Yukawa potential under weak magnetic field

Fábio L. Braghin,[1] Marcelo Loewe,[2,3] and Cristian Villavicencio[4]

[1]*Instituto de Física, Federal University of Goiás, Avenida Esperança, s/n, 74690-900, Goiânia, Goiás, Brazil*

[2]*Facultad de Ingeniería, Arquitectura y Diseño, Universidad San Sebastián, Santiago, Chile*

[3]*Centre for Theoretical and Mathematical Physics, and Department of Physics, University of Cape Town, Rondebosch 7700, South Africa*

[4]*Centro de Ciencias Exactas and Departamento de Ciencias Básicas, Facultad de Ciencias, Universidad del Bío-Bío, Casilla 447, Chillán, Chile*

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Weak magnetic field induced corrections for the Yukawa potential due to one pion exchange between two constituent quarks (nucleons) are presented. For that, the constant magnetic field effect on the pion propagator and on the pion form factor are taken into account. An effective gluon propagator parametrized with an effective gluon mass ($M_g \sim 0.5$ GeV) is considered. In the limit of magnetic field that is weak with respect to the constituent quark mass and pion mass, analytical and semianalytical expressions can be obtained. Different types of contributions are found, isotropic or anisotropic, dependent on the pion mass and also on the constituent quark and effective gluon masses. Overall the corrections are of the order of 2% to 5% of the Yukawa potential at distances close to 2 fm, and they decrease slower than the Yukawa potential. The anisotropic corrections are considerably smaller than the isotropic components. A sizable splitting between results due to the magnetic field dependent neutral or charged pion mass is found.

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I. INTRODUCTION

The Yukawa potential, due to one pion exchange [1], is a cornerstone of nuclear and particle physics being responsible, for example, for the long range nucleon potential. Besides its real importance for particle and nuclear physics, the Yukawa potential appears in other areas of physics under different names, such as static screened Coulomb potential in solid state and plasma physics [2]. More recently, bound states of dark matter components have been envisaged by considering this type of interaction [3]. Therefore, it is interesting to understand how it behaves under different external conditions such as magnetic fields.

In recent years many effects in hadron degrees of freedom due to strong magnetic fields have been investigated [4,5]. Initially, indirect effects were searched in relativistic heavy ion collisions [6–8], in dense stars/magnetars, including in low density outer crust regions, [9–12] and in the early Universe [13,14]. Expectations of strong magnetic fields in peripheral relativistic heavy ion collisions were somewhat diminished in recent years [15]. Modifications in the hadron's dynamics can occur both at the more fundamental level, for quarks and gluons, and for the hadron level. A magnetic field leads to the magnetic catalysis, due to the high degeneracy of the lowest Landau levels [4,16,17], and correspondingly it increases the quark effective masses. Less understood is the role of magnetic fields on hadron masses/structure and dynamics in general. In this respect, effective models can be very useful to provide a framework to perform feasible calculations, usually making it possible to reach reasonably correct results when compared to lattice QCD (LQCD) [18–21]. Usually these models are compatible with the framework of the constituent quark model. Both the Nambu-Jona-Lasinio (NJL) model and the sigma model are suitable frameworks for the investigation of the quark-antiquark mesons providing usually very good results for meson dynamics and other global properties of low energy hadrons, the list of works is extensive, to quote few examples [22–27]. Even in such simplified effective models, there are difficulties involved in this calculation of properties under strong magnetic fields. Different calculations indicate that neutral (charged) pions have their masses decreased (increased) with the magnetic field strength [18,19,23–25]. Effects of magnetic fields on hadron couplings can also be computed although are exploited less extensively in the literature. Some derivations of the contribution of magnetic fields to hadron or quark couplings are found in [28–31]. In spite of considering strong magnetic fields, they will be taken as relatively weak with respect to hadron mass scales, i.e., $eB \ll M_q^2$ (as a constituent quark mass) and/or $eB < m_\pi^2$ (pion mass). Therefore we shall fix $eB \sim 0.1 M_q^2$. In this limit, it is possible to perform expansions in the quark and pion propagators and to present analytical or semianalytical calculations for the resulting form factors and overall contributions for the Yukawa potential. The only contribution we do not take into account is the magnetic field effect on the gluon (effective) propagator and quark running coupling constant. The constituent quark model (CQM), in a wide variety of different versions, has provided a sound framework for the description of global properties of hadron structure and interactions, for example, in [32,33]. It is based in the idea that dressed valence quarks give rise to the hadron observables such as masses and coupling constants.

Pion constituent quark couplings and form factors have been derived at one loop level by considering standard techniques [34]. Full form factors can be obtained as Lagrangian interactions by considering an important term of the QCD effective action for quarks, which is a quark-antiquark interaction mediated by a (nonperturbative) one gluon exchange. This one gluon exchange is included by means of an (external) effective gluon propagator that accounts for nonperturbative effects of the gluon non-Abelian dynamics and it will be parametrized by a gluon mass. The pion pseudoscalar coupling can be obtained by standard analytical methods, which are represented in Fig 1. The coupling under magnetic field arises by considering the same method [28].

In this work, we investigate weak constant magnetic field corrections to the Yukawa potential. Different types of effects will be considered at this order of correction: the magnetic field correction to the pion propagator [35], the weak magnetic field contributions for the pion coupling to constituent quarks. These first two effects indicate the order of $(eB)^2$. We also consider that magnetic fields lead to corrections of quark and pion masses [16,22,23]. The work is organized as follows. In the next section, the corresponding corrections to the pion propagator and to the pion form factor, due to a constant weak magnetic field, are presented. In Sec. III the Fourier transformation of the momentum dependent potential is calculated mostly analytically. In Sec. IV, numerical estimations displayed and discussed. In the final part there is a Summary.

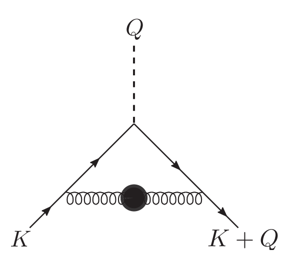

FIG. 1. One loop structure of the pion form factor: solid line for the quark with incoming and outgoing momenta K and $K+Q$, wiggly line for the gluon, and dashed line for the pion with momentum Q.

II. THE YUKAWA POTENTIAL AND CORRECTIONS

The standard Yukawa potential can then be written as [1,36]

$$V(R) = -\frac{g_{\rm ps}^2}{4\pi R} e^{-m_\pi R}, \tag{1}$$

where $g_{\rm ps}$ is the pseudoscalar coupling constant.

In the presence of a constant background magnetic field, the pion propagator will receive standard corrections from the magnetic field. Besides that, the relatively weak magnetic field will induce corrections to the pion pseudoscalar coupling constant. However, instead of considering a punctual coupling we shall consider the whole form factor. The following resulting potential can be calculated as

$$V(\vec{R}) = \int \frac{d^3Q}{(2\pi)^3} (G_{\rm ps}^B(\vec{Q}^2, Q_z^2))^2 e^{-i\vec{Q}\cdot\vec{R}} D_\pi^B(Q) \tag{2}$$

where the leading contributions from the magnetic field can be written as

$$G_{\rm ps}^B(\vec{Q}^2, Q_z^2) \simeq g_{\rm ps} + \left(\frac{eB}{M^2}\right)^2 F_{\rm ps}^B(\vec{Q}, Q_z),$$
$$D_\pi^B(Q) \simeq D_\pi(Q) + (eB)^2 D_1(Q). \tag{3}$$

The pseudoscalar pion coupling constant will be taken from the whole form factor with $Q^2 = 0$, as shown in the Appendix. Therefore, we will calculate quantities in the momentum space for the one pion exchange and then perform the Fourier transform.

