QCD with an infrared fixed point: The pion sector

Roman Zwicky*

Higgs Centre for Theoretical Physics, School of Physics and Astronomy, University of Edinburgh, Peter Guthrie Tait Road, Edinburgh EH9 3FD, United Kingdom and Theoretical Physics Department, CERN, Esplanade des Particules 1, Geneva CH-1211, Switzerland

(Received 30 October 2023; accepted 16 January 2024; published 8 February 2024)

The possibility that gauge theories with chiral symmetry breaking below the conformal window exhibit an infrared fixed point is explored. With this assumption three aspects of pion physics are reproduced if the quark mass anomalous dimension at the infrared fixed point is $\gamma_* = 1$. First, by matching the long-distance scalar adjoint correlation function. Second, by perturbing the fixed point by a small quark mass, the m_q-dependence of the pion mass is reproduced by renormalization group arguments. Third, consistency of the trace anomaly and the Feynman-Hellmann theorem, for small m_q, imply the same result once more. This suggests the following picture for the conformal window; close to its upper boundary γ_* is zero and grows as the number of fermions is reduced until its lower boundary $\gamma_* = 1$ is reached, where chiral symmetry breaking sets in. Below, the strongly coupled gauge theory with $\gamma_* = 1$ is infrared dual to the free theory of pions. A possible dilaton sector of the scenario will be addressed in a companion paper.

DOI: 10.1103/PhysRevD.109.034009

I. INTRODUCTION

The idea that spontaneous chiral symmetry breaking in the strong interaction induces scale spontaneous symmetry breaking (SSB) predates QCD [1–6]. The goal of this paper is to explore this idea within gauge theories, using parts of chiral perturbation theory (χPT) [7–10] and the renormalization group (RG). Whether this scenario corresponds to a new phase [11], or an unexplored feature of QCD has to be left open at this stage. The assumption of an infrared fixed point (IRFP) is nonstandard. The main point of the paper is that under this hypothesis aspects of pion physics are reproduced consistently.

An IRFP and scale SSB is accompanied by a (pseudo) Goldstone boson, known as the dilaton. Its features and interactions are less transparent than that of the pion as scale symmetry is only emergent in the IR. Since the results presented here are seemingly independent of dilaton aspects, its main discussion is postponed to a companion paper [12].[1] At the end of the paper, we briefly comment on how the addition of a dilaton does not alter the results.

The starting assumption is that the massless degrees of freedoms, to which we will refer to as IR-states, see the world as a conformal field theory (CFT) in the deep IR.[2] That is, the trace of the EMT on the IR states $\phi_{\rm IR}$

$$\langle \phi_{\rm IR}(p)|T^\rho{}_\rho|\phi'_{\rm IR}(p)\rangle \to 0, \tag{1.1}$$

vanishes for zero momentum transfer.[3] It is though reasonable to assume that there exists a scheme for which $\beta_* = 0$ if (1.1) holds. Amongst those degrees of freedom are the vacuum, the pions resulting from chiral SSB and possibly the dilaton [11,12]. Equation (1.1) may be regarded as the minimal form by which IR conformality manifests itself in the dilatation Ward identity. Technically

*roman.zwicky@ed.ac.uk

[1]Possibly the most spectacular aspect of a dilaton is that massive hadrons, such as a nucleon, and a traceless energy momentum tensor (EMT) $\langle N|T^\rho{}_\rho|N\rangle = 0$ are compatible with each other. [11]. The dilaton restores the dilatation Ward identity just as the pion does for chiral Ward identities. Another attractive feature is that the gauge theory contribution to the cosmological constant could be zero for $m_q = 0$ [11]. It is worthwhile to mention that the dilaton under discussion is not a gravity-scalar, such as in string theory, nor an accidentally light scalar but a genuine Goldstone resulting from SSB, e.g. [13] for a historical perspective. If QCD were to possess an IRFP and a dilaton, there is consensus that it corresponds to the σ-meson, known as the $f_0(500)$ in the Particle Data Group [14].

[2]See Appendix B for comments on scale versus conformal symmetry.

[3]This does not imply that any definition of a β-function assumes a zero in the IR as it is only the combination of β times the field strength tensor which is RG-invariant cf. Sec. II C 1. This aspect has for example been emphasized in the review [15].

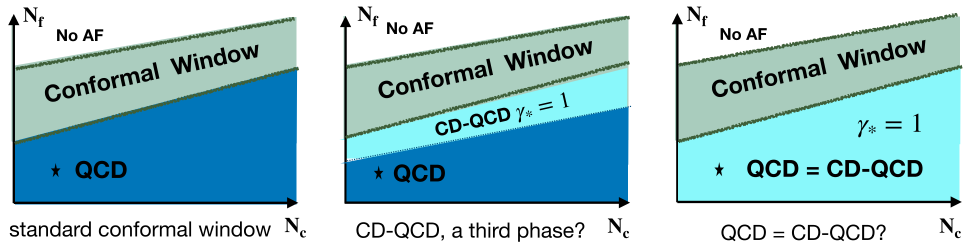


FIG. 1. Sketch of phase diagrams of gauge theories with quark matter as a function of the number of flavors N_f and colors N_c, as described in the main text. "No AF" stands for no asymptotic freedom and its boundary is known from the Caswell-Banks-Zaks analysis [23,24]. The lower dark green line marks the end of the conformal window and its precise location is unknown in the nonsupersymmetric case. In the lower dark blue phase chiral symmetry is broken, hadrons confine and $N_f = 3$ and $N_c = 3$ represents QCD. (left) Literature-standard conformal window scenario. (center) CD-QCD as a third phase as advertised in [11]. (right) QCD and CD-QCD are one and the same. We emphasize again that boundaries other than the one of AF are unknown and are shown for illustrative purposes only.

this means that correlation functions in the deep IR, and generally physical observables, are determined by the scaling dimension $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \propto (x^2)^{-\Delta_\mathcal{O}}$. The quark mass anomalous dimension γ_m, denoted by $\gamma_* = \gamma_m|_{\mu=0}$ at the IRFP, governs the scaling dimension of many important operators. The central result of this paper is that with the IRFP-assumption, this anomalous dimension must assume

$$\gamma_* = 1. \tag{1.2}$$

This is inferred in three different ways, by matching the pion low energy physics with the gauge theory. The value (1.2) is then important in two respects: it marks the lower boundary of the conformal window *and* it describes the pion physics in the chirally broken phase in terms of the strongly coupled IRFP of the gauge theory. Whereas the former is compatible with previous work and lattice Monte Carlo studies, as discussed in the conclusions, the latter is a new perspective.

The main part of the paper consists of Sec. II where $\gamma_* = 1$ (1.2) is derived from: a) a specific long-distance correlator; b) the hyperscaling relation of the pion mass; and c) the matching of the trace anomaly with the Feynman-Hellmann theorem, given in Secs. II A, II B, and II C. respectively. In Sec. III we comment on what happens when a dilaton is added. The paper ends with a summary and discussion in Sec. IV. Appendixes A, B, and C contain conventions, related discussion of scale versus conformal invariance and the soft-pion theorem in use.

