformal limit of QCD. The three-point interaction in Eq. (88) is of the same form as in the CFL phase, albeit with a different coupling [48]. Since the LO damping rate is calculated from the one-loop diagram arising from this term, the leading momentum dependence turns out to be the same. In Ref. [48], their damping rate was calculated in the CFL phase and it goes like p^5, for small p. This is exactly the same momentum dependence as the classic result by Beliav [37] calculated 65 years ago for the dilute Bose gas in the limit $p \to 0$, which therefore is recovered.

ACKNOWLEDGMENTS

APPENDIX: INTEGRALS

We use dimensional regularization to regulate ultraviolet divergences. The integrals in Euclidean space are defined as

$$\int_P = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \int_p, \tag{A1}$$

where

$$\int_p = \left(\frac{e^{\gamma_E} \Lambda^2}{4\pi}\right)^\epsilon \int \frac{d^d p}{(2\pi)^d}, \tag{A2}$$

with $P = (p_0, \mathbf{p})$, $d = 3 - 2\epsilon$, $p = |\mathbf{p}|$, and Λ is the renormalization scale associated with the $\overline{\text{MS}}$ scheme.

The following class of one-loop integrals appears

$$I_n(m^2) = \int_P \frac{1}{(P^2 + m^2)^n}, \tag{A3}$$

$$I_0'(m^2) = -\int_P \log\left[P^2 + m^2\right], \tag{A4}$$

where n is a non-negative integer and the prime denotes differentiation with respect to n. Evaluating the integrals yields

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[3]At large μ_I, quarks are interacting weakly due to asymptotic freedom. Since the dominant interaction is attractive, the Fermi surface is rendered unstable which leads to the formation of Cooper pairs.

$$I_n(m^2) = e^{\gamma_E \epsilon} \frac{m^{4-2n}}{(4\pi)^2} \left(\frac{\Lambda}{m}\right)^{2\epsilon} \frac{\Gamma(n-2+\epsilon)}{\Gamma(n)}. \quad \text{(A5)}$$

$I_0'(m^2)$ is divergent and $I_n(m^2)$ is divergent for $n = 1, 2$. Expanding them to order ϵ, we find

$$I_0'(m^2) = \frac{m^4}{2(4\pi)^2} \left(\frac{\Lambda}{m}\right)^{2\epsilon} \left[\frac{1}{\epsilon} + \frac{3}{2} + \mathcal{O}(\epsilon)\right], \quad \text{(A6)}$$

$$I_1(m^2) = -\frac{m^2}{(4\pi)^2} \left(\frac{\Lambda}{m}\right)^{2\epsilon} \left[\frac{1}{\epsilon} + 1 + \frac{\pi^2 + 12}{12}\epsilon + \mathcal{O}(\epsilon^2)\right], \quad \text{(A7)}$$

$$I_2(m^2) = \frac{1}{(4\pi)^2} \left(\frac{\Lambda}{m}\right)^{2\epsilon} \left[\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\right]. \quad \text{(A8)}$$

We also need to evaluate the setting-sun type integral appearing in the NNLO calculation of the pressure in the chiral limit,

$$J(m^2) = \int_{PQ} \frac{p_0^2}{P^2(Q^2 + m^2)[(P+Q)^2 + m^2]}$$

$$= \int_P \frac{p_0^2}{P^2} \Pi(P), \quad \text{(A9)}$$

where we have defined the self-energy

$$\Pi(P) = \int_Q \frac{1}{(Q^2 + m^2)[(P+Q)^2 + m^2]}. \quad \text{(A10)}$$

Using Feynman parameters and averaging over angles, we obtain

$$J(m^2) = \frac{1}{d+1} I_1^2(m^2). \quad \text{(A11)}$$

In the theory of dilute Bose gases, the following integrals appear [13]

$$I_{m,n}(M^2) = \int_p \frac{p^{2m}}{p^n(p^2 + M^2)^{\frac{n}{2}}}. \quad \text{(A12)}$$

They satisfy the recursion relation

$$\frac{dI_{m,n}}{dM^2} = -\frac{1}{2} n I_{m+1,n+2}(M^2), \quad \text{(A13)}$$

which follows directly from the definition Eq. (A12). Evaluating the integrals in dimensional regularization, we find

$$I_{m,n}(M^2) = e^{\gamma_E \epsilon} \frac{M^{3+2m-2n}}{(4\pi)^{\frac{3}{2}}} \left(\frac{\Lambda}{M}\right)^{2\epsilon}$$

$$\times \frac{\Gamma(\frac{3}{2} - \frac{n}{2} + m - \epsilon)\Gamma(n - m - \frac{3}{2} + \epsilon)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{3}{2} - \epsilon\right)}. \quad \text{(A14)}$$

We specifically need

$$I_{0,-1}(M^2) = \frac{16}{15} \frac{M^5}{(4\pi)^2} [1 + \mathcal{O}(\epsilon)], \quad \text{(A15)}$$

$$I_{1,1}(M^2) = \frac{16M^3}{3(4\pi)^2} [1 + \mathcal{O}(\epsilon)]. \quad \text{(A16)}$$

Note that both integrals are finite in $d = 3$.

In the nonrelativistic limit, a generalized hypergeometric function $_3F_2$ appears. Hypergeometric functions function of type $_pF_q$ are analytic functions of a single variable z with $p + q$ parameters. The generalized hypergeometric function has a power series representation in z

$$_pF_q \left[\begin{matrix} \alpha_1, \alpha_2, \ldots \alpha_p \\ \beta_1, \beta_2, \ldots \beta_q \end{matrix} \bigg| z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \ldots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \ldots (\beta_q)_n n!} z^n, \quad \text{(A17)}$$

where $(a)_b$ is the Pochhammer symbol

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}. \quad \text{(A18)}$$

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