A. The pion propagator under weak magnetic field

The scalar field propagator under strong magnetic fields has been derived in Ref. [35]. We will consider the same propagator for the pion by neglecting therefore the particular quark-antiquark structure. In the limit of very weak magnetic field, $eB \ll m^2$, the scalar field propagator can be written as

$$iD^B(Q) \simeq \frac{i}{Q^2 - m^2}\left[1 - (eB)^2\left(\frac{1}{(Q^2-m^2)^2} + \frac{2Q_\perp^2}{(Q^2-m^2)^3}\right)\right]$$
$$\equiv D_0(Q) + (eB)^2 D_1^B(Q, Q_\perp). \tag{4}$$

Note that the leading term is of the order of $(eB)^2$ and there is one B-dependent correction that is isotropic and one that is anisotropic.

B. Constituent quark-pion coupling under weak magnetic field

By starting with a quark-antiquark interaction mediated by a (nonperturbative) one gluon exchange, the pion pseudoscalar form factor in vacuum and under weak magnetic field were derived, respectively, in [28,34]. The mediation of the (dressed) gluon, by means of an effective gluon propagator, gives rise to a dressed quark current, which means that it can be parametrized in terms of a gluon effective mass of the order of magnitude of a constituent quark mass. The quark-antiquark interaction can be Fierz transformed leading to different Dirac and flavor types of quark currents, dressed by components of the gluon propagator, that can couple to any meson field. The resulting effective Lagrangian terms for the magnetic field induced contribution for the pseudoscalar field (P_1, P_2, P_3) coupling to the pseudoscalar quark current $j^i_{\rm ps}$ can be written as

$$\mathcal{L}_{\pi-Q(B)} = c_i F^B_{\rm ps}(Q,K) P_i(Q) (j^i_{\rm ps})^\dagger, \quad i = 1,2,3 \tag{5}$$

$j^i_{\rm ps} = \bar\psi i\gamma_5 \lambda_i \psi$, and $c_1 = c_2 = -4/9$ and $c_3 = 5/9$ are obtained from the trace in flavor indices. $c_{1,2}$ and c_3 give rise, respectively, to the charged and neutral pion couplings. The weak magnetic field couples to the internal quark lines of Fig. 1. Although it may seem that the leading term would be a magnetic field to a single internal quark line [which would yield a correction of the order of (eB)] it turns out that the two possible diagrams of this type lead to a tiny contribution that was not taken into account. The leading contribution for the pseudoscalar pion coupling is that of the one magnetic field insertion for each of the internal quark lines shown in Fig. 2, which is of the order of $(eB)^2$, is different from the axial coupling that goes with (eB) [28]. Another very small contribution is due to two magnetic field insertions on a single quark line, shown in Figs. 3 and 4, which is also or the order of $(eB)^2$, but is negligible when compared to the leading contribution. The leading Lagrangian term can be written in terms of two parts, isotropic and anisotropic, which are, respectively, given by

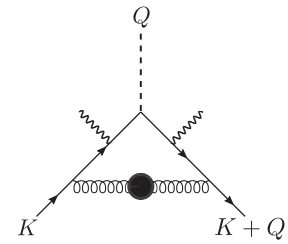

FIG. 2. Leading contribution: one magnetic field insertion for each internal quark line.

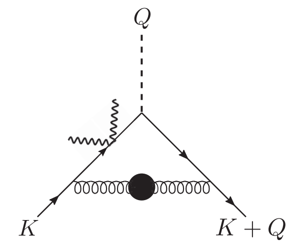

FIG. 3. Two magnetic field insertions on a single quark internal line.

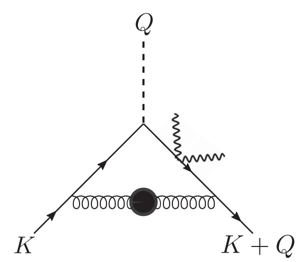

FIG. 4. Two magnetic field insertions on a single quark internal line.

$$F^B_{\rm ps}(Q,K) = \left(\frac{eB_0}{M^2}\right)^2 [F^{B,{\rm iso}}_{\rm ps}(Q,K) + F^{B,{\rm ani}}_{\rm ps}(Q,K)] \tag{6}$$

where Q, K are the pion and constituent quark momenta, and the relative weak strength of the magnetic field was factorized in the dimensionless factor. These form factors can be written as

$$\begin{aligned} F^{B,{\rm iso}}_{\rm ps}(Q,K) &= iC^B_{\rm PS} M^{*4} \int \frac{d^4k}{(2\pi)^4} \frac{-k\cdot(k+Q)+M^2}{[k^2-M^2]^2[(k+Q)^2-M^2]^2} \frac{R(-k-K)}{K_g} \\ F^{B,{\rm ani}}_{\rm ps}(Q,K) &= iC^B_{\rm PS} M^{*4} \int \frac{d^4k}{(2\pi)^4} \frac{-k_\perp\cdot(k_\perp+Q_\perp)}{(k^2-M^2)^2((k+Q)^2-M^2)^2} \frac{R(-k-K)}{K_g} \end{aligned} \tag{7}$$

where $R(-k)$ is the effective gluon propagator, with a normalization K_g discussed below, and the following constant was defined:

$$C^B_{\rm PS} = 8N_c \alpha K_g C_i.$$

In the present work we consider an effective gluon propagator inspired in [37] that is confining and leads to dynamical chiral symmetry breaking. It will be given by

$$R(k) = \frac{K_g}{(k^2 - M_G^2)^2} \tag{8}$$

where K_g, M_G are, respectively, the normalization constant and an effective (constant) mass. With this effective gluon propagator it is possible to carry the overall calculation analytically and free of UV divergences, in spite of the need of a renormalization of their equations to settle the scale of K_g. Besides that, this effective gluon propagator provides a string tension and it provides numerical results for meson-constituent quark form factors that are basically the same as results from another gluon propagator extracted from extensive calculations with Schwinger-Dyson equations for the hadron structure [33]. This comparison was shown in, for example, Refs. [28,34]. This normalization of the gluon propagator was fixed by the pseudoscalar pion coupling constant in vacuum, as a renormalization condition, such that it reproduces the phenomenological value $G_{\rm ps} \simeq 13$ [36,38]. The calculation of this coupling constant, by considering the same method employed to calculate Eq. (7), is shown in the Appendix. It is interesting to note that in the magnetic field contribution for the form factor above, the Schwinger phase from the quark propagators does not contribute since it can be gauged away for this type of diagrams [39].

By considering the charged and neutral pion fields with pseudoscalar currents associated, the pion-constituent quark coupling above can be written as

$$\mathcal{L}_{\pi-Q(B)} = F^B_{\rm ps}(Q,K)[\sqrt{2}c_1(\pi^+\bar{d}u + \pi^-\bar{u}d) + c_3\pi^0(\bar{u}u - \bar{d}d)]. \tag{9}$$

To calculate these form factors we considered the Feynman trick in terms of Feynman parameters although it is not possible to perform all integrals analytically. So, with the Feynman trick we perform the integral in momentum which makes it possible to perform the Fourier transform to calculate the Yukawa potential and corrections at the cost of introducing integrals in the Feynman parameters that are numerically simpler to be achieved.