II. CONSEQUENCES OF AN IRFP FOR QCD-LIKE THEORIES

The conformal window is reviewed as this work builds on it, and for further reading the reader is referred to [15–17]. The starting point is an asymptotically free gauge theory with gauge group G, e.g. $G = SU(N_c)$, and N_f massless quarks in a given representation of G. The point of study are the IR phases of these gauge theories as a function of N_c, N_f and the quark representation, cf. Fig. 1. The figure on the left depicts the standard picture for nonsupersymmetric gauge theories.[4] The boundary in the (N_c, N_f) plane of where asymptotic freedom is lost is known and for N_f below the boundary the theories admit a perturbative IRFP, the so-called Caswell-Banks-Zaks FP [23,24]. This phase, shown in green, continues until the coupling becomes strong enough for chiral symmetry to break via the formation of the quark condensate $\langle \bar{q}q\rangle \neq 0$, marked in dark blue and collectively referred to as QCD. This breaks the global flavor symmetry $SU(N_f)_L \otimes SU(N_f)_R \to SU(N_f)_V$, accompanied by $N_f^2 - 1$ massless pions as Goldstones and is believed to cause quarks and gluons to confine into hadrons. The exact boundary between the two phases is unknown and the matter of intensive debates in the literature. All evidence points towards a monotonically increasing γ_*, cf. the list of references in the conclusions. A large γ_* is important for the walking technicolor scenario, e.g. [16,25], and gave rise to efforts to determine it from lattice Monte Carlo simulations, e.g. [26–39] as reviewed in [15,17,40]. In [11] the conformal dilaton was advocated as a third phase as shown in the central figure and its domain and location should not be taken literally. In this work we refer to this phase as the conformal dilaton CD of QCD. It seemed reasonable to assume that this phase lies in between the others as it is the same for its properties. Clearly neither its existence nor its location are certainties. At least, any of the three cases shown in Fig. 1 are logical possibilities. This paper consists in analyzing the IRFP scenario, or the CD-QCD phase. It seems worthwhile to reemphasize that none of the results obtained directly depend on the presence of a dilaton.

[4]The conformal window of supersymmetric theories is well-understood due to the Seiberg dualities [18–21] and the exact NSVZ β-function [22]. The lower boundary becomes a perturbative FP in the dual theory. The value of γ_* at which the transition occurs is $\gamma_* = 1$, related to the unitarity bound of the squark composite operator $\Delta_{\tilde{Q}Q} = 2 - \gamma_* \geq 1$. The phases below the conformal window are richer in that there is a phase with confinement without chiral symmetry breaking. It is not believed that this is repeated for nonsupersymmetric gauge theories.

A. Deep-IR interpretation of the adjoint scalar correlator ($m_q = 0$)

For $m_q = 0$ the theory exhibits, the previously mentioned, scaling in correlation functions and this is what we will exploit in this section. The scalar operator, with $J^P = 0^+$ quantum numbers

$$S^a = \bar{q} T^a q, \tag{2.1}$$

where T^a generates the flavor symmetry, is an example that offers itself since it is not perturbed by a single Goldstone. Consistency of the IRFP interpretation means that

$$\langle S^a(x) S^a(0) \rangle_{\text{CD-QCD}} = \langle S^a(x) S^a(0) \rangle_{\chi\text{PT}}, \quad \text{for } x^2 \to \infty, \tag{2.2}$$

must hold, since they describe the same theory in the deep-IR limit $x^2 \to \infty$. The next two sections are devoted to this matching.[5]

1. The CD-QCD correlator in the deep IR

It is our assumption that QCD is described by an IRFP in the deep IR, which in turn means that CFT methods apply in that regime. In CFTs 2- and 3-point correlators [43–48] are entirely governed by their scaling dimensions, $\Delta_{\mathcal{O}} = d_{\mathcal{O}} + \gamma_{\mathcal{O}}$, which is the sum of the engineering dimension and the anomalous dimension. Concretely, for a Euclidean CFT

$$\langle \mathcal{O}(x) \mathcal{O}^\dagger(0) \rangle_{\text{CFT}} \propto (x^2)^{-\Delta_{\mathcal{O}}}, \tag{2.3}$$

where $x^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$ and $\langle \ldots \rangle$ denoting, hereafter, the vacuum expectation value. The behavior in (2.3) should be mirrored by the correlation function (2.2) in the deep IR. The only necessary ingredient is the scaling dimension of S^a which is

$$\Delta_{S^a} = d_{S^a} - \gamma_* = 3 - \gamma_*, \tag{2.4}$$

since $d_{S^a} = 3$. Equation (2.4) follows from $\Delta_{S^a} = \Delta_{P^a}$, which holds at least in perturbation theory since the γ_5 can be commuted through the diagram for $P^a = \bar{q} i \gamma_5 T^a q$ to recover S^a if $m_q = 0$ is assumed. In turn, $\Delta_{P^a} = 3 - \gamma_m$ follows from the Ward identity $\partial^\mu \langle A^a_\mu(x) P^b(0) \rangle \propto \delta^{(4)}(x) \delta^{ab} \langle \bar{q} q \rangle$ and the fact that A^a_μ and $m_q \bar{q} q$ and are RG invariants. This is true for the former since it is a softly conserved current and for the latter it follows for instance from the quantum action principle for which the reader is referred to [49], for a discussion in the perturbative context. With (2.3) and (2.4), one concludes that

$$\langle S^a(x) S^a(0) \rangle_{\text{CD-QCD}} \propto (x^2)^{-(3-\gamma_*)}, \qquad x^2 \to \infty. \tag{2.5}$$

2. Leading-order chiral perturbation theory

In order to compute the correlator (2.2) in χPT, the QCD operator S^a needs to be described in terms of pion fields. This can be done by the source method [7–10], starting from the leading-order (LO) mass Lagrangian

$$\delta \mathcal{L}_{m_q} = \frac{F_\pi^2 B_0}{2} \text{Tr}[\mathcal{M} U^\dagger + U \mathcal{M}^\dagger], \tag{2.6}$$

where $B_0 = -\langle \bar{q} q \rangle / F_\pi^2$ and the quark mass matrix is $\mathcal{M} = m_q \mathbb{1}_{N_f}$ in our case. The operator S^a is obtained by replacing $\mathcal{M} \to T^a J_{S^a}$ and differentiating the log of the Euclidean generating functional $\mathcal{Z}$ with respect to the source, $\langle S^a(x) \rangle \leftrightarrow \delta_{J_{S^a}(x)} \ln \mathcal{Z}$,

$$\begin{aligned} S^a|_{\text{LO}} &= -\frac{F_\pi^2 B_0}{2} \text{Tr}[T^a U^\dagger + U T^a] \\ &\propto B_0 d^{abc} \pi^b \pi^c + \mathcal{O}(1/F_\pi^2), \end{aligned} \tag{2.7}$$

where the $\mathcal{O}(1/F_\pi^2)$ terms are cutoff suppressed and thus next-LO.[6] The computation of the correlator in LO χPT is now straightforward

$$\langle S^a(x) S^a(0) \rangle_{\chi\text{PT}} \propto B_0^2 d^{abc} d^{abc} \langle \pi^e(x) \pi^e(0) \rangle^2 \propto \frac{1}{x^4}, \quad \text{for } x^2 \to \infty, \tag{2.8}$$

where as anticipated $m_q \to 0$ limit has been assumed. Above $\langle \pi^e(x) \pi^e(0) \rangle = \frac{1}{(4\pi)^2} \frac{1}{x^2}$, with e fixed, is the standard Euclidean propagator for a massless scalar field $\pi^e(x)$. Thus the LO χPT is just given by a free field theory computation as illustrated in Fig. 2.

3. IRFP matching and contemplation on $\gamma_* = 1$ in the wider picture

The matching of CD-QCD and χPT, as in (2.2), with (2.5) and (2.8) enforces

$$\gamma_* = 1, \tag{2.9}$$

which is the main result of this work.

[5] This correlator has been used in [41] to match χPT and the spectral representation. It was deduced that the correction to the Dirac eigenvalue density is $\rho(\lambda) - \rho(0) = C|\lambda|$, where C is known and $\rho(0) = -\langle \bar{q} q \rangle / \pi$ is the famous Banks-Casher relation [42].

[6] Generically, $d^{abc} \neq 0$ but for $N_f = 2$ it vanishes; $d^{abc} d^{abc} \propto N_f^2 - 4$. This accidentality is of no special concern to the argument made in this section.

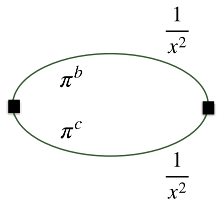
$\langle S^a(x)S^a(0)\rangle =$  $+ \quad \mathcal{O}(1/F_\pi^2)$

FIG. 2. Adjoint scalar correlation function in χPT (2.8) which behaves as $1/x^4$ for large distances.

Let us try to put this result into perspective, before rederiving it in two different ways. First it is noted that, $\gamma_* = 1$ is considerably below the unitarity bound $\gamma_* \leq 2$, which follows from $\Delta_{\bar{q}q} = 3 - \gamma_* \geq 1$ [50]. The result gives rise to the following picture. For $\gamma_* = 0$ or $\Delta_{S^a} = 3$ it corresponds to two free fermions, whereas for $\gamma_* = 1$ or $\Delta_{S^a} = 2$ it describes two free scalar pions and finally for $\gamma_* = 2$ or $\Delta_{S^a} = 1$, when reaching the unitarity bound, it is equivalent to one free scalar particle [51]. The message seems to be that for integer powers of the scaling dimension, the theory lends itself to a free particle interpretation, cf. Fig. 3. Note that the gauge theories only seem to make use of the [0, 1] range in γ_*, which corresponds to only a third of the allowed range $-1 \leq \gamma_* \leq 2$ in the nonsupersymmetric case.

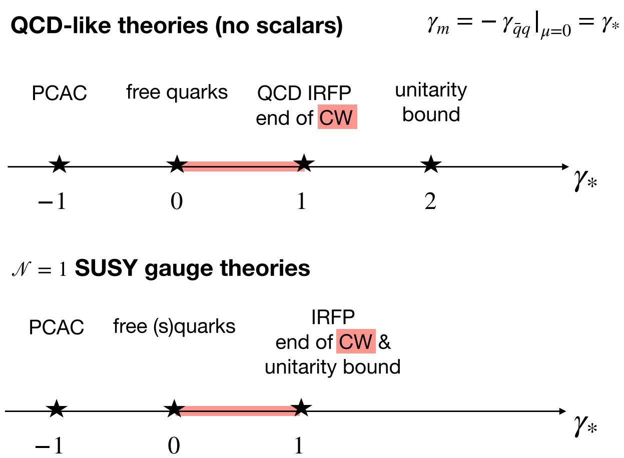


FIG. 3. Range of possible IRFP anomalous dimension γ_*. As emphasized in the main text, integer values seem to play a special role. The value of γ_* is bounded from above by the unitarity bound $\gamma_* \leq 2(1)$, $\Delta_{\bar{q}q} = 3 - \gamma_* \geq 1 (\Delta_{\bar{Q}Q} = 2 - \gamma_* \geq 1)$ [50] in QCD-like theories ($\mathcal{N} = 1$ SUSY). The lower bound $\gamma_* > -1$ comes from the requirement of soft breaking such that the PCAC is not spoiled [52]. The value $\gamma_* = 0$ corresponds to the trivial FP at the upper end of the conformal window cf. Fig. 1. As the number of flavors is lowered, γ_* raises and as it reaches $\gamma_* = 1$, chiral symmetry breaking sets in marking the lower end of the conformal window. This is true in $\mathcal{N} = 1$, cf. footnote 4, and in this paper this is conjectured to hold in QCD-like theories as well. The peculiarity of $\mathcal{N} = 1$ is that the unitarity bound and the end of the conformal window coalesce whereas this does not seem to be the case in QCD-like theories.

Of course, (2.8) cannot be viewed as novel from the χPT-viewpoint as it is simply the LO analysis. However, what is new is the way in which this is realized in the gauge theory. The free pions are IR-dual to a gauge theory with a strongly coupled IRFP; strongly coupled since the anomalous dimension is large. These types of interpretations hold in many EFT formulations of weakly coupled ultraviolet Lagrangians, and may be regarded as the very purpose of the EFT program when the microscopic formulation is known.

This suggest the following picture for the conformal window. At the upper end $\gamma_* = 0$ and then γ_* increases as N_f is lowered and when $\gamma_* = 1$ is reached chiral symmetry is broken and confinement sets in. The anomalous dimension γ_*, then remains one in the entire domain of the CD phase. As mentioned in the introduction the latter could be or not be identical to QCD itself. The result (2.9) is consistent with $\mathcal{N} = 1$ supersymmetric gauge theories, as mentioned previously.

B. Scaling of the pion mass implies $\gamma_* = 1$ ($m_q \neq 0$)

In what follows the IRFP is perturbed by a nonvanishing quark mass m_q. Even though the quark mass is scheme dependent, the physics can be analyzed by tracking powers of the rescaled bare mass. This is the standard method of hyperscaling extensively applied to the conformal window [53–57] where hadrons appear when a quark mass term is introduced.[7] The difference in the scenario at hand is that chiral symmetry is spontaneously broken, and this introduces a natural cutoff scale $\Lambda = 4\pi F_\pi$ [59]. The quantity $F_\pi \approx 93$ MeV in QCD is the pion decay constant and the order parameter of chiral symmetry breaking [8–10]. We assume that the χPT cutoff Λ does not affect the LO m_q behavior of the pion mass, which is natural from the viewpoint of χPT itself which is organized in a $1/\Lambda$-expansion. Under this assumption the behavior of the pion mass is governed by hyperscaling due to the RG, in the same way as in the conformal window. The result, perhaps most cleanly derived in [54], is

$$m_\pi^2|_{\mathrm{RG}} \propto m_q^{\frac{2}{1+\gamma_*}}, \tag{2.10}$$

where γ_* is the previously introduced mass anomalous dimension at the FP. In QCD the linear behavior

$$m_\pi^2|_{\mathrm{QCD}} \propto m_q, \tag{2.11}$$

is deducible in many ways such as from the GMOR relation [60], derived in Appendix C 1 from a double

[7]The idea is that for $m_q \neq 0$ the quarks decouple leaving behind pure Yang-Mills which is known to confine [58]. Hence there are hadrons and hadronic observables which however need to vanish when $m_q \to 0$. The way this happens is dictated by the RG [53,54,56].

soft-pion theorem. Since Eq. (2.11) holds in QCD, we assume it would in a CD-QCD phase as well. The dilaton is not affecting the LO pion mass. Hence, equating Eqs. (2.10) and (2.11) implies the central result in $\gamma_* = 1$ (2.9) once more.

C. Trace anomaly and Feynman-Hellmann theorem ($m_q \neq 0$)

The goal of this section is to show that the trace anomaly and the Feynman-Hellmann theorem are compatible if $\gamma_* = 1$ for an IRFP upon applying the formula

$$2m_\pi^2 = \langle \pi^a(p)|T^\rho{}_\rho|\pi^a(p)\rangle, \qquad a \text{ fixed.} \tag{2.12}$$

The validity of (2.12) when a dilaton is added will be commented on in Sec. III.