1. Form factor: Isotropic part $F^{B,\rm iso}_{\rm ps}$

The isotropic part $F^{B,\rm iso}_{\rm ps}$ (7) has the following integral:

$$I_4(Q^2) = \int \frac{d^3k}{(2\pi)^3} \int \frac{dk_0}{(2\pi)} \frac{-(k+K)\cdot(k+Q+K) + M^2}{[(k+K)^2 - M^2]^2[(k+K+Q)^2 - M^2]^2} R(-k). \tag{10}$$

For the gluon propagator of Eq. (8), this integral has six double poles, four of them from quark propagators and two from the gluon propagator. For an off shell pion, $Q_0 \to 0$, and on shell constituent quark in its rest frame, $K^2 = M^2$, and by using the Feynman parametrization, this integral can be written as

$$\frac{I_4(Q^2)}{K_g} = \frac{3}{8\pi^2}\int_0^1 dz \int_0^{z-1} dy \left(\frac{-2(1-y-z)yz}{3A^3} + \frac{D_{\rm iso}(1-y-z)yz}{A^4}\right), \tag{11}$$

$$\text{where } A = -[M^2(1-z)^2 + \vec{Q}^2 y(1-y) + M_g^2 z],$$
$$D_{\rm iso} = [M^2z^2 - \vec{Q}^2[y^2 - y] + M^2](1-y-z)yz. \tag{12}$$

2. Form factor: Anisotropic part $F^{B,\rm ani}_{\rm ps}$

The anisotropic part of the magnetic field correction to the form factor (7) is given by

$$\frac{I_5(Q_\perp^2, Q^2)}{K_g} = \int_k \frac{-(k_\perp + K_\perp)\cdot(k_\perp + K_\perp + Q_\perp)}{((k+K)^2 - M^2)^2((k+Q+K)^2 - M^2)^2} \frac{1}{(k^2 - M_g^2)^2}. \tag{13}$$

By employing the same Feynman trick, the momentum integration of the anisotropic part of the form factor, for $\vec{K} = 0$ and $K_0 = M$ and $Q_0 = 0$, this form factor can be written as

$$\frac{I_5(Q_\perp^2, Q^2)}{K_g} = \frac{3}{8\pi^2}\int_0^1 dz \int_0^{z-1} dy \left(-\frac{xyzQ_\perp^2[y^2 - y]}{A^4} + \frac{1}{3A^3}\right). \tag{14}$$

III. THE MAGNETIC FIELD CORRECTIONS TO THE YUKAWA POTENTIAL

By replacing the magnetic field dependent quantities in Eq. (2) the potential in the momentum space with corrections due to weak magnetic fields is

$$\begin{aligned}\tilde{V}(Q) &= G^B_{\mathrm{ps}}(K,Q)D^B_\pi(Q^2)G^B_{\mathrm{ps}}(K,Q)\\ &= \left(g_{\mathrm{ps}} + F^B_{\mathrm{ps}}(K,Q)\frac{(eB)^2}{M^4}\right)(D_0(Q) + (eB)^2 D^B_1(Q,Q_\perp))\left(g_{\mathrm{ps}} + F^B_{\mathrm{ps}}(K,Q)\frac{(eB)^2}{M^4}\right) + \cdots\\ &\simeq V_0(Q^2) + V^B_\pi(Q) + V^B_{FF}(Q^2) + \cdots \end{aligned} \tag{15}$$

where the first term yields the usual Yukawa potential, and the two types of corrections are due, respectively, to the pion propagator $[V^B_\pi(Q)]$ and to the pion form factor $[V^B_{FF}(Q^2)]$.

Note that both the B-dependent corrections to the pion propagator and to the pion coupling are of the order of $(eB)^2$. They will be computed separately as suggested above. For the corresponding potential in position space we take $Q_0 = 0$ and $K^2 = M^2$. The following Fourier transformation for each of the contributions shown above will be computed:

$$V(\vec{R}) = \int \tilde{V}(\vec{Q}^2)e^{-i\vec{Q}\cdot\vec{R}}\frac{d^3Q}{(2\pi)^3}. \tag{16}$$

By defining $E_Q = \sqrt{\vec{Q}^2 + m^2}$ we have the following magnetic field corrections:

$$V^B_{FF}(\vec{R}) = -i\left(\frac{eB}{M^2}\right)^2 g_{\mathrm{ps}} \int F^B_{\mathrm{ps}}(K,Q)\frac{1}{E^2_Q}e^{-i\vec{Q}\cdot\vec{R}}\frac{d\vec{Q}}{(2\pi)^3}, \tag{17}$$

$$\begin{aligned}V^B_\pi(\vec{R}) &\equiv I_1(R) + I_2(R_z, R_\perp)\\ &= -i(eB)^2 g^2_{\mathrm{ps}} \int \frac{1}{E^4_Q}\left(\frac{1}{E^2_Q} + \frac{2Q^2_\perp}{E^4_Q}\right)e^{-i\vec{Q}\cdot\vec{R}}\frac{d\vec{Q}}{(2\pi)^3},\end{aligned} \tag{18}$$

where the both parts, $V^B_\pi(\vec{R})$ and $V^B_{FF}(\vec{R})$, have two parts each, as calculated below—one isotropic and one anisotropic.

A. Contribution of the B-dependent meson propagator

By denoting $Q = |\vec{Q}|$ and $R = |\vec{R}|$, the first integral of Eq. (18)

$$I_1(R) = -\frac{g^2_{\mathrm{ps}}}{(2\pi)^2}(eB)^2\int_0^\infty dQ\, Q^2\frac{e^{iQ|\vec{R}|} - e^{-iQ|\vec{R}|}}{(iQ|\vec{R}|)(Q^2+m^2)^3}. \tag{19}$$

There are two triple poles at $Q = \pm im$, so that, by calculating the residues, it yields

$$I_1(R) = -\frac{g^2_{\mathrm{ps}}}{32\pi}\frac{(eB)^2}{m^4}(m^2|\vec{R}| + m)e^{-m|\vec{R}|}. \tag{20}$$

The next correction is anisotropic with an integral denoted by I_2, it can be written as

$$I_2(R_z,R_\perp) = -(eB)^2\frac{g^2_{\mathrm{ps}}}{(2\pi)^3}\int dQ_z d^2Q_\perp 2Q^2_\perp \frac{e^{i(Q_zR_z + Q_\perp\cdot R_\perp)}}{(Q^2+m^2)^4} \tag{21}$$

where

$$R_z = |z_1 - z_2|, \qquad R_\perp = |R^\perp_1 - R^\perp_2|.$$

The first integration in Eq. (21) is the angular one, for which one obtains the Bessel function since

$$2\int_0^\pi d\theta e^{\beta\cos(\theta)} = 2\pi J_0(\beta) \tag{22}$$

where $\beta = R_\perp Q_\perp$. The second integration that can be done analytically (dQ_z) can be solved by recurring to the residue theorem by considering the fourth order poles. With an integration in the upper semiplane, the following pole is taken into account: $Q_z = i\sqrt{Q^2_\perp + m^2} \equiv iE_p$. It yields

$$\begin{aligned}I_2(R_z,R_\perp) &= -\frac{g^2_{\mathrm{ps}}}{\pi 2^4}(eB)^2\mathcal{J}_2(R,R_z)\\ &\equiv -\frac{g^2_{\mathrm{ps}}}{\pi 2^4}(eB)^2\int dQ_\perp Q^3_\perp e^{-R_zE_p}J_0(R_\perp k_\perp)\\ &\quad\times\left(\frac{R^3_z}{6E^4_p} + \frac{R^2_z}{E^5_p} + \frac{5R_z}{2E^6_p} + \frac{5}{2E^7_p}\right)\end{aligned} \tag{23}$$

which is the remaining integral solved numerically. This result is quite anisotropic.

The resulting magnetic field dependent correction due to the pion propagator can be written as

$$V_{\pi}^{B}(R,R_z) = -\frac{g_{\rm ps}^2 (eB)^2}{32}\left[\left(\frac{|\vec{R}|}{m^2}+\frac{1}{m^3}\right)e^{-m|\vec{R}|}+2\mathcal{J}_2(R,R_z)\right]. \tag{24}$$

B. Contribution of the pion form factor

The pion form factor contribution for the magnetic field dependent Yukawa potential (25) has two terms and, accordingly, the corresponding contributions for the Yukawa potential splits into two terms:

$$\begin{aligned} V_{FF}^{B}(R) &= V_{\rm iso}(R) + V_{\rm ani}(R_z, R_\perp) \\ &= 2i\left(\frac{eB_0}{M^2}\right)^2 g_{\rm ps} C_{\rm PS}^{B} C_i \int \frac{d^3Q}{(2\pi)^3} e^{-i\vec{Q}\cdot\vec{R}} \frac{-iI_4(\vec{Q}^2)}{\vec{Q}^2+m^2} + 2i\left(\frac{eB_0}{M^2}\right)^2 g_{\rm ps} C_{\rm PS}^{B} C_i \int \frac{d^2Q_\perp dQ_z}{(2\pi)^3} e^{-i(Q_zR_z+Q_\perp\cdot R_\perp)} \frac{-iI_5(Q_\perp^2, Q^2)}{\vec{Q}^2+m^2}, \end{aligned} \tag{25}$$

which is $\vec{Q}^2 = Q_z^2 + Q_\perp^2$.

1. Isotropic term

For the Fourier transform of $V_{\rm iso}$, first the angular integration is done and it leads to the following integral in 3-momentum spherical coordinates:

$$\begin{aligned} V_{\rm iso} &= C_{\rm iso}\int \frac{Q^2dQ}{(2\pi)^2}\frac{(e^{-iQR}-e^{iQR})}{iQR}\frac{I_4(Q^2)}{\vec{Q}^2+m_\pi^2} \\ &\equiv C_{\rm iso}[\mathcal{F}_{4a}(R)+\mathcal{F}_{4b}(R)] \end{aligned} \tag{26}$$

where $C_{\rm iso} = 2i(\frac{eB_0}{M^2})^2 g_{\rm ps} C_{\rm PS}^{B} C_i K_g$. In this integral there are the two simple poles in the complex plane from the pion propagator and two double or triple poles (for each of the parts of integral I_4) from the form factor Eq. (11). These poles can be written as

$$|\vec{Q}| = \pm i\phi \equiv \pm i\sqrt{\frac{M^2(1-z)^2+M_g^2 z}{y(1-y)}}. \tag{27}$$

By performing the integrals with the residue theorem, by taking into account all the poles above and below the real axis, the following equations are obtained:

$$\mathcal{F}_{4a} = -\frac{1}{16\pi^3}\int_{y,z}[(1-y-z)yz]\left[\frac{2(e^{-\phi R}-e^{-m_\pi R})}{R(\phi^2-m_\pi^2)^3}+\frac{Re^{-\phi R}}{4\phi^2(\phi^2-m_\pi^2)}+\frac{e^{-\phi R}}{4\phi^3(\phi^2-m_\pi^2)}+\frac{e^{-\phi R}}{\phi(\phi^2-m_\pi^2)^2}\right] \tag{28}$$

$$\mathcal{F}_{4b} = \frac{-1}{32\pi^3}\int_y\int_z \frac{(1-y-z)yz}{[(y(1-y)]^4}F_{4b} \tag{29}$$

where

$$\begin{aligned} F_{4b} &= \frac{R^2e^{-\phi R}}{8\phi^3(m_\pi^2-\phi^2)}+\frac{3Re^{-\phi R}}{8\phi^2(m_\pi^2-\phi^2)}+\frac{3e^{-\phi R}}{8\phi^5(m_\pi^2-\phi^2)}+\frac{3Re^{-\phi R}}{4\phi^2(m_\pi^2-\phi^2)^2}-\frac{3e^{-\phi R}}{4\phi^3(m_\pi^2-\phi^2)^2} \\ &\quad +\frac{6e^{-\phi R}}{2\phi(m_\pi^2-\phi^2)^3}+\frac{6(-e^{-\phi R}+e^{-m_\pi R})}{(m_\pi^2-\phi^2)^4}. \end{aligned} \tag{30}$$

There is only one term (the last one) in each of these equations ($\mathcal{F}_{4a}$ and $\mathcal{F}_{4b}$) due to the poles from the pion propagator, which is all the others due to the structure of the form factor.

2. Anisotropic term

The anisotropic contribution of the form factor is written in terms of the momentum integral equation (13) that, for $\vec{K}=0$ and $K_0 = M$ and $Q_0 = 0$, is given by

$$I_5(Q^2,Q_\perp) = -K_g\frac{6i}{12\pi^2}\int_0^1 dy\int_0^{1-y}dz\left(\frac{-i}{4}\frac{D(1-y-z)yz}{(y(1-y))^4\tilde{A}^4}+\frac{ix(1-x-z)z}{(y(1-y))^3\tilde{A}^3\times 12}\right)$$
$$D = \vec{Q}_\perp^2 y(1-y)$$
$$\tilde{A}(x,y) = -[\vec{Q}^2+\phi]. \qquad (31)$$

The Fourier transform of the potential can be written as

$$V_{\rm ani}(R_z,R_\perp)\equiv\mathcal{F}(V_{\rm ps}^{B,{\rm ani}}(Q)) = C_{\rm iso}\int\frac{d^2Q_\perp dQ_z}{(2\pi)^3}e^{-i(Q_z\cdot R_z+Q_\perp\cdot R_\perp)}\frac{I_5(\vec{Q}^2,\vec{Q}_\perp^2)}{\vec{Q}^2+M_\pi^2}. \qquad (32)$$

By considering the cylindrical coordinate system, the angular integration leads to a Bessel function of the first kind, $J_0(Q_\perp R_\perp)$. After that, the integration in dQ_z can be done with the residue theorem for the following imaginary poles:

$$E^\perp\equiv\sqrt{\phi+\vec{Q}_\perp^2},\qquad E_\pi^\perp\equiv\sqrt{m_\pi^2+\vec{Q}_\perp^2}. \qquad (33)$$

The following equation is obtained:

$$V_{\rm ani}(R) = \frac{C_{\rm iso}}{(2\pi)}\int dQ_\perp Q_\perp J_0(Q_\perp R_\perp)\frac{[y(1-y)]^3}{i(1-y-z)yz}I_p(Q_\perp^2), \qquad (34)$$

where the following equation for $I_p(Q_\perp^2)$ was defined:

$$\begin{aligned}I_p(Q_\perp^2) = &\frac{\pi Q_\perp^2 R_z e^{-(E^\perp)R_z}}{2(E^\perp)^2((E_\pi^\perp)^2-(E^\perp)^2)^3}-\frac{\pi Q_\perp^2 e^{-(E^\perp)R_z}}{(E^\perp)((E_\pi^\perp)^2-(E^\perp)^2)^4}+\frac{\pi Q_\perp^2 e^{-E_\pi^\perp R_z}}{(E_\pi^\perp)((E_\pi^\perp)^2-(E^\perp)^2)^4}+\frac{\pi R_z e^{-(E^\perp)R_z}}{6(E^\perp)^2((E_\pi^\perp)^2-(E^\perp)^2)^2}\\
&-\frac{\pi e^{-(E^\perp)R_z}}{3e((E_\pi^\perp)^2-(E^\perp)^2)^3}+\frac{\pi e^{-(E_\pi^\perp)R_z}}{3(E_\pi^\perp)((E_\pi^\perp)^2-(E^\perp)^2)^3}+\frac{5\pi Q_\perp^2 e^{-(E^\perp)R_z}}{16(E^\perp)^7((E_\pi^\perp)^2-(E^\perp)^2)}\\
&+\frac{5\pi Q_\perp^2 R_z e^{-(E^\perp)R_z}}{16(E^\perp)^6((E_\pi^\perp)^2-(E^\perp)^2)}+\frac{\pi Q_\perp^2 R_z^2 e^{-(E^\perp)R_z}}{8(E^\perp)^5((E_\pi^\perp)^2-(E^\perp)^2)}-\frac{3\pi Q_\perp^2 e^{-(E^\perp)R_z}}{8(E^\perp)^5((E_\pi^\perp)^2-(E^\perp)^2)^2}-\frac{\pi e^{-(E^\perp)R_z}}{8(E^\perp)^5((E_\pi^\perp)^2-(E^\perp)^2)}\\
&+\frac{\pi Q_\perp^2 R_z^3 e^{-(E^\perp)R_z}}{48(E^\perp)^4((E_\pi^\perp)^2-(E^\perp)^2)}-\frac{3\pi Q_\perp^2 R_z e^{-(E^\perp)R_z}}{8(E^\perp)^4((E_\pi^\perp)^2-(E^\perp)^2)^2}-\frac{\pi R_z e^{-(E^\perp)R_z}}{8(E^\perp)^4((E_\pi^\perp)^2-(E^\perp)^2)}-\frac{\pi Q_\perp^2 R_z^2 e^{-(E^\perp)R_z}}{8(E^\perp)^3((E_\pi^\perp)^2-(E^\perp)^2)^2}\\
&+\frac{\pi Q_\perp^2 e^{-eR_z}}{2(E^\perp)^3((E_\pi^\perp)^2-(E^\perp)^2)^3}-\frac{\pi R_z^2 e^{-(E^\perp)R_z}}{24(E^\perp)^3((E_\pi^\perp)^2-(E^\perp)^2)}+\frac{\pi e^{-(E^\perp)R_z}}{6(E^\perp)^3((E_\pi^\perp)^2-(E^\perp)^2)^2}.\end{aligned} \qquad (35)$$

The poles of the pion propagator are responsible for the third and fifth terms and all the others come from the pion form factor structure.

Note that in all the contributions from the pion form factor $[V_{FF}^B(R_z,R)\sim V_{\rm iso}(R)+V_{\rm ani}(R_z,R)]$ there is a strong dependence on the constituent quark and gluon effective masses that are not negligible.

IV. NUMERICAL RESULTS

The complete leading weak magnetic field correction to the Yukawa potential can be written in terms of three terms:

$$V^B(R_z,R) = V_\pi^B(R_z,R)+V_{FF}^B(R_z,R),$$
$$V_{FF}^B(R_z,R) = V_{\rm iso}^B(R)+V_{\rm ani}^B(R_z,R). \qquad (36)$$

These terms were given in Eqs. (24), (26), and (34) and they will be shown below compared to the Yukawa potential $[V_{\rm Yuk}(R)]$ by means of the ratios:

$$\frac{V^B(R_z,R)}{V_{\rm Yuk}} = \frac{V_{FF}^B(R_z,R)+V_\pi^B(R_z,R)}{V_{\rm Yuk}(R)}, \qquad (37)$$

$$\frac{V_{4,5}(Rz,R)}{V_{\rm Yuk}(R)} = \frac{V_{FF}^B(R_z,R)}{V_{\rm Yuk}(R)}, \qquad (38)$$

$$\frac{I_{12}(R_z,R)}{V_{\mathrm{Yuk}}(R)} = \frac{V_\pi^B(R_z,R)}{V_{\mathrm{Yuk}}(R)}, \tag{39}$$

$$V_{\mathrm{Yuk}}(R) = -g_{\mathrm{ps}}^2 \frac{e^{-m_\pi R}}{4\pi R}. \tag{40}$$

The following phenomenological or experimental numerical values for the parameters will be considered for the pseudoscalar pion coupling [36,38], quark effective mass, gluon effective mass, and pion mass (whenever degenerate):

$$g_{\mathrm{ps}} = 13, \qquad M_q = 0.35\ \mathrm{GeV},$$
$$M_g = 0.5\ \mathrm{GeV}, \qquad m_\pi = 0.137\ \mathrm{GeV}. \tag{41}$$

This effective gluon mass is slightly larger than most recent results in lattice QCD and Schwinger-Dyson equation calculations suggest [40]. For values $M_g \sim 0.35$ GeV, numerical results from the form factor contributions are increased nearly by a factor $(0.5/0.35)^4 \sim 4$. However, the large quark mass expansion performed to obtain the form factors above [28,34] also have to be compatible with a reasonable large gluon effective mass, since the gluon effective propagator dresses the quark currents. Therefore, such a smaller gluon mass may invalidate the expansion and it would lead to too large magnetic field corrections as it was verified numerically. Besides that, there are indications, from lattice QCD for exotic mesons, of constituent gluons with a mass close to 1 GeV [41]. Therefore, we keep the value $M_g = 0.5$ GeV to provide a first calculation for which the gluon is attached to a (constituent) quark.

Besides the effects of the magnetic field on the shape of the Yukawa potential discussed along this work, we also present the effect of the magnetic field dependence of the hadron masses: pions and constituent quarks. Quark effective masses increase with increasing magnetic fields [16] and neutral and charged pions were found to have decreasing and increasing masses, respectively, as found from different calculations with LQCD and with the NJL model [18,19,22–25]. To present estimations of these effects on the overall corrections to the Yukawa potential, we consider the numerical results obtained in these just quoted for the range of weak magnetic field considered in the present work ($eB \sim 0.01\ \mathrm{GeV}^2$). The following numerical values will be adopted:

$$M_q(B) \sim M_q + 0.020\ \mathrm{GeV}, \qquad (MqB)$$
$$m_{\pi^0}(B) \sim m_{\pi^0} - 0.020\ \mathrm{GeV}, \qquad (\mathrm{pne}B)$$
$$m_{\pi^\pm}(B) \sim m_{\pi^\pm} + 0.020\ \mathrm{GeV}, \qquad (\mathrm{pch}B), \tag{42}$$

where the electromagnetic parts of the pion masses are considered in these cases specifically: $m_{\pi^\pm} = 0.139$ GeV (pch) and $m_{\pi^0} = m_\pi = 0.135$ GeV (pne). However, it is important to point out that the difference in the results of using $m_{\pi^\pm}$ or m_{π^0} in the curves for the magnetic field independent pion mass is not noted in the figures below.

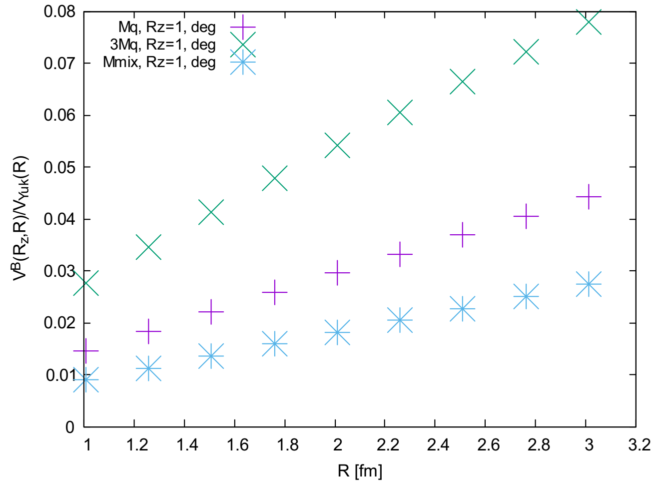

FIG. 5. Total ratio $V^B(R_z,R))/V_{\mathrm{yuk}}(R)$ for $R_z = 1$ fm for the following cases: $M = Mq = 0.35$ GeV, $M = 3Mq$ and the mixed calculation $Mmix$, for degenerate m_π by reducing $G_{\mathrm{ps}}^B c_i$ with unique coupling $c^i = 1$ (deg).