1. The $T^\rho{}_\rho$-anomaly and renormalization group invariant combinations

The part relevant to physical matrix elements of the trace anomaly reads[8]

$$T^\rho{}_\rho|_{\text{phys}} = \frac{\beta}{2g}G^2 + \sum_q m_q(1+\gamma_m)\bar{q}q, \tag{2.13}$$

where all quantities including the composite operators are renormalized.

An important aspect is that $m\bar{q}q$ is an RG invariant as mentioned previously. Since $T^\rho{}_\rho$ is an RG invariant, the following two combinations:

$$O_1 = \frac{\beta}{2g}G^2 + \sum_q \gamma_m m_q \bar{q}q, \qquad O_2 = \sum_q m_q\bar{q}q, \tag{2.14}$$

are RG invariants, or equally so, with $\delta\gamma \equiv \gamma_m - \gamma_*$

$$O_1' = \frac{\beta}{2g}G^2 + \delta\gamma\sum_q m_q\bar{q}q, \qquad O_2' = (1+\gamma_*)\sum_q m_q\bar{q}q, \tag{2.15}$$

since the FP value γ_* is an RG invariant. Using (2.12) to (2.13), the $\mathcal{O}(m_q)$ contribution then follows

$$2m_\pi^2 = (1+\gamma_*)\sum_q m_q\langle\pi|\bar{q}q|\pi\rangle + \mathcal{O}(m_q^2), \tag{2.16}$$

The statement of (2.16) is that O_2' is the leading operator in the quark mass and that O_1' must be suppressed.[9] Some more insight into this matter is provided in Sec. II C 3 through the final matching.

2. The $T^0{}_0 = H$ viewpoint: Feynman-Hellmann theorem

The Feynman-Hellmann theorem [67,68] offers a way to obtain the LO quark mass dependence directly from the Hamiltonian $\delta H_m = \sum_q m_q\bar{q}q$ by differentiation in m_q. It is technically convenient to use states, $\langle\hat{\pi}(p')|\hat{\pi}(p)\rangle = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}')$, which are normalized in nonrelativistic manner. One can switch to the usual states by $|\pi\rangle = |\hat{\pi}\rangle\sqrt{2E_\pi}$ after the m_q differentiation. The Feynman-Hellmann formula implies

$$\partial_{\ln m_q}E_\pi = \sum_q m_q\langle\hat{\pi}|\bar{q}q|\hat{\pi}\rangle + \mathcal{O}(m_q^2), \tag{2.17}$$

where $E_\pi = \langle\hat{\pi}|H|\hat{\pi}\rangle$, $V \leftrightarrow (2\pi)^3\delta^{(3)}(0)$ and $\partial_{m_q}\langle\hat{\pi}(p')|\hat{\pi}(p)\rangle = 0$ have been used. Switching back to standard pion states $|\pi\rangle$ and using $\partial_{m_q}E_\pi^2 = \partial_{m_q}m_\pi^2$, which follows from the m_q independence of the 3-momentum $\vec{p}$, one obtains

$$\partial_{\ln m_q}2m_\pi^2 = 2\sum_q m_q\langle\pi|\bar{q}q|\pi\rangle + \mathcal{O}(m_q^2). \tag{2.18}$$

Further assuming $m_\pi^2 = \mathcal{O}(m_q)$ then gives[10]

$$2m_\pi^2 = 2\sum_q m_q\langle\pi|\bar{q}q|\pi\rangle + \mathcal{O}(m_q^2), \tag{2.19}$$

the formula linear in the quark mass. The formula (2.19) in itself is not new and has been used and derived in the literature frequently e.g. [70,71]. The correctness of (2.19) is verified in Appendix C 1 by reproducing the GMOR relation [60]. This is important as an incorrect numerical prefactor, by matching to the trace anomaly below, would give an incorrect γ_*.

3. Matching the two mass formulas

The two mass formulas, (2.16) and (2.19), are, once more, compatible with each other if and only if $\gamma_* = 1$. We

[8]The trace anomaly was first observed in correlation functions [61–63] and subsequently worked out in detail [49,64–66] including equation of motions and BRST exact terms arising upon gauge fixing.

[9]From the viewpoint of χPT the relative corrections are of order $\mathcal{O}(m_q \ln m_q)$, e.g. [9,10], and thus it is possible that the RG analysis on its own does not reveal the true next-to-LO behavior. This is not relevant for the point we are making but worthwhile to investigate further.

[10]The use of the Feynman-Hellmann theorem and its derivative is crucial. If one were to use the Hamiltonian, written schematically as $H = \vec{E}^2 + \vec{B}^2 + \sum_q m_q\bar{q}q$, then applying the states would result in $2E_\pi = \langle\pi|\vec{E}^2 + \vec{B}^2|\pi\rangle + \langle\pi|\sum_q m_q\bar{q}q|\pi\rangle$, where the momentum dependence of E_π has to reside in the electromagnetic $\vec{E}^2 + \vec{B}^2$ matrix element, cf. [69] for related discussions.

consider this an important result since the assumption is weaker than in Sec. II B.[11] In that section we assumed that the renormalization behavior (hyperscaling), in the pion sector at LO in the quark mass, is unaffected by the presence of the χPT cutoff $\Lambda = 4\pi F_\pi$. Here we merely assumed that the β- and $\delta\gamma$-terms can be neglected in the vicinity of the FP. As the RG scale μ can be made arbitrarily small seems a lesser assumption and thus more satisfying in our view. Since $\beta_* = 0$ and $\gamma_* = 1$ are formally correct in that it matches the Feynman-Hellmann expression this also provides indirect justification for the earlier statement that the operator $\langle\pi|O_1'|\pi\rangle$ is suppressed.

It seems worthwhile to point out that independent of whether there is an IRFP or not, $\langle\pi|O_1|\pi\rangle = \langle\pi|O_2|\pi\rangle + \mathcal{O}(m_q^2)$. This is the case since both (2.13) and (2.19) derive from first principles and are not related to any of the specific FP assumptions made in this paper. In more familiar notation the relation reads

$$\langle\pi|\frac{\beta}{2g}G^2 + \sum_q m_q\gamma_m\bar{q}q|\pi\rangle = \langle\pi|\sum_q m_q\bar{q}q|\pi\rangle + \mathcal{O}(m_q^2), \tag{2.20}$$

assures that the trace anomaly and Feynman-Hellmann derivation of the LO pion mass are consistent with each other. The solution for an IRFP $\beta \to \beta_* = 0$ and $\gamma_m \to \gamma_* = 1$ is a straightforward one. Other solutions, not related to an IRFP, demand a specific interplay between the β- and γ-term. This is though perfectly possible since O_1 in (2.14) is an RG invariant.

III. BRIEF COMMENTS ON THE ADDITION OF A DILATON

The results obtained did not make use of the presence of a dilaton. Conversely, if pion physics can be interpreted by an IRFP then this suggests that a dilaton could be present and it is a valid question whether the latter would impact on any of the results obtained. Let us refer for practical reasons to the dilaton as the lightest state in the $J^{\mathrm{PC}} = 0^{++}$ flavor singlet channel. If the dilaton remains massive in the limit where the explicit symmetry breaking is removed, $m_q \to 0$, then it can simply be integrated out in the deep IR and everything remains the same. If on the other hand it becomes massless in that limit then a closer inspection is needed. We proceed case by case:

(i) For the long-distance correlator in Sec. II A there would be no relevant changes. The dilaton would alter the correlation function at $\mathcal{O}(1/x^6)$ which is subleading with respect to $1/x^4$ behavior as in (2.8). Its quantum numbers do not allow a $1/x^4$-contribution. Hence the conclusions remain unchanged.

(ii) The hyperscaling argument of Sec. II B is also unaltered but it implies in turn $m_D^2 \propto m_q$ in the same way as it does for the pion.

(iii) The matching of the trace anomaly and the Feynman-Hellmann theorem in Sec. II C is more subtle and requires some care. In the case of a massless dilaton the standard formula $2m_\phi^2 = \langle\phi|T^\rho{}_\rho|\phi\rangle$, where ϕ is a physical state, cannot be used because of the dilaton pole [11]. However, in the case of the pion (2.12) holds since the effect of the dilaton pole for massless states, such as the pion, is undone by its coupling to pions. Concretely,

$$\langle\pi^a(p')|T_{\mu\nu}|\pi^a(p)\rangle \supset c(q_\mu q_\nu - q^2\eta_{\mu\nu})\frac{g_{D\pi\pi}}{q^2 - m_D^2}, \qquad q \equiv p - p', \tag{3.1}$$

where $c = \text{const} \times F_D$ and $g_{D\pi\pi}$ is given by [3,12,72,73]

$$g_{D\pi\pi} = \frac{1}{F_D}\left(q^2 + (1-\gamma_*)m_\pi^2 + \mathcal{O}(m_q^2)\right), \tag{3.2}$$

where F_D is the dilaton decay constant as defined in [11]. The q^2 dependence originates from the pion kinetic term to which the dilaton couples. Taking the trace in (3.1) and the limit $q^2 \to 0$ we learn that this term does not contribute. The pion and the dilaton are both massive due to $m_q \neq 0$, which would be enforced in a systematic EFT approach. The behavior for massive hadrons is qualitatively different since $g_{D\phi\phi} \propto m_\phi^2/F_D = \mathcal{O}(\Lambda)$, here for a scalar ϕ, does not vanish any limit. It is precisely this behavior that gives rise to the vanishing of the trace of the EMT for a massless dilaton [11].

A further point of concern is that the dilaton could alter the evaluation of the matrix element $\langle\pi^a|\bar{q}q|\pi^a\rangle$ in Appendix C 1. The dilaton contribution to this matrix element is analogous to (3.1) is $\propto g_{D\pi\pi}/(q^2 - m_D^2)$ [without the $(q_\mu q_\nu - q^2\eta_{\mu\nu})$ prefactor]. Assuming $\gamma_* \to 1$ and $q^2 \to 0$ we learn that this term does not contribute. In coming to this conclusion it is important to use $m_D \neq 0$ (due to $m_q \neq 0$) and $\gamma_* = 1$. The latter is legitimate since it has already been concluded by equating (2.16) and (2.19).

We infer from our considerations that the addition of a (massless) dilaton does not alter the results.

IV. SUMMARY AND CONCLUSIONS

In this work we have offered an interpretation of low-energy pion physics in terms of a strongly coupled infrared

[11]In the setting of the conformal window, with quark mass deformation (cf. [56] and footnote 7), both approaches lead to $m_\phi^2 \propto (m_q)^{2/(1+\gamma_*)}$ where ϕ stands for any hadron. The situation is though different in the case at hand because of the cutoff scale mentioned above.

fixed point of a QCD-like gauge theory. Colloquially speaking, this means that the infrared states such as the pions experience the world as a conformal field theory in the deep infrared. Comparing observables in the conformal or renormalization group picture with standard pion physics we deduced in three ways that the quark-mass anomalous dimension takes on the value $\gamma_* = 1$, at the fixed point. Namely:

(i) By requiring consistency between the leading order χPT and the CFT interpretation of the adjoint scalar correlator $\langle S^a(x) S^a(0)\rangle$ in Sec. II A;

(ii) by renormalization group arguments and assuming that the χPT cutoff $\Lambda = 4\pi F_\pi$ does not affect leading quark-mass behavior of the pion in Sec. II B;

(iii) by requiring consistency between the trace anomaly and the Feynman-Hellmann theorem in Sec. II C.[12]

These arguments are largely independent and thus any of the three could have served as a starting point for the paper. Perhaps, the third point is the strongest as it only relies on the near fixed point behavior. The important point is, though, that by assuming an infrared fixed point we were able to derive internally consistent results. This is no substitute for a proof. In fact there are at least the three possibilities shown in Fig. 1. The scenario is not realized in any gauge theory (left), it is realized in some area outside the conformal window (center) or it is identical to the standard QCD-type theories (right). Alternatively, it is conceivable that the matching with pion physics and the fixed-point viewpoint are only valid at the lower edge of the conformal window suggesting that the transition is smooth. It could be that fixed point scenario is related to the presence of a massless dilaton. We cannot answer these question in any definite form but to reemphasize that the goal of this paper was to explore the consequences of assuming an infrared fixed point.

The smooth matching of pion physics with the conformal window led us to conjecture that $\gamma_* = 1$ marks the end of the conformal window. The value $\gamma_* = 1$ as the end of the conformal window is consistent with lattice Monte Carlo computations [15,17,40], in particular the dilaton-EFT fits in [74,75], perturbative computations [76,77], gap equations [78–80], walking technicolor phenomenology [16,25,81], holographic approaches [82,83] and $\mathcal{N} = 1$ supersymmetry [18–20]. However, these works do not interpret the pion physics below the boundary by an infrared fixed point, which is the main point of our work. From a certain perspective our work is more closely related to the pre-QCD work [1–6] or its revival a decade ago [72,84]. The difference to these papers is that there is a definite statement about the scaling of the most important operators.

Another way to look at the proposal is to notice that QCD in the deep infrared is described by the free-field theory of pions and is thus scale invariant.[13] This makes the fixed-point interpretation look natural, and is indeed assumed in the context of the a-theorem, e.g. [85]. The χPT gauge-theory matching can be seen as infrared duality of weak and strong coupling theories, cf. Sec. II A 3, which are often the motivation for an effective-field theory program. That these types of dualities are more fundamental, might be related to the Seiberg dualities [18–20], which in turn gave new motivation to the fascinating idea of hidden local symmetry, e.g. [86–88].

There are other factors supporting the infrared-fixed point picture; dense nuclear interactions [89–91] or the Goldstone improvement [73].[14] An indirect hint is coming from the fact that lattice gauge theories close to the conformal window [e.g. $N_f = 8$ and $SU(3)$ gauge group] can be fitted with a dilaton EFT for $\gamma_* \approx 1$ [75,94]. In view of this fact understanding the origin of such an EFT and finding the correct picture invites further investigations e.g. [12,74,75,95]. Or, the addition of the dilaton sector to be discussed in [12] will offer other ways to test the scenario. As mentioned earlier the dilaton candidate in QCD is the broad $f_0(500)$ meson. Importantly, if the Higgs sector is to be replaced by a gauge theory then its dilaton can take on the role of a Higgs which is hard to distinguish from the Standard Model one. This has been appreciated since a long time within the gauge theory setting e.g. [96] or without a specific ultraviolet completion e.g. [97,98]. Our work strengthens this case considerably and identifies in $S = \bar{q}q$ the presumably most relevant operator as its scaling dimension assumes $\Delta_{\bar{q}q} = 2$ as a consequence of $\gamma_* = 1$. A further advantage of gauge theories for a dilaton sector is that they can be explored with analytic tools and lattice Monte Carlo simulations serving as a laboratory to further ideas in a concrete setting.