In Fig. 5 the total ratio $V^B(R_z,R))/V_{\mathrm{Yuk}}(R)$ is presented by fixing $R_z = 1$ fm for a unique constant value of the [degenerate (deg)] pion mass ($m_\pi = 0.137$ GeV) in different situations for the quark effective mass:

$$M = Mq \equiv 0.35\ \mathrm{GeV}, \qquad M = 3Mq,$$
$$Mmix = \mathrm{Mixed}\, Mq \text{ and } 3Mq. \tag{43}$$

The first case, Mq, is a typical constituent (up or down) quark mass and the second one, $3Mq$, is a large quark effective mass, of the order of magnitude of the nucleon mass, to test the resulting behavior. This may be seen as a partial account of the whole nucleon, since the constituent quark is considered to carry the whole nucleon mass tightly but not necessarily the entire nucleon size. In this case, the normalization of the effective gluon propagator (K_g) was found to increase considerably, although the form factor contribution is reduced. The overall effect is to increase the strength of the magnetic field correction to the Yukawa potential because of the increase in K_g. To make clearer the effect of a larger quark mass in the form factor and consequently in the Yukawa potential, the $Mmix$ case was done as follows. The value of the effective gluon propagator normalization K_g was fixed to reproduce g_{ps}, as shown in the Appendix, with usual quark mass $Mq = 0.35$ GeV and the whole estimation of the effects of the magnetic field for the Yukawa potential was done by considering a large mass $3Mq$ (possibly as if one had a full nucleon mass). In this case, the contribution of the form factor should be trivial (toward punctual particles and coupling) and the magnetic field correction to the Yukawa potential reduces basically to the simpler (punctual) pion

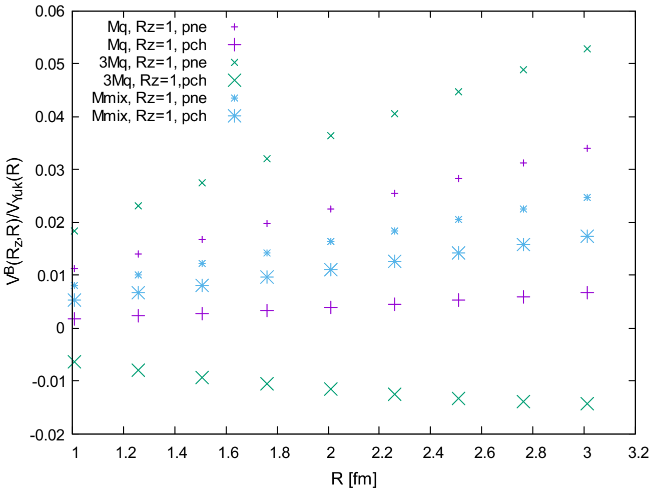

FIG. 6. Total ratio $V^B(R_z,R))/V_{\rm yuk}(R)$ for $R_z = 1$ fm for the following cases: $M = Mq = 0.35$ GeV, $M = 3Mq$ and the mixed calculation $Mmix$, for unique m_π and $G^B_{\rm ps}c_i$ for charged and neutral pions ($c^1 = -4\sqrt{2}/9$ and $c^3 = 5/9$, respectively) denoted by pch and pne.

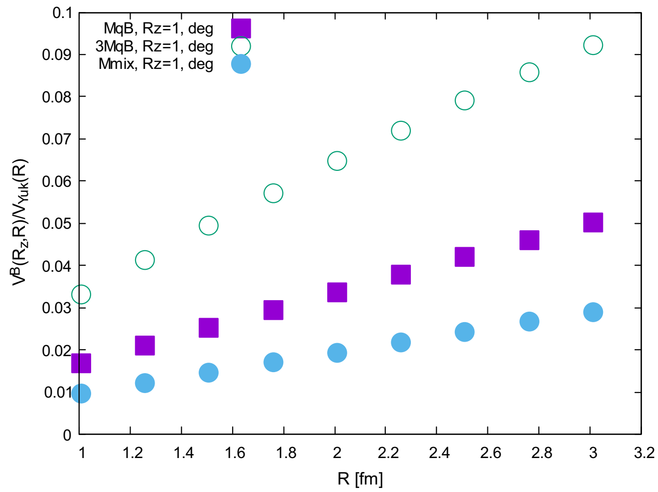

FIG. 7. Total ratio $V^B(R_z,R))/V_{\rm yuk}(R)$, at $R_z = 1$ fm, for magnetic field dependent quark effective mass (MqB), for the following cases: $M = Mq = 0.35$ GeV, $M = 3Mq$ and the mixed calculation $Mmix$, for unique m_π, and $G^B_{\rm ps}c_i$ by reducing $c^i = 1$ (deg).

exchange. Indeed, it is seen in the figure that the strength of the curves $Mmix$ (the mixed case) is reduced and the term due to the pion propagator is dominant.

The role of the magnetic field corrections to the charged and neutral pion (pch and pne) form factors, by considering the different couplings according to Eqs. (5) and (9), is presented in Fig. 6 by keeping the unique constant pion mass $m_\pi = 0.137$ GeV. These $c_1 = c_2$ and c_3 factors are given, respectively, by

$$\frac{-4\sqrt{2}}{9} \times G^B_{\rm ps}({\rm pch}), \qquad \frac{5}{9} \times G^B_{\rm ps}({\rm pne}). \qquad (44)$$

Whereas the neutral pion form factor provides a positive correction (i.e., more attractive potential), the charged pion form factor leads to a reduction of the total magnetic field correction (slightly less attractive potential). In fact, there is a cancellation of contributions from the charged pion propagator and the charged pion form factor that leads to a much smaller overall modulus than the neutral pion. For the large quark mass limit, $3Mq$, the overall correction to the (charged pion exchange) Yukawa potential becomes repulsive. However, most of the other magnetic field corrections are basically attractive, and they make the Yukawa potential more and more negative with increasing distances—this helps to increase the range of the interaction. Note that by the distance typical of the stability of the deuteron, $R \sim 2$ fm, the neutral and charged pion lead to different strengths of the order of 2%–3% without taking into account further effects as discussed below.

Different values for R_z, 1 fm and 2 fm, were used to test the contributions of the anisotropic terms V_5 and I_2. It turns out that the resulting anisotropies were found to be very small, of the order of few percent of the isotropic terms. Therefore, by fixing another value, for example, $R_z = 2$ fm for $R > 2$ fm, it leads to very tiny negligible differences in the figures.