ACKNOWLEDGMENTS

R. Z. is supported by a CERN associateship and an STFC Consolidated Grant No. ST/P0000630/1. I am grateful to Steve Abel, Tim Cohen, Gilberto Colangelo, Matthew McCullough, Poul Damgaard, Luigi Del Debbio John Donoghue, John Ellis, Nick Evans, Max Hansen, Shoji Hashimoto, Andreas Jüttner, Daniel Litim, Tom Melia, Kohtaroh Miura, Agostino Patella, Jérémie Quevillon, Mannque Rho, Francesco Sannino, Misha Shifman, Christopher Smith, Lewis Tunstall, and Koichi Yamawaki for correspondence or discussions.

[12]Note that Eq. (2.20) must hold in the chirally broken phase irrespective of the fixed point interpretation.

[13]If the dilaton is added then the theory becomes at least classically conformal [73], circumventing the Goldstone improvement problem.

[14]A dilaton ameliorates the convergence of the integrals of the a-theorem [92,93] for QCD-like theories [73]. A sufficiently fast vanishing $T^\rho{}_\rho$ in the IR is a necessary condition for the integral formulas to converge.

APPENDIX A: CONVENTIONS

The Minkowski metric $\eta_{\mu\nu}$ reads $\mathrm{diag}(1,-1,-1,-1)$. The Lagrangian of the gauge theory is given by

$$\mathcal{L} = -\frac{1}{4}G^2 + \sum_q m_q \bar{q}(iD - m_q)q, \tag{A1}$$

where $G^2 = G^A_{\mu\nu}G^{A\mu\nu}$ is the field strength tensor and A the adjoint index of the gauge group. The N_f quark flavors are assumed to be degenerate in mass. The beta function is defined by $\beta = \frac{d}{d\ln\mu}g$ and the mass anomalous dimension is given by $\gamma_m = -\frac{d}{d\ln\mu}\ln m_q$. Quantities at the FP are designated by a star e.g. $\gamma_* = \gamma_m|_{\mu=0}$ (1.2). QED is omitted even though the massless photon is definitely an IR degree of freedom but it does not change the picture considerably as it is weakly coupled in the IR. The $SU(N)$ flavor symmetry generators T^a are normalized as $T^aT^b = \frac{1}{2N_c}\delta^{ab}\mathbb{1}_{N_c} + \frac{1}{2}d^{abc}T^c + \frac{i}{2}f^{abc}T^c$, $\mathrm{Tr}[T^aT^b] = \frac{1}{2}\delta^{ab}$ and f/d^{abc} are the totally anti/symmetric tensors.

APPENDIX B: CONFORMAL VERSUS SCALE INVARIANCE

Scale and conformal invariance are not distinguished in this work as it is widely believed that the former implies the latter for theories like QCD (and most nonexotic $d = 4$ theories) cf. Ref. [99] for a review. A scale invariant theory is one where $T^\rho{}_\rho = \partial\cdot V$ such that $J^D_\mu = x^\nu T_{\mu\nu} - V_\mu$ is conserved. Since the scaling dimension of the trace of the EMT is d, the one of the virial current has to be $d-1$ which is highly nongeneric as it, usually, requires the protection of a symmetry.

APPENDIX C: SOFT-PION THEOREM

Since the soft-pion theorem is important in the main text, we reproduce its form from the textbook [8]

$$\langle\pi^a(q)\beta|\mathcal{O}(0)|\alpha\rangle = -\frac{i}{F_\pi}\langle\beta|[Q_5^a,\mathcal{O}(0)]|\alpha\rangle + \lim_{q\to0} iq\cdot R^a, \tag{C1}$$

where the square brackets denote the commutator. Above α and β are other physical states and R^a is the so-called remainder

$$R^a_\mu = -\frac{i}{F_\pi}\int d^dx e^{iq\cdot x}\langle\beta|TJ^a_{5\mu}(x)\mathcal{O}(0)|\alpha\rangle, \tag{C2}$$

which vanishes unless there are intermediate states degenerate with either α or β.[15] Equation (C1) is straightforward to derive from correlation functions using a dispersive representation.

1. The GMOR-relation from double soft-pion theorem

In Sec. II C 3 it was concluded that the trace anomaly and the Feynman-Hellmann theorem imply $\gamma_* = 1$ but this relies in particular that the *prefactor* in Eq. (2.19) is correct. This can be verified by making the link to the celebrated GMOR-relation [60] of QCD. The procedure is to apply the soft theorem, summarized above, twice to eliminate the pions. Applying it once results in

$$\begin{aligned} m_\pi^2 &= \sum_q m_q\langle\pi^a|\bar{q}q|\pi^a\rangle = \frac{-m_q}{F_\pi}\langle0|i[Q_5^a,\bar{q}\mathbb{1}_{N_f}q]|\pi^a\rangle \\ &= \frac{2m_q}{F_\pi}\langle0|P^a|\pi^a\rangle, \end{aligned} \tag{C3}$$

where $P^a = \bar{q}i\gamma_5T^aq$ as previously, and $\sum_q\bar{q}q\to\bar{q}\mathbb{1}_{N_f}q$ as it is a more suitable notation to evaluate the commutator. The remainder (C2) can be omitted since it is zero. This is not obvious when a dilaton is present as commented on in Sec. III. Applying the soft theorem to (C3) once more, using $d^{abc}\langle\bar{q}T^cq\rangle = 0$, one gets

$$\langle0|P^b|\pi^a\rangle = -\frac{1}{F_\pi}\langle0|i[Q_5^a,P^b]|0\rangle = -\frac{1}{F_\pi}\langle\bar{q}q\rangle\delta^{ab}, \tag{C4}$$

which combines into

$$m_\pi^2F_\pi^2 = -2m_q\langle\bar{q}q\rangle, \tag{C5}$$

the GMOR-relation [8–10,60]. This completes the task of this appendix.

[15]We have checked that it vanishes in the cases at hand and will therefore not discuss it any further. A case where the remainder is relevant is the matrix element $\langle N^a\pi^b(q)|J^c_\mu|N^{*d}\rangle$. The $|N^{a'}\rangle$ is degenerate and one can use the Callan-Treiman relation, due to the chiral Ward identity, to infer $\lim_{q\to0}q^\mu\langle N^a|J^b_{5\mu}|N^d\rangle\neq0$, implying the nonvanishing of the remainder.

[1] C. J. Isham, A. Salam, and J. A. Strathdee, Broken chiral and conformal symmetry in an effective-Lagrangian formalism, Phys. Rev. D **2**, 685 (1970).

[2] C. J. Isham, A. Salam, and J. A. Strathdee, Spontaneous breakdown of conformal symmetry, Phys. Lett. **31B**, 300 (1970).

[3] J. R. Ellis, Aspects of conformal symmetry and chirality, Nucl. Phys. **B22**, 478 (1970); Nucl. Phys. **B25**, 639(E) (1971).

[4] J. R. Ellis, Phenomenological actions for spontaneously-broken conformal symmetry, Nucl. Phys. **B26**, 536 (1971).

[5] R. J. Crewther, Broken scale invariance in the width of a single dilaton, Phys. Lett. **33B**, 305 (1970).

[6] R. J. Crewther, Spontaneous breakdown of conformal and chiral invariance, Phys. Rev. D **3**, 3152 (1971); Phys. Rev. D **4**, 3814(E) (1971).

[7] J. Gasser and H. Leutwyler, Chiral perturbation theory to one loop, Ann. Phys. (N.Y.) **158**, 142 (1984).

[8] J. F. Donoghue, E. Golowich, and B. R. Holstein, *Dynamics of the Standard Model* (Cambridge University Press, Cambridge, England, 2014), Vol. 2.

[9] H. Leutwyler, On the foundations of chiral perturbation theory, Ann. Phys. (N.Y.) **235**, 165 (1994).

[10] S. Scherer and M. R. Schindler, *A Primer for Chiral Perturbation Theory* (Springer, Berlin, Heidelberg, 2012), Vol. 830.

[11] L. Del Debbio and R. Zwicky, Dilaton and massive hadrons in a conformal phase, J. High Energy Phys. 08 (2022) 007.

[12] R. Zwicky, QCD with an infrared fixed point and a dilaton, arXiv:2312.13761.

[13] R. J. Crewther, Genuine dilatons in gauge theories, Universe **6**, 96 (2020).

[14] R. L. Workman *et al.* (Particle Data Group Collaboration), Review of particle physics, Prog. Theor. Exp. Phys. **2022**, 083C01 (2022).

[15] D. Nogradi and A. Patella, Strong dynamics, composite Higgs and the conformal window, Int. J. Mod. Phys. A **31**, 1643003 (2016).

[16] F. Sannino, Conformal dynamics for TeV physics and cosmology, Acta Phys. Pol. B **40**, 3533 (2009).

[17] L. Del Debbio, The conformal window on the lattice, Proc. Sci. Lattice2010 (**2014**) 004 [arXiv:1102.4066].

[18] K. A. Intriligator and N. Seiberg, Lectures on supersymmetric gauge theories and electric-magnetic duality, Nucl. Phys. B, Proc. Suppl. **45BC**, 1 (1996).

[19] M. A. Shifman, *ITEP Lectures on Particle Physics and Field Theory Vol. 1, 2* (World Scientific, 1999), vol. 62.

[20] J. Terning, *Modern Supersymmetry: Dynamics and Duality* (Oxford Academic, Oxford, 2006).

[21] T. A. Ryttov and R. Shrock, Scheme-independent calculations of physical quantities in an $N = 1$ supersymmetric gauge theory, Phys. Rev. D **96**, 105018 (2017).

[22] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Exact Gell-Mann-Low function of supersymmetric Yang-Mills theories from instanton calculus, Nucl. Phys. **B229**, 381 (1983).

[23] W. E. Caswell, Asymptotic behavior of non-Abelian gauge theories to two loop order, Phys. Rev. Lett. **33**, 244 (1974).

[24] T. Banks and A. Zaks, On the phase structure of vector-like gauge theories with massless fermions, Nucl. Phys. **B196**, 189 (1982).

[25] C. T. Hill and E. H. Simmons, Strong dynamics and electroweak symmetry breaking, Phys. Rep. **381**, 235 (2003); Phys. Rep. **390**, 553(E) (2004).

[26] A. Hasenfratz, C. Rebbi, and O. Witzel, Gradient flow step-scaling function for SU(3) with ten fundamental flavors, Phys. Rev. D **101**, 114508 (2020).

[27] J. Kuti, Z. Fodor, K. Holland, and C. H. Wong, From ten-flavor tests of the β-function to α_s at the Z-pole, Proc. Sci. LATTICE2021 (**2022**) 321 [arXiv:2203.15847].

[28] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. H. Wong, Case studies of near-conformal β-functions, Proc. Sci. LATTICE2019 (**2019**) 121 [arXiv:1912.07653].

[29] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. H. Wong, Extended investigation of the twelve-flavor β-function, Phys. Lett. B **779**, 230 (2018).

[30] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. H. Wong, Is SU(3) gauge theory with 13 massless flavors conformal?, Proc. Sci. LATTICE2018 (**2018**) 198 [arXiv:1811.05024].

[31] Z. Fodor, K. Holland, J. Kuti, and C. H. Wong, Tantalizing dilaton tests from a near-conformal EFT, Proc. Sci. LATTICE2018 (**2019**) 196 [arXiv:1901.06324].

[32] Z. Fodor, K. Holland, J. Kuti, and C. H. Wong, Dilaton EFT from p-regime to RMT in the ϵ-regime, Proc. Sci. LATTICE2019 (**2020**) 246 [arXiv:2002.05163].

[33] T.-W. Chiu, Improved study of the β-function of $SU(3)$ gauge theory with $N_f = 10$ massless domain-wall fermions, Phys. Rev. D **99**, 014507 (2019).

[34] T. Appelquist *et al.* (Lattice Strong Dynamics Collaboration), Nonperturbative investigations of SU(3) gauge theory with eight dynamical flavors, Phys. Rev. D **99**, 014509 (2019).

[35] L. Del Debbio, B. Lucini, A. Patella, C. Pica, and A. Rago, Large volumes and spectroscopy of walking theories, Phys. Rev. D **93**, 054505 (2016).

[36] R. C. Brower, A. Hasenfratz, C. Rebbi, E. Weinberg, and O. Witzel, Composite Higgs model at a conformal fixed point, Phys. Rev. D **93**, 075028 (2016).

[37] T. Appelquist *et al.* (LSD Collaboration), Lattice simulations with eight flavors of domain wall fermions in SU(3) gauge theory, Phys. Rev. D **90**, 114502 (2014).

[38] Y. Aoki *et al.* (LatKMI Collaboration), Light composite scalar in eight-flavor QCD on the lattice, Phys. Rev. D **89**, 111502 (2014).

[39] Y. Aoki *et al.* (LatKMI Collaboration), Light flavor-singlet scalars and walking signals in $N_f = 8$ QCD on the lattice, Phys. Rev. D **96**, 014508 (2017).

[40] O. Witzel, Review on composite Higgs models, Proc. Sci. LATTICE2018 (**2019**) 006 [arXiv:1901.08216].

[41] A. V. Smilga and J. Stern, On the spectral density of Euclidean Dirac operator in QCD, Phys. Lett. B **318**, 531 (1993).

[42] T. Banks and A. Casher, Chiral symmetry breaking in confining theories, Nucl. Phys. **B169**, 103 (1980).

[43] G. Mack, Introduction to conformal invariant quantum field theory in two-dimensions and more dimensions, in *NATO Advanced Summer Institute on Nonperturbative Quantum Field Theory (Cargese Summer Institute)* (1988), p. 8.

[44] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics (Springer-Verlag, New York, 1997).

[45] V. M. Braun, G. P. Korchemsky, and D. Müller, The uses of conformal symmetry in QCD, Prog. Part. Nucl. Phys. **51**, 311 (2003).

[46] H. Osborn, Lectures on conformal field theories in more than two dimensions, https://www.damtp.cam.ac.uk/user/ho/CFTNotes.pdf.

[47] S. Rychkov, EPFL Lectures on Conformal Field Theory in D >= 3 Dimensions, 1st ed., *SpringerBriefs in Physics* (Springer, Cham, 2016).

[48] D. Poland, S. Rychkov, and A. Vichi, The conformal bootstrap: Theory, numerical techniques, and applications, Rev. Mod. Phys. **91**, 015002 (2019).

[49] J. C. Collins, A. Duncan, and S. D. Joglekar, Trace and dilatation anomalies in gauge theories, Phys. Rev. D **16**, 438 (1977).

[50] G. Mack, All unitary ray representations of the conformal group SU(2,2) with positive energy, Commun. Math. Phys. **55**, 1 (1977).

[51] R. Jost, *The General Theory of Quantized Fields* (American Mathematical Society, Providence, RI, 1965), Vol. 4.

[52] K. G. Wilson, Nonlagrangian models of current algebra, Phys. Rev. **179**, 1499 (1969).

[53] L. Del Debbio and R. Zwicky, Hyperscaling relations in mass-deformed conformal gauge theories, Phys. Rev. D **82**, 014502 (2010).