Figures 7 and 8 present the same cases of the previous two figures by considering constituent quark mass under magnetic field according to Eq. (42). The strengths of the magnetic field corrections increase slightly as compared to the previous Figs. 5 and 6. As can be seen, also from the previous figures, the difference between the mixed case

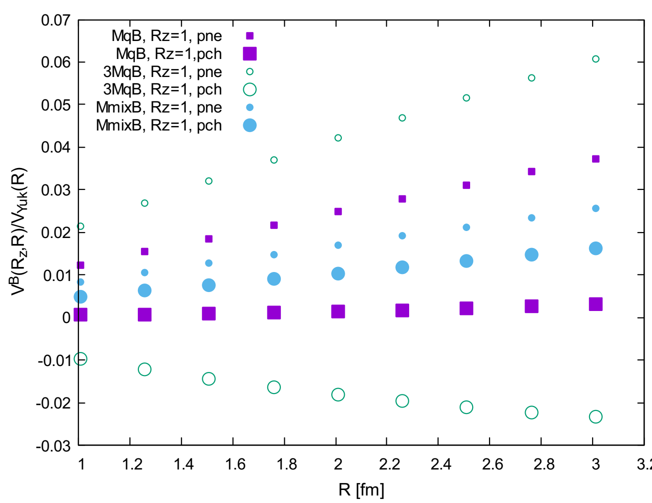

FIG. 8. Total ratio $V^B(R_z,R))/V_{\rm yuk}(R)$, at $R_z = 1$ fm, for magnetic field dependent quark effective mass (MqB), for the following cases: $M = Mq = 0.35$ GeV, $M = 3Mq$ and the mixed calculation $Mmix$, by considering an unique constant m_π and $G^B_{\rm ps}c_i$ for charged and neutral pions ($c^1 = -4\sqrt{2}/9$ and $c^3 = 5/9$, respectively) denoted by pch and pne.

$Mmix$ (or $MmixB$ in Fig. 7) and Fig. 8, respectively, for degenerate pion mass (deg) and neutral pion (via the form factor) (pne), is not large.

It is important to make clear the difference between the cases with larger quark masses, $3Mq$ and $Mmix$. In the first case, the quark mass is large in all calculations, i.e. it leads to a more trivial punctual interaction but it makes the normalization of the effective gluon propagator larger. This effect in K_g manifests in all the resulting curves for $V^B(R_z, R)$. The second case, $Mmix$, helps to identify the role of particular corrections to the Yukawa potential. For large quark mass, the form factor tends to be trivial and the most important contribution for the Yukawa potential is the one from the pion propagator. Having shown this point, $Mmix$ will not be exploited much further below.

Figure 9 presents similar curves to Fig. 6, by considering the neutral and charged pion masses to be nondegenerate and with magnetic field correction according to (42) in all the terms—by keeping the quark effective mass in the vacuum. Constituent quark mass was kept as Mq or $3Mq$. The difference in the behavior of these magnetic field corrections for neutral or charged pions becomes slightly larger than in the cases presented in Fig. 6 for which the pion masses were taken for $B = 0$.

Figure 10 presents the same cases of the previous Fig. 9, by considering the magnetic field dependent quark effective mass, Eq. (42). Again we see that, by taking into account the magnetic field dependent masses, the strength of the corrections to the Yukawa potential increases for both neutral and charged pions.

In the next Fig. 11, the separate contributions V^B_{FF} (from V_4 and V_5) and V^B_π (from $I_1 + I_2$), Eqs. (38) and (39), are shown for the degenerate numerical factors in the form factors and degenerate pion mass. Three cases of quark masses, Mq, $3Mq$, and $Mmix$ were considered. In Fig. 12 similar curves are presented for two cases for the magnetic field dependent quark mass, MqB and $3MqB$, by considering the degenerate numerical factors of the form factors c_i. The effect due to nondegenerate pion mass is stronger in the contribution from the pion exchange. The nondegenerate pion mass effect in the form factor is small although it is amplified if the nondegenerate coupling to magnetic field, Eq. (9), is taken into account. As seen before, these two effects increase the strength of the magnetic field corrections to the Yukawa potential and they make the

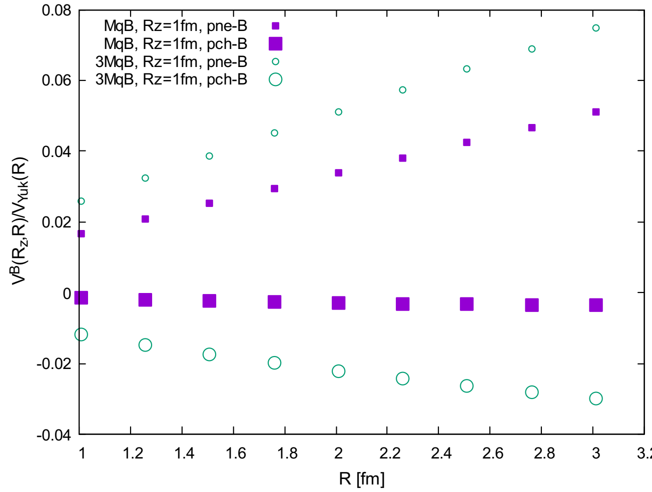

FIG. 10. Total ratio $V^B(R_z, R))/V_{\rm yuk}(R)$ for $R_z = 1$ fm for the following cases: quark effective mass corrected by the magnetic field, $M = MqB$, $M = 3MqB$, and $G^B_{\rm ps} c_i$ for charged and neutral pions ($c^1 = -4\sqrt{2}/9$ and $c^3 = 5/9$, respectively) (pch and pne) with the magnetic field dependent pion mass.

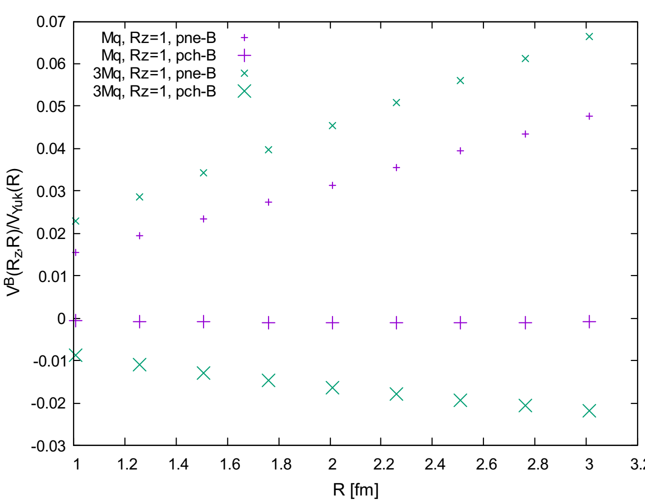

FIG. 9. Total ratio $V^B(R_z, R))/V_{\rm yuk}(R)$ for $R_z = 1$ fm for the following cases: $M = Mq = 0.35$ GeV, $M = 3Mq$, and $G^B_{\rm ps} c_i$ for charged and neutral pions ($c^1 = -4\sqrt{2}/9$ and $c^3 = 5/9$, respectively) denoted by pch and pne with the magnetic field dependent pion mass.

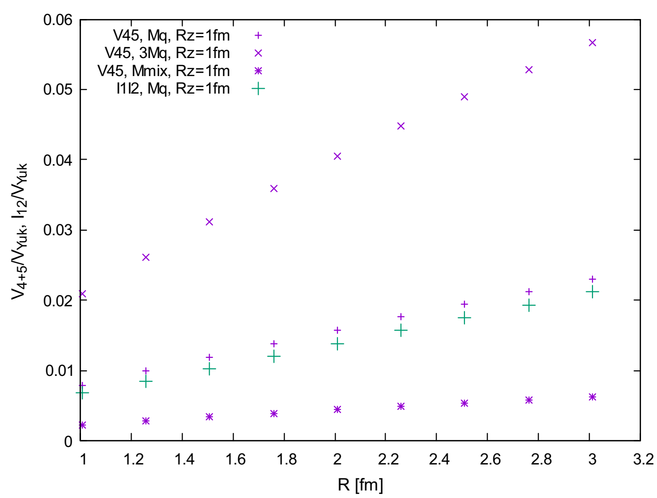

FIG. 11. Separate contributions for $V^B_{FF}(R_z, R)$ from isotropic and anisotropic corrections V_4 and V_5, and for $V^B_\pi(R_z, R)$, from $I_1 + I_2$. $R_z = 1$ fm for Mq, 3Mq, $Mmix$, m_π, and $G^B_{\rm ps}$ without the isospin factor c_i.