[54] L. Del Debbio and R. Zwicky, Scaling relations for the entire spectrum in mass-deformed conformal gauge theories, Phys. Lett. B **700**, 217 (2011).

[55] A. Patella, A precise determination of the $\bar{\psi}\psi$ anomalous dimension in conformal gauge theories, Phys. Rev. D **86**, 025006 (2012).

[56] L. Del Debbio and R. Zwicky, Conformal scaling and the size of m-hadrons, Phys. Rev. D **89**, 014503 (2014).

[57] R. Marcarelli, N. Miesch, and E. T. Neil, Mass-induced confinement near the sill of the conformal window, Phys. Rev. D **107**, 076011 (2023).

[58] V. A. Miransky, Dynamics in the conformal window in QCD like theories, Phys. Rev. D **59**, 105003 (1999).

[59] A. Manohar and H. Georgi, Chiral quarks and the non-relativistic quark model, Nucl. Phys. **B234**, 189 (1984).

[60] M. Gell-Mann, R. J. Oakes, and B. Renner, Behavior of current divergences under $\mathrm{SU}(3) \times \mathrm{SU}(3)$, Phys. Rev. **175**, 2195 (1968).

[61] R. J. Crewther, Nonperturbative evaluation of the anomalies in low-energy theorems, Phys. Rev. Lett. **28**, 1421 (1972).

[62] M. S. Chanowitz and J. R. Ellis, Canonical anomalies and broken scale invariance, Phys. Lett. **40B**, 397 (1972).

[63] M. S. Chanowitz and J. R. Ellis, Canonical trace anomalies, Phys. Rev. D **7**, 2490 (1973).

[64] P. Minkowski, On the Anomalous Divergence of the Dilatation Current in Gauge Theories, Report No. PRINT-76-0813 (BERN), 1976.

[65] S. L. Adler, J. C. Collins, and A. Duncan, Energy-momentum-tensor trace anomaly in spin $1/2$ quantum electrodynamics, Phys. Rev. D **15**, 1712 (1977).

[66] N. K. Nielsen, The energy momentum tensor in a non-Abelian quark gluon theory, Nucl. Phys. **B120**, 212 (1977).

[67] Clusius, Einführung in die quantenchemie. von h. hellmann. 350 s.,43 abb., 35 tab. franz deuticke, leipzig u. wien 1937. pr. geh. rm. 20,. geb. rm. 22,-, Angew. Chem. **54**, 156 (1941).

[68] R. P. Feynman, Forces in molecules, Phys. Rev. **56**, 340 (1939).

[69] V. Prochazka and R. Zwicky, Gluon condensates from the Hamiltonian formalism, J. Phys. A **47**, 395402 (2014).

[70] J. Gasser and H. Leutwyler, Quark masses, Phys. Rep. **87**, 77 (1982).

[71] L. Del Debbio and R. Zwicky, Renormalisation group, trace anomaly and Feynman–Hellmann theorem, Phys. Lett. B **734**, 107 (2014).

[72] R. J. Crewther and L. C. Tunstall, Status of chiral-scale perturbation theory, Proc. Sci. CD15 (**2015**) 132 [arXiv:1510.01322].

[73] R. Zwicky, The dilaton improves Goldstones, arXiv:2306.12914.

[74] T. Appelquist, J. Ingoldby, and M. Piai, Dilaton EFT framework for lattice data, J. High Energy Phys. 07 (2017) 035.

[75] T. Appelquist, J. Ingoldby, and M. Piai, Dilaton potential and lattice data, Phys. Rev. D **101**, 075025 (2020).

[76] T. A. Ryttov and R. Shrock, Scheme-independent calculation of $\gamma_{\bar{\psi}\psi,IR}$ for an SU(3) gauge theory, Phys. Rev. D **94**, 105014 (2016).

[77] T. A. Ryttov and R. Shrock, Higher-order scheme-independent series expansions of $\gamma_{\bar{\psi}\psi,IR}$ and β'_{IR} in conformal field theories, Phys. Rev. D **95**, 105004 (2017).

[78] K. Yamawaki, M. Bando, and K.-i. Matumoto, Scale invariant technicolor model and a technidilaton, Phys. Rev. Lett. **56**, 1335 (1986).

[79] T. Appelquist, K. D. Lane, and U. Mahanta, On the ladder approximation for spontaneous chiral symmetry breaking, Phys. Rev. Lett. **61**, 1553 (1988).

[80] A. G. Cohen and H. Georgi, Walking beyond the rainbow, Nucl. Phys. **B314**, 7 (1989).

[81] T. Appelquist, M. Piai, and R. Shrock, Fermion masses and mixing in extended technicolor models, Phys. Rev. D **69**, 015002 (2004).

[82] M. Jarvinen and E. Kiritsis, Holographic models for QCD in the Veneziano limit, J. High Energy Phys. 03 (2012) 002.

[83] R. Alvares, N. Evans, and K.-Y. Kim, Holography of the conformal window, Phys. Rev. D **86**, 026008 (2012).

[84] R. J. Crewther and L. C. Tunstall, $\Delta I = 1/2$ rule for kaon decays derived from QCD infrared fixed point, Phys. Rev. D **91**, 034016 (2015).

[85] J. L. Cardy, Is there a c-theorem in four-dimensions?, Phys. Lett. B **215**, 749 (1988).

[86] K. Yamawaki, Dynamical gauge boson of hidden local symmetry within the standard model, arXiv:1803.07271.

[87] Z. Komargodski, Vector mesons and an interpretation of Seiberg duality, J. High Energy Phys. 02 (2011) 019.

[88] S. Abel and J. Barnard, Seiberg duality versus hidden local symmetry, J. High Energy Phys. 05 (2012) 044.

[89] Y.-L. Ma and M. Rho, Towards the hadron–quark continuity via a topology change in compact stars, Prog. Part. Nucl. Phys. **113**, 103791 (2020).

[90] M. Rho and Y.-L. Ma, Manifestation of hidden symmetries in baryonic matter: From finite nuclei to neutron stars, Mod. Phys. Lett. A **36**, 2130012 (2021).

[91] G. E. Brown and M. Rho, Scaling effective Lagrangians in a dense medium, Phys. Rev. Lett. **66**, 2720 (1991).

[92] Z. Komargodski and A. Schwimmer, On renormalization group flows in four dimensions, J. High Energy Phys. 12 (2011) 099.

[93] M. A. Luty, J. Polchinski, and R. Rattazzi, The *a*-theorem and the asymptotics of 4D quantum field theory, J. High Energy Phys. 01 (2013) 152.

[94] T. Appelquist *et al.* (LSD Collaboration), Hidden conformal symmetry from the lattice, Phys. Rev. D **108**, L091505 (2023).

[95] M. Golterman and Y. Shamir, Low-energy effective action for pions and a dilatonic meson, Phys. Rev. D **94**, 054502 (2016).

[96] O. Catà, R. J. Crewther, and L. C. Tunstall, Crawling technicolor, Phys. Rev. D **100**, 095007 (2019).

[97] W. D. Goldberger, B. Grinstein, and W. Skiba, Distinguishing the Higgs boson from the dilaton at the Large Hadron Collider, Phys. Rev. Lett. **100**, 111802 (2008).

[98] B. Bellazzini, C. Csaki, J. Hubisz, J. Serra, and J. Terning, A Higgslike dilaton, Eur. Phys. J. C **73**, 2333 (2013).

[99] Y. Nakayama, Scale invariance vs conformal invariance, Phys. Rep. **569**, 1 (2015).