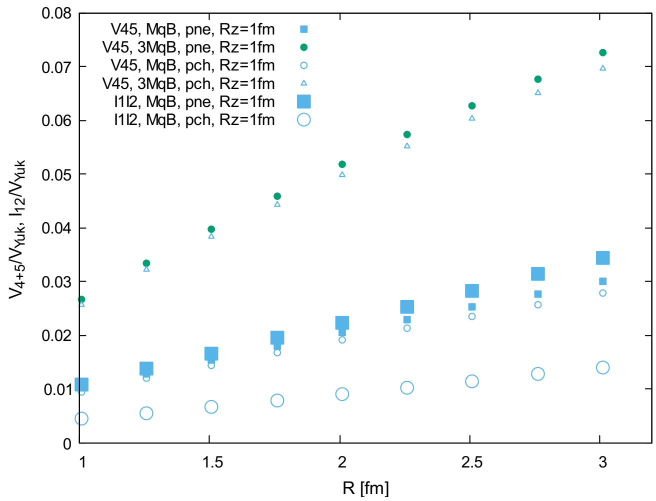

FIG. 12. Similar curves to the previous figure: Separate contributions for $V^B_{FF}(R_z, R)$, from isotropic and anisotropic corrections V_4 and V_5, and for $V^B_\pi(R_z, R)$, from $I_1 + I_2$. $R_z = 1$ fm for quark masses MqB and 3MqB, nondegenerate $m_\pi^{0,\pm}$, and $G^B_{\rm ps}$ without the isospin factor c_i.

range of the potential to change different for neutral or charged pions. For neutral pion exchange the range of the Yukawa potential increases, whereas for the charged pion exchange it is nearly unchanged depending on the range of the parameters such as constituent quark mass.

V. SUMMARY

Magnetic field induced corrections to the Yukawa potential were presented for $(eB) \sim 0.1 M_q^2 \sim 50 \times 10^{12}$ T. Most of the calculations are basically analytical, with remaining numerical integrals mostly in Feynman parameters and, in some cases, one integration in a component of the pion three-momentum. Three types of effects were investigated, two of them concerning the shape of the potential, besides the role of masses of pion and of constituent quarks. First, the (relatively) weak magnetic field contribution for the pion propagator. Second, the contribution of the weak magnetic field for the pseudoscalar pion form factor, $G^B_{\rm ps}(Q^2)$, in a one loop calculation. This second contribution leads to a dependence on the constituent quark effective mass and on a gluon effective mass which parametrizes the gluon propagator. Finally, the effects of the magnetic field on the pion and constituent quark masses were also considered. The first two effects are of the order of magnitude of $(eB)^2$ and the third ones go with (eB). Considering the values of the (phenomenological) parameters of masses, three different situations were analyzed. First, the constituent quark mass was taken to be $Mq = 0.35$ GeV for which results are well inside the perturbative regime for $M_g = 0.5$ GeV. Second, a larger quark mass, $3Mq$, corresponding to nearly the nucleon mass was considered. However, in this calculation the normalization of the gluon propagator was modified accordingly to keep $g_{\rm ps} = 13$, the physical value. Results from the form factor contributions increase accordingly by nearly 3 times. Finally a mixed calculation ($Mmix$) to keep the quark mass $Mq = 0.35$ GeV to fit $g_{\rm ps}$ was done, but increasing the quark mass of the form factors to $3Mq$. This yielded a strong reduction of the contribution of the form factors leading to a pointlike interaction.

The overall modification of the potential for the weak magnetic field limit is not large in the range of distances $R \sim 1$–3 fm, the long range component of the nucleon interactions. These magnetic field contributions become larger at larger distances, mainly for the neutral pion exchange. Anisotropic components are, in general, quite smaller than the isotropic ones in the range of distances exploited in this work. However, for larger distances, the overall Yukawa potential becomes tiny. Perturbative corrections due to the magnetic field are, therefore, of the order of around 5%. Charged and neutral pion exchanges receive different contributions from the coupling to the magnetic field, and therefore nucleon (and more generally baryons) interactions should manifest this splitting associated to the neutral or charged pion exchange.

In most of the cases, the Yukawa potential becomes more attractive with slightly longer range, given that the magnetic field correction increases the strength of the Yukawa potential. The exception to this increase was found for the charged pion exchange that may lead to a less attractive Yukawa potential and a shorter range interaction. The magnetic field contributions to constituent quark and pion masses lead overall to amplification of the effects due to the form factor and to the pion propagator. The inclusion of the form factor in the calculation of the potential has brought several issues into discussion. For the limit of very large constituent quark mass and effective gluon mass these contributions from the form factor tend to zero. Modifications due to magnetic fields can be expected to occur in the neighborhood (low density outer crust) of dense stars when magnetic fields may reach such values we consider [12]. Our results suggest nontrivial contributions to the deuteron formed in such environment, or alternatively, for the equation of state and other related observables at low baryon density, may be expected. In the outer crust of dense stars, the density is estimated to be less than the nuclear saturation density. Note, however, that the long range pion exchange term is suppressed in the high density regime. Besides that, currently, the expected values of magnetic fields in heavy ions collisions are quite reduced [15]. However, if magnetic fields with such strengths are reached by the hadronization time scales or in the spectator region of the collisions, in peripheral heavy ions collisions [8] or in experiments where low density matter is obtained, Yukawa long range nucleon interaction may also be probed. Accordingly, deuterons formed in such magnetic fields possibly undergo to excited states when passing to a region of zero magnetic field. Along the same lines, other meson exchanges must be verified. This will be developed elsewhere. Magnetic field corrections to the effective gluon propagator and quark-gluon running coupling constant were not considered. This kind of calculation may possibly help to identify the validity of the CQM in what concerns the hadron couplings by means of such internal degrees of freedom. A detailed investigation of the role of the form factor for the potential in vacuum will be performed elsewhere by one of the authors. The role of different choices of the gluon effective propagator and the limit of very strong magnetic fields are intended to be scrutinized further in a different work.

ACKNOWLEDGMENTS

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APPENDIX: PSEUDOSCALAR PION COUPLING TO CONSTITUENT QUARKS

The one loop pseudoscalar pion form factor, for the gluon effective propagator (8), can be written as

$$G_{\mathrm{PS}} = C_{\mathrm{ps0}} \int \frac{d^4k}{(2\pi)^4} \frac{k\cdot(k+Q)+M^2}{(k^2-M^2)((k+Q)^2-M^2)((k-K)^2-M_g^2)^2} \tag{A1}$$

where $C_{\mathrm{ps0}} = 8N_c(\alpha K_g)$. By employing the Feynman trick to carry the momentum integral, it can be written as

$$G_{\mathrm{PS}} = C_{\mathrm{ps0}} \frac{i}{6(4\pi)^2} \int_0^1 dz \int_0^{1-z} dyz\left(\frac{-2}{E} + \frac{F}{E^2}\right) \tag{A2}$$

$$\begin{aligned} E &= Q^2y(y-1) + K^2z(z-1) - 2K\cdot Qyz + M^2(1-y-z) + M^2y + M_g^2z \\ F &= Q^2(y-1)y + K^2z^2 + K\cdot Qz(1-2y) + M^2. \end{aligned} \tag{A3}$$

For the numerical calculation of this coupling constant, we considered off shell pions and quarks: $Q^2 = 0$ and $K^2 = 0$.

The value of K_g is fixed with

$$K_g = \frac{13}{\frac{G_{\mathrm{PS}}}{K_g}}.$$

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